Maximum Norm Error Estimators For Three Dimensional Elliptic Problems

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Abstract. *In this paper we define an a posteriori error estimator for finite element approximations of 3-d elliptic problems. We prove that the estimator is equivalent, up to logarithmic factors of the meshsize, to the maximum norm of the error. The results are valid for an arbitrary polyhedral domain and rather general meshes. We also obtain analogous results for the non conforming method of Crouzeix-Raviart. Finally, we present some numerical results comparing adaptive procedures based on controlling the error in different norms.*

Keywords. A posteriori error estimators, maximum norm, adaptivity.

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1 Introduction

This paper deals with a posteriori error estimators in the maximum norm for finite element approximations of elliptic problems in three dimensions. For both conforming and non conforming methods we define error estimators of the residual type and prove that they are equivalent to the maximum norm of the error under rather general assumptions.

The literature on a posteriori error estimators for 2-d problems is very extensive and many of the results can be extended straightforward to 3-d. However this is not always the case and, in particular, the extension of the results on estimators in the uniform norm requires some technical modifications that will be developed here.

For conforming elements, two different approaches to prove the equivalence of an error estimator with the maximum norm of the error have been introduced. One, due to Nochetto [15], is based on a priori estimates of a regularized Green function while the other one, due to Eriksson [10] uses pointwise estimates of the Green function and its derivatives near singularities. For our extension to 3-d problems we use the first approach. We derive an a priori estimate for a regularized Green function by doing an appropriate modification of the arguments given in [15]. The other approach could also be extended but it would require a precise knowledge of the

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behaviour of the Green function near singularities which, for the case of a general polyhedral domain, is not found in the literature and its derivation would involve rather technical difficulties due to the different kind of singularities arising in 3-d.

We show that the techniques applies also to obtain analogous results for non conforming approximations. We consider the well known Crouzeix-Raviart elements [6] consisting of piecewise linear elements continuous at the barycenter of the faces of the tetrahedral elements. In the 2-d case, error estimators for the energy norm for this method were introduced and analyzed in [7, 8, 13, 16]. Our results are the first on a posteriori error estimators for non conforming methods in 3-d.

In Section 1 we derive an a priori estimate for a regularized Green function. In Section 2 we define the error estimator and prove its equivalence with the error. In Section 3 we extend the results to non conforming approximations and finally, in Section 4 we present some numerical computations in which we compare adaptivity procedures based on error estimators for different norms.

2 A priori estimate for a regularized Green function

Let $\Omega \in \mathbb{R}^3$ be a bounded polyhedral domain. We consider the standard elliptic model problem, i.e., the Poisson equation. The results could be extended to other equations provided some a priori estimates in L^p norms holds for their solutions (this will be clear from our proofs). We will use standard notation for Sobolev spaces, i.e., $W^{m,p}(\Omega)$ is the space of $L^p(\Omega)$ functions with derivatives up to order m in $L^p(\Omega)$ and $W_0^{1,p}$ is the space of $W^{1,p}$ functions with vanishing trace on the boundary of Ω. For $p = 2$ we will write $H^m(\Omega)$ and $H_0^m(\Omega)$ instead of $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$.

Assume that we have a regular family of partitions $\{\mathcal{T}\}$ of Ω into tetrahedra (i.e., the family satisfies the standard minimum angle condition). Denote by h_T the diameter of a tetrahedron T and let $h_{\mathcal{T}} = \max_{T\in\mathcal{T}} h_T$ and $\underline{h}_{\mathcal{T}} = \min_{T\in\mathcal{T}} h_T$. We assume also that there exists $\gamma \geq 1$ independent of $\mathcal{T}$ such that

$$\underline{h}_{\mathcal{T}} \geq C h_{\mathcal{T}}^{\gamma} \tag{2.1}$$

which is a minor restriction and is satisfied in practical situations.

For $x_0 \in \Omega$ let $\delta \in C_0^\infty$ be a regularization of the Dirac measure on x_0, i.e., δ is such that $\int_\Omega \delta = 1$ and it satisfies

$$\operatorname{supp}\delta \subset B = \{x \in \Omega : |x - x_0| \leq \rho\}$$

and

$$0 \leq \delta \leq \frac{C}{\rho^3}$$

where $\rho = h_{\mathcal{T}}^{\beta}$ for some $\beta > 1$ which depends on γ and on the Hölder exponent of the solution and will be chosen below.

Note that from the properties above it follows that

$$\|\delta\|_{L^p(\Omega)} \leq C\rho^{(-3/p)(p-1)} \tag{2.2}$$

We introduce the regularized Green function for the Poisson equation $G \in H_0^1(\Omega)$ defined by

$$-\Delta G = \delta \quad \text{in} \quad \Omega$$

Under appropriate assumptions on the boundary it is known that for $p > 1$, the following estimate holds

$$\|D^2 v\|_{L^p(\Omega)} \leq C_p \|\Delta v\|_{L^p(\Omega)} \qquad \forall v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \tag{2.3}$$

where $D^2 v$ denotes the matrix of second derivatives of v.

For example, for a smooth domain, this is a classical estimate (see for example [1, 11]) and it is well known that $C_p = C/(p-1)$ when $p \to 1$ and therefore, from (2.2) we can see that

$$\|D^2 G\|_{L^p(\Omega)} \leq C \frac{\rho^{(-3/p)(p-1)}}{p-1} \tag{2.4}$$

which can be extrapolated to $p = 1$ by taking $1 - p = 1/log\,\rho$ to obtain

$$\|D^2 G\|_{L^1(\Omega)} \leq C|log\,\rho| \tag{2.5}$$

On the other hand, for polyhedral domains a similar a-priori estimate holds for $1 < p < 4/3$ [9]. However, the dependence on p of the constant is not known explicitly and it seems a rather technical problem to obtain its precise dependence on p from the proofs. We will find a slightly weaker estimate than (2.5) by modifying the argument of Nochetto [15].

The following Lemma gives some useful bounds for G which follows from known a priori estimates and Sobolev theorems.

Lemma 2.1 *For $1 < p < 2$ the following estimates hold*

$$\|\nabla G\|_{L^{3/2}(\Omega)} \leq C(p-1)^{-2/3} \|\delta\|_{L^p(\Omega)} \tag{2.6}$$

and,

$$\|G\|_{L^3(\Omega)} \leq C(p-1)^{-2/3} \|\delta\|_{L^p(\Omega)} \tag{2.7}$$

Proof. First we observe that for any function $v \in W_0^{1,3}(\Omega)$ we have

$$\|v\|_{L^{p'}(\Omega)} \leq C(p-1)^{-2/3} \|\nabla v\|_{L^3(\Omega)} \tag{2.8}$$

where $p' = p/(p-1)$ is the dual exponent of p. Indeed, this is a standard Sobolev inequality. The dependence on p for the constant follows easily from the proof (see for example [11, page 159]).

On the other hand, it is known that [9]

$$\|\nabla G\|_{L^{3/2}(\Omega)} \leq C\|\delta\|_{W^{-1,3/2}(\Omega)}$$

but,

$$\|\delta\|_{W^{-1,3/2}(\Omega)} = \sup_{v\in W_0^{1,3}(\Omega)} \frac{\int_\Omega \delta v}{\|\nabla v\|_{L^3(\Omega)}} \le C\|\delta\|_{L^p(\Omega)} \sup_{v\in W_0^{1,3}(\Omega)} \frac{\|v\|_{L^{p'}(\Omega)}}{\|\nabla v\|_{L^3(\Omega)}}$$

and therefore, (2.6) follows from (2.8), and (2.7) is an immediate consequence of (2.6) and a standard Sobolev inequality.□

The following lemma gives an estimate similar to (2.4) valid for a polyhedral domain.

Lemma 2.2 *There are p_0 and C such that for $1 < p < p_0$ the regularized Green function satisfies*

$$\|D^2 G\|_{L^p(\Omega)} \le C\frac{\rho^{-6(p-1)}}{(p-1)^{4/3}} \tag{2.9}$$

Proof. We define $d_j = 2^j\rho$ for $j \ge 0$ an integer number and $d_{-1} = 0$ and let A_j and B_j be the subsets of Ω defined by

$$A_j = \{x \in \Omega : d_{j-1} \le |x - x_0| \le d_j\}$$

and

$$B_j = \{x \in \Omega : \frac{d_{j-1}}{2} \le |x - x_0| \le 2d_j\}$$

Let $\eta_j \in C_0^\infty(B_j)$ be a cut function such that $\eta_j = 1$ in a neighborhood of A_j and $|D^k\eta_j| \le Cd_j^{-k}$.

Let $p_0 < 4/3$ fixed now on (so we will drop the dependence on p_0 from the constant in the estimates below). For $p < p_0$ we have,

$$\|D^2G\|^p_{L^p(\Omega)} = \sum_j \|D^2G\|^p_{L^p(A_j)} \le \sum_j \|D^2(\eta_j G)\|^p_{L^p(B_j)}$$

and we are going to estimate each term of the sum. To do that, we use the a priori estimate (2.3) with $v = \eta_j G$ and $p = p_0$. Then, we have

$$\|D^2(\eta_j G)\|^p_{L^p(B_j)} \le C|B_j|^{1-(p/p_0)}\|\Delta(\eta_j G)\|^p_{L^{p_0}(\Omega)}$$
$$\le Cd_j^{3(1-(p/p_0))}\left(\|\eta_j\delta\|^p_{L^{p_0}(B_j)} + \|\nabla\eta_j\nabla G\|^p_{L^{p_0}(B_j)} + \|\Delta\eta_j G\|^p_{L^{p_0}(B_j)}\right)$$

and we have to bound each of the three terms on the right hand side. First, since δ has support in B_0 we have

$$\sum_j d_j^{3(1-(p/p_0))}\|\eta_j\delta\|^p_{L^{p_0}(B_j)} \le \rho^{3(1-(p/p_0))}\|\delta\|^p_{L^{p_0}(\Omega)} \le C\rho^{3(1-p)}$$

where we have used (2.2). For the second part using the Hölder inequality with the exponent $3/2p_0$ and the fact that $|\nabla\eta_j| \le Cd_j^{-1}$ we have

$$\|\nabla\eta_j\nabla G\|^p_{L^{p_0}(B_j)} \le Cd_j^{3(p/p_0)(1-p_0)}\|\nabla G\|^p_{L^{3/2}(B_j)}$$

therefore,

$$\sum_j d_j^{3(1-(p/p_0))}\|\nabla\eta_j\nabla G\|^p_{L^{p_0}(B_j)} \le \sum_j d_j^{3(1-p)}\|\nabla G\|^p_{L^{3/2}(B_j)}$$

and finally, proceeding in the same way, using now that $|\Delta\eta_j| \le C d_j^{-2}$ we have

$$\|\Delta\eta_j G\|_{L^{p_0}(B_j)}^p \le C d_j^{3(p/p_0)(1-p_0)} \|G\|_{L^3(B_j)}^p$$

and so,

$$\sum_j d_j^{3(1-(p/p_0))} \|\Delta\eta_j G\|_{L^{p_0}(B_j)}^p \le \sum_j d_j^{3(1-p)} \|G\|_{L^3(B_j)}^p$$

Now, using the Hölder inequality for the sum with the exponent $3/2p$ we have

$$\sum_j d_j^{3(1-p)} \|\nabla G\|_{L^{3/2}(B_j)}^p \le C \|\nabla G\|_{L^{3/2}(\Omega)}^p \Big(\sum_j d_j^{3(1-p)3/(3-2p)} \Big)^{(3-2p)/3}$$

The sum on the last term is a geometric one and so it is easy to see that it behaves like (see details in [15])

$$C\rho^{3(1-p)}(p-1)^{(2p/3)-1}$$

and therefore, using Lemma 2.1 and the inequality (2.2) it follows that

$$\sum_j d_j^{3(1-p)} \|\nabla G\|_{L^{3/2}(B_j)}^p \le C(p-1)^{-1} \rho^{-6(p-1)}$$

In the same way we can see that

$$\sum_j d_j^{3(1-p)} \|G\|_{L^3(B_j)}^p \le C(p-1)^{-4/3} \rho^{-6(p-1)}$$

and collecting all the estimates we conclude the proof.□

Corollary 2.3 *For any polyhedral domain and for any $\beta > 1$, there is a positive constant C such that*

$$\|D^2 G\|_{L^1(\Omega)} \le C |log\, h_{\mathcal{T}}|^{4/3}$$

Proof. It follows by taking $1 - p = 1/log\,\rho$ and recalling that $\rho = h_{\mathcal{T}}^{\beta}$.□

3 A posteriori error estimator

Let $V_{\mathcal{T}}$ be the space of continuous piecewise linear functions associated with $\mathcal{T}$, vanishing on $\partial\Omega$. Let u be the solution of the Poisson's equation with homogeneous Dirichlet boundary condition in Ω and $u_{\mathcal{T}}$ its finite element approximation defined by

$$\int_\Omega \nabla u_{\mathcal{T}} \nabla v = \int_\Omega f v \qquad , \qquad \forall v \in V_{\mathcal{T}}$$

and we want to estimate the error $e = u - u_{\mathcal{T}}$. Let $x_0 \in \Omega$ be such that $\|e\|_{L^\infty(\Omega)} = |e(x_0)|$ and δ be the regularized Dirac mass of Section 2.

Assume that for some element $\hat{T}$, the maximum interpolation error on $\hat{T}$ is bounded from below by a constant times $h_{\hat{T}}^2$ (this is a minor restriction satisfied essentially always but in trivial cases). Then, since $u \in C^\alpha$ for some α, it can be seen that choosing $\rho = h_{\mathcal{T}}^\beta$ with $\beta > 2\gamma/\alpha$ we have that for $h_{\mathcal{T}} \leq h^*$

$$\|e\|_{L^\infty(\Omega)} \leq C\Big|\int_\Omega e\delta\Big| \tag{3.1}$$

where C and h^* depends on f and Ω but are independent of u and $\mathcal{T}$, provided (2.1) is satisfied. Indeed, this can be shown in the same way as in the two dimensional case and so, we omit the details and refer to [15].

To define the error estimator we use the standard notation: F_T denotes the set of faces of T which are in the interior of Ω, $\left[\frac{\partial u_{\mathcal{T}}}{\partial n}\right]_S$ is the jump of the normal derivative across the face S (we will drop the subscript S when no confusion arises) and h_S is the diameter of S. Then, the estimator is defined by

$$\eta = \max_{T\in\mathcal{T}} \eta_T$$

where the local error estimator is given by

$$\eta_T = h_T^2\|f\|_{L^\infty(T)} + \max_{S\in F_T} h_S\Big|\Big[\frac{\partial u_{\mathcal{T}}}{\partial n}\Big]\Big|$$

In the next two theorems we prove the equivalence between the maximum error and the estimator.

Theorem 3.1 *There exist constants C and $h^* > 0$ such that for $h_{\mathcal{T}} \leq h^*$ the following estimate holds*

$$\|e\|_{L^\infty(\Omega)} \leq C|log\, h_{\mathcal{T}}|^{4/3}\eta \tag{3.2}$$

Proof. Let x_0 be a point such that $\|e\|_{L^\infty(\Omega)} = |e(x_0)|$, δ be the regularized Dirac mass on x_0 and G be the regularized Green function defined in Section 2. In view of (3.1) we have

$$\|e\|_{L^\infty(\Omega)} \leq C\Big|\int_\Omega e(-\Delta G)\Big| = C\Big|\int_\Omega \nabla e\nabla G\Big| = C\Big|\int_\Omega \nabla e\nabla(G-G^I)\Big|$$

for any $G^I \in V_{\mathcal{T}}$. Now we take G^I as an approximation to G satisfying

$$\|G - G^I\|_{L^1(T)} \leq Ch_T^2\|D^2G\|_{L^1(\Omega_T)} \tag{3.3}$$

and,

$$\|G - G^I\|_{L^1(S)} \leq Ch_S\|D^2G\|_{L^1(\Omega_T)} \tag{3.4}$$

where Ω_T denotes the union of all elements sharing some vertex with T (these estimates are satisfied, for example, by the usual average interpolants (see [5, 18])). Then, integrating by parts on each element we have

$$\Big|\int_\Omega \nabla e\nabla(G-G^I)\Big| = \Big|\sum_T\Big\{\int_T f(G-G^I) + \sum_{S\in F_T}\int_S\Big[\frac{\partial u_{\mathcal{T}}}{\partial n}\Big](G-G^I)\Big\}\Big|$$

$$\le C\eta \|D^2 G\|_{L^1(\Omega)} \le C|log\, h_T|^{4/3}\eta$$

where we have used (3.3), (3.4) and Corollary 2.3.□

To bound the estimator by the error we use a variant of the known argument of Verfürth [20]. We assume that the right hand side is constant on each element. If not, an extra term, which is of higher order than the error under minor assumptions on f, appears in the inequality (see [15, 20] for details).

Theorem 3.2 *If $f|_T = f_T$ is constant for every T then, there exists a constant C such that for any element T the following estimate holds*

$$\eta_T \le C\|e\|_{L^\infty(\tilde{\Omega}_T)} \tag{3.5}$$

where $\tilde{\Omega}_T$ denotes the union of all elements sharing a face with T.

Proof. For all $v \in H_0^1(\Omega)$ we have

$$\sum_T \Big\{ \int_T fv + \frac{1}{2}\sum_{S\in F_T}\int_S \Big[\frac{\partial u_h}{\partial n}\Big] v\Big\} = \int_\Omega \nabla e \nabla v = \sum_T \Big\{ \int_T e(-\Delta v) + \frac{1}{2}\sum_{S\in F_T}\int_S e\Big[\frac{\partial v}{\partial n}\Big]\Big\} \tag{3.6}$$

Now, we will bound each term of the estimator separately. First we take $v = b_T$ in (3.6), where b_T is the usual bubble function, i.e., the polynomial of degree four obtained as the product of the barycentric coordinates.Then,

$$-\int_T e\Delta b_T + \frac{1}{2}\sum_{S\in F_T}\int_S e\Big[\frac{\partial b_T}{\partial n}\Big] = \int_T f_T b_T$$

so,

$$\begin{aligned} |f_T||\textstyle\int_T b_T| &\le \|e\|_{L^\infty(T)}|\textstyle\int_T \Delta b_T| + \frac{1}{2}\sum_{S\in F_T}\|e\|_{L^\infty(T)}|\int_S [\frac{\partial b_T}{\partial n}]| \\ &\le \|e\|_{L^\infty(T)}\{|\textstyle\int_T \Delta b_T| + \frac{1}{2}\sum_{S\in F_T}|\int_S \frac{\partial b_T}{\partial n}|\} \end{aligned}$$

and therefore,

$$h_T^2|f_T| \le C\|e\|_{L^\infty(T)} \tag{3.7}$$

Now, given a face $S \in F_T$ we denote by T_1 the neighbor element sharing this face with T and take $v = \phi_S$ in (3.6) where ϕ_S is piecewise cubic on $T \cup T_1$, is one on the barycenter of S and vanishes on the other faces of $T \cup T_1$. Then, we have

$$-\int_T e\Delta\phi_S - \int_{T_1} e\Delta\phi_S + \frac{1}{2}\int_S e\Big[\frac{\partial \phi_S}{\partial n}\Big] = \int_{T\cup T_1} f\phi_S + \frac{1}{2}\int_S \Big[\frac{\partial u_h}{\partial n}\Big]\phi_S$$

so,

$$\frac{1}{2}\Big|\Big[\frac{\partial u_h}{\partial n}\Big]\Big|\,\Big|\int_S \phi_S\Big| \le \|e\|_{L^\infty(T\cup T_1)}\Big\{\Big|\int_S \Delta\phi_S\Big| + \Big|\int_S \Big[\frac{\partial \phi_S}{\partial n}\Big]\Big|\Big\} + \|f\|_{L^\infty(T\cup T_1)}\Big|\int_{T\cup T_1}\phi_S\Big|$$

and therefore,

$$h_S\Big|\Big[\frac{\partial u_h}{\partial n}\Big]\Big| \le Ch_T^2\|f\|_{L^\infty(T\cup T_1)} + C\|e\|_{L^\infty(T\cup T_1)}$$

and the proof concludes by using (3.7).□

4 Extension to nonconforming elements

Now we extend the results of the previous section to nonconforming approximations. Consider the finite element approximation of the Poisson equation obtained with piecewise linear functions continuous at the barycenter of the interior edges and vanishing at the barycenter of boundary faces. We will call $V_{\mathcal{T}}^{NC}$ the corresponding finite element space associated with a triangulation $\mathcal{T}$. Then the approximate solution $u_{\mathcal{T}}^{NC} \in V_{\mathcal{T}}^{NC}$ is defined by

$$\sum_{T\in\mathcal{T}} \int_T \nabla u_{\mathcal{T}}^{NC} \nabla v = \int_\Omega f v \qquad , \qquad \forall v \in V_{\mathcal{T}}^{NC}$$

and we denote the error with $e^{NC} = u - u_{\mathcal{T}}^{NC}$. To simplify notation we will drop the superscript NC from now on.

The arguments given for the conforming approximation can be extended to this case. However, some modifications are needed in order to take care of the discontinuity of the approximate solution $u_{\mathcal{T}}$.

First, in the following lemma we will prove a modification of (3.1) valid for the nonconforming case. In order to do that we introduce a regularization of $u_{\mathcal{T}}$, $\tilde{u}_{\mathcal{T}}$ continuous and piecewise linear defined in the following way:

For any vertex y of $\mathcal{T}$ let T_i, $i = 1, \cdots, M_y$ be all the elements sharing it. Then, calling $u_{\mathcal{T},T}$ the restriction of $u_{\mathcal{T}}$ to T we define

$$\tilde{u}_{\mathcal{T}}(y) = \sum_{i=1}^{M_y} \frac{1}{M_y} u_{\mathcal{T},T_i}(y)$$

Now, calling $\tilde{e} = u - \tilde{u}_{\mathcal{T}}$ we can decompose the error as

$$e = \tilde{e} + \tilde{u}_{\mathcal{T}} - u_{\mathcal{T}} \tag{4.1}$$

In the proof of the next lemma we will use that two elements T_1 and T_2 with a common vertex can be connected by a sequence of elements $T_1 = \tilde{T}_1, ..., \tilde{T}_k = T_2$ sharing the same vertex and such that $\tilde{T}_j$ and $\tilde{T}_{j+1}$ have a common face. This requires a very mild assumption on the boundary of the domain and we will assume it from now on. Observe that k can be bounded by a constant depending only on the minimum angle of the triangulation. We will denote with F the set of all faces of $\mathcal{T}$.

Lemma 4.1 *Let x_0 be such that $|\tilde{e}(x_0)| = \|\tilde{e}\|_{L^\infty(\Omega)}$ and δ be the regularized Dirac mass on x_0. Then, under the same assumptions needed for (3.1) we have*

$$\|e\|_{L^\infty(\Omega)} \le C\Big|\int_\Omega e\delta\Big| + C \max_{S\in F} \|[u_{\mathcal{T}}]\|_{L^\infty(S)}$$

Proof. Proceeding exactly as in [15] we can prove that

$$\|\tilde{e}\|_{L^\infty(\Omega)} \le C\Big|\int_\Omega \tilde{e}\delta\Big|$$

and so, in view of (4.1) we have

$$\|e\|_{L^\infty(\Omega)} \le C\Big|\int_\Omega e\delta\Big| + \|\tilde{u}_{\mathcal{T}} - u_{\mathcal{T}}\|_{L^\infty(\Omega)}$$

therefore, in order to conclude the proof, we have to show that

$$\|\tilde{u}_{\mathcal{T}} - u_{\mathcal{T}}\|_{L^\infty(\Omega)} \le C \max_{S\in F} \|[u_{\mathcal{T}}]\|_{L^\infty(\Omega)}$$

Let T be the element where the maximum of $\tilde{u}_{\mathcal{T}} - u_{\mathcal{T}}$ is attained and y be the vertex such that

$$\|\tilde{u}_{\mathcal{T}} - u_{\mathcal{T}}\|_{L^\infty(\Omega)} = |\tilde{u}_{\mathcal{T}}(y) - u_{\mathcal{T},T}(y)|$$

¿From the definition of $\tilde{u}_{\mathcal{T}}$ we have

$$\tilde{u}_{\mathcal{T}}(y) - u_{\mathcal{T},T}(y) = \sum_{i=1}^{M_y} \frac{1}{M_y}(u_{\mathcal{T},T_i}(y) - u_{\mathcal{T},T}(y))$$

and so, if for each i we call $T_i = \tilde{T}_1, \cdots, \tilde{T}_k = T$ a sequence of elements sharing y and connecting T_i and T (in the sense that $\tilde{T}_j$ and $\tilde{T}_{j+1}$ have a common face) we can write

$$u_{\mathcal{T},T_i}(y) - u_{\mathcal{T},T}(y) = \sum_{j=1}^{k-1}(u_{\mathcal{T},\tilde{T}_j}(y) - u_{\mathcal{T},\tilde{T}_{j+1}}(y))$$

and therefore, since $k \le M_y$ we obtain

$$|\tilde{u}_{\mathcal{T}}(y) - u_{\mathcal{T},T}(y)| \le M_y \max_{S\in F} \|[u_{\mathcal{T}}]\|_{L^\infty(S)}$$

and the proof concludes by observing that, under the regularity assumption on the meshes, M_y is uniformly bounded. □

Now, we define the estimator as

$$\eta = \max_{T\in\mathcal{T}} \eta_T$$

where the local estimator is given by

$$\eta_T = h_T^2\|f\|_{L^\infty(T)} + \max_{S\in F_T} h_S\Big|\Big[\frac{\partial u_{\mathcal{T}}}{\partial n}\Big]\Big| + \max_{S\in F_T} \|[u_{\mathcal{T}}]\|_{L^\infty(S)} \tag{4.2}$$

The next theorem proves the equivalence between the maximum error and the estimator.

Theorem 4.2 *There exist constants C_1, C_2 and $h^* > 0$ such that for $h_{\mathcal{T}} \le h^*$ the following estimate holds*

$$\|e\|_{L^\infty(\Omega)} \le C_1|logh_{\mathcal{T}}|^{4/3}\eta \tag{4.3}$$

and, if $f|_T = f_T$ is constant for every T then,

$$\eta_T \le C_2\|e\|_{L^\infty(\tilde{\Omega}_T)} \tag{4.4}$$

where $\tilde{\Omega}_T$ *is as in Theorem 3.2.*

Proof. Let G^I be the piecewise linear conforming interpolant of G introduced in Section 3 (i.e. an average interpolant) then,

$$\|\frac{\partial(G-G^I)}{\partial n}\|_{L^1(S)} \leq C\|D^2G\|_{L^1(\Omega_T)} \tag{4.5}$$

Integrating by parts on each element we have

$$\begin{aligned} \int_\Omega e \quad (-\Delta G) &= \textstyle\sum_{T\in\mathcal{T}}\{\int_T \nabla e \nabla G - \int_{\partial T} u_{\mathcal{T}} \frac{\partial G}{\partial n}\} \\ &= \textstyle\sum_{T\in\mathcal{T}}\{\int_T \nabla e \nabla(G-G^I) - \frac{1}{2}\sum_{S\in F_T}\int_S [u_{\mathcal{T}}]\frac{\partial(G-G^I)}{\partial n}\} \\ &= \textstyle\sum_{T\in\mathcal{T}}\{\int_T f(G-G^I) - \frac{1}{2}\sum_{S\in F_T}\int_S [\frac{\partial u_{\mathcal{T}}}{\partial n}](G-G^I) - \frac{1}{2}\sum_{S\in F_T}\int_S [u_{\mathcal{T}}]\frac{\partial(G-G^I)}{\partial n}\} \end{aligned} \tag{4.6}$$

Therefore, proceeding as in the conforming case, the estimate (4.3) follows from (4.6), (4.5) and Lemma 4.1.

To obtain (4.4) we observe that the first two terms in (4.2) can be bounded as in the conforming case. Therefore, it remains to estimate the last term in (4.2) But this follows immediately from the observation that the jump of u is zero and so

$$\|[u_{\mathcal{T}}]\|_{L^\infty(S)} \leq \|[e]\|_{L^\infty(S)} \leq 2\|e\|_{L^\infty(\tilde{\Omega}_T)}$$

. □

Remark: It can be seen that the term corresponding to the jumps of the normal derivative in the definition of η_T can be bounded by the term corresponding to the f (we omit the details since the argument is analogous to that in [7]). Therefore, the following simpler estimator is also equivalent to the norm of the error

$$\tilde{\eta} = \max_{T\in\mathcal{T}} \tilde{\eta}_T$$

where

$$\tilde{\eta}_T = h_T^2\|f\|_{L^\infty(T)} + \max_{S\in F_T} \|[u_{\mathcal{T}}]\|_{L^\infty(S)}$$

5 Numerical results

In this section we present some numerical results for standard conforming linear elements. Our objective is to compare automatically generated meshes obtained by adaptive approaches based on different norms error estimators, namely, the H^1 and L^∞ norms.

Although it is difficult to state what would be an optimal mesh, it seems natural to look for meshes such that the error be almost equidistributed in some norm. In general, meshes with this property will be different for different norms. Consider for example the following two dimensional model problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \ \Omega = \{(r,\theta) : r<1\,, 0<\theta<2\pi\} \\ u|_{\theta=0} = 0 \\ u|_{r=1} = \sin\,(\theta/4) \\ \frac{\partial u}{\partial n}|_{\theta=2\pi} = 0 \end{cases} \tag{5.1}$$

The exact solution in polar coordinates is given by

$$u = r^{1/4}\sin\,(\theta/4)$$

For this case it has been shown, by using a priori estimates, that optimal order convergence with respect to the number of elements can be obtained by using appropriately graded meshes (see [2, 12] for the H^1 norm and [19] for the L^∞ norm). Indeed, if d_T denotes the distance of an element T to the singularity and h_T is the diameter of T, the optimal order in H^1, i.e., $\|u - u_{\mathcal{T}}\|_{H^1} = O(N^{-1/2})$, where N is the number of elements in $\mathcal{T}$, is obtained if the meshes satisfy (see [2, 12])

$$h_T \sim d_T^\alpha \qquad \text{with} \qquad 3/4 < \alpha < 1 \tag{5.2}$$

while the optimal order convergence in L^∞, i.e., $\|u - u_{\mathcal{T}}\|_{L^\infty} = O(N^{-1})$, holds if (see [19])

$$h_T \sim d_T^\beta \qquad \text{with} \qquad 7/8 < \beta < 1 \tag{5.3}$$

Note that, although the same order with respect to N is obtained for a range of α in (5.2) and β in (5.3), the constant involved deteriorates when α (resp.: β) increases. So, one could say that the optimal mesh would be that with $\alpha = 3/4 + \epsilon$ (resp.: $\beta = 7/8 + \epsilon$).

On the other hand, in order to equidistribute the error one should have

$$h_T \sim d_T^{7/8}$$

both for the L^∞ and the H^1 norms.

In fact, this can be shown by the following heuristic argument, which can be formalized by using local estimates (see [19, 21])

If $e = u - u_{\mathcal{T}}$ denotes the error then,

$$\|e\|_{L^\infty(T)} \sim h_T^2 \|D^2 u\|_{L^\infty(T)}$$

but $D^2 u \sim r^{-7/4}$ and so

$$\|e\|_{L^\infty(T)} \sim h_T^2 d_T^{-7/4}$$

and therefore, the error will be equidistributed if $h_T = d_T^{7/8}$.

Analogously,

$$\|e\|_{H^1(T)} \sim h_T \|D^2 u\|_{L^2(T)}$$

and so, the error will be equidistributed in the H^1 norm if $h_T \sim d_T^{7/8}$.

Remark: The fact that we have obtained the same power for both norms is a coincidence for the 2-d case. Indeed, for a singularity of the same order in 3-d we would obtain the same power for the L^∞ norm while for the H^1 case the condition for equidistribution of the error would be $h_T \sim d_T^{7/10}$.

Remark: With a similar argument, one can see that the meshes satisfying the condition (5.2) with the limit power $\alpha = 3/4$ (which provides the best constant in the error estimate) corresponds to equidistribute the error in annular regions. This shows in particular, that it is not necessary to equidistribute the error in order to obtain optimal order convergence in H^1.

Now we present the results obtained with two different adaptive methods. One based on the H^1 error estimator and the other based on the L^∞ error estimator. The numerical experiments show that meshes with the graduation predicted are obtained in both cases. For each norm, we have used the following adaptive procedure: start with an initial quasi uniform mesh $\mathcal{T}_0$ and compute the approximate solution u_0. Then, given a solution u_k corresponding to a mesh $\mathcal{T}_k$, the following mesh is obtained by refining those elements T such that the error indicator η_T satisfies

$$\eta_T \geq \gamma \eta_{max} \qquad \text{, with} \qquad \eta_{max} = \max_{T \in \mathcal{T}_k} \eta_T$$

for some fixed constant γ (we have taken γ=0.5).

In order to obtain an admissible mesh, the refinement has been propagated by using the method given by Rivara [17] which guarantee that, at every step, the minimum angle is always greater or equal than 0.5 times the minimum angle of the starting mesh.

Figure 1-(a) and 1-(b) below show the size of the elements as a function of their distance from the singularity for the described adaptive procedures. As it can be seen from these graphs, in both cases, the graduation of the mesh corresponds to that needed for equidistribution of the error.

Figure 2 shows the order of convergence with respect to the number of elements in the L^∞-norm obtained when each of the estimators is used. As it can be seen, the optimal order convergence $O(N^{-1})$ is obtained in both cases.

Finally, Figure 3 shows the order of convergence in the H^1-norm obtained with both procedures. The optimal order $O(N^{-1/2})$ is also obtained in both cases.

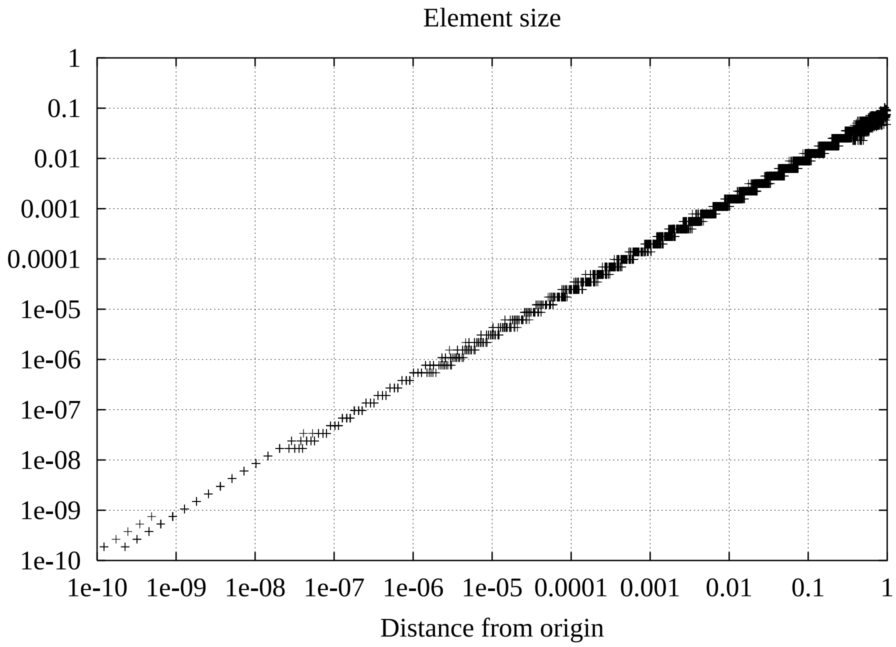
Now we present some results for 3-d problems. Again we want to compare the results obtained with the two different error estimators. The adaptive procedure we have used is similar to that described above for the 2-d case. However, in 3-d, the mesh $\mathcal{T}_{k+1}$ is obtained from the $\mathcal{T}_k$ using a mesh optimization algorithm, an extension to 3-d of the algorithm described in [3].

Consider the following problem described in spherical coordinates r, θ (see figure 4-(a))

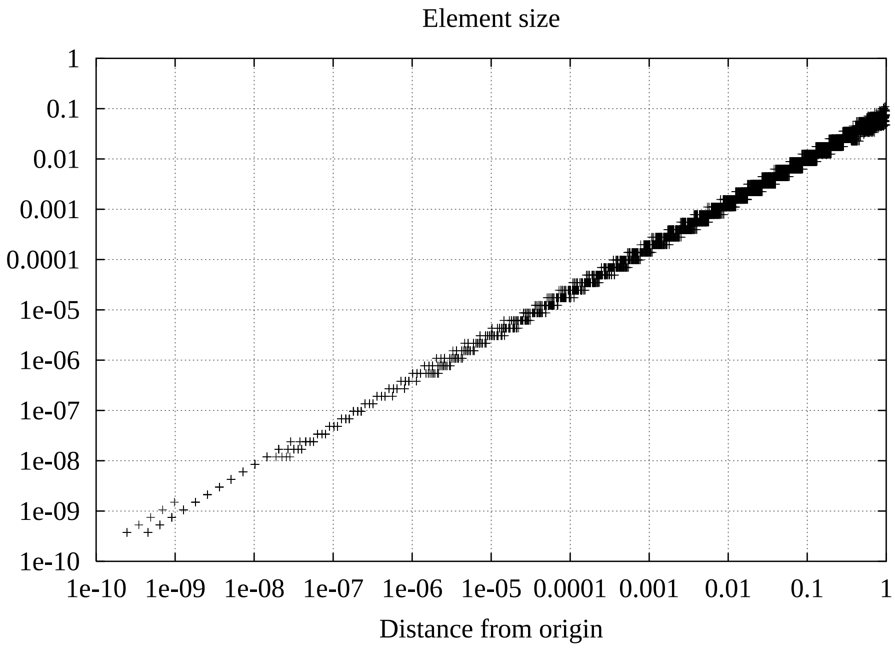
$$\begin{cases} -\Delta u = 0 & \text{in } \Omega = \{(r,\theta) : r < 1\,, \theta > \beta\} \\ u|_{\theta=\beta} = 0 & \\ u|_{r=1} = P_\nu(\cos\,\theta) & \end{cases} \tag{5.4}$$

where P_ν is the first class Legendre function of order ν. The exact solution of this problem is given by

$$u = r^\nu P_\nu(\cos\,\theta)$$



(a)



(b)

Figure 1: Element sizes of the meshes obtained applying the adaptive procedure with (a) the energy norm and (b) the maximum norm.

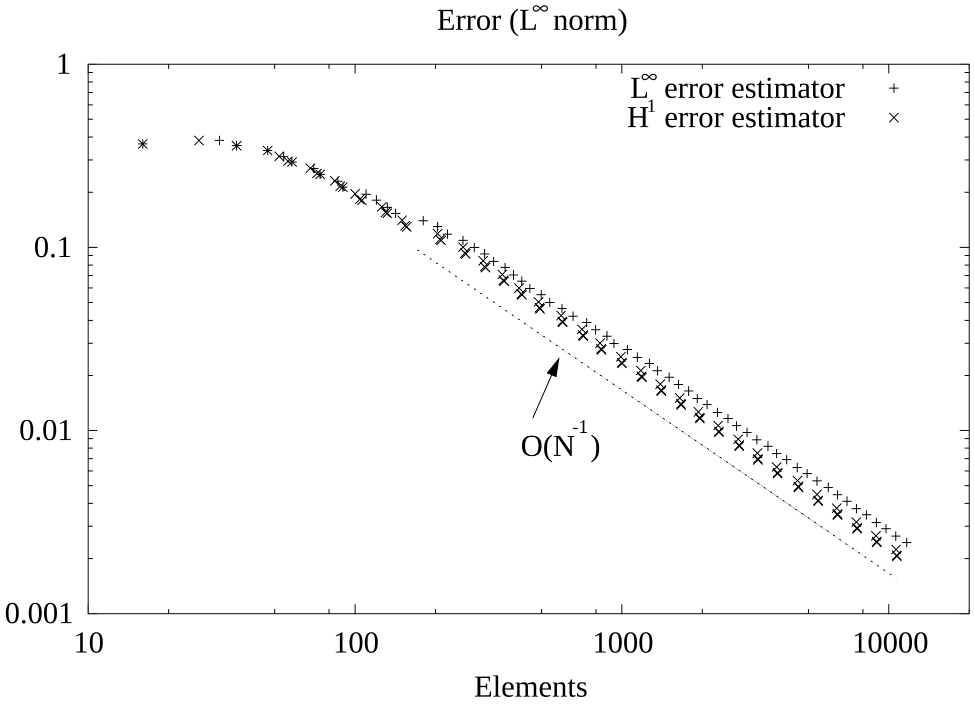


Figure 2: Error in L^{∞}-norm norm using both estimators.

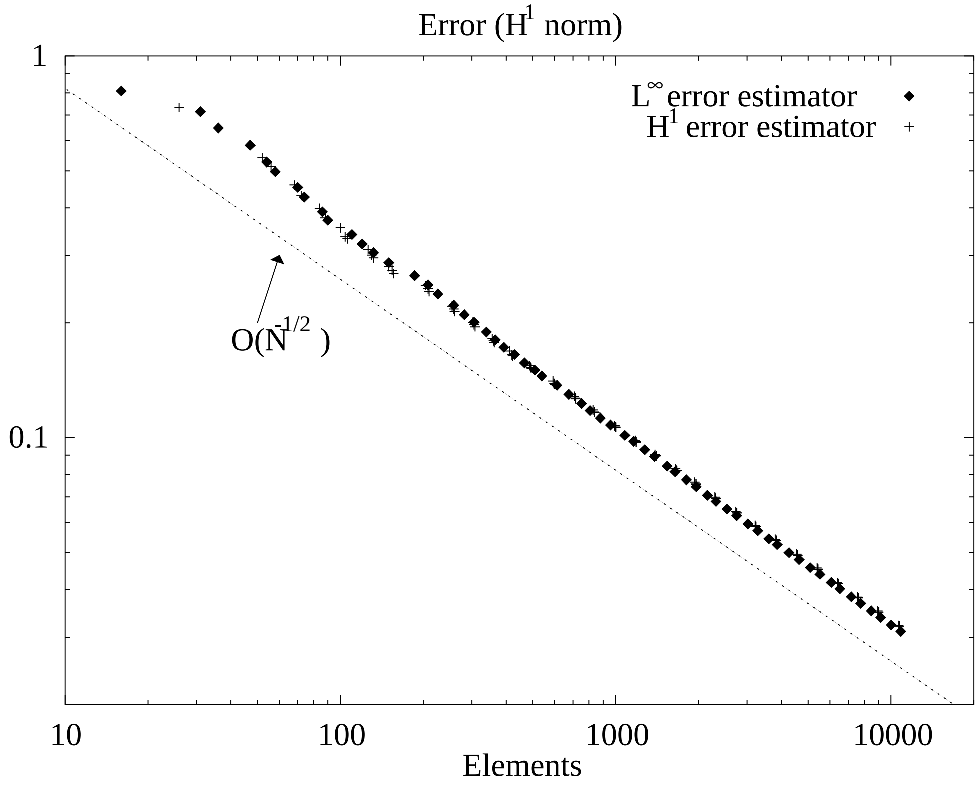


Figure 3: Error in H^1-norm using both estimators.

with ν depending on β. We will take $\beta = \pi/16$ and in this case $\nu \simeq 0.215$ (see [14]). We start with an initial mesh with 2704 elements, 605 nodes and $h \simeq 0.25$ as shown in figure 4-(b).

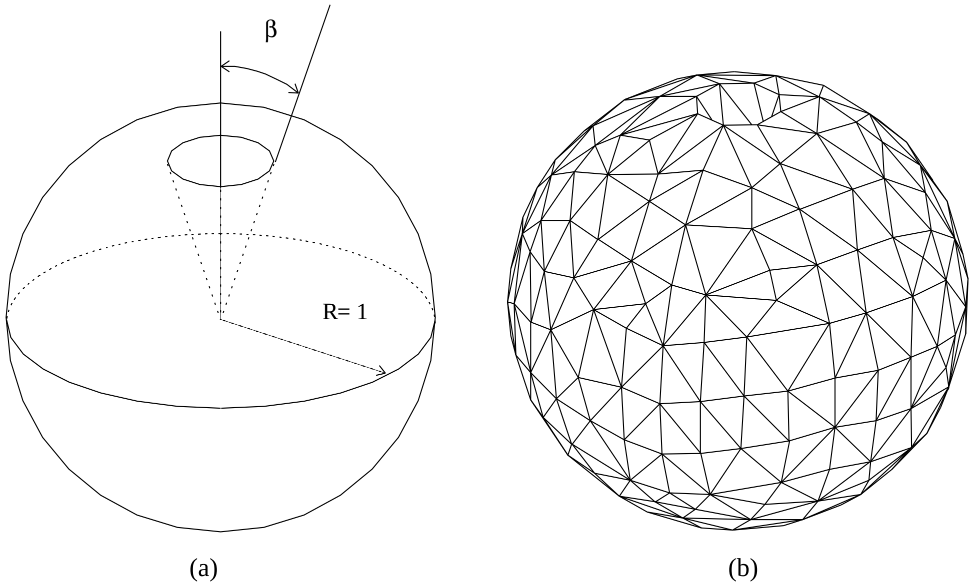


Figure 4: (a) geometry and (b) initial mesh for the first 3d test problem.

As in the 2-d example we perform adaptivity by using the error estimators in the L^∞ and H^1 norms. We observe that the meshes obtained with both procedures are quite different. In fact, after 28 steps we obtain a similar number of elements but the mesh obtained with the L^∞ error estimator is much more refined near the singularity. This is shown in Figures 5 and 6 below.

Figure 7 shows the behavior of the L^∞-norm of the error for both procedures. The expected optimal order in L^∞ is $O(N^{-2/3})$. The graphic shows that this order is obtained when the L^∞ error estimator is obtained to generate the meshes. Instead, if the H^1 error estimator is used, the optimal order is not obtained. In conclusion, if one is interested in reducing the L^∞ error, it is important to use the appropriate error estimator.

Figure 8 shows the behavior of the H^1-norm of the error for both procedures. We observe that the order is quite similar for both cases, but better results are obtained with the H^1 error estimator for which the optimal order $O(N^{-1/3})$ is clearly obtained.

Finally, we consider a problem which has a stronger singularity which is indeed not located only at a point. The exact solution of this problem is not known but, in view of the theoretical results showing the equivalence between the error and the estimator, we can compute the order of convergence. The domain Ω is defined as

$$\Omega = \{|x_j| < 1, j = 1, 2, 3\} - \{(x_1, x_2, 0) : 0 \leq x_1 \leq 1\,, 0 \leq x_2 \leq 1\}$$

and the problem we solve is

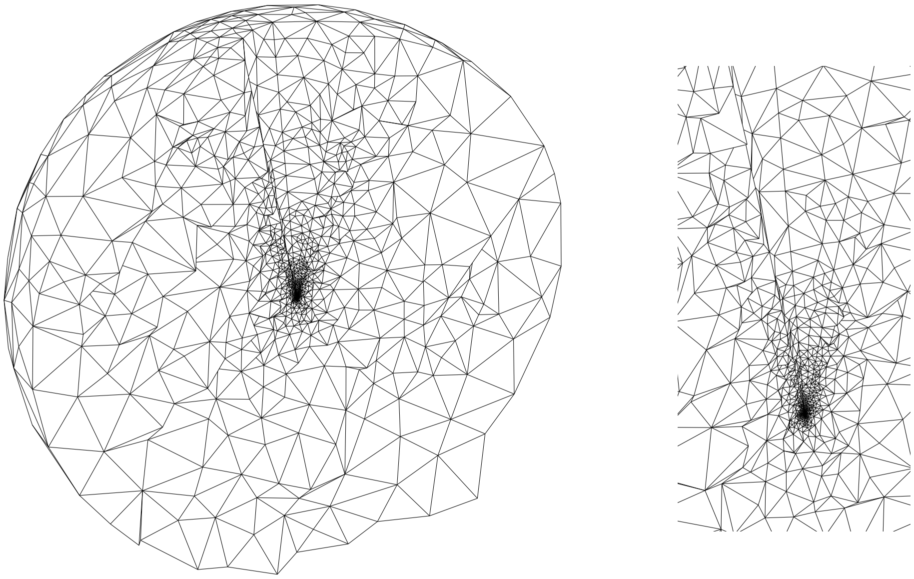


Figure 5: Mesh obtained after 28 steps using the adaptive procedure with the L^∞ norm.

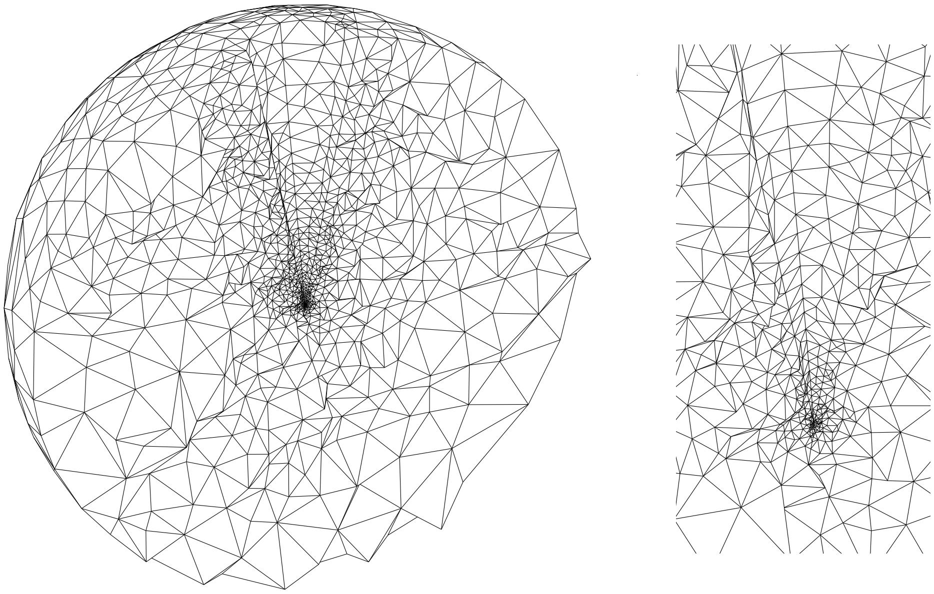


Figure 6: Mesh obtained after 28 steps using the adaptive procedure with the H^1 norm.

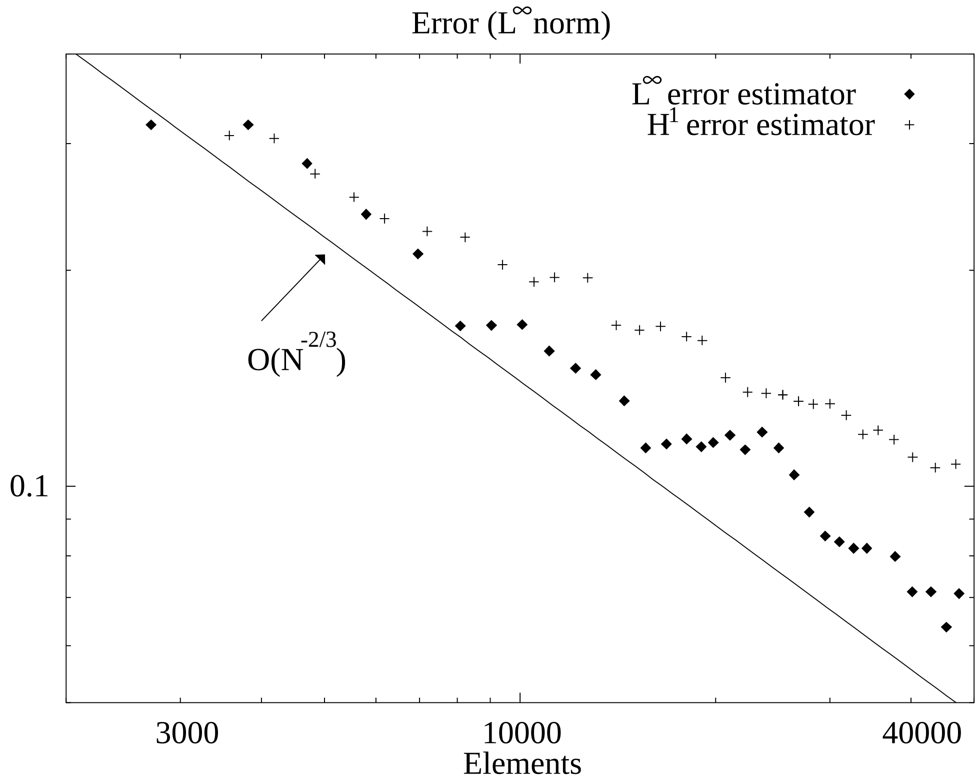


Figure 7: Convergence of the L^{∞}-norm of the error for both adaptive procedures.

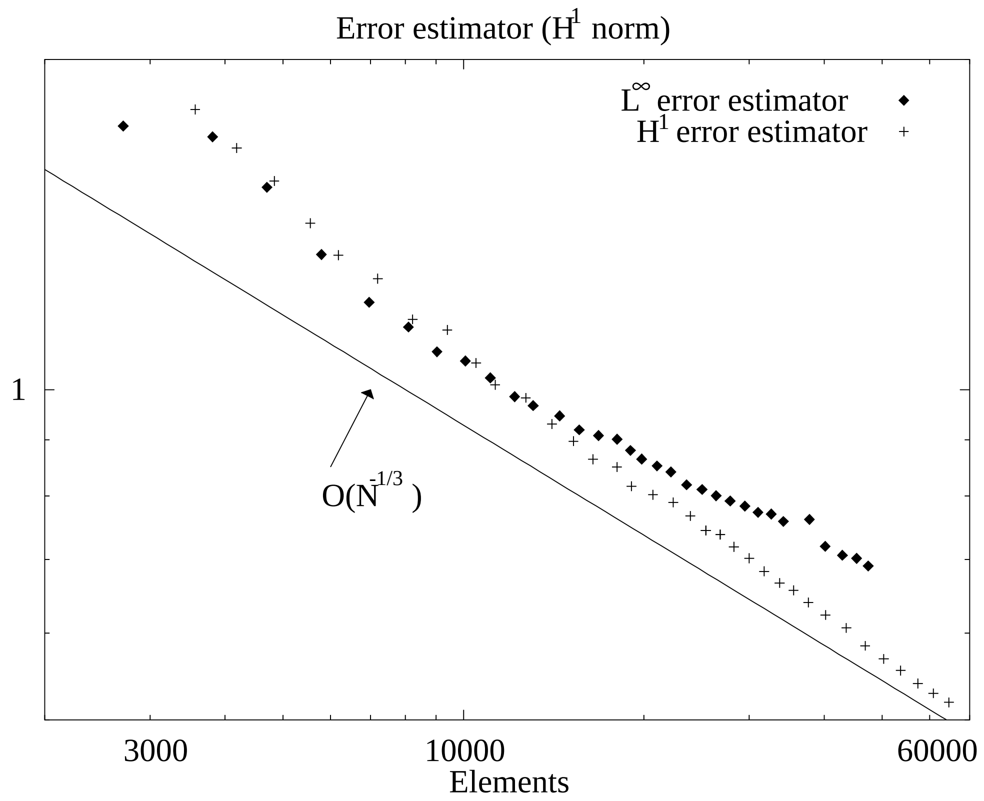


Figure 8: Convergence of the H^1-norm of the error for both adaptive procedures.

$$
\begin{cases}
-\Delta u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \{x_1 = 1\}, \{x_2 = 1\} \text{ and both sides of the crack} \\
\frac{\partial u}{\partial n} = 0 & \text{on } \{x_1 = -1\} \text{ and } \{x_2 = -1\} \\
\frac{\partial u}{\partial n} = \cos\left(\frac{\pi}{4}(x_1+1)\right)\cos\left(\frac{\pi}{4}(x_2+1)\right) & \text{on } \{x_3 = 1\} \text{ and } \{x_3 = -1\}
\end{cases}
\tag{5.5}
$$

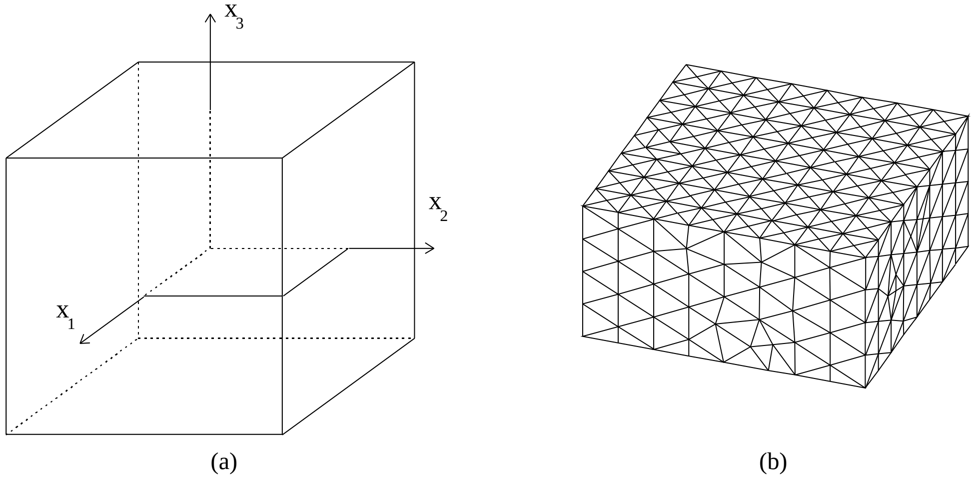


Figure 9: (a) geometry and (b) initial mesh for the second 3d test problem.

Figure 9 shows the domain definition and the initial mesh (by symmetry only the half cube with $x_3 < 0$ is considered).

As in the previous example, we perform adaptivity using the two different error estimators. 24 steps were performed with the H^1 error estimator and 49 with the L^∞ based one. The meshes obtained are shown in Figure 10 below. As before, the number of elements in both meshes is similar but the mesh obtained with the L^∞ error estimator is much more refined near the singularities.

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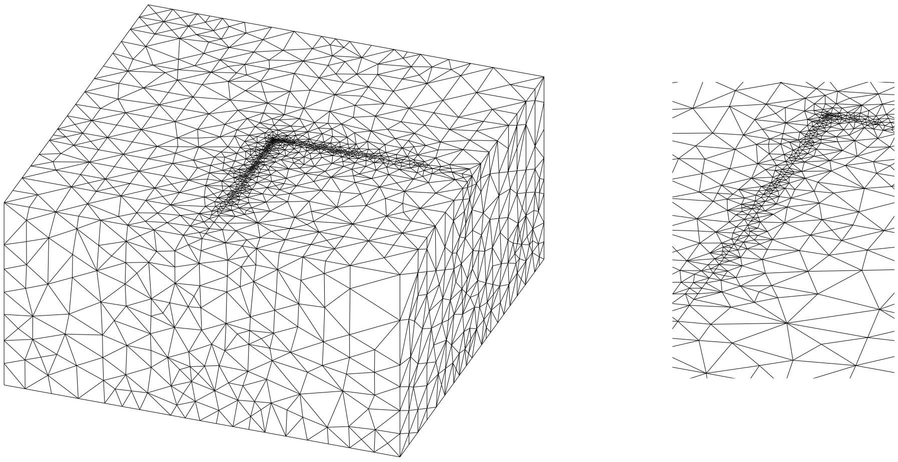
References

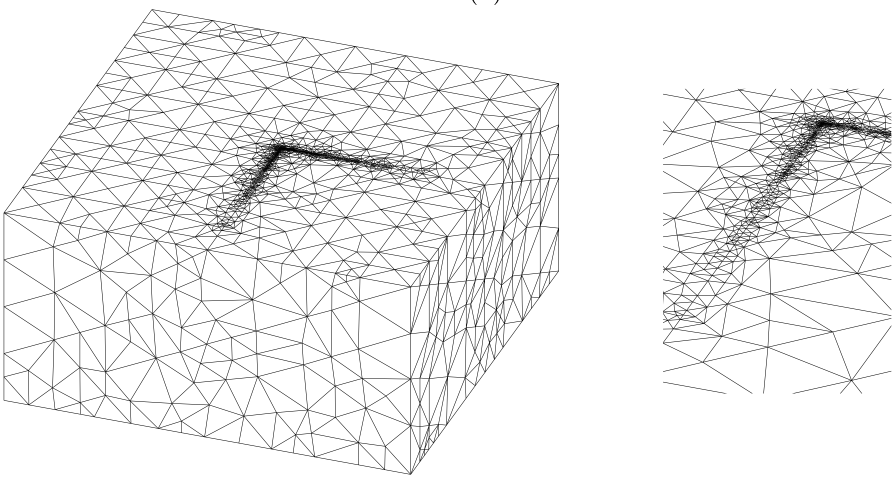
[1] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Commun. Pure Appl. Math. **12** 623-727, 1959.

[2] I. Babuška, R. B. Kellogg, J. Pitkäranta, *Direct and inverse error estimatesfor finite elements with mesh refinements*, Numer. Math. **33**, 447-471, 1972.

[3] G. Buscaglia, E. A. Dari *Anisotropic mesh optimization and its application in adaptivity*, Int. J. Numer. Meth. in Eng. **40**, 4119-4136, 1997.



(a)



(b)

Figure 10: Meshes obtained using (a) the H^1 based error estimator and (b) the L^∞ based error estimator.

[4] P.G. Ciarlet, *The finite element method for elliptic problems*, North Holland, 1978.

[5] P. Clément, *Approximation by finite element function using local regularization*, RAIRO **R-2**, 77-84, 1975.

[6] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*, R.A.I.R.O., Anal. Numér. **7 R-3**, 33-76, 1973.

[7] E. Dari, R. Durán, C. Padra, V.Vampa, *A posteriori error estimators for nonconforming finite element methods*, Math. Model. and Numer. Anal., **30**, 385-400, 1996.

[8] E. Dari, R. Durán, C. Padra, *Error estimators for nonconforming finite elements approximations of the Stokes problem*, Math. Comp. **64**, 1017-1033, 1995.

[9] M. Dauge, *Problèmes de Neumann et de Dirichlet sur un polyèdre dans* $\mathbb{R}^3$*: régularité dans des espaces de Sobolev* L_p, C.R. Acad. Sci. Paris, **307-I**, 27-32, 1988.

[10] K. Eriksson, *An adaptive finite element method with efficient maximum norm error control for elliptic problems*, **4**, 313-329, 1994.

[11] D.Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[12] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.

[13] R. Hoppe and B. Wohlmuth, *Element-oriented and edge-oriented local error estimators for nonconforming finite element methods, Math. Model. Numer. Anal.*, **30**, 237-263, 1996.

[14] J. D. Jackson, *Classical Electrodynamics*, 2nd. ed., John Wiley & Sons., 1975.

[15] R. Nochetto, *Pointwise a posteriori error estimates for elliptic problems on highly graded meshes*, Math. Comp. **64**, 1-22, 1995.

[16] C. Padra, *A posteriori error estimators for nonconforming approximation of some quasi-newtonian flows*, SIAM J. Numer. Anal., **34**, 1600-1615, 1997.

[17] M. C. Rivara, *Mesh refinement processes based on the generalized bisection of simplices*, **21**, 604-613, 1984.

[18] L.R. Scott, S. Zhang, *Finite element interpolation of non-smooth functions satisfying boundary conditions*, Math. Comp. **54**, 483-493, 1990.

[19] A. H. Schatz , L. B. Wahlbin, *Maximum norm estimates in the finite element method on plane polygonal domains, Part 2, Refinements*, Math. Comp. **33**, 465-492, 1979.

[20] R. Verfürth, *A posteriori error estimators for the Stokes equations*, Numer. Math. **55**, 309-325, 1989.

[21] L. B. Wahlbin, *Local behavior in finite element methods*, in Handbook of Numerical Analysis vol.II, (P.G.Ciarlet, J.L. Lions eds.), 352-522, 1991.