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A WEIGHTED SETTING FOR THE NUMERICAL APPROXIMATION OF THE POISSON PROBLEM WITH SINGULAR SOURCES*

IRENE DRELICHMAN†, RICARDO G. DURÁN†, AND IGNACIO OJEA‡

Abstract. We consider the approximation of Poisson type problems where the source is given by a singular measure and the domain is a convex polygonal or polyhedral domain. First, we prove the well-posedness of the Poisson problem when the source belongs to the dual of a weighted Sobolev space where the weight belongs to the Muckenhoupt class. Second, we prove the stability in weighted norms for standard finite element approximations under the quasi-uniformity assumption on the family of meshes.

Key words. finite element methods, Poisson problem, weighted Sobolev spaces

AMS subject classifications. Primary, 65N30; Secondary, 65N15, 35B45

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1. Introduction.

This paper is motivated by the analysis of numerical approximations of elliptic problems with singular sources. The standard finite element analysis is based on the variational formulation in Sobolev spaces. For example, for the classic Poisson problem in a bounded domain $\Omega \in \mathbb{R}^n$, it is known that the problem is well posed in $H_0^1(\Omega)$ whenever the right-hand side is in the dual space $H^{-1}(\Omega)$.

However, the finite element method can be applied in many situations where the right-hand side is not in $H^{-1}(\Omega)$, and consequently, the solution is not in $H_0^1(\Omega)$. Interesting examples of this situation arise when the right-hand side is given by a singular measure μ.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, we consider the Poisson problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

To perform a variational analysis, suitable in particular for finite element approximations, it is natural to work with weighted Sobolev spaces. This approach has been used in several papers (see, for example, [2, 3, 8, 9]).

Associated with a locally integrable function $w \geq 0$ we define the space $L_w^p(\Omega)$ as the usual $L^p(\Omega)$ space with measure $w(x)dx$ and $\mathbf{L}_w^p(\Omega) = L_w^p(\Omega)^n$. We will also work with the Sobolev spaces $W_w^{1,p}(\Omega) = \{v \in L_w^p(\Omega) : \nabla v \in \mathbf{L}_w^p(\Omega)\}$, which is a Banach space with the norm given by

$$\|v\|_{W_w^{1,p}(\Omega)} = \|v\|_{L_w^p(\Omega)} + \|\nabla v\|_{\mathbf{L}_w^p(\Omega)},$$

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†IMAS (UBA-CONICET), Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina (irene@drelichman.com, rduran@dm.uba.ar).
‡Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina (iojea@dm.uba.ar).

and $W^{1,p}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}$, where the closure is taken with respect to $\|\cdot\|_{W^{1,p}_w(\Omega)}$. As it is usual, we replace $W^{1,2}_w(\Omega)$ by $H^1_w(\Omega)$.

Consider, for example, the simple situation where μ is the Dirac δ and $0 \in \Omega$. In this case,

$$|\nabla u(x)| \sim |x|^{1-n} \notin L^2(\Omega)$$

but

$$|\nabla u| \in L^2_w(\Omega)$$

if $w(x) = |x|^\alpha$ with $\alpha > n-2$. Therefore, to analyze this problem one can work with a Sobolev space associated with w. More generally, in [9] the authors consider an application which leads to a problem like (1.1) with a measure μ supported in a curve Γ contained in a three-dimensional Ω. They propose to work with $w(x) = dist(x,\Gamma)^\alpha$, $0 < \alpha < 1$, and prove the well-posedness of the problem in the associated weighted Sobolev space when α is small enough. Afterwards, in [8], the author gives a more general stability result for the continuous as well as for the discrete problem obtained by the standard finite element method. However, his proof is not correct. Indeed, the argument given in that paper is based on a Helmholtz decomposition in weighted spaces. The author introduces a saddle point formulation of the problem and tries to prove the usual inf-sup conditions that imply the existence and uniqueness of solution. The flaw lies on the fact that (using the notation of that paper) the inf-sup conditions needed are

$$\sup_{\boldsymbol{\tau}\neq 0} \frac{a(\boldsymbol{\sigma},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{K}_2}} \geq \alpha_1 \|\boldsymbol{\sigma}\|_{\mathbf{K}_1}, \sup_{\boldsymbol{\sigma}\neq 0} \frac{a(\boldsymbol{\sigma},\boldsymbol{\tau})}{\|\boldsymbol{\sigma}\|_{\mathbf{K}_1}} \geq \alpha_2 \|\boldsymbol{\tau}\|_{\mathbf{K}_2},$$

where $\mathbf{K}_i = \{\boldsymbol{\sigma} : b_i(w,\boldsymbol{\sigma}) = 0\}$ and not those proved in [8] where these inequalities are proved but with $\mathbf{K}_i$ replaced by $\mathbf{M}_i$ (see Lemma 2.1 in that paper).

Recall that to obtain a Helmholtz decomposition for a vector field $\mathbf{q}$ one has to solve

$$\begin{cases} -\Delta u = \operatorname{div} \mathbf{q} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

with a control of ∇u in terms of $\mathbf{q}$. For example, for $\mathbf{q} \in \mathbf{L}^p_w(\Omega)$, we want to have the weighted a priori estimate

$$\|\nabla u\|_{\mathbf{L}^p_w(\Omega)} \leq C\|\mathbf{q}\|_{\mathbf{L}^p_w(\Omega)}. \tag{1.3}$$

The first goal of our paper is to prove this estimate for convex polygonal or polyhedral domains and for $w \in A_p$, $1 < p < \infty$ (see section 2 for the definition of the Muckenhoupt classes A_p). These kinds of domains are very important in finite element applications. Analogous estimates have been proved in [5, Theorem 2.5] for the case of C^1-domains.

Although this part of the paper concerns the continuous problem, it is important to remark that this a priori estimate plays an important role in the analysis of finite element approximations. Indeed, (1.3) is the starting point for the analysis of a posteriori error estimators (see [2]). On the other hand, we need this a priori estimate for a duality argument used to prove the stability in weighted norms of the discrete problem.

For nonsmooth domains the convexity assumption is necessary as it is shown by the following example. Consider a polygonal domain with an interior angle $\omega > \pi$ at the origin. It is known (see [15]) that the solution u can have a singularity such that

$|\nabla u| \sim |x|^{s-1}$, with $s = \pi/\omega < 1$, even if the right-hand side is very smooth. In such a case $|\nabla u|^p |x|^\alpha \sim |x|^{ps-p+\alpha}$, but $|x|^\alpha \in A_p$ for $-2 < \alpha < 2(p-1)$ (see, for example, the remark after Theorem 7.7 in [11]) and $|\nabla u| \notin L^p_{|x|^\alpha}$ whenever $-2 < \alpha \leq -2+p(1-s)$. On the other hand, assuming that the weight singularities are far from the boundary, as it is the case of the model problem considered in [8], the weighted a priori estimates can be generalized for nonconvex Lipschitz polytopes (see [25]).

As we mentioned at the beginning, our main motivation comes from the analysis of finite element methods. Usually, singular problems require the use of appropriate adapted meshes to obtain good numerical approximations efficiently. One way to produce this kind of meshes is based on the use of a posteriori error estimators. As it is known, efficient and reliable estimators can be derived by using the stability of the continuous problem. Therefore, these kinds of results could be obtained using (1.3). This was done for the case of $\mu = \delta$ in [2].

Another way to produce adapted meshes in problems where the location of the singularities is known a priori, like those considered here, is by using stability results in order to bound the approximation error by an interpolation one and then designing the meshes in such a way that this last error is of optimal order (see, for example, [3] and [24]).

To prove stability results in weighted norms for general meshes seems to be a very difficult task. Indeed, the problem is closely related with stability in $W^{1,p}$ norms for $2 < p \leq \infty$, a problem that has received great attention by people working in the theory of finite element methods in the last forty years (see, for example, the books [7, 4] and references therein). More precisely, as a consequence of a celebrated Rubio de Francia's extrapolation theorem, stability in H^1_w for all $w \in A_1$, would imply stability in $W^{1,p}$ for $2 < p < \infty$ as well as almost stability (i.e., up to a logarithmic factor) in $W^{1,\infty}$. As far as we know, this kind of results have not been proved for general meshes (not even assuming regularity of the family of triangulations).

The second goal of our paper is to prove stability results in weighted norms for standard finite element approximations under the assumption that the family of meshes is quasi-uniform. Although this is a severe restriction for the problems considered here, our result seems to be the first one on stability for a general family of weights, including those given by appropriate powers of the distance to a closed subset $\Gamma \subset \bar{\Omega}$ arising in the analysis of these problems. Further research is needed to improve the results in order to allow more realistic meshes. Our proof of the stability results make use of an estimate proved by Rannacher and Scott [26]. Roughly speaking, their result says that, if u_h denotes the finite element approximation to the solution u of a regular problem then, for any $z \in \Omega$, the value $|\nabla u_h(z)|$ is bounded by a local contribution given by the average of $|\nabla u|$ in the element containing z plus a decay estimate which is small away from z. It is interesting to remark that this is the only part of our argument where the restriction on the meshes is needed. It is worth noting that, since our arguments are based on estimates for the Green function, the same techniques may be applied to more general equations provided those estimates hold true.

The rest of the paper is organized as follows. In section 2 we recall the Muckenhoupt classes and prove the well posedness of the Poisson problem in weighted Sobolev spaces for convex polygonal or polyhedral domains. Section 3 deals with the stability in weighted norms for finite element approximations.

2. The continuous case. In this section we prove the weighted a priori estimate (1.3) for (1.2). We will follow the arguments given in [6] which are a generalization of techniques used to prove continuity of singular integral operators. The difference with [6] is that now we are interested in bounding first derivatives when the right-hand side is in a weaker space than those considered in that paper. Therefore, we need to use different estimates for the Green function.

As mentioned in the introduction, our motivation comes from the analysis of finite element approximations, and, therefore, it is important to consider polygonal or polyhedral domains. In our proofs we will use estimates for the Green function which, for these kinds of domains, have been proved only for the Poisson equation. On the other hand, if the domain is smooth enough, the estimates for the Green function that we are going to use are known to hold for general elliptic equations (see [20, Theorem 3.3]) and, therefore, our results apply in that case.

We will make also use of the Hardy–Littlewood maximal operator defined as

$$\mathcal{M}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes containing x. A useful well-known bound involving this operator that we will use is the following: if $\beta, \delta > 0$, then

$$\int_{|y-x|\geq\delta} \frac{|f(y)|}{|x-y|^{n+\beta}}\, dy \leq C\delta^{-\beta}\mathcal{M}f(x) \tag{2.1}$$

(see [17, Lemma (b)]).

A weight is a non-negative measurable function w defined in $\mathbb{R}^n$. Let us recall that, for $1 < p < \infty$, the Muckenhoupt A_p class is defined by the condition

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty$$

where the supremum is taken over all cubes Q. It is well known that $\mathcal{M}$ is bounded in $L^p_w(\mathbb{R}^n)$, for $1 < p < \infty$, if and only if $w \in A_p$ (see, for example, [11, Theorem 7.3]).

In the next section we will also work with the A_1 class. Recall that a weight is in A_1 if

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n} \frac{\mathcal{M}w(x)}{w(x)} < \infty. \tag{2.2}$$

In our proofs we will make use of the well-known property: if $p > 1$, then $A_1 \subset A_p$ and $[w]_{A_p} \leq [w]_{A_1}$ (see [12, Theorem 2.1(ii)]).

We will also need the local sharp maximal operator, namely,

$$\mathcal{M}^{\#}_{\Omega} f(x) = \sup_{\Omega\supset Q\ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy,$$

where now the supremum is taken over all cubes containing x and contained in Ω, and $f_Q = \frac{1}{|Q|}\int_Q f$. It is easy to see that $\mathcal{M}^{\#}_{\Omega}$ is a sublinear operator and that $\mathcal{M}^{\#}_{\Omega}|f| \leq \mathcal{M}^{\#}_{\Omega} f$. Moreover, to estimate $\mathcal{M}^{\#}_{\Omega} f$ one can replace the average f_Q by any constant a. Indeed,

$$\begin{aligned}\int_Q |f(x) - f_Q|\, dx &\leq \int_Q |f(x) - a|\, dx + |Q||a - f_Q| \\ &\leq \int_Q |f(x)-a|\, dx + |Q|\frac{1}{|Q|}\int_Q |a - f(x)|\, dx \leq 2\int_Q |f(x) - a|\, dx.\end{aligned}$$

It is known that the solution of the Poisson problem (1.1) is given by

$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy,$$

where $G(x,y)$ is the Green function (for its existence see, for example, [28, Chapter III, section 29]).

We need a Hölder-type estimate for the derivatives of G. We prove it in Lemma 2.1. This result is known for smooth domains (see point b. in the Corollary of Theorem 3.3 in [20]), and it is also stated for polyhedral domains in [16, equation (1.4)]. Hence, we only need to prove it for polygonal domains.

We begin stating some known estimates for the derivatives of G in polygonal domains.

Let $x^{(k)}$, $k = 1, \dots, K$ be the vertices of Ω. We denote $\rho_k(x) = \operatorname{dist}(x, x^{(k)})$ and $\mathcal{V}_k = B(x^{(k)}, \eta) \cap \Omega$ a neighborhood of $x^{(k)}$ for some fixed η sufficiently small, to guarantee that $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ whenever $i \neq j$. If ω_k is the interior angle on $x^{(k)}$, we take $\tau_k = \frac{\pi}{\omega_k}$. Observe that the convexity of Ω implies $\tau_k > 1$ for every k. This fact is crucial all along the proof.

We will make use of the following results:

$$|D_x^\alpha D_y^\beta G(x,y)| \leq C|x-y|^{-|\alpha|-|\beta|} \quad \text{for } |\alpha| = |\beta| = 1. \tag{2.3}$$

Moreover, for $x, y \in \mathcal{V}_k$, we have that (2.3) holds for every α and β such that $|\alpha| + |\beta| > 0$, provided that $\rho_k(y)/2 < \rho_k(x) < 2\rho_k(y)$. Also

$$|D_x^\alpha D_y^\beta G(x,y)| \leq C\rho_k(x)^{\tau_k - |\alpha| - \varepsilon} \rho_k(y)^{-\tau_k - |\beta| + \varepsilon} \quad \text{if } \rho_k(x) < \rho_k(y)/2, \tag{2.4}$$

and

$$|D_x^\alpha D_y^\beta G(x,y)| \leq C\rho_k(x)^{-\tau_k - |\alpha| + \varepsilon} \rho_k(y)^{\tau_k - |\beta| - \varepsilon} \quad \text{if } \rho_k(x) > 2\rho_k(y). \tag{2.5}$$

If $x \in \mathcal{V}_k$ and $y \in \mathcal{V}_\ell$ for $\ell \neq k$

$$|D_x^\alpha D_y^\beta G(x,y)| \leq C\rho_k(x)^{\tau_k - |\alpha| - \varepsilon} \rho_\ell(y)^{\tau_\ell - |\beta| - \varepsilon}, \tag{2.6}$$

and, finally, if $x \in \mathcal{V}_k$ and y is far from all vertices,

$$|D_x^\alpha D_y^\beta G(x,y)| \leq C\rho_k(x)^{\tau_k - |\alpha| - \varepsilon}. \tag{2.7}$$

In [14, Proposition 1] the reader can find the proof of (2.3) for every convex domain. If $x, y \in \mathcal{V}_k$ and $\rho_k(x)$ and $\rho_k(y)$ are similar to each other, (2.3) is also stated for derivatives of higher order in [22, page 286]. Estimates (2.4) and (2.5) can be found there, too. All these estimates are also stated in [23, Theorem 3 (c)], with the addition of (2.6). Estimate (2.7) is not explicitly stated in [22, 23], but it can be easily derived using the same arguments. Let us remark that it can also be obtained using the conformal transformation of the unit disk onto a convex polygon and how it acts on the Green function associated to the Laplacian (see [27, section 3]). Observe that in both [22, 23], general elliptic operators of order $2m$ are considered; in our case, set $m = 1$. The parameters τ_k are described in [22, 23] in a very general sense: certain pencil operators $\mathcal{A}_k$ associated with the elliptic problem near the vertices $x^{(k)}$ are considered, and it is proved that the line $Im(\lambda) = 0$ (for $\lambda \in \mathbb{C}$) is free of eigenvalues of $\mathcal{A}_k$. τ_k is then defined as the greater positive number such that $|Im(\lambda)| < \tau_k$ is free of eigenvalues of $\mathcal{A}_k$. However, in [19, section 2.1] it is shown that, for the case of an angle ω_k in a two-dimensional domain, the eigenvalues of $\mathcal{A}_k$ are exactly $\lambda_j = \pm\frac{j\pi}{\omega_k}$ for $j \in \mathbb{N}$. Hence we can take $\tau_k = \frac{\pi}{\omega_k}$.

LEMMA 2.1. *If Ω is a convex polygonal or polyhedral domain, there exist positive constants C and γ, depending on the geometry of Ω such that*

$$|\partial_{x_i}\partial_{y_j}G(x,y) - \partial_{x_i}\partial_{y_j}G(\bar{x},y)| \le C|x-\bar{x}|^{\gamma}(|x-y|^{-n-\gamma} + |\bar{x}-y|^{-n-\gamma}). \tag{2.8}$$

Proof. As we mentioned above, we only need to consider the case in which Ω is a convex polygon, since the three-dimensional case is proved in [16, equation (1.4)]. The proof is rather technical, and it is based on Lemmas 3.1 to 3.3 in [16].

We fix $\varepsilon > 0$ such that $\tau_k - 1 - \varepsilon > 0$ for every k, and take γ such that $0 < \gamma < \tau_k - 1 - \varepsilon$ for every k.

Observe that, since the singularities lie on the corners of the domain, it is enough to prove the result for $x \in B(x^{(k)}, \frac{\eta}{2}) \cap \Omega \subset \mathcal{V}_k$. Here, $B(x^{(k)}, \frac{\eta}{2})$ denotes the ball of radius $\frac{\eta}{2}$ centered at $x^{(k)}$. We take $M > 0$ a fixed constant satisfying some restrictions that we shall state later.

In what follows, set $I := |\partial_{x_i}\partial_{y_j}G(x,y) - \partial_{x_i}\partial_{y_j}G(\bar{x},y)|$. We consider three main cases, depending on the relationship between $x, \bar{x}$, and y, that will be also branched in several subcases.

Case 1: $|x-y| \le M|x-\bar{x}|$.

Applying the triangle inequality and (2.3) we obtain:

$$\begin{aligned} I &\le |\partial_{x_i}\partial_{y_j}G(x,y)| + |\partial_{x_i}\partial_{y_j}G(\bar{x},y)| \\ &\le C\big(|x-y|^{-2} + |\bar{x}-y|^{-2}\big) \\ &\le C|x-\bar{x}|^{\gamma}\left\{|x-y|^{-2}|x-\bar{x}|^{-\gamma} + |\bar{x}-y|^{-2}|x-\bar{x}|^{-\gamma}\right\}. \end{aligned}$$

Then, (2.8) follows by observing that $|x-y| \le M|x-\bar{x}|$, $|\bar{x}-y| \le (M+1)|x-\bar{x}|$ and $\gamma > 0$.

Case 2: $|x-y| > M|x-\bar{x}| > \rho_k(x)$.

Observe that, in this case,

$$\rho_k(\bar{x}) \le |x-\bar{x}| + \rho_k(x) \le M^{-1}|x-y| + \eta/2 \le M^{-1}diam(\Omega) + \eta/2,$$

hence, taking M sufficiently large, we may assume that $\bar{x} \in \mathcal{V}_k$.

Now, we have to distinguish two different situations, according to whether or not $y \in \mathcal{V}_k$:

If $y \in \mathcal{V}_\ell$ for some $\ell \ne k$ we may use that $\rho_\ell(y) \le diam(\Omega)$, the triangle inequality, and (2.6) to obtain

$$\begin{aligned} I &\le C\rho_\ell(y)^{\tau_\ell - 1-\varepsilon}\big(\rho_k(x)^{\tau_k-1-\varepsilon} + \rho_k(\bar{x})^{\tau_k-1-\varepsilon}\big) \\ &\le C|x-\bar{x}|^{\tau_k-1-\varepsilon} \\ &\le C|x-\bar{x}|^{\gamma}, \end{aligned}$$

where we have used that $\tau_\ell - 1 - \varepsilon > 0$, that $|x-\bar{x}| < 1$, and that $\tau_k - 1 - \varepsilon > \gamma$. The estimate follows by observing that $|x-y| \le diam(\Omega)$, so

$$\begin{aligned} I &\le C|x-\bar{x}|^{\gamma}|x-y|^{-2-\gamma}|x-y|^{2+\gamma} \\ &\le C|x-\bar{x}|^{\gamma}|x-y|^{-2-\gamma}. \end{aligned}$$

If y is far from all corners, it is immediate that the same estimate holds using (2.7) instead of (2.6).

If $y \in \mathcal{V}_k$ a more complex analysis is needed. We consider three subcases, according to the relation between $\rho_k(x)$ and $\rho_k(y)$:

- If $\rho_k(x) < \rho_k(y)/4$, we can apply (2.4) to (x, y). However, we need to show that this can be done for $(\bar{x}, y)$ too. Indeed, we have that $|x-y| < \rho_k(x) + \rho_k(y) < \frac{5}{4}\rho_k(y)$ and that $\rho_k(\bar{x}) \le |x-\bar{x}| + \rho_k(x) \le M^{-1}|x-y| + \rho_k(x) \le \left(\frac{5}{4M} + \frac{1}{4}\right)\rho_k(y)$. If we take $M > 5$, we obtain $\rho_k(\bar{x}) < \rho_k(y)/2$ and, therefore, (2.4) holds for $(\bar{x}, y)$.
  Finally, observe that $|x-\bar{x}| \le \rho_k(x) + \rho_k(\bar{x}) \le \frac{3}{4}\rho_k(y)$ and recall that $\rho_k(x), \rho_k(\bar{x}) \le C|x-\bar{x}|$. Then, applying (2.4) we obtain
  $$\begin{aligned} I &\le C\rho_k(y)^{-\tau_k-1+\varepsilon}\big(\rho_k(x)^{\tau_k-1-\varepsilon} + \rho_k(\bar{x})^{\tau_k-1-\varepsilon}\big) \\ &\le C\rho_k(y)^{-2-\gamma}\rho_k(y)^{\gamma-(\tau_k-1-\varepsilon)}|x-\bar{x}|^{\tau_k-1-\varepsilon} \\ &\le C|x-y|^{-2-\gamma}|x-\bar{x}|^{\gamma-(\tau_k-1-\varepsilon)}|x-\bar{x}|^{\tau_k-1-\varepsilon} \\ &= C|x-y|^{-2-\gamma}|x-\bar{x}|^{\gamma}, \end{aligned}$$
  where we have used that $\gamma - (\tau_k - 1 - \varepsilon) < 0$.
- If $\rho_k(x) > 4\rho_k(y)$, we can apply (2.5) to (x, y). As in the previous subcase, we need to prove that the same can be done for $(\bar{x}, y)$. Indeed, we have that $|x-y| \le \rho_k(x) + \rho_k(y) \le \frac{5}{4}\rho_k(x)$ and that $\rho_k(x) \le |x-\bar{x}| + \rho_k(\bar{x}) \le M^{-1}|x-y| + \rho_k(\bar{x}) \le \frac{5}{4}M^{-1}\rho_k(x) + \rho_k(\bar{x})$. Now we recall that we assumed $M \ge 5$, which allows us to kick back the $\rho_k(x)$ term, obtaining $\rho_k(x) \le \frac{4M}{4M-5}\rho_k(\bar{x})$, and, consequently,
  $$\rho_k(y) \le \frac{M}{4M-5}\rho_k(\bar{x}) \le \frac{1}{2}\rho_k(\bar{x}),$$
  which implies that (2.5) holds for $\bar{x}$.
  Hence, using that $\rho_k(y) \le C\rho_k(x)$, that $|x-y| \le C\rho_k(x)$, and that $\rho_k(x) \le C|x-\bar{x}|$ we have
  $$\begin{aligned} I &\le C\rho_k(y)^{\tau_k-1-\varepsilon}\big(\rho_k(x)^{-\tau_k-1+\varepsilon} + \rho_k(\bar{x})^{-\tau_k-1+\varepsilon}\big) \\ &\le C\rho_k(x)^{-2} \\ &\le C\rho_k(x)^{-2-\gamma}\rho_k(x)^{\gamma} \\ &\le C|x-y|^{-2-\gamma}|x-\bar{x}|^{\gamma}. \end{aligned}$$
- If $\rho_k(y)/4 \le \rho_k(x) \le 4\rho_k(y)$ we only need (2.3). We have that $|x-y| \le \rho_k(x) + \rho_k(y) \le 5\rho_k(x) \le 5M|x-\bar{x}|$. Applying (2.3),
  $$\begin{aligned} I &\le C\big(|x-y|^{-2} + |\bar{x}-y|^{-2}\big) \\ &\le C|x-y|^{-2-\gamma}|x-y|^{\gamma} \\ &\le C|x-y|^{-2-\gamma}|x-\bar{x}|^{\gamma}, \end{aligned}$$
  where we have eliminated the term $|\bar{x}-y|$ thanks to the fact that $|x-y| \le |x-\bar{x}| + |\bar{x}-y| \le M^{-1}|x-y| + |\bar{x}-y|$, which implies that $|x-y| \le C|\bar{x}-y|$.

Case 3: $|x-y| > M|x-\bar{x}|$ and $\rho_k(x) > M|x-\bar{x}|$.
We use a mean value argument, obtaining

$$I \le |x-\bar{x}||\nabla_x \partial_{x_i}\partial_{y_j} G(z, y)| \tag{2.9}$$

for $z = x + s(\bar{x} - x)$, $0 \le s \le 1$. Moreover, we have that

$$|x-\bar{x}| \le M^{-1}\rho_k(x) \le M^{-1}(\rho_k(z) + |z-x|) \le M^{-1}(\rho_k(z) + |x-\bar{x}|),$$

and, consequently, $|x-\bar{x}| \le \frac{1}{M-1}\rho_k(z)$.

As in Case 2, *if* $y \notin \mathcal{V}_k$, consider first $y \in \mathcal{V}_\ell$ for some $\ell \neq k$. In this case, (2.6) applied to (2.9) gives (assuming $\gamma \le 1$)

$$\begin{aligned}
I &\le C|x-\bar{x}|\rho_k(z)^{\tau_k-2-\varepsilon}\rho_\ell(y)^{\tau_\ell-1-\varepsilon}\\
&\le C|x-\bar{x}|^\gamma\rho_k(z)^{\tau_k-2-\varepsilon+1-\gamma}\\
&\le C|x-\bar{x}|^\gamma\\
&\le C|x-\bar{x}|^\gamma|x-y|^{-2-\gamma}|x-y|^{2+\gamma}\\
&\le C|x-\bar{x}|^\gamma|x-y|^{-2-\gamma},
\end{aligned}$$

where we have used that $\rho_\ell(y), \rho_k(z), |x-y| \le diam(\Omega)$.

The same estimate can be obtained when y is far from all vertices using (2.7) instead of (2.6).

If $y \in \mathcal{V}_k$, we can easily check that $\rho_k(z) \le |x-\bar{x}|+\rho_k(x) \le (1+M^{-1})\rho_k(x) \le (1+M^{-1})\eta/2$, so that $z \in \mathcal{V}_k$. Once again, we split the proof in three subcases:

- If $\rho_k(z) < \rho_k(y)/4$, applying (2.4) to the right-hand side (RHS) of (2.9) and recalling that $|x-\bar{x}| \le C\rho_k(z)$ we obtain

$$\begin{aligned}
I &\le C|x-\bar{x}|\rho_k(z)^{\tau_k-2-\varepsilon}\rho_k(y)^{-\tau_k-1+\varepsilon}\\
&\le C|x-\bar{x}|^\gamma\rho_k(z)^{\tau_k-1-\varepsilon-\gamma}\rho_k(y)^{-\tau_k-1+\varepsilon}\\
&\le C|x-\bar{x}|^\gamma\rho_k(y)^{-2-\gamma}\\
&\le C|x-\bar{x}|^\gamma|x-y|^{-2-\gamma},
\end{aligned}$$

  where in the last step we have used the estimate $|x-y| < |x-z| + \rho_k(z)+\rho_k(y) \le C\rho_k(z)+\rho_k(y) \le C\rho_k(y)$.
- If $\rho_k(z) > 4\rho_k(y)$, observe that we have $|x-y| \le |x-z|+\rho_k(z)+\rho_k(y) \le C\rho_k(z)$. Applying (2.5) to the RHS of (2.9) we obtain

$$\begin{aligned}
I &\le C|x-\bar{x}|\rho_k(z)^{-\tau_k-2+\varepsilon}\rho_k(y)^{\tau_k-1-\varepsilon}\\
&\le C|x-\bar{x}|^\gamma\rho_k(z)^{-\tau_k-1+\varepsilon-\gamma}\rho_k(y)^{\tau_k-1-\varepsilon}\\
&\le C|x-\bar{x}|^\gamma\rho_k(z)^{-1-\gamma} \le C|x-\bar{x}|^\gamma|x-y|^{-1-\gamma}\\
&\le C|x-\bar{x}|^\gamma|x-y|^{-2-\gamma},
\end{aligned}$$

  where we have used again $|x-y| \le diam(\Omega)$.
- If $\rho_k(y)/4 \le \rho_k(z) \le 4\rho_k(y)$, as in the last step of Case 2, we only need (2.3) (but now $|\alpha|+|\beta| > 2$). We have that $|x-y| \le |x-z|+|z-y| \le M^{-1}|x-y|+|z-y|$, which leads to $|x-y| \le \frac{M}{M-1}|z-y|$. Therefore, applying (2.3) to the RHS of (2.9)

$$\begin{aligned}
I &\le C|x-\bar{x}||z-y|^{-3}\\
&\le C|x-\bar{x}|^\gamma|z-y|^{-2-\gamma}
\end{aligned}$$

  and the result follows. □

Given $w \in A_p$ we consider (1.2) with $\mathbf{q} \in \mathbf{L}^p_w(\Omega)$. Recalling that $G(x,y)=0$, for $y \in \partial\Omega$, we have

$$u(x) = \int_\Omega G(x,y)\operatorname{div}\mathbf{q}(y)\,dy = -\int_\Omega \nabla_y G(x,y)\cdot\mathbf{q}(y)\,dy. \tag{2.10}$$

We will use the following known unweighted a priori estimate.

Lemma 2.2. *Let Ω be a convex domain and u be the solution of* (1.2). *Then, for* $1 < p < \infty$, *we have*

$$\|\nabla u\|_{\mathbf{L}^p(\Omega)} \leq C\|\mathbf{q}\|_{\mathbf{L}^p(\Omega)}. \tag{2.11}$$

Proof. In [14], it is stated that for $1 < p < \infty$, Ω a bounded convex domain and $f \in W^{-1,p}(\Omega) = (W_0^{1,p'}(\Omega))'$, there is a unique solution $u \in W_0^{1,p}(\Omega)$ of the problem $\Delta u = f$, and that $\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{W^{-1,p}(\Omega)}$ (see [14, Corollary 1], taking $s = 1$). Now, take $f = \operatorname{div}\mathbf{q}$. To estimate $\|f\|_{W^{-1,p}(\Omega)}$, take $g \in W_0^{1,p'}(\Omega)$:

$$f(g) = \int_\Omega fg = \int_\Omega \operatorname{div}\mathbf{q}\; g = \int_\Omega \mathbf{q}\cdot\nabla g \leq \|\mathbf{q}\|_{\mathbf{L}^p(\Omega)}\|g\|_{W_0^{1,p'}(\Omega)}.$$

Hence, $\|f\|_{W^{-1,p}(\Omega)} \leq \|\mathbf{q}\|_{\mathbf{L}^p(\Omega)}$, and the result follows. ∎

The argument given in [6] makes use of the following inequality proved in [10, Theorem 5.23].

Lemma 2.3. *For $f \in L^1_{loc}(\Omega)$, $w \in A_p$ and f_Ω the mean value of f over Ω, we have*

$$\|f - f_\Omega\|_{L^p_w(\Omega)} \leq C\|\mathcal{M}^\#_\Omega f\|_{L^p_w(\Omega)}.$$

In what follows we make use of the fact that $C_0^\infty(\Omega)$ is dense in $L^p_w(\Omega)$ (see [18, Corollary 1.7]) and, therefore, we can assume that $\mathbf{q}$ is smooth. Hence, pointwise values of the derivatives of u are well defined.

Lemma 2.4. *Let Ω be a convex polygonal or polyhedral domain and u be the solution of* (1.2). *Then, for any $s > 1$, we have*

$$\mathcal{M}^\#_\Omega(|\nabla u|)(\bar{x}) \leq C(\mathcal{M}|\mathbf{q}|^s)^{\frac{1}{s}}(\bar{x})$$

for all $\bar{x} \in \Omega$.

Proof. We extend $\mathbf{q}$ by zero outside Ω. Given $\bar{x} \in \Omega$, let $Q \subset \Omega$ be a cube such that $\bar{x} \in Q$ and let Q^* be an expansion of Q by a factor 2. We decompose $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$, where $\mathbf{q}_1 = \chi_{Q^*}\mathbf{q}$, where χ_{Q^*} denotes the characteristic function of Q^*, and call u_i the solution of (1.2) with RHS given by $\operatorname{div}\mathbf{q}_i$.

By sublinearity it is enough to bound $\mathcal{M}^\#_\Omega(\partial_{x_i}u(\bar{x}))$ for any i. Also, as mentioned after the definition of $\mathcal{M}^\#_\Omega$, we may replace the average by any constant. We take $a = \partial_{x_i}u_2(\bar{x})$ to obtain

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |\partial_{x_i}u(x) - \partial_{x_i}u_2(\bar{x})|\,dx\\ &\leq \frac{1}{2}\frac{1}{|Q|}\int_Q |\partial_{x_i}u_1(x)|\,dx + \frac{1}{|Q|}\int_Q |\partial_{x_i}u_2(x) - \partial_{x_i}u_2(\bar{x})|\,dx =: (i) + (ii).\end{aligned}$$

Given $s > 1$, using Hölder's inequality, the unweighted estimate (2.11) in L^s, and recalling that $\mathbf{q}_1$ vanishes outside $\Omega \cap Q^*$, we have

$$(i) \leq \left(\frac{1}{|Q|}\int_Q |\partial_{x_i}u_1(x)|^s\,dx\right)^{\frac{1}{s}} \leq C\left(\frac{1}{|Q|}\int_{Q^*} |\mathbf{q}_1(x)|^s\,dx\right)^{\frac{1}{s}} \leq C(\mathcal{M}|\mathbf{q}|^s)^{\frac{1}{s}}(\bar{x}).$$

To bound (ii), since x and $\bar{x}$ are outside the support of $\mathbf{q}_2$, we can take the derivative inside the integral in the expression for u_2 given by (2.10), and using (2.8), we obtain

$$\begin{aligned}(ii) &\leq \frac{1}{|Q|}\int_Q \int_{\Omega\cap(Q^*)^c} |\partial_{x_i}\nabla_y G(x,y) - \partial_{x_i}\nabla_y G(\bar{x},y)||\mathbf{q}_2(y)|\,dy\,dx \\ &\leq \frac{C}{|Q|}\int_Q \int_{(Q^*)^c} |x-\bar{x}|^\gamma(|x-y|^{-n-\gamma} + |\bar{x}-y|^{-n-\gamma})|\mathbf{q}(y)|\,dy\,dx.\end{aligned}$$

Now, since $x, \bar{x} \in Q$, and $y \in (Q^*)^c$, we have $|x-y| \sim |\bar{x}-y| \geq \frac{\ell(Q)}{2}$, where $\ell(Q)$ denotes the length of the edges of Q, and therefore,

$$\begin{aligned}(ii) &\leq C\frac{\ell(Q)^\gamma}{|Q|}\int_Q \int_{(Q^*)^c} \frac{|\mathbf{q}(y)|}{|\bar{x}-y|^{n+\gamma}}\,dy\,dx \\ &\leq C\int_{\ell(Q)/2<|\bar{x}-y|} \frac{\ell(Q)^\gamma|\mathbf{q}(y)|}{|\bar{x}-y|^{n+\gamma}}\,dy \leq C\mathcal{M}|\mathbf{q}|(\bar{x}),\end{aligned}$$

where the last inequality follows from (2.1).

But, by Hölder's inequality, $\mathcal{M}|\mathbf{q}|(\bar{x}) \leq (\mathcal{M}|\mathbf{q}|^s)^{\frac{1}{s}}(\bar{x})$ and so the lemma is proved. ∎

Now, we are able to prove our main result, namely, the weighted estimate for ∇u.

THEOREM 2.5. *Let Ω be a convex polygonal or polyhedral domain. Given $1 < p < \infty$ and $w \in A_p$, if $\mathbf{q} \in \mathbf{L}^p_w(\Omega)$ and u is the solution of* (1.2)*, there exists a constant C depending on p, Ω, and w such that*

$$\|\nabla u\|_{\mathbf{L}^p_w(\Omega)} \leq C\|\mathbf{q}\|_{\mathbf{L}^p_w(\Omega)}.$$

Proof. Let $(\nabla u)_\Omega := \frac{1}{|\Omega|}\int_\Omega |\nabla u(x)|\,dx$. We have

$$\|\nabla u\|_{\mathbf{L}^p_w(\Omega)} \leq \|\nabla u - (\nabla u)_\Omega\|_{\mathbf{L}^p_w(\Omega)} + \|(\nabla u)_\Omega\|_{\mathbf{L}^p_w(\Omega)} =: I + II.$$

Now, it is known that if $w \in A_p$, then $w \in A_{\frac{p}{s}}$ for some s such that $1 < s < p$ (see, for example, [11, Corollary 7.6]). Then, using Lemmas 2.3 and 2.4, and that $\mathcal{M}$ is bounded on $L^{\frac{p}{s}}_w$ ([11, Theorem 7.3]), we obtain

$$I \leq C\|\mathcal{M}^\#_\Omega(|\nabla u|)\|_{L^p_w(\Omega)} \leq C\|(\mathcal{M}|\mathbf{q}|^s)^{\frac{1}{s}}\|_{L^p_w(\Omega)} \leq C\|\mathbf{q}\|_{\mathbf{L}^p_w(\Omega)}.$$

Then, to finish the proof it is enough to bound $|(\nabla u)_\Omega|$. Using Hölder's inequality, with exponent s in the first inequality and with exponent p/s in the third one, and the a priori estimate (2.11) for the second inequality, we obtain

$$\begin{aligned}|(\nabla u)_\Omega| &\leq \left(\frac{1}{|\Omega|}\int_\Omega |\nabla u(x)|^s\,dx\right)^{\frac{1}{s}} \leq C\left(\frac{1}{|\Omega|}\int_\Omega |\mathbf{q}(x)|^s\,dx\right)^{\frac{1}{s}} \\ &\leq C\left(\frac{1}{|\Omega|}\int_\Omega |\mathbf{q}(x)|^p w(x)\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|\Omega|}\int_\Omega w(x)^{-\frac{s}{p-s}}\,dx\right)^{\frac{p-s}{ps}},\end{aligned}$$

and the last integral is finite since $w \in A_{\frac{p}{s}}$. ∎

Now we can prove the well-posedness of (1.1). This result follows from Theorem 2.5 by standard functional analysis arguments. We give it here for the sake of completeness. In the proof we will use the following weighted Poincaré inequality (see [21, Chapter 2 section 15]): if $w \in A_p$, then there exists a constant C such that

$$\|v\|_{L^p_w(\Omega)} \le C\|\nabla v\|_{\mathbf{L}^p_w(\Omega)} \qquad \forall v \in W^{1,p}_{w,0}(\Omega). \tag{2.12}$$

Given $w \in A_p$ we introduce its dual weight $w' := w^{-1/(p-1)}$. It is known that $w' \in A_{p'}$ (see, for example, [12, Theorem 2.1(i)]) and that $\mathbf{L}^p_w(\Omega)' = \mathbf{L}^{p'}_{w'}(\Omega)$.

REMARK 2.6. *The proof of Theorem* 2.5 *also applies to any second order elliptic operator for which one has estimate* (2.8) *for the corresponding Green function and Lemma* 2.2.

COROLLARY 2.7. *If* Ω *is a convex polygonal or polyhedral domain,* $1 < p < \infty$ *and* $w \in A_p$, *then, given* $\mu \in (W^{1,p'}_{w',0}(\Omega))'$, *there exists a unique solution of problem* (1.1) *satisfying,*

$$\|u\|_{W^{1,p}_w(\Omega)} \le C\|\mu\|_{(W^{1,p'}_{w',0}(\Omega))'}. \tag{2.13}$$

Proof. We define $\mathcal{L}(\nabla v) := -\langle \mu, v\rangle$ which is a linear functional over the subspace of $\mathbf{L}^{p'}_{w'}(\Omega)$ given by gradient fields of $W^{1,p'}(\Omega)$. Using the Poincaré inequality (2.12) we have

$$|\mathcal{L}(\nabla v)| \le \|\mu\|_{(W^{1,p'}_{w',0}(\Omega))'}\|v\|_{L^{p'}(\Omega)} \le C\|\mu\|_{(W^{1,p'}_{w',0}(\Omega))'}\|\nabla v\|_{\mathbf{L}^{p'}_{w'}(\Omega)}.$$

Therefore, $\mathcal{L}$ defines a continuous linear functional on the gradient fields of functions in $W^{1,p'}_{w',0}(\Omega)$ and so, by the Hahn–Banach theorem, it can be extended to all $\mathbf{L}^{p'}_{w'}(\Omega)$. Therefore, there exists $\mathbf{q} \in \mathbf{L}^p_w(\Omega)$ such that $\|\mathbf{q}\|_{\mathbf{L}^p_w(\Omega)} = \|\mu\|_{(W^{1,p'}_{w',0}(\Omega))'}$ and $\langle \mathbf{q}, \nabla v\rangle = -\langle \mu, v\rangle$. Then $\operatorname{div}\mathbf{q} = \mu$, and therefore, the existence of u and the estimate (2.13) are immediate consequences of Theorem 2.5 and (2.12). ◻

The results obtained above can be applied to the problem considered in [8]. In that paper the author considers a problem like (1.1) with μ supported in a curve contained in a three-dimensional domain. He works with a weighted space where the weight is a power of the distance to the curve. More generally one can consider $\Gamma \subset \overline{\Omega} \subset \mathbb{R}^n$ where Γ is a compact set. We will assume that Γ is a k-regular set for some $0 \le k < n$, namely, there exist constants $C_1, C_2 > 0$ such that $C_1 r^k \le \mathcal{H}^k(B(x,r)\cap\Gamma) \le C_2 r^k$ for every $x \in \Gamma$ and $0 < r \le diam(\Gamma)$, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. Let us remark that k is not necessarily an integer. However, if Γ is smooth, then k is the usual dimension.

To simplify notation we introduce $w_\lambda := dist(x,\Gamma)^\lambda$. It is known that, if Γ is a k-regular set, then, for $1 \le p < \infty$,

$$-(n-k) < \lambda < (n-k)(p-1) \Longrightarrow w_\lambda \in A_p \tag{2.14}$$

(see [13, Lemma 2.3,vi] or [1, Appendix B]).

THEOREM 2.8. *If* Ω *is a convex polygonal or polyhedral domain,* $\Gamma \subset \overline{\Omega}$ *is a* k*-regular set and* $1 < p < \infty$, *then, for* $-(n-k) < \lambda < (n-k)(p-1)$, *given* $\mu \in (W^{1,p'}_{w_{-\lambda/(p-1)},0}(\Omega))'$ *there exists a unique solution* $u \in W^{1,p}_{w_\lambda}(\Omega)$ *of* (1.1) *satisfying,*

$$\|u\|_{W^{1,p}_{w_\lambda}(\Omega)} \le C\|\mu\|_{\left(W^{1,p'}_{w_{-\lambda/(p-1)},0}(\Omega)\right)'}.$$

Proof. In view of (2.14) the result is an immediate consequence of Corollary 2.7. ∎

In particular, taking $n = 3$, $k = 1$, and $p = 2$ we obtain the result stated in [8, Corollary 2.2].

3. The discrete case. The goal of this section is to prove weighted stability estimates for finite element approximations of the Poisson equation.

Given a convex polygonal or polyhedral domain Ω and a family of triangulations $\mathcal{T}_h$, where as usual $h > 0$ denotes the maximum of the diameters of the elements, let V_h^k be the space of continuous piecewise polynomial functions of degree $k \geq 1$. The finite element approximation $u_h^k \in V_h^k$ of u is given by

$$\int_\Omega \nabla u_h^k \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v \qquad \forall v \in V_h^k.$$

Observe that u_h^k is well defined for any $u \in W^{1,1}(\Omega)$, in particular, for any $u \in L^1(\Omega)$ such that $\nabla u \in \mathbf{L}_w^p(\Omega)$ for some $w \in A_p$.

Since k will be fixed, we will drop it from now on and will write simply V_h and u_h.

LEMMA 3.1. *Let Ω be a convex polygonal or polyhedral domain and assume that the family of partitions $\mathcal{T}_h$ is quasi-uniform. Then, for $u \in C_0^\infty(\Omega)$, there exist positive constants C and σ such that, if T_z is an element containing z, then*

$$|\nabla u_h(z)|^2 \leq C \left\{ \left(\frac{1}{h^n} \int_{T_z} |\nabla u(x)|\, dx \right)^2 + \int_\Omega \frac{h^\sigma}{(|x-z|^2 + h^2)^{\frac{n+\sigma}{2}}} |\nabla u(x)|^2\, dx \right\}.$$

Proof. Following [4, section 8.2] we introduce a regularized delta function $\delta^z \in C_0^\infty(T_z)$ satisfying

$$\int_\Omega \delta^z(x) P(x)\, dx = P(z) \quad \forall P \in \mathcal{P}_k$$

and

$$\|D^k \delta^z\|_{L^\infty(\Omega)} \leq C h^{-n-k}, \quad k = 0, 1, \ldots \tag{3.1}$$

Since z is arbitrary but fixed, we drop the z and write simply δ.

An immediate consequence of [4, Corollary 8.2.8] is that there exist positive constants C and σ such that

$$\left| \frac{\partial u_h}{\partial x_j}(z) \right| \leq C \left\{ \int_{T_z} \left| \frac{\partial u}{\partial x_j}(x) \right| \delta\, dx + \left(\int_\Omega \frac{h^\sigma}{(|x-z|^2 + (Kh)^2)^{\frac{n+\sigma}{2}}} |\nabla u(x)|^2\, dx \right)^{\frac{1}{2}} \right\}$$

where $K > 1$ is a constant. Using the support of δ and (3.1), we obtain the desired result.

However, the proof of [4, Corollary 8.2.8] requires a more restrictive condition on the angles in the three-dimensional case. But, by a slight modification of the arguments in [16] it was shown by the third author in [24] that the result is still true for general convex polyhedral domains. We include the proof here for the sake of completeness.

We define $g \in H_0^1(\Omega)$ as the solution of $-\Delta g = \frac{\partial \delta}{\partial x_j}$ and $g_h \in V_h$ as its Galerkin projection. Then, it is easy to see that

$$\begin{aligned}\frac{\partial u_h}{\partial x_j}(z) &= \int_\Omega \frac{\partial u_h}{\partial x_j}\,\delta\,dx \\ &= \int_\Omega \frac{\partial u}{\partial x_j}\delta\,dx - \int_\Omega \frac{\partial(u-u_h)}{\partial x_j}\delta\,dx \\ &= \int_\Omega \frac{\partial u}{\partial x_j}\delta\,dx + \int_\Omega (u-u_h)\frac{\partial\delta}{\partial x_j}\,dx \\ &= \int_\Omega \frac{\partial u}{\partial x_j}\delta\,dx + \int_\Omega \nabla(u-u_h)\cdot\nabla g\,dx \\ &= \int_\Omega \frac{\partial u}{\partial x_j}\delta\,dx + \int_\Omega \nabla u\cdot\nabla(g-g_h)\,dx.\end{aligned}$$

Therefore,

$$|\nabla u_h(z)| \le C\left\{\frac{1}{h^n}\int_{T_z}|\nabla u(x)|\,dx + \int_\Omega |\nabla u(x).\nabla(g-g_h)(x)|\,dx\right\},$$

and, for any $K>0$,

$$\begin{aligned}\int_\Omega |\nabla u(x)\cdot\nabla(g-g_h)(x)|\,dx \le &\left(\int_\Omega \frac{h^\sigma}{(|x-z|^2+(Kh)^2)^{\frac{3+\sigma}{2}}}|\nabla u(x)|^2\,dx\right)^{\frac12} \\ &\cdot\left(\int_\Omega \frac{(|x-z|^2+(Kh)^2)^{\frac{3+\sigma}{2}}}{h^\sigma}|\nabla(g-g_h)(x)|^2\,dx\right)^{\frac12}.\end{aligned}$$

Hence, it suffices to see that, for convex polyhedral $\Omega\subset\mathbb{R}^3$ and for sufficiently large K to be chosen below,

$$\int_\Omega (|x-z|^2+(Kh)^2)^{\frac{3+\sigma}{2}}|\nabla(g-g_h)(x)|^2\,dx \le Ch^\sigma. \tag{3.2}$$

To see this, observe first that we may assume by rescaling that $diam(\Omega)=1$. Following the proof of [16, Theorem 2], we set $d_j := 2^{-j}$ and split Ω into the subdomains

$$\Omega^* := \{x\in\Omega : |x-z|\le Kh\}$$

and

$$\Omega_j := \{x\in\Omega : d_{j+1} < |x-z| \le d_j\} \quad (j=1,\dots,J)$$

where J is such that $2^{-J}\le Kh\le 2^{-J+1}$. Then, we have

$$\int_\Omega (|x-z|^2+(Kh)^2)^{\frac{3+\sigma}{2}}|\nabla(g-g_h)(x)|^2\,dx \le \left(\int_{\Omega^*} + \sum_{j=0}^{J}\int_{\Omega_j}\right)\dots\,dx = I+II.$$

I can be bounded as follows:

$$\begin{aligned}I &\le (Kh)^{3+\sigma}\|\nabla(g-g_h)\|^2_{L^2(\Omega)} \\ &\le C(Kh)^{3+\sigma}h^2\|D^2g\|^2_{L^2(\Omega)} \\ &\le C(Kh)^{3+\sigma}h^2\|\nabla\delta\|^2_{L^2(\Omega)} \\ &\le C(Kh)^{3+\sigma}h^2\|\nabla\delta\|^2_{L^\infty(\Omega)}h^3 \\ &\le CK^{3+\sigma}h^\sigma,\end{aligned}$$

where in the third step we have used a well-known a priori estimate valid for convex domains.

Before we bound II, as in [16, Theorem 2, Step 1], we define $M_j = d_j^{\frac{3}{2}} \|\nabla(g - g_h)\|_{L^2(\Omega_j)}$. By the last inequality of [16, Step 3] and [16, equation (4.6)], there holds

$$M_j \leq C\left(\frac{h}{d_j}\right)^{\sigma} + Chd_j^{\frac{1}{2}}\|\nabla(g-g_h)\|_{L^2(\Omega_j'')},$$

where $\Omega_j'' = \{x \in \Omega : d_{j+3} \leq |x - z| \leq d_{j-2}\}$. Therefore, observing that in each Ω_j, $|x - z| + Kh \sim d_j$, we have

$$\begin{aligned} II &\leq C\sum_{j=0}^{J} d_j^{3+\sigma}\|\nabla(g-g_h)\|^2_{L^2(\Omega_j)} \\ &= C\sum_{j=0}^{J} d_j^{\sigma} M_j^2 \\ &\leq C\sum_{j=0}^{J} d_j^{\sigma}\left(\frac{h}{d_j}\right)^{2\sigma} + C\sum_{j=0}^{J} h^2 d_j^{1+\sigma}\|\nabla(g-g_h)\|^2_{L^2(\Omega_j'')} \\ &\leq C\sum_{j=0}^{J} d_j^{\sigma}\left(\frac{h}{d_j}\right)^{2\sigma} + C\sum_{j=0}^{J}\left(\frac{h}{d_j}\right)^{2} d_j^{3+\sigma}\|\nabla(g-g_h)\|^2_{L^2(\Omega_j'')} \\ &\leq C\sum_{j=0}^{J} d_j^{\sigma}\left(\frac{h}{d_j}\right)^{2\sigma} + C\sum_{j=0}^{J}\frac{1}{K^2} d_j^{3+\sigma}\|\nabla(g-g_h)\|^2_{L^2(\Omega_j'')}. \end{aligned}$$

Since the last term on the right is a multiple of II, for sufficiently large K we may kick-back the last term on the RHS to obtain

$$II \leq C\sum_{j=0}^{J} d_j^{\sigma}\left(\frac{h}{d_j}\right)^{2\sigma} \leq Ch^{2\sigma}\sum_{j=0}^{J} d_j^{-\sigma} \leq Ch^{2\sigma}2^{J\sigma} \leq C\frac{h^{\sigma}}{K^{\sigma}}$$

which finishes the proof of (3.2). ∎

THEOREM 3.2. *Let Ω be a convex polygonal or polyhedral domain and assume that the family of partitions $\mathcal{T}_h$ is quasi-uniform. If $w \in A_1$ and $u \in H^1_{w,0}(\Omega)$, then there exists a constant C, depending only on $[w]_{A_1}$, such that*

$$\|\nabla u_h\|_{\mathbf{L}^2_w(\Omega)} \leq C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}.$$

Proof. Assume first that $u \in C_0^\infty(\Omega)$. Using Lemma 3.1 we obtain

$$|\nabla u_h(z)|^2 \leq C\left\{\mathcal{M}(|\nabla u(z)|)^2 + \int_\Omega \frac{h^\sigma}{(|x-z|^2+h^2)^{\frac{n+\sigma}{2}}}|\nabla u(x)|^2\,dx\right\}.$$

Then, multiplying by $w(z)$ an integrating we obtain

$$\begin{aligned} \int_\Omega |\nabla u_h(z)|^2 w(z)\,dz \leq C\Bigg\{&\int_\Omega \mathcal{M}(|\nabla u(z)|)^2 w(z)\,dz \\ &+ \int_\Omega\int_\Omega \frac{h^\sigma|\nabla u(x)|^2 w(z)}{(|x-z|^2+h^2)^{\frac{n+\sigma}{2}}}\,dx\,dz\Bigg\}. \end{aligned} \tag{3.3}$$

But

$$\int_\Omega \frac{h^\sigma w(z)}{(|x-z|^2+h^2)^{\frac{n+\sigma}{2}}}\,dz \le \frac{1}{h^n}\int_{|x-z|\le h} w(z)\,dz + \int_{|x-z|>h}\frac{h^\sigma w(z)}{|x-z|^{n+\sigma}}\,dz.$$

Using the definiton of $\mathcal{M}$ to bound the first term and (2.1) to bound the second one, we have

$$\int_\Omega \frac{h^\sigma w(z)}{(|x-z|^2+h^2)^{\frac{n+\sigma}{2}}}\,dz \le C\mathcal{M}w(x),$$

and, therefore, interchanging the order of integration in (3.3),

$$\int_\Omega |\nabla u_h(z)|^2 w(z)\,dz \le C\left\{\int_\Omega \mathcal{M}(|\nabla u(z)|)^2 w(z)\,dz + \int_\Omega |\nabla u(x)|^2 \mathcal{M}w(x)\,dx\right\}.$$

In particular, if $w \in A_1$, using (2.2) and recalling that $A_1 \subset A_2$ and so the maximal operator is bounded in $L^2_w(\Omega)$, we conclude that

$$\|\nabla u_h\|_{\mathbf{L}^2_w(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}$$

for $u \in C_0^\infty(\Omega)$.

The following density argument finishes the proof: let $(u_j)_{j\in\mathbb{N}}$ be a sequence of functions in $C_0^\infty(\Omega)$ such that $u_j \to u$ in $H^1_w(\Omega)$. By the above inequality

$$\|\nabla(u_{j,h}-u_{k,h})\|_{\mathbf{L}^2_w(\Omega)} \le C\|\nabla(u_j-u_k)\|_{\mathbf{L}^2_w(\Omega)},$$

whence $(\nabla u_{j,h})_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbf{L}^2_w(\Omega)$ for each h. By Poincaré's inequality (2.12), it follows that $(u_{j,h})_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{w,0}(\Omega)$ and, therefore, it exists $\tilde{u}_h := \lim_{j\to\infty} u_{j,h}$, $\tilde{u}_h \in V_h$. It remains to see that $\tilde{u}_h = u_h$, but, since for all $v \in V_h$,

$$\int_\Omega \nabla u_{j,h}(x)\cdot\nabla v(x)\,dx = \int_\Omega \nabla u_j(x)\cdot\nabla v(x)\,dx,$$

we obtain

$$\int_\Omega \nabla\tilde{u}_h(x)\cdot\nabla v(x)\,dx = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx$$

for all $v \in V_h$, which implies that $\tilde{u}_h = u_h$, as we wanted to see. □

From a known extrapolation theorem we obtain the following result.

Corollary 3.3. *Under the hypotheses of the previous theorem, for $2 < p < \infty$ there exists a constant C depending only on p and $[w]_{A_1}$, such that,*

$$\|\nabla u_h\|_{\mathbf{L}^p_w(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^p_w(\Omega)}.$$

Proof. By the previous theorem, we know that

$$\|\nabla u_h\|_{\mathbf{L}^2_w(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}$$

for every $w \in A_1$, where C depends on $[w]_{A_1}$ only. Therefore, it follows from [12, Corollary 3.5] (choosing $s_0 = 1$ and $p_0 = 2$) that

$$\|\nabla u_h\|_{\mathbf{L}^p_w(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^p_w(\Omega)}$$

for any $p > 2$ and every $w \in A_{\frac{p}{2}}$, where C depends on $[w]_{A_{\frac{p}{2}}}$. Since $A_1 \subset A_{\frac{p}{2}}$ and $[w]_{A_{\frac{p}{2}}} \le [w]_{A_1}$, the result follows. □

Next, using a standard duality argument combined with the weighted a priori estimates given in the previous section, we extend the stability result for weights with inverse in A_1.

COROLLARY 3.4. *Under the hypotheses of the previous theorem, if* $w^{-1} \in A_1$, *then*

$$\|\nabla u_h\|_{\mathbf{L}^2_w(\Omega)} \leq C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}.$$

Proof. Take $\mathbf{q} = w\nabla u_h$, and let v be the solution of $-\Delta v = \operatorname{div} \mathbf{q}$ vanishing on $\partial\Omega$. From Theorems 3.2 and 2.5 we know that

$$\|\nabla v_h\|_{\mathbf{L}^2_{w^{-1}}(\Omega)} \leq C\|\nabla v\|_{\mathbf{L}^2_{w^{-1}}(\Omega)} \leq C\|\mathbf{q}\|_{\mathbf{L}^2_{w^{-1}}(\Omega)}.$$

Then

$$\begin{aligned}\|\nabla u_h\|^2_{\mathbf{L}^2_w(\Omega)} &= \int_\Omega \nabla u_h \cdot \mathbf{q} = \int_\Omega \nabla u_h \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v_h \\ &\leq C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}\|\nabla v_h\|_{\mathbf{L}^2_{w^{-1}}(\Omega)} \\ &\leq C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}\|\mathbf{q}\|_{\mathbf{L}^2_{w^{-1}}(\Omega)} \\ &= C\|\nabla u\|_{\mathbf{L}^2_w(\Omega)}\|\nabla u_h\|_{\mathbf{L}^2_w(\Omega)}.\end{aligned}$$

□

As we have done in the continuous case we can apply these results to the problem considered in [8] as well as to the generalization introduced at the end of the previous section. With the notation used there we have the following.

THEOREM 3.5. *Under the hypotheses of Theorem* 3.2, *if* $\Gamma \subset \overline{\Omega}$ *is a* k*-regular set, then, for* $-(n-k) < \lambda < n-k$,

$$\|\nabla u_h\|_{\mathbf{L}^2_{w_\lambda}(\Omega)} \leq C\|\nabla u\|_{\mathbf{L}^2_{w_\lambda}(\Omega)}.$$

Proof. It is an immediate consequence of Theorem 3.2 and Corollary 3.4 because either $w_\lambda \in A_1$ or $w_{-\lambda} \in A_1$ by (2.14). □

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