A posteriori error estimates for non conforming approximation of eigenvalue problems

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Abstract. We consider the approximation of eigenvalue problem for the laplacian by the Crouzeix-Raviart non conforming finite elements in two and three dimensions.

Extending known techniques for source problems, we introduce a posteriori error estimators for eigenvectors and eigenvalues. We prove that the error estimator is equivalent to the energy norm of the eigenvector error up to higher order terms. Moreover, we prove that our estimator provides an upper bound for the error in the approximation of the first eigenvalue, also up to higher order terms.

We present numerical examples of an adaptive procedure based on our error estimator in two and three dimensions. These examples show that the error in the adaptive procedure is optimal in terms of the number of degrees of freedom.

1 Introduction

A posteriori error estimates for non conforming Crouzeix-Raviart approximations of second order elliptic problems [8], as well as for the closely related (see [5, 13]) Raviart-Thomas mixed method of lowest order [14], have been developed and analyzed in several papers. The first results proving the equivalence between the error and a residual type estimator were based on the use of a Helmholtz type decomposition of the error [9]. See also [3, 7, 11] where similar techniques were applied for mixed finite element approximations.

A slightly different argument avoiding the use of the Helmholtz decomposition was introduced in [12] and further developed in [2] to obtain upper estimators without involving unknown constants. The goal of this paper is to extend this approach to the case of eigenvalue problems. For simplicity we consider the Laplace operator although similar arguments can be applied to more general second order elliptic problems.

For $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a polygonal or polyhedral domain, our model problem is

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

As it is well known, this problem has a sequence of eigenpairs (λ_j, u_j), with positive eigenvalues λ_j diverging to $+\infty$.

Given a family $\{\mathcal{T}_h\}$, $0 < h < h_0$ of triangulations of Ω made of triangles or tetrahedra, we define $h = \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. We assume that we have a family of triangulations which is regular in the classic sense, i. e., $h_T/\rho_T \leq \sigma$, where ρ_T is the diameter of the largest ball contained in T and σ is a positive constant. For a face (*resp. edge in the 2d case*) F of an element T we denote with h_F its diameter. Given T_1 and T_2 in $\mathcal{T}_h$ such that $T_1 \cap T_2 = F$ and a function $v \in L^2(T_1 \cup T_2)$ such that $v|_{T_i} \in H^1(T_i)$ we define $[v]_F$ as the jump of v across F (with an arbitrary election of the sign). Moreover, for a face (*resp. edge*) F contained in $\partial\Omega$ we set $[v]_F = 0$. Finally, $\mathcal{F}_h$ denotes the set of all the faces (*resp. edges*) of elements in $\mathcal{T}_h$.

Now, associated with a triangulation $\mathcal{T}_h$, the Crouzeix-Raviart non conforming finite element space V_h^{NC} is defined as

$$V_h^{NC} = \left\{ v \in L^2(\Omega) : \ v|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_h \quad \text{and} \quad \int_F [v]_F = 0 \ \forall F \in \mathcal{F}_h \right\}$$

where we have used the standard notation $\mathcal{P}_1(T)$ for affine functions on T.

In our analysis we will also make use of the standard conforming $\mathcal{P}_1$-elements associated with the triangulation $\mathcal{T}_h$. We denote this space by V_h^C.

The Crouzeix-Raviart finite element approximation of problem (1.1) is given by

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \lambda_h \int_\Omega u_h \, v_h \quad \forall v_h \in V_h^{NC} \tag{1.2}$$

where

$$\nabla_h u_h|_T := \nabla(u_h|_T).$$

The rest of the paper is as follows. In Section 2 we explain the ideas leading to the definition of the estimator and state one of the main results concerning the error estimation for the eigenvectors approximation. The proof of this result is given in Section 3. Also in that section we introduced a locally computable error estimator based on a postprocessing of the numerical solution and prove the reliability of this estimator. In Section 4 we prove an a posteriori error estimate for the approximation of the first eigenvalue. Section 5 deals with the efficiency of the estimator. We conclude the paper giving some numerical examples in Section 6.

2 Motivation and definition of the error estimator

Let us give a heuristic idea for the definition of our error estimator in the case of the first eigenvalue $\lambda = \lambda_1$. We will use standard notations for Sobolev norms and we will denote with $\|u\|$ the L^2-norm of u and analogously for vector fields. The letter C will denote a generic constant which can change from line to line and may depend on the regularity of the meshes (i.e., on the constant σ defined in the previous section).

The first eigenvalue is given by

$$\lambda = \inf_{v \in H_0^1(\Omega)} \frac{\|\nabla v\|^2}{\|v\|^2} \tag{2.3}$$

In many cases the Crouzeix-Raviart approximation provides lower bounds of the eigenvalues. Indeed, it was proved in [4] that, for singular eigenfunctions, $\lambda_h \leq \lambda$ for h small enough. Let us give here an argument which is simpler than that given in [4]. With this goal we will make use of the edge average interpolant of u [8], $u_I \in V_h^{NC}$ given by

$$\int_F u_I = \int_F u \quad \forall F \in \mathcal{F}_h.$$

It is well known, and easy to check, that $\nabla_h u_I$ is the L^2-projection of ∇u onto the piecewise constant vector fields, and therefore,

$$\|\nabla_h u_I\| \leq \|\nabla u\| \tag{2.4}$$

Take u as the positive eigenfunction associated with λ normalized such that $\|u\| = 1$. Then, using (2.3) and (2.4) (which actually is a strict inequality because u is not in the finite element space), we have

$$\lambda_h = \inf_{v \in V_h^{NC}} \frac{\|\nabla_h v\|^2}{\|v\|^2} \leq \frac{\|\nabla_h u_I\|^2}{\|u_I\|^2} < \frac{\|\nabla u\|^2}{\|u_I\|^2} = \lambda \left(1 - \frac{\|u_I\|^2 - \|u\|^2}{\|u_I\|^2}\right) \tag{2.5}$$

Now, it is known (see [10]) that, for any polygonal or polyhedral domain Ω, there exists some $p > 1$ and a constant C depending only on Ω and p such that

$$\|u\|_{W^{2,p}} \leq C\lambda \|u\|_{L^p}$$

in particular,

$$\|u\|_{W^{2,1}} \leq C$$

for a constant C which depends on λ, p and Ω. Therefore, standard arguments give

$$\|u - u_I\|_{L^1} \leq Ch^2$$

Now, recall that $u \in L^\infty(\Omega)$. Indeed, for $d = 2$, this follows from a Sobolev imbedding theorem since $u \in W^{2,p}(\Omega)$, for some $p > 1$. On the other hand for $d = 3$ the boundedness of u follows easily using that $u \in L^2(\Omega)$ and the known estimate for the Green function in Lipschitz domains $G(x, y) \leq C|x - y|^{2-d}$ (see for example [6]). Then,

$$\left| \|u_I\|^2 - \|u\|^2 \right| = \left| \int_\Omega (u_I - u)(u_I + u) \right| \leq C\|u\|_{L^\infty} \|u_I - u\|_{L^1} \leq Ch^2$$

where we have used that $\|u_I\|_{L^\infty} \leq C\|u\|_{L^\infty}$. Therefore, it follows from (2.5), using also that $\|u_I\| \to 1$ when $h \to 0$, that

$$\lambda_h < \lambda \left(1 - O(h^2)\right)$$

But, for the singular eigenfunctions u arising when the polygonal or polyhedral domain is not convex, we have

$$|\lambda - \lambda_h| = O(h^{2r}) \qquad r < 1$$

and then

$$\lambda_h \leq \lambda$$

for h small enough.

On the other hand, upper bounds for the first eigenvalue are easy to find. Indeed, taking any $v \in H_0^1(\Omega)$ such that $\|v\| = 1$, we obtain from (2.3) that

$$\lambda \leq \tilde{\lambda} := \|\nabla v\|^2 \tag{2.6}$$

So, if $\lambda_h \leq \lambda$, we would have the following explicit bound for the eigenvalue error,

$$0 \leq \lambda - \lambda_h \leq \tilde{\lambda} - \lambda_h = \|\nabla v\|^2 - \|\nabla_h u_h\|^2$$

$$= \|\nabla v - \nabla_h u_h\|^2 + 2 \int_\Omega \nabla_h u_h \nabla_h (v - u_h)$$

In particular we can choose $v \in V_h^C$. In this case, using (1.2) and that $\|v\| = \|u_h\| = 1$, it is easy to see that

$$2 \int_\Omega \nabla_h u_h \nabla_h (v - u_h) = -\lambda_h \|v - u_h\|^2$$

Then, using that $\lambda_h \geq 0$, we obtain

$$0 \leq \lambda - \lambda_h \leq \|\nabla v - \nabla_h u_h\|^2$$

and therefore, since $v \in V_h^C$ with $\|v\| = 1$ is arbitrary, we conclude that

$$0 \leq \lambda - \lambda_h \leq d(\nabla_h u_h, \nabla V_{h,\|v\|=1}^C)^2 \tag{2.7}$$

where

$$\nabla V_h^C = \{G \in L^2(\Omega)^d \, : \, G = \nabla v \ , \quad \text{for some} \quad v \in V_h^C\}$$

and $\nabla V_{h,\|v\|=1}^C$ is the subset of ∇V_h^C such that v can be taken with $\|v\| = 1$.

On the other hand, using analogous notations with V_h^C replaced by H_0^1, we have

$$d(\nabla_h u_h, \nabla H_{0,\|v\|=1}^1)^2 \leq \|\nabla u - \nabla_h u_h\|^2 \sim |\lambda - \lambda_h|$$

where the last equivalence is known from a priori error estimates.

In conclusion, if $d(\nabla_h u_h, \nabla H_{0,\|v\|=1}^1)^2$ and $d(\nabla_h u_h, \nabla V_{h,\|v\|=1}^C)^2$ are of the same order (we will show that this is the case!), any of them seem to be reasonable estimators.

Afterwards, in order to obtain a computable estimator one can bound this distances by the distance to an appropriate function constructed by post-processing the discrete solution u_h (a procedure already used in [2, 12]).

Unfortunately, as far as we know, it is not known whether $\lambda_h \leq \lambda$ is always true (this was only proved for singular eigenvectors and h small enough). Therefore, our heuristic argument cannot be formalized.

However, we are able to prove a slightly weaker result which shows that the proposed estimator is correct if we add appropriate element interior residual terms.

We will use the following well known results. For $w \in H^1(T)$,

$$\|w - w_I\|_{L^2(T)} \leq C_1 h_T \|\nabla w\|_{L^2(T)} \tag{2.8}$$

and, for $w \in H_0^1(\Omega)$,

$$\|w\|_{L^2(\Omega)} \leq C_2 \|\nabla w\|_{L^2(\Omega)} \tag{2.9}$$

If λ is any eigenvalue of the continuous problem (1.1) with eigenfunction u and λ_h and u_h are the corresponding discrete approximations defined by (1.2) then, we have

Theorem 2.1 *If C_1 and C_2 are the constants in (2.8) and (2.9), then*

$$\|\nabla_h e\|_{L^2(\Omega)} \leq d(\nabla_h u_h, \nabla H_0^1) + C_1 \left\{ \sum_T h_T^2 \|\lambda_h u_h\|_{L^2(T)}^2 \right\}^{\frac{1}{2}} + h.o.t.$$

where

$$|h.o.t.| \leq C_2 \left\{ (\lambda - \lambda_h) + (\lambda \lambda_h)^{\frac{1}{2}} \|e\|_{L^2(\Omega)} \right\}$$

The proof of this theorem will be given in the following section.

Remark 2.1 *Observe that we have replaced $d(\nabla_h u_h, \nabla H^1_{0,\|v\|=1})$ by $d(\nabla_h u_h, \nabla H_0^1)$, which is better from a practical point of view.*

Remark 2.2 *According to known a priori estimates, the term h.o.t. given in Theorem 2.1 is a higher order term.*

Remark 2.3 *It is known that C_1 is independent of the element shape (see for example [1]). Therefore, the constants in the a posteriori error estimate given in the theorem depend only on Ω. In particular they are independent of the elements shape.*

Remark 2.4 *The argument used in (2.6) cannot be applied for other eigenvalues and this is why part of our analysis is restricted to the approximation of the first one. It would be possible to obtain upper bounds for other eigenvalues using the min-max characterization. However, this generalization is not straightforward and it will be the subject of further research.*

3 Error estimates for the eigenfunctions

Let u, $\|u\| = 1$, be an eigenfunction of the continuous problem (1.1) and u_h, $\|u_h\| = 1$, a corresponding solution of (1.2) (i. e., u_h is an approximation of u). The goal of this section is to estimate the error $e := u - u_h$.

For an element $T \in \mathcal{T}_h$, $\mathcal{F}_T$ denotes the set of faces (*resp. edge in the 2d case*) of $T \in \mathcal{T}_h$ which are not on $\partial\Omega$. For $F \in \mathcal{F}_T$ we introduce the jump of the normal derivative of u_h across F, $J_F^n := \left[\frac{\partial u_h}{\partial n}\right]$ (where we have eliminated the dependence on h to simplify notation).

Let

$$P : L^2(\Omega)^d \to \nabla H^1_0(\Omega)$$

be the L^2-orthogonal projection. Then, if $P(\nabla_h e) = \nabla\tilde{e}$ with $\tilde{e} \in H^1_0(\Omega)$, we have

$$\|\nabla_h e\|^2_{L^2(\Omega)} = \|\nabla_h u_h - P(\nabla_h u_h)\|^2_{L^2(\Omega)} + \|\nabla\tilde{e}\|^2_{L^2(\Omega)} \tag{3.10}$$

where we have used that $P(\nabla u) = \nabla u$.

The main part of the error analysis is the estimate for $\|\nabla\tilde{e}\|^2_{L^2(\Omega)}$ which is given in the next lemma. We define the local and global error estimators as follows,

$$\eta_T^2 = h_T^2 \|\lambda_h u_h\|^2_{L^2(T)} \quad , \quad \eta^2 = \sum_T \eta_T^2 \tag{3.11}$$

Lemma 3.1 *If C_1 and C_2 are the constants in (2.8) and (2.9), then*

$$\|\nabla\tilde{e}\|_{L^2(\Omega)} \leq C_1\eta + C_2\{(\lambda - \lambda_h) + (\lambda\lambda_h)^{\frac{1}{2}}\|e\|_{L^2(\Omega)}\}$$

Proof. Using (1.1), (1.2) and integrating by parts element by element we obtain, for any $w \in H^1_0(\Omega)$,

$$\int_\Omega P(\nabla_h e)\cdot\nabla w = \int_\Omega \nabla_h e\cdot\nabla w = \sum_T \left\{\int_T \lambda u w + \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F J_F^n w\right\}.$$

Then, taking $w = \tilde{e}$ and recalling that $\nabla\tilde{e} = P(\nabla_h e)$, we have

$$\|\nabla\tilde{e}\|^2_{L^2(\Omega)} = \sum_T \left\{\int_T \lambda u\tilde{e} + \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F J_F^n\tilde{e}\right\}.$$

On the other hand, integrating by parts on each element in (1.2) we obtain, for any $v_h \in V_h^{NC}$,

$$\sum_T \left\{\int_T \lambda_h u_h v_h + \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F J_F^n v_h\right\} = 0$$

and then,

$$\|\nabla\tilde{e}\|^2_{L^2(\Omega)} = \sum_T \left\{\int_T (\lambda u\tilde{e} - \lambda_h u_h v_h) + \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F J_F^n(\tilde{e} - v_h)\right\}.$$

Choosing $v_h = \tilde{e}_I$, the last term on the right hand side vanishes and therefore,

$$\|\nabla\tilde{e}\|^2_{L^2(\Omega)} = \sum_T \int_T (\lambda u\tilde{e} - \lambda_h u_h\tilde{e}_I).$$

Consequently,

$$\|\nabla\tilde{e}\|^2_{L^2(\Omega)} = \int_\Omega (\lambda u - \lambda_h u_h)\tilde{e} + \int_\Omega \lambda_h u_h(\tilde{e} - \tilde{e}_I)$$

and using (2.8) we obtain

$$\|\nabla\tilde{e}\|^2_{L^2(\Omega)} \leq \|\lambda u - \lambda_h u_h\|_{L^2(\Omega)}\|\tilde{e}\|_{L^2(\Omega)} + C_1\eta\|\nabla\tilde{e}\|_{L^2(\Omega)}.$$

Then, using now (2.9) we conclude that

$$\|\nabla\tilde{e}\|_{L^2(\Omega)} \le C_1\eta + C_2\|\lambda u - \lambda_h u_h\|_{L^2(\Omega)}.$$

But,

$$\|\lambda u - \lambda_h u_h\|^2_{L^2(\Omega)} = \lambda^2 + \lambda_h^2 - 2\lambda\lambda_h \int_\Omega u u_h$$

and using $2\int u u_h = 2 - \|e\|^2_{L^2(\Omega)}$ we obtain

$$\|\lambda u - \lambda_h u_h\|^2_{L^2(\Omega)} = (\lambda - \lambda_h)^2 + \lambda\lambda_h\|e\|^2_{L^2(\Omega)}$$

concluding the proof. □

We can now give the proof of the theorem providing the upper bound of the error.

Proof of Theorem 2.1. The result follows immediately from the previous lemma and the decomposition (3.10), observing that

$$\|\nabla_h u_h - P(\nabla_h u_h)\|_{L^2(\Omega)} = d(\nabla_h u_h, \nabla H_0^1). \qquad \square$$

Now, we want to introduce a computable error estimator for the eigenvector approximation. In view of the previous theorem it is enough to find a good estimate for the term $d(\nabla_h u_h, \nabla H_0^1)$. Extending the ideas of [12, 2] we construct an approximation $\tilde{u}_h \in V_h^C$ of u by postprocessing u_h.

It is enough to define $\tilde{u}_h$ at the vertices of the triangulation. A natural way to define $\tilde{u}_h$ is by averaging the values of u_h. Namely, for each interior vertex P we consider all the elements T_i, $i = 1, \dots, N$ containing P (where N depends on P) and define

$$\tilde{u}_h(P) = \sum_{i=1}^{N} w_i u_h|_{T_i}(P)$$

where the weights w_i are such that $\sum_{i=1}^N w_i = 1$. For example, we can take

$$w_i = \frac{1}{N} \qquad \text{or} \qquad w_i = \frac{|T_i|}{|\Omega_P|}$$

with $\Omega_P = \cup_{i=1}^N T_i$. If P is a boundary vertex we set $\tilde{u}_h(P) = 0$. Define now

$$\mu_T^2 = \|\nabla\tilde{u}_h - \nabla u_h\|^2_{L^2(T)} \quad , \quad \mu^2 = \sum_T \mu_T^2 \tag{3.12}$$

Then, the following theorem is an immediate consequence of Theorem 2.1.

Theorem 3.2 *If C_1 and C_2 are the constants in (2.8) and (2.9), then*

$$\|\nabla_h e\|_{L^2(\Omega)} \le \mu + C_1\eta + h.o.t.$$

with

$$|h.o.t.| \le C_2\left\{(\lambda - \lambda_h) + (\lambda\lambda_h)^{\frac{1}{2}}\|e\|_{L^2(\Omega)}\right\}.$$

4 Error estimates for the first eigenvalue

In this section we prove an a posteriori error estimate for the error $|\lambda_h - \lambda|$ in the case of the first eigenvalue.

Lemma 4.1 *For the case $\lambda = \lambda_1$ we have*

$$|\lambda_h - \lambda| \le 2\|\nabla_h e\|^2_{L^2(\Omega)} + 2d(\nabla_h u_h, \nabla V^C_{h,\|v\|=1})^2$$

Proof. If $\lambda_h \le \lambda$ we have already proved the stronger estimate (2.7).

So, it only remains to consider the case $\lambda < \lambda_h$. Take $v \in V^C_{h,\|v\|=1}$. Since $V^C_h \subset V^{NC}_h$ we have $\lambda_h \le \|\nabla v\|^2$.

Then, using that $\|u\|_{L^2(\Omega)} = 1$, we have

$$\lambda + \lambda_h \le \|\nabla u\|^2 + \|\nabla v\|^2 \le \|\nabla(u - v)\|^2_{L^2(\Omega)} + 2\int_\Omega \nabla u \cdot \nabla v$$

and so, using (1.1), we obtain

$$\lambda + \lambda_h = \|\nabla(u - v)\|^2_{L^2(\Omega)} + 2\lambda \int_\Omega uv = \|\nabla(u - v)\|^2_{L^2(\Omega)} - \lambda\|u - v\|^2_{L^2(\Omega)} + 2\lambda$$

and subtracting 2λ from both sides it follows that

$$\lambda_h - \lambda \le \|\nabla(u - v)\|^2_{L^2(\Omega)}$$

Therefore,

$$\lambda_h - \lambda \le \left(\|\nabla_h(u - u_h)\|_{L^2(\Omega)} + \|\nabla_h(u_h - v)\|_{L^2(\Omega)}\right)^2$$

and, since $v \in V^C_{h,\|v\|=1}$ is arbitrary we conclude the proof. $\square$

The above lemma together with Theorem 2.1 gives the following estimate for the error in the approximation of the first eigenvalue.

Theorem 4.2 *For $\lambda = \lambda_1$ there exists a constant C, which depends only on C_1 defined in (2.8), such that*

$$|\lambda_h - \lambda| \le C\left\{d(\nabla_h u_h, \nabla V^C_{h,\|v\|=1})^2 + \sum_T h_T^2\|\lambda_h u_h\|^2_{L^2(T)}\right\} + h.o.t.$$

with

$$|h.o.t.| \le C\left\{(\lambda - \lambda_h) + (\lambda\lambda_h)^{\frac{1}{2}}\|e\|_{L^2(\Omega)}\right\}^2.$$

Proof. The result follows immediately from Theorem 2.1, Lemma 4.1 and the obvious inequality

$$d(\nabla_h u_h, \nabla H^1_0) \le d(\nabla_h u_h, \nabla V^C_{h,\|v\|=1}). \quad \square$$

5 Efficiency of the error estimator

For positive quantities A and B, $A \sim B$ will mean that the ratio between A and B is bounded by above and below by positive constants. Let us recall that, since we are assuming regularity of the family of meshes, we have $h_F \sim h_T$ for F a face of T and also $h_{T_1} \sim h_{T_2}$, whenever T_1 and T_2 are neighbor elements. We will use these equivalences several times in what follows. Given a vertex P of a face $F \in \mathcal{F}_h$ and a function v we denote with $[v(P)]_F$ the jump of v across F evaluated in P. For an element T, v_T and $v_{h,T}$ mean the restriction to T of v and v_h respectively. Analogous notation will be used for the restriction to a face F.

Lemma 5.1 *Let $v \in H_0^1(\Omega)$ and $v_h \in V_h^{NC}$. If P is a vertex of a face $F \in \mathcal{F}_h$, where $F = T_1 \cap T_2$, we have*

$$|[v_h(P)]_F| \leq \frac{C}{h_F^{d/2-1}} \|\nabla_h(v_h - v)\|_{L^2(T_1 \cup T_2)}. \tag{5.13}$$

Analogously, if $P \in \partial\Omega$ and T is an element containing P,

$$|v_{h,T}(P)| \leq \frac{C}{h_T^{d/2-1}} \|\nabla_h(v_h - v)\|_{L^2(T)} \tag{5.14}$$

Where the constant C depends on the regularity of the elements.

Proof. We consider the case in which P is an interior point. The proof for the other case is analogous. Since $[v_h]_F$ is an affine function, we can see by standard scaling arguments and using the equivalence of norms in finite dimensional spaces, we have

$$[v_h(P)]_F \leq \frac{C}{h_F^{(d-1)/2}} \|[v_h]_F\|_{L^2(F)}. \tag{5.15}$$

Since $[v]_F = 0$, we have

$$[v_h]_F = [v_h - v]_F. \tag{5.16}$$

Define now

$$m_F = \frac{1}{|F|}\int_F (v_{h,T_1} - v) = \frac{1}{|F|}\int_F (v_{h,T_2} - v)$$

and write

$$[v_h - v]_F = (v_{h,T_1} - v)_F - m_F - ((v_{h,T_2} - v)_F - m_F). \tag{5.17}$$

Applying a standard trace theorem we have

$$\|v_{h,T_1} - v - m_F\|_{L^2(F)} \leq C\{h_{T_1}^{-1/2}\|v_h - v - m_F\|_{L^2(T_1)} + h_{T_1}^{1/2}\|\nabla(v_h - v)\|_{L^2(T_1)}\},$$

and by a Poincaré inequality for functions with vanishing mean value on F, we obtain

$$\|v_{h,T_1} - v - m_F\|_{L^2(F)} \leq C h_{T_1}^{1/2}\|\nabla(v_h - v)\|_{L^2(T_1)}.$$

Analogously,

$$\|v_{h,T_2} - v - m_F\|_{L^2(F)} \leq C h_{T_2}^{1/2}\|\nabla(v_h - v)\|_{L^2(T_2)},$$

and therefore, the statement follows from (5.15), (5.16), and (5.17). $\square$

In the next two theorems we prove the so called efficiency of the error estimator. Namely, we prove that both parts of the estimator, defined in (3.11) and (3.12), are bounded by a constant times the error (plus a higher order term in the case of η). Given an element $T \in \mathcal{T}_h$ we denote with T^* the union of all the elements in $\mathcal{T}_h$ sharing a vertex with T.

Theorem 5.2 *For all $v \in H_0^1(\Omega)$ we have*

$$\mu_T \leq C\|\nabla v - \nabla_h u_h\|_{L^2(T^*)}$$

where the constant C depends only on the regularity of the elements. In particular,

$$\mu_T \leq C\|\nabla_h e\|_{L^2(T^*)}$$

Proof. Given $T \in \mathcal{T}_h$, let N_i be the standard conforming Lagrange basis of $\mathcal{P}_1(T)$. calling P_i the vertices of T we have, in T,

$$\nabla(\tilde{u}_h - u_h) = \sum_{i=1}^{d+1} (\tilde{u}_h(P_i) - u_{h,T}(P_i))\, \nabla N_i$$

and, since $\|\nabla N_i\|_{L^2(T)} \leq C h_T^{d/2-1}$, we obtain

$$\|\nabla(\tilde{u}_h - u_h)\|_{L^2(T)} \leq C h_T^{d/2-1} \sum_{i=1}^{d+1} |\tilde{u}_h(P_i) - u_{h,T}(P_i)|. \tag{5.18}$$

Therefore, it is enough to estimate $|\tilde{u}_h(P_i) - u_{h,T}(P_i)|$.

If $P_i \in \partial\Omega$ we have, by definition, that $\tilde{u}_h(P_i) = 0$, and therefore, using (5.14) we obtain

$$|\tilde{u}_h(P) - u_{h,T}(P)| = |u_{h,T}(P)| \leq \frac{C}{h_T^{d/2-1}} \|\nabla_h(u_h - v)\|_{L^2(T)}, \tag{5.19}$$

Consider now a vertex P of T such that $P \notin \partial\Omega$. Recall that Ω_P denotes the union of all the elements containing P. We can numerate these elements, T_i, $i = 0, 1, 2, \cdots, M$, in such a way that $T_0 = T$, T_i and T_{i+1} have a common face for all i, and T_M shares a face with T. Observe that, in the three dimensional case, the T_i are not necessarily all different. However, we can choose the numeration in such a way that M is bounded by a constant which depends only on the regularity of the meshes. We also define $T_0 = T_{M+1} = T$.

Then, we have

$$\tilde{u}_h(P) - u_{h,T}(P) = \sum_{i=0}^{M} w_i (u_{h,T_i}(P) - u_{h,T}(P)),$$

and therefore,

$$|\tilde{u}_h(P) - u_{h,T}(P)| \leq \sum_{i=0}^{M} w_i |u_{h,T_i}(P) - u_{h,T}(P)|.$$

Defining $F_1 = T \cap T_1$ and using (5.13), we have, for all $v \in H_0^1(\Omega)$,

$$|u_{h,T_1}(P) - u_{h,T}(P)| = |[u_h(P)]_{F_1}| \leq \frac{C}{h_{F_1}^{d/2-1}} \|\nabla_h(u_h - v)\|_{L^2(T\cup T_1)}.$$

Analogously, calling now $F_i = T_{i-1} \cap T_i$, we obtain

$$\begin{aligned}|u_{h,T_2}(P) - u_{h,T}(P)| &\leq |u_{h,T_2}(P) - u_{h,T_1}(P)| + |u_{h,T_1}(P) - u_{h,T}(P)| \\ &= |[u_h(P)]_{F_2}| + |[u_h(P)]_{F_1}| \leq \frac{C}{h_{F_2}^{d/2-1}} \|\nabla_h(u_h - v)\|_{L^2(T\cup T_1\cup T_2)},\end{aligned}$$

and in general,

$$|u_{h,T_i}(P) - u_{h,T}(P)| \leq \sum_{j=1}^{i} |[u_h(P)]_{F_j}| \leq \frac{C}{h_{F_i}^{d/2-1}} \|\nabla_h(u_h - v)\|_{L^2(\cup_{j=0}^{i} T_j)}.$$

Consequently,

$$|\tilde{u}_h(P) - u_{h,T}(P)| \leq \frac{C}{h_T^{d/2-1}} \|\nabla_h(u_h - v)\|_{L^2(\Omega_P)},$$

and therefore, using this estimate for all the vertices of T together with (5.18) and (5.19) we conclude the proof. $\square$

Remark 5.1 *Since $\tilde{u}_h \in H_0^1(\Omega)$, we have*

$$d(\nabla_h u_h, \nabla H_0^1(\Omega)) \leq \|\nabla \tilde{u}_h - \nabla_h u_h\|_{L^2(\Omega)} = \mu$$

and so, the result of the previous theorem says that

$$\mu \sim d(\nabla_h u_h, \nabla H_0^1(\Omega)).$$

Therefore, $\tilde{u}_h$ is a reasonable election to define a computable estimator for $d(\nabla_h u_h, \nabla H_0^1(\Omega))$

Theorem 5.3 *There exists a constant C, depending only on the regularity of the elements, such that*

$$\eta_T \leq C\|\nabla_h e\|_{L^2(T)} + h.o.t.,$$

where

$$|h.o.t.| \leq C h_T \|\lambda u - \lambda_h u_h\|_{L^2(T)}$$

Proof. Let $b_T \in H_0^1(T) \cap \mathcal{P}_{d+1}$ be a bubble function which is equal to one at the barycenter of T. By standard arguments we can prove that

$$\|u_h b_T\|_{L^2(T)} \leq \|u_h\|_{L^2(T)} \leq C \left(\int_T |u_h|^2 b_T \right)^{1/2} \tag{5.20}$$

and

$$\|\nabla(u_h b_T)\|_{L^2(T)} \leq \frac{C}{h_T} \|u_h\|_{L^2(T)}. \tag{5.21}$$

Take $v = u_h b_T$. Using (1.1) and that $\int_T \nabla u_h \nabla v = 0$, we obtain

$$\lambda_h \int_T u_h v = \int_T \nabla e \cdot \nabla v - \int_T (\lambda u - \lambda_h u_h) v.$$

Therefore, applying the Schwarz inequality and using (5.20) and (5.21) we conclude the proof. $\square$

6 Numerical examples

Now we present the results obtained with adaptive methods based on the H^1 error estimator defined by

$$\xi^2 = \sum_{T \in \mathcal{T}_k} \xi_T^2,$$

where $\xi_T^2 = \eta_T^2 + \mu_T^2$. We have used the following standard adaptive procedure: start with an initial quasi uniform mesh $\mathcal{T}_0$ and compute the approximate solution u_0. Then, given a solution u_k corresponding to a mesh $\mathcal{T}_k$, the following mesh is obtained by refining those elements T such that the error indicator ξ_T satisfies

$$\xi_T \geq \gamma\, \xi_{max}\,, \qquad \text{with} \qquad \xi_{max} = \max_{T \in \mathcal{T}_k} \xi_T$$

for some fixed constant γ (we have taken γ=0.7).

In the two dimensional case we use the refinement propagation method given by Rivara [15] which guarantee that, at every step, the minimum angle is always greater than or equal 0.5 times the minimum angle of the starting mesh. In 3-d, the mesh $\mathcal{T}_{k+1}$ is obtained from the $\mathcal{T}_k$ using a recursive largest edge partition procedure that limits the propagation of the refinement [16].

6.1 Two dimensional example

We start solving the problem (1.1) in the classical L-shaped domain, namely,

$$\Omega = [-1,1] \times [-1,1] \setminus [0,1] \times [-1,0].$$

The first eigenvalue is $\lambda_1 \simeq 9.64$ as computed numerically using a very refined mesh. In figure 1 the domain with the initial mesh and the mesh obtained after 8 steps are presented.

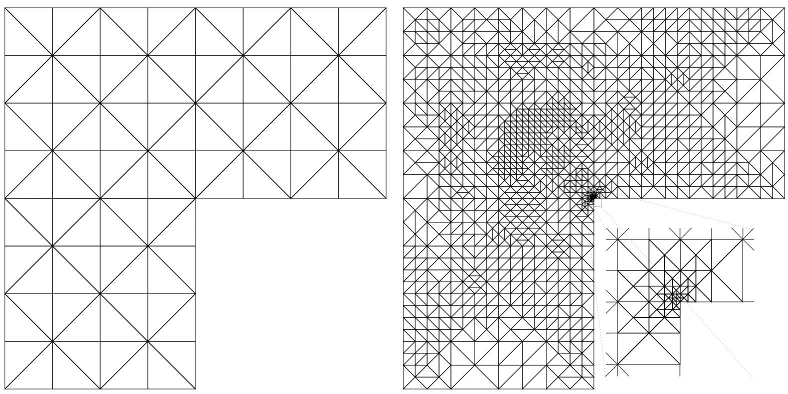

Figure 1: Domain, inital mesh, mesh after 8 adaption steps, and zoom (10x)

Figure 2 shows the eigenvalues obtained in each mesh of the adaptive procedure, together with the Rayleigh quotient for the function $\tilde{u}_h$ defined in Section 3, plotted against the number of unknowns N. We can see that these approximations provide suitable upper and lower bounds for the extrapolated eigenvalue.

In figure 3 we show the convergence of the adaptive procedure by showing the squared global estimator ξ^2 and the error $\lambda - \lambda_h$ against the number of unknowns. It is also shown that the two parts η^2 and μ^2 achieve the same optimal order of convergence N^{-1} as the total estimator ξ^2.

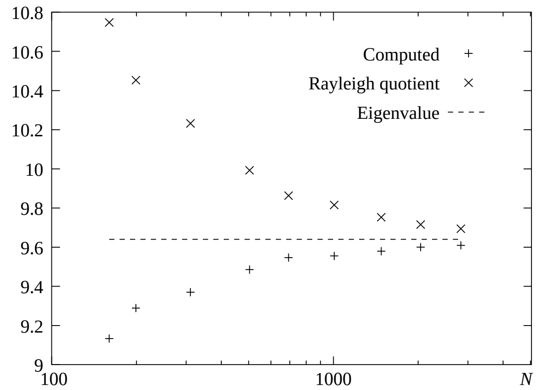

Figure 2: Sucessive approximations of the eigenvalue

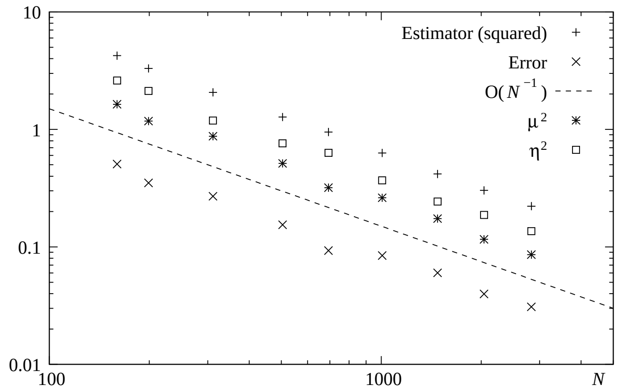

Figure 3: Convergence of the computed eigenvalue

6.2 Three dimensional example

We also considered the eigenvalue problem (1.1) in the three dimensional domain:

$$\Omega = \{|x_j| < 1, j = 1, 2, 3\} - \{0 \leq x_j \leq 1\,, j = 1, 2, 3\}$$

Starting with a uniform mesh we performed 8 adaptive steps and obtained the mesh shown in Figure 4.

The numerical approximation of the eigenvalue, and the Rayleigh quotient are shown in figure 5, where the monotonic convergence of these values can be observed. Finally, figure 6 shows the total estimated error and the eigenvalue error (in this case, the exact value is obtained by extrapolation of the numerical approximations).

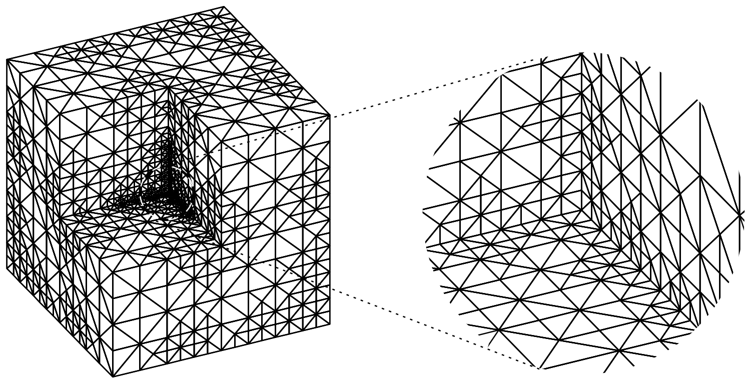

Figure 4: Mesh after 8 adaptive steps.

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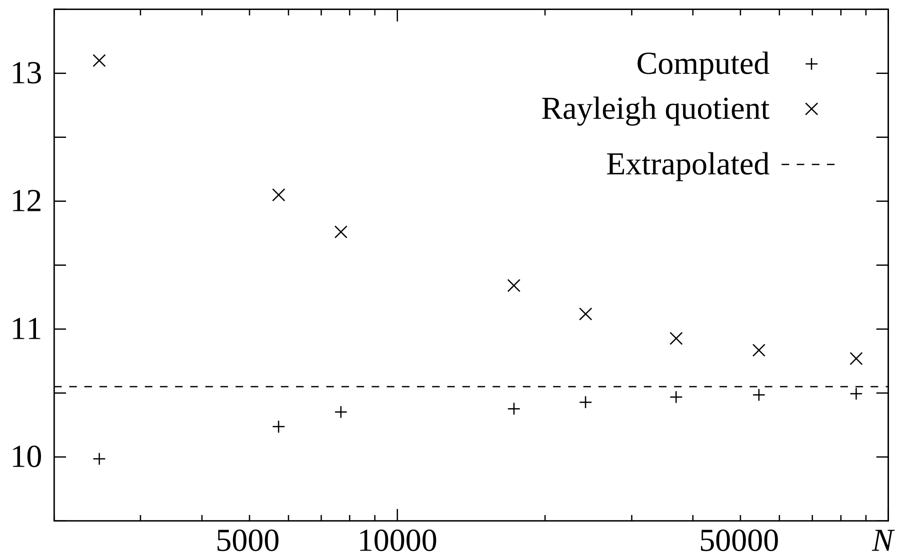

Figure 5: Eigenvalue and Rayleigh quotient

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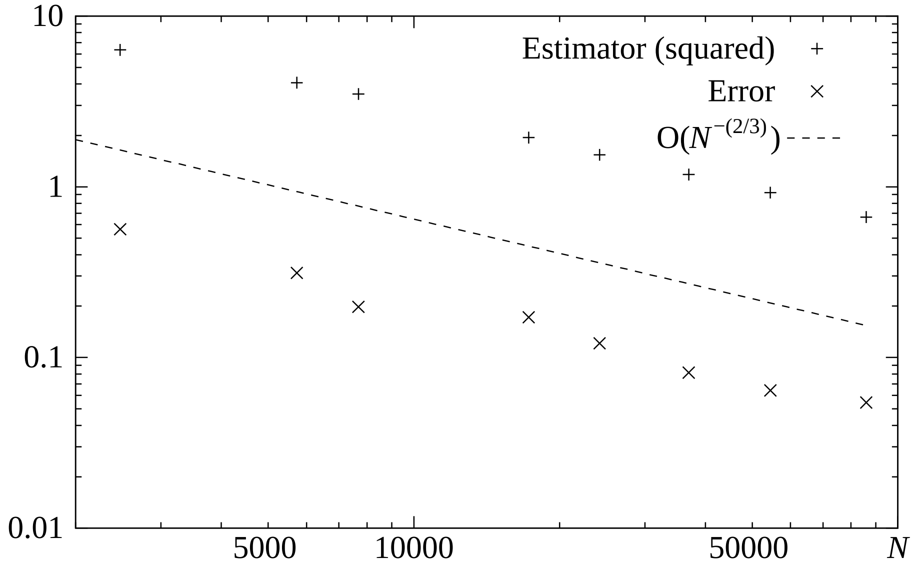

Figure 6: Global error estimator (squared) and error of the eigenvalue