NUMERICAL APPROXIMATION OF A PARABOLIC PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. We analyze numerical approximations of positive solutions of a heat equation with a nonlinear flux condition which produces blow up of the solutions. By a semidiscretization using finite elements in the space variable we obtain a system of ordinary differential equations which is expected to be an approximation of the original problem. Our objective is to analyze whether this system has a similar behaviour than the original problem. We find a necessary and sufficient condition for blow up of this system. However, this condition is sligthly different than the one known for the original problem, in particular, there are cases in which the continuous problem has blow up while its semidiscrete approximation does not.

Under certain assumptions we also prove that the numerical blow up time converges to the real blow-up time when the meshsize goes to zero.

Our proofs are given in one space dimension. Similar arguments could be applied for higher dimensions but a further analysis is required.

I. Introduction.

For many differential equations (or systems) modelling transient problems the solutions blow up at a finite time (typical examples are problems involving reaction terms) and it is an interesting question what can be said about the corresponding numerical discretizations.

In this work we consider the following model problem involving a nonlinear boundary flux condition:

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial n} = f(u), & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, f is C^2, increasing and positive in $\mathbb{R}_+$, u_0 is $C^{2+\alpha}(\overline{\Omega})$, positive and verifies the compatibility condition $\frac{\partial u_0}{\partial n} = f(u_0)$.

This problem has been extensively analyzed, in particular, existence and uniqueness of a classical solution up to some time T was proved in [5] under the hypothesis above. On the other hand it is known that for each f the existence of global solutions only depends on the behavior of f at infinity. More precisely, Rial and Rossi [7] shows that if f satisfies:

$$\int^{+\infty} \frac{1}{f} < \infty \tag{1.2}$$

(a typical example is $f(u) = u^p$ for $p > 1$) then every positive solution of (1.1) has a finite interval of existence, $[0, T)$, and

$$\limsup_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty$$

In this case we say that the solution $u(x, t)$ blows-up at time T. (see also [4], [8], [9])

As a first step to understand the blow-up properties of numerical approximations of this kind of problems, we analyze here semidiscretizations in space obtained applying the lines method to equation (1.1). Therefore we replace the problem by a system of ordinary differential equations and our objective is to show that this system has a similar behavior to the original problem. We give the proofs for the one dimensional case. Part of our arguments can be easily extended to several variables, as we will point out, provided the numerical solutions satisfy a maximum principle (which is the case, for example, for linear finite elements with triangles having only acute angles). However, there are other points that require a different analysis. This possible extension will be analyzed in the last section.

We prove that the semidiscrete approximation also blows up for any initial positive data provided condition (1.2) is satisfied. We also prove that this condition is necessary to have blow up. It is interesting to note that, when f is convex, (1.2) is not necessary to have blow up in the continuous case. In fact it is known [8] that blow up is equivalent to the following weaker condition on f:

$$\int^{+\infty} \frac{1}{ff'} < \infty$$

Therefore, it follows from our results, that there are examples in which all the positive solutions of the original differential equation blow up but their semidiscrete approximations do not.

When f is convex, we also prove that, under suitable assumptions on the initial data, the blow up time of the approximation converges to the one of the original problem when the meshsize goes to 0.

II. Analysis of the numerical approximations.

We now restrict ourselves to the one dimensional case, i.e., we consider the problem:

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } (0,1)\times(0,T) \\ u_x(1,t) = f(u(1,t)), & \text{in } (0,T) \\ u_x(0,t) = 0, & \text{in } (0,T) \\ u(x,0) = u_0(x), & \text{in } (0,1) \end{cases} \tag{2.1}$$

where u_0 is positive and compatible with the boundary conditions.

Consider a uniform partition of $(0,1)$ of size h and its associated standard piecewise linear finite element space V_h (we take uniform partitions in order to simplify notation but the proofs can be trivially extended to arbitrary partitions). Let $\{\varphi_j\}_{1\leq j\leq N}$ be the usual Lagrange basis of V_h. The semidiscrete approximation $u_h(t)\in V_h$ obtained by the finite element method with mass lumping is defined by:

$$\int_0^1 (u_h' v)^I dx + \int_0^1 u_{hx} v_x dx = f(u(1,t))v(1,t), \qquad \forall v\in V_h, \forall t\in(0,T)$$

where the superindex I denotes the Lagrange interpolation.

Writing,

$$u_h(t) = \sum_{j=1}^N U_j(t)\varphi_j$$

a simple computation shows that U_j satisfy the system of ordinary differential equations:

$$\begin{cases} U_1' = \frac{2(U_2-U_1)}{h^2} \\ U_i' = \frac{(U_{i+1}-2U_i+U_{i-1})}{h^2} & (i=2,...,N-1) \\ U_N' = \frac{2(U_{N-1}-U_N)}{h^2} + \frac{2f(U_N)}{h} \end{cases} \tag{2.2}$$

In the following two lemmas we give minimum and comparison principles and in the third one we give a result concerning monotonicity of the numerical solutions. Their proofs are standard but we outline them for the sake of completeness.

Lemma 1. *(Minimum principle) If U satisfies,*

$$\begin{cases} U_1' \geq \frac{2(U_2-U_1)}{h^2} \\ U_i' \geq \frac{(U_{i+1}-2U_i+U_{i-1})}{h^2} & (i=2,...,N-1) \\ U_N' \geq \frac{2(U_{N-1}-U_N)}{h^2} + \frac{2f(U_N)}{h} \end{cases} \tag{2.3}$$

then, it attains its minimum at $t=0$. In particular u_h is positive if the initial data is.

Proof. Let

$$0 < \beta < \delta = \min_{i=1,\dots,N} (U_0)_i$$

and suppose that there exists a first time t_0 such that $U_i(t_0) = \beta$ for some $i = 1,...,N$. Then $U_i' \leq 0$, from (2.3) we obtain that, if $i\neq N$, $U_i = U_{i-1} = U_{i+1}$

$(i \neq 1)$ or $U_1 = U_2$ and so on (all the U_i must be equal to β at t_0 and $U_i(t_0)' \leq 0$). But at $i = N$ we have that $U_N' \geq \frac{2f(U_N)}{h} > 0$ a contradiction. $\Box$

Lemma 2. *(Comparison principle) Let U be a supersolution (i.e., a solution of (2.3)) and W be a solution of system (2.2) such that: $U_i(0) > W_i(0)$ for $i = 1, ..., N$. Then $U_i(t) > W_i(t)$ for $i = 1, ..., N$ $\forall t$.*

Proof. The argument is similar to the one used in the proof of Lemma 1. Let

$$0 < \beta < \delta = \min_{i=1,\dots,N} (U_i(0) - W_i(0))$$

and suppose that there exists a first time t_0 such that $U_i(t_0) - W_i(t_0) = \beta$ for some $i = 1, ..., N$. We have that $(U_i - W_i)' \leq 0$; from (2.3) we obtain that, if $i \neq N$, $U_i - W_i = U_{i-1} - W_{i-1} = U_{i+1} - W_{i+1}$ $(i \neq 1)$ or $U_1 - W_1 = U_2 - W_2$ and so on (all the $U_i - W_i$ must be equal to β and $(U_i - W_i)' \leq 0$ at t_0). But at $i = N$ we get that $(U_N - W_N)' \geq 2\left(fracf(U_N)h - \frac{f(W_N)}{h}\right) > 0$ a contradiction. $\Box$

Lemma 3. *Let U be a solution of system (2.2) such that $U_i(0) > U_{i-1}(0)$ $(i = 2, ..., N)$. Then $U_i(t) > U_{i-1}(t)$ for all t. (i.e., if $u_h(0)$ is an increasing function of x then $u_h(t)$ results also increasing in x for $t > 0$).*

Proof. Suppose that the result is false and let t_0 be the first time such that $U_i(t_0) - U_{i-1}(t_0) = 0$ for some $i = 1, ..., N$. Then $(U_i - U_{i-1})' \leq 0$; from (2.2) we obtain that, if $i \neq N$, $U_i = U_{i-1} = U_{i+1}$ $(i \neq 1)$ or $U_1 = U_2$. But at $i = N$ we get that $(U_N - U_{N-1})' \geq \frac{2f(U_N)}{h} > 0$ a contradiction (t_0 is the first time such that $U_N - U_{N-1} = 0$ and hence $U_N - U_{N-1}$ has a minimum there). $\Box$

The following Theorem gives the main result of this section. Its proof is a discrete analogous of the one given in [7] for the continuous case.

Theorem 1. *Every positive solution of (2.2) blows-up if and only if (1.2) holds.*

Proof. Assume that (1.2) holds and define

$$I(t) = \frac{h}{2} \int_{U_1(t)}^{+\infty} \frac{1}{f(s)}\, ds + \frac{h}{2} \int_{U_N(t)}^{+\infty} \frac{1}{f(s)}\, ds + h \sum_{j=2}^{N-1} \int_{U_j(t)}^{+\infty} \frac{1}{f(s)}\, ds$$

We observe that $I(t)$ is well defined (thanks to (1.2)) and is positive as long as $u_h(t)$ exists. We are going to see that if $u_h(t)$ were defined for every t then $I(t)$ should vanish for some t, which is a contradiction. With this goal we compute the derivative of I. We have,

$$I'(t) = -h \sum_{j=2}^{N-1} \frac{U_j'(t)}{f(U_j(t))} - \frac{h}{2} \frac{U_1'(t)}{f(U_1(t))} - \frac{h}{2} \frac{U_N'(t)}{f(U_N(t))} =$$

$$= -1 - \frac{(U_{N-1}(t) - U_N(t))}{hf(U_N(t))} - \frac{(U_2(t) - U_1(t))}{hf(U_1(t))} -$$

$$- \sum_{j=2}^{N-1} \frac{(U_{j+1}(t) - U_j(t)) - (U_j(t) - U_{j-1}(t))}{hf(U_j(t))}$$

Now we observe that,

$$\sum_{j=2}^{N-1} (a_{j+1} - a_j) b_j = a_N b_N - a_2 b_2 - \sum_{j=2}^{N-1} a_{j+1}(b_{j+1} - b_j)$$

and taking $a_i = U_i - U_{i-1}$ and $b_i = 1/f(U_i)$ we obtain,

$$I'(t) = -1 + \sum_{j=1}^{N-1} \frac{U_{j+1}(t) - U_j(t)}{h} \left(\frac{1}{f(U_{j+1}(t))} - \frac{1}{f(U_j(t))}\right)$$

but, since f is monotone increasing and positive,

$$\frac{U_{j+1}(t) - U_j(t)}{h} \left(\frac{1}{f(U_{j+1}(t))} - \frac{1}{f(U_j(t))}\right) =$$

$$\left(\frac{(U_{j+1}(t) - U_j(t))^2}{h}\right) \left(\frac{(f(U_j(t)) - f(U_{j+1}(t))}{(U_{j+1}(t) - U_j(t))}\right) \left(\frac{1}{f(U_{j+1}(t)) f(U_j(t))}\right) \leq 0$$

and hence,

$$I'(t) \leq -1$$

Since $I(t)$ is positive, the solution can exist at most up to time $t = I(0)$.

Now suppose that,

$$\int^{+\infty} \frac{1}{f} = +\infty$$

Assume first that u_h is increasing, then (thanks to Lemma 3) the maximum of u_h at time t is attained at U_N. Now we observe that U_N verifies

$$U_N' \leq \frac{2f(U_N)}{h}$$

and hence U_N can not blow-up because of the divergence of the integral of $1/f$. For a general initial data $W_i(0)$, $i = 1, .., N$ we can use the comparison Lemma (Lemma 2) with a solution u_h such that $U_i(0) > W_i(0)$ and $u_h(0)$ increasing. $\square$

III. Convergence of the blow up time.

When f is convex, and the initial data satisfies,

$$(u_0)_{xx} \geq \alpha > 0 \tag{3.1}$$

it is known that there exists $\delta > 0$ such that the solution of the differential equation (2.1) satisfies, $u_t \geq \delta f(u)$ (see [1]).

From this last inequality a bound for $T-t$ can be easily obtained, indeed, integrating we have,

$$\int_t^T \frac{u_t}{f(u)} \geq \delta(T-t)$$

therefore, changing variable we get

$$\int_{u(x,t)}^{+\infty} \frac{1}{f(s)} ds \geq \delta(T-t) \qquad , \qquad \forall x \in [0,1])$$

and so,

$$(T-t) \leq \frac{1}{\delta} \int_{\|u(\cdot,t)\|_{L^\infty}}^{+\infty} \frac{1}{f(s)} ds \tag{3.2}$$

This last inequality implies that if $\|u(\cdot,t)\|_{L^\infty}$ is large enough then t is close to the blow-up time.

In the next Lemma we obtain a similar estimate for the semi-discrete approximation which, in particular, shows in another way that the semidiscrete solution blows-up in finite time. Moreover, we will show that this inequality can be used to prove that the discrete blow-up time converge to T when the mesh size goes to zero.

Lemma 4. *If U is a solution of (2.2) with f convex and U_0 satisfies*

$$\begin{cases} \frac{2(U_{0,2}-U_{0,1})}{h^2} \geq \beta > 0 \\ \frac{(U_{0,i+1}-2U_{0,i}+U_{0,i-1})}{h^2} \geq \beta > 0 \qquad (i=2,...,N-1) \\ \frac{2(U_{0,N-1}-U_{0,N})}{h^2} + \frac{2f(U_{0,N})}{h} \geq \beta > 0 \end{cases} \tag{3.3}$$

β independent of h, then there exists $\gamma > 0$, independent of h, such that

$$U_i' \geq \gamma f(U_i) \qquad (i=1,...,N)$$

__Proof__. Let $W_i = U_i' - \gamma f(U_i)$. We want to use the minimum principle to show that W_i is positive for every $i = 1, ..., N$ if γ is small enough. Using the fact that f is convex we have,

$$W_1' + \frac{2(W_1-W_2)}{h^2} = U_1'' - \gamma f'(U_1)U_1' + \frac{2(U_1'-U_2')}{h^2} - \gamma \frac{2(f(U_1)-f(U_2))}{h^2} =$$

$$= \frac{2(U_2'-U_1')}{h^2} - \gamma f'(U_1)U_1' + \frac{2(U_1'-U_2')}{h^2} - \gamma \frac{2(f(U_1)-f(U_2))}{h^2} =$$

$$= -\frac{2\gamma}{h^2}\{f'(U_1)(U_2-U_1) - (f(U_2)-f(U_1))\} \geq 0$$

And for $i = 2, ..., N-1$,

$$W_i' - \frac{W_{i+1}-2W_i+W_{i-1}}{h^2} =$$

$$= U_i'' - \gamma f'(U_i)U_i' - \frac{U_{i+1}' - 2U_i' + U_{i-1}'}{h^2} + \gamma \frac{f(U_{i+1}) - 2f(U_i) + f(U_{i-1})}{h^2} =$$

$$= -\frac{\gamma}{h^2}(f'(U_i)(U_{i+1} - 2U_i + U_{i-1}) - f(U_{i+1}) + 2f(U_i) - f(U_{i-1})) \geq 0$$

Finally for $i = N$ we get,

$$W_N' + \frac{2(W_N - W_{N-1})}{h^2} =$$

$$= U_N'' - \gamma f'(U_N)U_N' + \frac{2(U_N' - U_{N-1}')}{h^2} - \gamma \frac{2(f(U_N) - f(U_{N-1}))}{h^2} =$$

$$= \frac{2f'(U_N)W_N}{h} + 2\gamma \frac{f'(U_N)(U_N - U_{N-1})}{h^2} - \gamma \frac{2(f(U_N) - f(U_{N-1}))}{h^2} \geq \frac{2f'(U_N)W_N}{h}$$

and as $W_i(0) > 0$ $(i = 1, ..., N)$ for γ small enough (independent of h) we can apply the minimum principle (Lemma 1) to conclude the theorem. $\square$

As a consequence of Lemma 4 and (1.2) we conclude that U has blow-up in finite time, T_h which satisfies the following inequality analogous to (3.2)

$$(T_h - t) \leq \frac{1}{\gamma} \int_{U_i(t)}^{+\infty} \frac{1}{f(s)} ds \quad \forall i \tag{3.4}$$

From (3.2) and (3.4) it follows that both the exact and approximate solutions blow up in a short time if they are large enough. Therefore, if they are close while u is regular they will blow up at close times. So, all what we need to obtain accurately the blow up time is that the numerical method provides a good approximation to the exact solution up to large enough values of u. In other words, for any $\tau > 0$ we want that $u_h \to u$ (when $h \to 0$) uniformly in $[0, T - \tau]$. This is a natural requirement since on such an interval the exact solution is regular (see [5]). Approximations of regular problems with a nonlinear flux condition like ours have been analyzed in [3],[6]. In those papers optimal order error estimates in the L^2 and H^1 norms are proven. In particular, uniform convergence can be obtained by using standard inverse inequalities. However, they do not consider numerical integration (i.e. the mass lumping) as we are doing. In the following Theorem we give a proof of the L^2 convergence for a problem like (2.1) with f replaced by a globally Lipschitz function g and considering mass lumping. As a corollary, we will obtain uniform convergence for problem (2.1).

Theorem 2. *Let u be the solution of a problem like (2.1) with f replaced by a globally Lipschitz function g and let u_h its semidiscrete approximation obtained by finite elements with mass lumping. If $u \in C^{2,1}([0, 1] \times [0, T_1])$ for some $T_1 > 0$ then, there exists a constant C depending on u and T_1 such that:*

$$\|u - u_h\|_{L^\infty([0,T_1],L^2)} \leq Ch^2$$

Proof.

First, observe that $e = u - u_h$ satisfies the following error equation,

$$\int_0^1 (e'v)^I \, dx + \int_0^1 e_x v_x \, dx = (g(u(1,t)) - g(u_h(1,t)))v(1,t) + \int_0^1 ((u'v)^I - u'v) \, dx$$

for all $v \in V_h$. Writing

$$e = u - u_h + u^I - u^I = u - u^I + \eta$$

and using known error estimates for Lagrange interpolation it rest to estimate η.

First, it is easy to see that,

$$\int_0^1 (u - u^I)_x v_x \, dx = 0 \qquad \forall v \in V_h$$

and therefore, replacing in the error equation we have,

$$\int_0^1 (\eta' v)^I \, dx + \int_0^1 \eta_x v_x \, dx =$$

$$= (g(u(1,t)) - g(u_h(1,t)))v(1,t) + \int_0^1 ((u'v)^I - u'v) \, dx - \int_0^1 ((u' - (u^I)')v)^I \, dx$$

In particular if we choose $v = \eta \in V_h$ we obtain

$$\frac{1}{2}\frac{d}{dt}(\int_0^1 (\eta^2)^I \, dx) + \int_0^1 (\eta_x)^2 \, dx =$$

$$(g(u^I(1,t)) - g(u_h(1,t)))\eta(1,t)$$

$$+ \int_0^1 ((u'\eta)^I - u'\eta) \, dx =$$

$$= I + II$$

Using the known inequality,

$$|\eta(1,t)|^2 \le C_\varepsilon \|\eta\|^2_{L^2((0,1))} + \varepsilon \|\eta_x\|^2_{L^2((0,1))}$$

and the fact that g is globally Lipschitz we have,

$$|I| \le |(g(u^I(1,t)) - g(u_h(1,t)))\eta(1,t)| \le C|\eta(1,t)|^2 \le$$
$$\le C\|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))}$$

where the notation $L^2((0,1))$ refers to the norm in the x variable for each t (we will use analogous notations for other norms below) and, since ε will be fixed later on, we write C instead of C_ε).

In order to bound II we decompose it in the following form,

$$II = \int_0^1 ((u'\eta)^I - u'\eta)\,dx = \int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx + \int_0^1 ((u')^I\eta - u'\eta)\,dx$$

For each subinterval I_j of the partition we know that,

$$\|((u')^I\eta)^I - (u')^I\eta\|_{L^1(I_j)} \leq Ch^2\|((u')^I\eta)_{xx}\|_{L^1(I_j)} \leq \|((u')^I)_x\eta_x\|_{L^1(I_j)}$$

because $(u')^I$ and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|((u')^I)_x\|_{L^2} \leq C\|u'\|_{H^1}$ we obtain,

$$\begin{aligned}\int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx &\leq Ch^2\int_0^1 |((u')^I)_x||\eta_x|\,dx \\ &\leq Ch^2\|((u')^I)_x\|_{L^2((0,1))}\|\eta_x\|_{L^2((0,1))} \\ &\leq Ch^4\|u'\|^2_{H^1((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))}\end{aligned}$$

It rests to estimate the second sumand of II. We have,

$$\begin{aligned}\int_0^1 ((u')^I - u')\eta\,dx &\leq \|(u')^I - u'\|_{L^2((0,1))}\|\eta\|_{L^2((0,1))} \\ &\leq \|(u')^I - u'\|^2_{L^2((0,1))} + \|\eta\|^2_{L^2((0,1))} \\ &\leq Ch^4\|u'\|^2_{H^2((0,1))} + \|\eta\|^2_{L^2((0,1))}\end{aligned}$$

Collecting all the bounds we obtain,

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I\,dx + \int_0^1 |\eta_x|^2\,dx \\ \leq C\left\{\|u - u^I\|^2_{L^\infty((0,1))} + h^4\|u'\|_{H^2((0,1))} + \|u' - (u^I)'\|_{L^\infty((0,1))}\right\} \\ +C\|\eta\|^2_{L^2((0,1))} + 3\varepsilon\|\eta_x\|^2_{L^2((0,1))}\end{gathered}$$

We choose ε such that $3\varepsilon = 1/2$ and we obtain,

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I\,dx + \int_0^1 |\eta_x|^2\,dx \\ \leq Ch^4\left\{\|u\|^2_{W^{2,\infty}((0,1))} + \|u'\|_{H^2((0,1))}\right\} + C\|\eta\|^2_{L^2((0,1))}\end{gathered}$$

Since

$$\int_0^1 (\eta^2)^I\,dx \sim \|\eta\|^2_{L^2((0,1))}$$

we can apply Gronwall's Lemma to obtain for $t \in [0, T_1]$

$$\|\eta(t)\|_{L^2((0,1))} + (\int_0^{T_1} \|\eta_x\|^2_{L^2((0,1))}\,dt)^{1/2} \leq e^{C(T_1)}\|u\|_{W^{1,\infty}([0,T_1],W^{2,\infty})}h^2$$

In particular,

$$\|\eta\|_{L^2((0,1))} \leq C(u, T_1)h^2$$

and hence,

$$\|e\|_{L^2((0,1))} \leq \|u - u^I\|_{L^2((0,1))} + \|\eta\|_{L^2((0,1))} \leq C(u, T_1)h^2$$

□

Corollary 1. *Let u be the solution of (2.1) with u_0 compatible with the boundary data (i.e., $(u_0)_x(0) = 0$ and $(u_0)_x(1) = f(u_0(1))$) and u_h its approximation defined by (2.2). Given $\tau > 0$ there exists a constant C depending on τ and u such that, for h small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{\frac{3}{2}}$$

Proof. It is known that before the blow up time u is regular, more precisely, $u \in C^{2,1}([0,1] \times [0, T - \tau])$. Let g be a globally Lipschitz function which agrees with f for $u \leq 2M$ where $M = \|u\|_{L^\infty([0,1]\times[0,T-\tau])}$. Let $\overline{u}$ and $\overline{u}_h$ be the exact and approximate solutions of a problem like (2.1) with f replaced by g. By uniqueness $u = \overline{u}$ in $[0,1]\times[0,T-\tau]$. A bound for $\|\overline{u}-\overline{u}_h\|_{L^\infty}$ can be obtained from Theorem 2. Indeed, it is enough to bound $\|\overline{u}^I - \overline{u}_h\|_{L^\infty}$, and using a standard inverse inequality (see [2]) we have,

$$\|\overline{u}^I - \overline{u}_h\|_{L^\infty} \leq Ch^{-\frac{1}{2}}\|\overline{u}^I - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}$$

$$\leq Ch^{-\frac{1}{2}}\{\|\overline{u}^I - \overline{u}\|_{L^\infty([0,T-\tau],L^2)} + C\|\overline{u} - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}\} \leq Ch^{\frac{3}{2}}$$

with C depending on u and the constant in Theorem 2 and so on τ.

Consequently, for h small enough $|\overline{u}_h| \leq 2M$. Therefore $f(\overline{u}_h) = g(\overline{u}_h)$ and so $\overline{u}_h$ is the finite element approximation of u and, by uniqueness $\overline{u}_h = u_h$ which concludes the proof. □

Now we can state the result concerning the convergence of the blow up time.

Theorem 3. *If T and T_h are the blow up times of problems (2.1) and (2.2) respectively, f is convex and u_0 is compatible with the boundary data and satisfies $(u_0)_{xx} \geq \alpha > 0$ then $T_h \to T$ $(h \to 0)$.*

Proof. By using the compatibility of the initial data and the hypothesis on u_{0xx} it is easy to see that the hypothesis of Lemma 4 are satisfied for $u_h(0) = u_0^I$ with $\beta = \alpha/2$, at least for h small enough.

Now, given $\varepsilon > 0$, we choose M large enough to ensure

$$\frac{1}{\gamma}\int_M^{+\infty} \frac{1}{f(s)}ds \leq \frac{\varepsilon}{2}$$

Next we choose $\tau < \frac{\varepsilon}{2}$ such that

$$\|u(T - \tau)\|_{L^\infty((0,1))} \geq 2M$$

If h is small enough then by Theorem 1

$$\|u_h - u\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{3/2} \leq M$$

and hence

$$\|u_h(T - \tau)\|_{L^\infty((0,1))} \geq M$$

and by (3.3)

$$|T_h - T| \leq |T_h - (T - \tau)| + |\tau| < \varepsilon$$

□

IV. Extensions to higher dimensions.

Some of the results obtained in Section III can be extended to higher dimensions. Given a regular family of triangulations of Ω, $\{\mathcal{T}_h\}$, h being the mesh size (see [2]) (we admit curved sides on the boundary in order to cover exactly Ω), we call V_h the space of continuous piecewise linear functions associated with $\mathcal{T}_h$. Then, the semidiscrete finite element approximation of problem (1.1) using mass lumping is $u_h(t) \in V_h$ defined by:

$$\int_\Omega (u_h' v)^I + \int_\Omega \nabla u_h \nabla v = \int_{\partial\Omega} f(u_h) v, \qquad \forall v \in V_h, \forall t \in (0,T)$$

In order to make computations we have to integrate approximately the term on the boundary. We use the trapezoidal rule, or in other words, replace the function integrated on the boundary by its piecewise linear Lagrange interpolation. So u_h is defined by:

$$\int_\Omega (u_h' v)^I + \int_\Omega \nabla u_h \nabla v = \int_{\partial\Omega} (f(u_h) v)^I, \qquad \forall v \in V_h, \forall t \in (0,T)$$

As in the one dimensional case, writing u_h in terms of the standard Lagrange basis of V_h we obtain a system of ordinary differential equations of the following form:

$$\begin{cases} MU' + AU = Bf(U) \\ U(0) = U_0 \end{cases} \tag{4.1}$$

where we denote with $f(U)$ a vector containing $f(U_j)$ in the positions corresponding to the boundary and zeros elsewhere.

From our numerical integrations it follows that M and B are diagonal matrices. Also they satisfy $m_{ii} > 0$ and $b_{ii} \geq 0$. Under these conditions and if A satisfies $a_{ii} > 0$, $a_{ij} \leq 0$ for $i \neq j$ and, $\sum_{i \neq j} a_{ij} = a_{ii}$ the proof of Lemma 4 can be extended to dimension n. As a consequence we obtain (3.3) for this case and, in particular, the system (4.1) has blow up if f is convex and satisfies (1.2) and the initial data satisfies $-AU_0 + Bf(U_0) \geq \beta > 0$. The hypothesis on A are verified in 2d for acute meshes, i.e., meshes such that all their angles are acute.

Therefore, Theorem 3 could be extended straightforward to higher dimensions if we could prove a result similar to Corollary 1 (or at least uniform convergence). Although the arguments given in Theorem 2 can be generalized, we loose a power of h because of the numerical integration on the boundary. Therefore, uniform convergence cannot be obtained by inverse inequalities for $n \geq 2$ and so, uniform convergence should be proved in a different way. On the other hand, if we use a more precise integration rule on the boundary, the matrix B is not be diagonal and we were not able to prove (3.3) in this case.

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