IMPROVED CAFFARELLI-KOHN-NIRENBERG AND TRACE INEQUALITIES FOR RADIAL FUNCTIONS

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Abstract. We show that Caffarelli-Kohn-Nirenberg first order interpolation inequalities as well as weighted trace inequalities in $\mathbb{R}^n \times \mathbb{R}_+$ admit a better range of power weights if we restrict ourselves to the space of radially symmetric functions.

1. Introduction

The aim of this paper is to show that inequalities of Caffarelli-Kohn-Nirenberg type hold for a wider class of exponents if we restrict ourselves to the space of radially symmetric functions. To make this precise, recall the classical first-order interpolation inequality obtained in [2]:

Theorem (Caffarelli-Kohn-Nirenberg). *Assume*

$$p, q \geq 1, \quad r > 0, \quad 0 \leq a \leq 1 \tag{1.1}$$

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{1.2}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{1.3}$$

Then, there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\| |x|^\gamma u \|_{L^r} \leq C \| |x|^\alpha \nabla u \|_{L^p}^a \| |x|^\beta u \|_{L^q}^{1-a} \tag{1.4}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha-1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right) \tag{1.5}$$

$$0 \leq \alpha - \sigma \qquad \text{if } a > 0, \tag{1.6}$$

and

$$\alpha - \sigma \leq 1 \qquad \text{if } a > 0 \quad \text{and} \quad \frac{1}{p} + \frac{\alpha-1}{n} = \frac{1}{r} + \frac{\gamma}{n}. \tag{1.7}$$

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Although the conditions of the above theorem cannot be improved in general, we will prove that if we require u to be radially symmetric, inequality (1.4) holds true for certain negative values of $\alpha - \sigma$ also. Indeed, the following improvement holds in this particular case:

Theorem 1.1. *Assume conditions* (1.1), (1.2), (1.3) *and* (1.5) *hold. Then there exists a positive constant* C *such that inequality* (1.4) *holds for all radially symmetric* $u \in C_0^\infty(\mathbb{R}^n)$ *and all*

$$\frac{1-a}{q} \leq \frac{1}{r} \leq \frac{a}{p} + \frac{1-a}{q} \tag{1.8}$$

provided that, if $a > 0$,

$$(n-1)\left[\frac{1}{a}\left(\frac{1}{r} - \frac{1}{q}\right) + \frac{1}{q} - \frac{1}{p}\right] \leq \alpha - \sigma \leq 0 \tag{1.9}$$

and

$$-\frac{\sigma}{n} < \frac{1}{a}\left(\frac{1}{r} - \frac{1}{q}\right) + \frac{1}{q}, \tag{1.10}$$

with strict inequality in (1.9) *if* $p = 1$.

Remark 1.1. *If* $\sigma > 0$ *condition* (1.10) *trivially holds because of* (1.8), *and thus our result admits a simpler statement in this case.*

The key to our proof is to use the well-known inequality relating $u \in C_0^\infty(\mathbb{R}^n)$ with the fractional integral (also called Riesz potential) of its gradient, namely

$$|u(x)| \leq C \int_{\mathbb{R}^n} \frac{|\nabla u|(y)}{|x-y|^{n-1}}\, dy =: T_{n-1}(|\nabla u|)(x) \tag{1.11}$$

together with improved weighted estimates for fractional integrals of radial functions from [4] and the observation that inequality (1.4) enjoys a certain self-improving property. It is worth noting that this method of proof is different from that of the original proof by L. Caffarelli, R. Kohn and L. Nirenberg [2], and also from a different approach developed by F. Catrina and Z-Q. Wang in [3].

We then show that also improved trace inequalities can be obtained in a similar way, but with a slightly different operator involved in the formula (1.11), for which we prove the required weighted estimates, that play the same role as that played by the result of [4] for the Caffarelli-Kohn-Nirenberg inequalities.

To be more precise, we are interested in showing that the following trace inequality (see, e.g. [1]) can be improved for radially symmetric functions (in the first n variables):

Theorem. *Let* $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$. *Then, the following inequality holds*

$$\| |x|^{-\beta} f(x,0) \|_{L^q(\mathbb{R}^n)} \leq C \| |(y,z)|^\alpha \nabla f(y,z) \|_{L^2(\mathbb{R}^n \times \mathbb{R}^+)}$$

provided that:

$$0 \leq \alpha + \beta \leq \frac{1}{2}, \tag{1.12}$$

$$\alpha > -\frac{n+1}{2} + 1, \tag{1.13}$$

and

$$\frac{n}{q} - \frac{n+1}{2} = \alpha + \beta - 1. \tag{1.14}$$

Indeed, we will show that the following refinement is possible in the case of radially symmetric functions:

Theorem 1.2. *Let $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ be a radially symmetric function in the first n variables. Then, the following inequality holds*

$$\|f(x,0)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} \leq C\||(y,z)|^\alpha \nabla f(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \tag{1.15}$$

provided that:

$$-\frac{n}{q'} \leq \alpha + \beta \leq \frac{1}{p'}, \tag{1.16}$$

$$\alpha > -\frac{n+1}{p} + 1, \tag{1.17}$$

and

$$\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1. \tag{1.18}$$

Remark 1.2. *Using condition* (1.18)*, condition* (1.16) *can be seen to be equivalent to $1 \leq p \leq q < \infty$.*

As a preliminary result for the proof of Theorem 1.2, we will first prove the following theorem, of independent interest.

Theorem 1.3. *Let $x \in \mathbb{R}^n$ and*

$$Tf(x) := \int_{\mathbb{R}^n \times \mathbb{R}_+} \frac{f(y,z)}{[(x-y)^2 + z^2]^{\frac{n}{2}}} \, dy \, dz. \tag{1.19}$$

Assume $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ is such that $f(y,z) = f_0(|y|, z)$. Then, the inequality

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} \leq C\||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \tag{1.20}$$

holds provided that

$$1 \leq p \leq q < \infty \tag{1.21}$$

$$\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1 \tag{1.22}$$

and

$$-\frac{n}{q'} < \beta < \frac{n}{q}. \tag{1.23}$$

The rest of this paper is organized as follows. In Section 2 we recall some necessary preliminaries. In Section 3 we prove Theorem 1.1. In Section 4 we explain the relation between the operator Tf defined by (1.19) and the weighted trace inequalities we are interested in, and find a convenient expression for this operator when acting on radially symmetric functions (in the first n variables). In Section 5 we prove Theorem 1.3 and, finally, in Section 6, we use Theorem 1.3 to prove Theorem 1.2.

2. Notation and Preliminaries

As it is usual, C will denote a positive constant, independent of relevant parameters, that may change even within a single string of estimates.

To prove Theorem 1.1 we will make use of a theorem proved in [4], that we recall here for the sake of completeness.

Theorem 2.1 ([4], Theorem 1.2)**.** *For $n \geq 1$ define*

$$(T_\gamma v)(x) := \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n. \tag{2.1}$$

Let

$$1 \leq p \leq q < \infty, \tag{2.2}$$

$$\alpha < \frac{n}{p'} \tag{2.3}$$

$$\beta < \frac{n}{q} \tag{2.4}$$

$$\alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p}) \tag{2.5}$$

and

$$\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1 \tag{2.6}$$

with strict inequality in (2.5) *if $p = 1$. Then, the inequality*

$$\| |x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \| |x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

holds for all radially symmetric $v \in L^p(\mathbb{R}^n, |x|^{p\alpha}dx)$, where C is independent of v.

For the proof of Theorem 1.3 we will use the main idea in the proof of Theorem 2.1, that is, to write the operator (in this case, the operator given by (1.19) instead of the Riesz potential (2.1)) as a convolution with respect to the Haar measure in $\mathbb{R}_+$. To make this precise, recall that if G is a locally compact group, then G posseses a Haar measure, that is, a positive Borel measure μ which is left invariant (i.e., $\mu(At) = \mu(A)$ whenever $t \in G$ and $A \subseteq G$ is measurable) and nonzero on nonempty open sets. In particular, if $G = \mathbb{R} - \{0\}$, then $\mu = \frac{dx}{|x|}$, and if $G = \mathbb{R}_+$, then $\mu = \frac{dx}{x}$.

The convolution of two functions $f, g \in L^1(G)$ is defined as:

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\, d\mu(y)$$

where y^{-1} denotes the inverse of y in the group G.

The following version of Young's inequality holds in this setting:

Theorem 2.2. [5, Theorem 1.2.12] *Let G be a locally compact group with left Haar measure μ that satisfies $\mu(A) = \mu(A^{-1})$ for all measurable $A \subseteq G$. Let $1 \leq p, q, s \leq \infty$ satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{s}.$$

Then for all $f \in L^p(G,\mu)$ and $g \in L^s(G,\mu)$ we have

$$\|f * g\|_{L^q(G,\mu)} \leq \|g\|_{L^s(G,\mu)} \|f\|_{L^p(G,\mu)}. \tag{2.7}$$

3. Proof of theorem 1.1

Clearly, when $a = 0$ the theorem is completely trivial. Therefore, we will split the proof into two cases, namely, when $a = 1$ and when $0 < a < 1$.

3.1. **Case $a = 1$.** Notice that in this case, $\sigma = \gamma$ by (1.3).

Observing that for $u \in C_0^\infty(\mathbb{R}^n)$

$$|u(x)| \leq C \int_{\mathbb{R}^n} \frac{|\nabla u|(y)}{|x-y|^{n-1}}\,dy := T_{n-1}(|\nabla u|)(x)$$

we see that

$$\||x|^\gamma u\|_{L^r} \leq C\||x|^\gamma T_{n-1}(|\nabla u|)\|_{L^r}$$

but, since we are assuming that u is a radial function, then so is $|\nabla u|$ and we can use Theorem 2.1 to deduce that

$$\||x|^\gamma T_{n-1}(|\nabla u|)\|_{L^r} \leq C\||x|^\alpha \nabla u\|_{L^p}$$

provided that

$$1 \leq p \leq r < \infty \tag{3.1}$$

$$\frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha - 1}{n} \tag{3.2}$$

$$\alpha < \frac{n}{p'} \tag{3.3}$$

$$-\gamma < \frac{n}{r} \tag{3.4}$$

and

$$(n-1)\left(\frac{1}{r} - \frac{1}{p}\right) \leq \alpha - \gamma, \tag{3.5}$$

with strict inequality in (3.5) if $p = 1$.

Clearly, the scaling condition (3.2) equals condition (1.5) when $a = 1$; and using (3.2), condition (3.1) can be seen to be equivalent to $\gamma - \alpha \leq 1$, which holds because of hypothesis (1.9) (recall that in this case $\gamma = \sigma$). Condition (3.4) equals condition (1.10) (in this case it is also included in (1.2)); and (3.5) follows from (1.9) since $a = 1$.

We claim that condition (3.3) can be removed if we only wish to consider the inequality

$$\||x|^\gamma u\|_{L^r} \leq C\||x|^\alpha \nabla u\|_{L^p} \tag{3.6}$$

(this is not the case if the operator T_{n-1} is not acting on $|\nabla u|$). Indeed, we will prove that if (3.6) holds for α and γ, then it also holds for $\alpha+1$ and $\gamma+1$, provided that $\alpha p \neq -1$. To this end, we apply the inequality to $|x|u$ (strictly speaking, this function is not C_0^∞, but it suffices to take a regularized distance function to the origin, see e.g. [6], and apply the same argument).

Then,

$$\||x|^{\gamma+1} u\|_r \leq C\||x|^\alpha \nabla(|x|u)\|_p \sim C(\||x|^{\alpha+1}\nabla u\|_p + \||x|^\alpha u\|_p)$$

and, therefore, it suffices to see that $\||x|^\alpha u\|_p \le C\||x|^{\alpha+1}\nabla u\|_p$. To this end write

$$\begin{aligned}\||x|^\alpha u\|_p^p &= \int |x|^{p\alpha}|u|^p\,dx\\ &\le C\int |\nabla |x|^{p\alpha+1}||u|^p\,dx\\ &\le C\int |x|^{p\alpha+1}|\nabla|u|^p|\,dx\\ &\le C\int |x|^{p\alpha+1}|u|^{p-1}|\nabla u|\,dx\\ &\le C\left(\int |x|^{p\alpha}|u|^p\,dx\right)^{\frac{1}{p'}}\left(\int |x|^{p(\alpha+1)}|\nabla u|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

Thus, we have proved that

$$\||x|^\alpha u\|_p^p \le C\||x|^\alpha u\|_p^{\frac{p}{p'}}\||x|^{\alpha+1}\nabla u\|_p$$

whence it follows immediately that

$$\||x|^\alpha u\|_p \le C\||x|^{\alpha+1}\nabla u\|_p.$$

Iterating the same argument, we can see that if (3.6) holds for γ and α, then it also holds for $\gamma+k$ and $\alpha+k$ with $k\in\mathbb{N}_0$ provided that $(\alpha-k)p\neq -1$. Therefore, to see that we can remove condition (3.3), it suffices to observe that any $\alpha \ge \frac{n}{p'}$ can be written as $(\alpha-k)+k$, with $-\frac{n}{p}<\alpha-k<\frac{n}{p'}$, and $(\alpha-k)p\neq -1$. Indeed, since $\frac{n}{p'}-(-\frac{n}{p})=n$, such a k exists except when $n=1$ and $\alpha=\frac{1}{p'}$. But this is impossible, since in that case, by (3.2) we should have $\frac{1}{r}+\gamma=\frac{1}{p}+\frac{1}{p'}-1$, that is, $\frac{1}{r}+\gamma=0$, which contradicts (1.2).

3.2. **Case** $0<a<1$**.** Write

$$\left(\int |x|^{\gamma r}|u|^{ra+(1-a)r}\,dx\right)^{\frac{1}{r}} = \left(\int |x|^{r\beta(1-a)}|u|^{(1-a)r}|x|^{r\gamma(1-\frac{\beta(1-a)}{\gamma})}|u|^{ar}\,dx\right)^{\frac{1}{r}}$$

$$\le \||x|^\beta u\|_{L^q}^{1-a}\||x|^{\frac{\gamma}{a}(1-\frac{\beta(1-a)}{\gamma})}u\|^a_{L^{\frac{arq}{q-r(1-a)}}} \tag{3.7}$$

$$= \||x|^\beta u\|_{L^q}^{1-a}\||x|^{\sigma}u\|^a_{L^{\frac{arq}{q-r(1-a)}}} \tag{3.8}$$

where in (3.7) we have used Hölder's inequality with exponent $\frac{q}{r(1-a)}$ (which is greater than 1 by (1.8)) and in (3.8) we have used the definition of σ, given in (1.3).

Applying now the result obtained in the case $a=1$, we deduce that

$$\||x|^\gamma u\|_{L^r} \le C\||x|^\beta u\|_{L^q}^{1-a}\||x|^\alpha \nabla u\|_{L^p}^a$$

provided that

$$1\le p\le \frac{arq}{q-r(1-a)}<\infty \tag{3.9}$$

$$\frac{q-r(1-a)}{arq}+\frac{\sigma}{n}=\frac{1}{p}+\frac{\alpha-1}{n} \tag{3.10}$$

$$-\sigma<\frac{n(q-r+ar)}{arq} \tag{3.11}$$

and

$$\alpha - \sigma \geq (n-1)\left(\frac{q - r(1-a)}{arq} - \frac{1}{p}\right), \tag{3.12}$$

where in (3.12) the inequality is strict if $p = 1$.

Clearly, condition (3.9) holds because of (1.8), and condition (3.10) is easily seen to be equivalent to (1.5) using the definition of σ given in (1.3). Finally, condition (3.11) equals conditon (1.10) while (3.12) is the same as (1.9). This concludes the proof.

4. The operator associated to trace inequalities

Before we can proceed to the proof of the announced trace inequality, we first need to obtain an expression analogous to (1.11) and, then, a convenient expression for the involved operator when acting on radial functions.

To this end, given u and a unitary vector ξ, consider $g(s) = u(s\xi, 0)$. Then, $g(0) = -\int_0^\infty g'(s)\,ds = -\int_0^\infty \nabla u(s\xi)\cdot\xi\,ds$.

Consider now $\varphi \in C_0^\infty(S^n)$ supported in $\mathbb{R}^n \times \mathbb{R}_+$ and such that $\int_{S^n} \varphi(\xi)\,d\sigma(\xi) = 1$. Then

$$u(0,0) = -\int_0^\infty \int_{S_n} \nabla u(s\xi)\cdot\xi\,\varphi(\xi)\,d\xi\,ds.$$

For $(y,z) \in \mathbb{R}^{n+1}$ let $\phi(y,z) = \varphi((y,z)/\|(y,z)\|)$. Therefore, $\phi(s\xi) = \varphi(\xi)$ for all $s \in \mathbb{R}^+, \xi \in S^n$, and the above identity becomes

$$\begin{aligned} u(0,0) &= -\int_0^\infty \int_{S_n} \nabla u(s\xi)\cdot s\xi\,\phi(s\xi)\frac{1}{s^{n+1}}s^n\,ds\,d\xi \\ &= -\int_{\mathbb{R}^n\times\mathbb{R}_+} \nabla u(y,z)\cdot(y,z)\,\phi(y,z)\frac{1}{\|(y,z)\|^{n+1}}\,dy\,dz \end{aligned}$$

More generally,

$$\begin{aligned} |u(x,0)| &\leq \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla u(y,z)|\frac{1}{\|(x-y,z)\|^n}\,dy\,dz \\ &= \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla u(y,z)|\frac{1}{[(x-y)^2+z^2]^{\frac{n}{2}}}\,dy\,dz \end{aligned}$$

Then, we have to study the behavior of the operator

$$Tf(x) = \int_{\mathbb{R}^n\times\mathbb{R}_+} \frac{f(y,z)}{[(x-y)^2+z^2]^{\frac{n}{2}}}\,dy\,dz$$

for $x \in \mathbb{R}^n$.

Since we are interested in the radial case, assume f is a radially symmetric function in the first variable (by an abuse of notation we will still call it f).

Using polar coordinates

$$\begin{aligned} y &= ry', \quad r = |y|, \quad y' \in S^{n-1} \\ x &= \rho x', \quad \rho = |x|, \quad x' \in S^{n-1} \end{aligned}$$

if $n \geq 2$ we may write:

$$Tf(x) = \int_0^\infty \left[\int_0^\infty \int_{S^{n-1}} \frac{f(r,z)\, r^{n-1}}{(\rho^2 - 2\rho r x' \cdot y' + r^2 + z^2)^{\frac{n}{2}}}\, dy'\, dr \right] dz$$
$$= \int_0^\infty \int_0^\infty f(r,z) r^{n-1} \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(\rho^2 - 2\rho r t + r^2 + z^2)^{\frac{n}{2}}}\, dt\, dr\, dz$$

where the second equality can be justified integrating in the sphere (see, e.g., Lemma 4.1 from [4]).

Making the change of variables $z = r\bar{z}$, $dz = r\, d\bar{z}$ we obtain

$$Tf(x) = \int_0^\infty \int_0^\infty f(r, r\bar{z}) r^n \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{r^n \left[1 - 2\left(\frac{\rho}{r}\right) t + \left(\frac{\rho}{r}\right)^2 + \bar{z}^2\right]^{\frac{n}{2}}}\, dt\, dr\, d\bar{z}$$

$$(4.1) \qquad = \int_0^\infty \int_0^\infty f(r, rz) I\left(\frac{\rho}{r}, z\right) dr\, dz$$

where, for $a > 0$,

$$I(a,z) := \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(1 - 2at + a^2 + z^2)^{\frac{n}{2}}}\, dt.$$

Expression (4.1) will allow us to write Tf as convolution operator and to obtain Theorem 1.3, that we proceed to prove next.

5. Proof of Theorem 1.3

If $n = 1$ recall that we want to prove

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R})} \leq C \||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}\times\mathbb{R}^+)}$$

Since in this case (4.1) does not hold, we remark that

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R},dx)} = \||x|^{-\beta+\frac{1}{q}} Tf\|_{L^q(\mathbb{R}, \frac{dx}{|x|})}$$

and write

$$|x|^{-\beta+\frac{1}{q}} Tf(x) = \int_{-\infty}^\infty \int_0^\infty \frac{f(y,z)|x|^{-\beta+\frac{1}{q}}}{[(x-y)^2 + z^2]^{\frac{1}{2}}}\, dz\, dy$$
$$= \int_{-\infty}^\infty \int_0^\infty \frac{f(y, |y|\bar{z})|x|^{-\beta+\frac{1}{q}}|y|}{(|\frac{x}{y} - 1|^2 + \bar{z}^2)^{\frac{1}{2}}}\, d\bar{z}\, \frac{dy}{|y|}$$
$$= \int_0^\infty \int_{-\infty}^\infty \frac{f(y, |y|\bar{z})(\frac{|x|}{|y|})^{-\beta+\frac{1}{q}}|y|^{1-\beta+\frac{1}{q}}}{(|\frac{x}{y} - 1|^2 + \bar{z}^2)^{\frac{1}{2}}}\, \frac{dy}{|y|}\, d\bar{z}$$
$$= \int_0^\infty (f(y, |y|\bar{z})|y|^{1-\beta+\frac{1}{q}}) * \left(\frac{|y|^{-\beta+\frac{1}{q}}}{(|y-1|^2 + \bar{z}^2)^{\frac{1}{2}}} \right) d\bar{z}$$

where the convolution is taken with respect to the first variable in the multiplicative group $\mathbb{R} - \{0\}$ with Haar measure $dx/|x|$.

Let $g(y) = \frac{|y|^{-\beta+\frac{1}{q}}}{(|y-1|^2+\bar{z}^2)^{\frac{1}{2}}}$. Then, by Young's inequality (Theorem 2.2), if

$$(5.1) \qquad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{s}$$

$$
\begin{aligned}
&\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R},dx)} \\
&\qquad \leq \int_0^\infty \|f(y,|y|\bar{z})|y|^{1-\beta+\frac{1}{q}}\|_{L^p(\frac{dy}{|y|})}\|g\|_{L^s(\frac{dy}{|y|})}\,d\bar{z} \\
&\qquad = \int_0^\infty \left(\int_{-\infty}^\infty |f(y,|y|\bar{z})|^p|y|^{(1-\beta+\frac{1}{q})p-1}(1+\bar{z}^2)^{\frac{\alpha p}{2}}\right)^{\frac{1}{p}} \left(\frac{\|g\|_{L^s(\frac{dy}{|y|})}}{(1+\bar{z}^2)^{\frac{\alpha}{2}}}\right) d\bar{z}
\end{aligned} \tag{5.2}
$$

Observing now that

$$
\begin{aligned}
\||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}\times\mathbb{R}_+)} &= \int_0^\infty\int_{-\infty}^\infty (y^2+z^2)^{\frac{\alpha p}{2}}|f(y,z)|^p\,dy\,dz \\
&= \int_0^\infty\int_{-\infty}^\infty (y^2+y^2\bar{z}^2)^{\frac{\alpha p}{2}}|f(y,|y|\bar{z})|^p|y|\,dy\,d\bar{z} \\
&= \int_0^\infty\int_{-\infty}^\infty (1+\bar{z}^2)^{\frac{\alpha p}{2}}|y|^{\alpha p+1}|f(y,|y|\bar{z})|^p\,dy\,d\bar{z}
\end{aligned}
$$

and that $(1-\beta+\frac{1}{q})p-1=\alpha p+1$ (by (1.22)), we can apply Hölder's inequality to (5.2) to obtain

$$
\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} \leq \||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} \left(\int_0^\infty \frac{\|g\|^{p'}_{L^s(\frac{dy}{|y|})}}{(1+z^2)^{\frac{\alpha p'}{2}}}\,dz\right)^{\frac{1}{p'}}
$$

Therefore, to conclude the proof of the one-dimensional case it suffices to see that

$$
\int_0^\infty \frac{\|g\|^{p'}_{L^s(\frac{dy}{|y|})}}{(1+z^2)^{\frac{\alpha p'}{2}}}\,dz < +\infty
$$

provided that (1.22), (1.23) and (5.1) hold. We omit the details since the computations are analogous to those that we will do in the higher dimensional case.

Now we proceed to the case $n\geq 2$. In this case, remark that,

$$
\begin{aligned}
\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} &= C\left(\int_0^\infty |Tf(\rho)|^q\rho^{-\beta q+n}\frac{d\rho}{\rho}\right)^{\frac{1}{q}} \\
&= C\|\rho^{-\beta+\frac{n}{q}}Tf\|_{L^q(\frac{d\rho}{\rho})}
\end{aligned}
$$

We claim that $\rho^{-\beta+\frac{n}{q}}Tf$ can be written as a convolution in the multiplicative group $(\mathbb{R}_+,\cdot)$. Indeed,

$$
\begin{aligned}
\rho^{-\beta+\frac{n}{q}}Tf &= \int_0^\infty\int_0^\infty f(r,rz)\,I\left(\frac{\rho}{r},z\right)\rho^{-\beta+\frac{n}{q}}\,drdz \\
&= \int_0^\infty\int_0^\infty f(r,rz)\,I\left(\frac{\rho}{r},z\right)\left(\frac{\rho}{r}\right)^{-\beta+\frac{n}{q}} r^{-\beta+\frac{n}{q}+1}\,\frac{dr}{r}dz \\
&= \int_0^\infty (f(r,rz)r^{-\beta+\frac{n}{q}+1})*(I(r,z)r^{-\beta+\frac{n}{q}})\,dz
\end{aligned}
$$

where $*$ denotes the convolution with respect to the Haar measure dr/r in the first variable.

Therefore, using Young's inequality, for

$$
\frac{1}{q}+1=\frac{1}{p}+\frac{1}{s}, \tag{5.3}
$$

we obtain

$$\begin{aligned}
&\|Tf(\rho)\rho^{-\beta+\frac{n}{q}}\|_{L^q(\frac{d\rho}{\rho})}\\
&\quad\leq \int_0^\infty \|(f(r,rz)r^{-\beta+\frac{n}{q}+1})*(I(r,z)r^{-\beta+\frac{n}{q}})\|_{L^q(\frac{dr}{r})}\,dz\\
&\quad\leq \int_0^\infty \|f(r,rz)r^{-\beta+\frac{n}{q}+1}\|_{L^p(\frac{dr}{r})}\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}\,dz\\
&\quad= \int_0^\infty \left(\int_0^\infty |f(r,rz)|^p r^{(-\beta+\frac{n}{q}+1)p}\frac{dr}{r}\right)^{\frac{1}{p}}\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}\,dz\\
&\quad= \int_0^\infty \left(\int_0^\infty |f(r,rz)|^p r^{(-\beta+\frac{n}{q}+1)p}(1+z^2)^{\frac{\alpha p}{2}}\frac{dr}{r}\right)^{\frac{1}{p}}\frac{\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}}{(1+z^2)^{\frac{\alpha}{2}}}\,dz
\end{aligned}$$

Now, since

$$\begin{aligned}
\||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} &= \left(\int_0^\infty\int_0^\infty (r^2+z^2)^{\frac{\alpha p}{2}}|f(r,z)|^p r^{n-1}\,drdz\right)^{\frac{1}{p}}\\
&= \left(\int_0^\infty\int_0^\infty (r^2+r^2\bar{z}^2)^{\frac{\alpha p}{2}}|f(r,r\bar{z})|^p r^{n}\,d\bar{z}dr\right)^{\frac{1}{p}}\\
&= \left(\int_0^\infty\int_0^\infty r^{\alpha p}(1+\bar{z}^2)^{\frac{\alpha p}{2}}|f(r,r\bar{z})|^p r^{n}\,d\bar{z}dr\right)^{\frac{1}{p}},
\end{aligned}$$

observing that $n+\alpha p = p(-\beta+\frac{n}{q}+1)-1$ and applying Hölder's inequality to the above expression, we obtain

$$\begin{aligned}
&\|Tf(\rho)\rho^{-\beta+\frac{n}{q}}\|_{L^q(\frac{d\rho}{\rho})}\\
&\qquad\leq \||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\left(\int_0^\infty \frac{\|I(r,z)r^{-\beta+\frac{n}{q}}\|^{p'}_{L^s(\frac{dr}{r})}}{(1+z^2)^{\frac{\alpha p'}{2}}}\,dz\right)^{\frac{1}{p'}}
\end{aligned}$$

Therefore, to conclude the proof of the theorem it suffices to see that

$$\int_0^\infty \|I(r,z)r^{-\beta+\frac{n}{q}}\|^{p'}_{L^s(\frac{dr}{r})}(1+z^2)^{-\frac{\alpha p'}{2}}\,dz<+\infty. \tag{5.4}$$

Observe first that the denominator of

$$I(r,z)=\int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(1-2rt+r^2+z^2)^{\frac{n}{2}}}\,dt$$

can be rewritten as $[(a-t)^2+(1-t^2)+z^2]^{\frac{n}{2}}$ and, therefore, it vanishes for $r=t=1$ and $z=0$ only.

To bound $\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}$, consider $\varphi\in C^\infty(\mathbb{R})$ such that $supp(\varphi)\subseteq[\frac{1}{2},\frac{3}{2}]$, $0\leq\varphi\leq 1$ and $\varphi\equiv 1$ in $(\frac{3}{4},\frac{5}{4})$. We can then split $I(r,z)r^{-\beta+\frac{n}{q}} = I(r,z)r^{-\beta+\frac{n}{q}}\varphi(r)+I(r,z)r^{-\beta+\frac{n}{q}}(1-\varphi(r)) = g_1(r)+g_2(r)$ and bound both terms separately. To this end, we will study first the behavior of g_1 and g_2 and then estimate (5.4).

Consider first g_2. For $r\to 0$, we have

$$I(0,z)=(1+z^2)^{-\frac{n}{2}}\int_{-1}^1(1-t^2)^{\frac{n-3}{2}}\,dt\sim(1+z^2)^{-\frac{n}{2}}.$$

Therefore, $\|g_2\|_{L^s(\frac{dr}{r})}$ behaves like $(1+z^2)^{-\frac{n}{2}}$ when $r \to 0$, provided that $\beta < \frac{n}{q}$.

When $r \to \infty$,

$$I(r,z) \sim \frac{1}{(r^2+z^2)^{\frac{n}{2}}}.$$

In this case, if z is bounded, say $z \le 2$, $\|g_2\|_{L^s(\frac{dr}{r})}$ is also bounded provided that $\beta > -\frac{n}{q'}$. On the other hand, when $z \to \infty$, we need to estimate

$$\left(\int_2^\infty \frac{r^{s(-\beta+\frac{n}{q})}}{(r^2+z^2)^{\frac{ns}{2}}} \frac{dr}{r}\right)^{\frac{1}{s}} = \left(z^{s(-\beta+\frac{n}{q}-n)} \int_{\frac{2}{z}}^\infty \frac{r^{s(-\beta+\frac{n}{q})}}{(r^2+1)^{\frac{ns}{2}}} \frac{dr}{r}\right)^{\frac{1}{s}}$$
$$\sim z^{-\beta-\frac{n}{q'}}$$

assuming again that $\beta > -\frac{n}{q'}$.

We can proceed now to $\|g_1\|_{L^s(\frac{dr}{r})}$. We consider first the case $k = \frac{n-3}{2} \in \mathbb{N}_0$, that is $n \ge 3$ and odd.

If z is sufficiently large, then $I(r,z) \sim z^{-n}$ and, therefore, $\|g_1\|_{L^s(\frac{dr}{r})} \sim z^{-n}$.

If, on the contrary, $z \to 0$, we may write

$$I(r,z) \sim \int_{-1}^1 (1-t^2)^k \frac{d^k}{dt^k} \left\{(1-2rt+r^2+z^2)^{-\frac{n}{2}+k}\right\} dt$$

and integrating by parts k-times (the boundary terms vanish), we obtain

$$I(r,z) \le C_k[(1-r)^2+z^2]^{-\frac{n}{2}+k+1}.$$

Since we are assuming that $-\frac{n}{2}+k+1 = -\frac{1}{2}$, we conclude that

$$\begin{aligned}
\|g_1\|_{L^s(\frac{dr}{r})} &\sim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dr}{[(1-r)^2+z^2]^{\frac{s}{2}}}\right)^{\frac{1}{s}} \\
&\sim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dr}{(|1-r|+z)^s}\right)^{\frac{1}{s}} \\
&\sim \frac{1}{z^{1-\frac{1}{s}}}
\end{aligned}$$

We can consider now $k = m + \frac{1}{2}, m \in \mathbb{N}_0$. In this case

$$\begin{aligned}
&|\frac{d}{dz} I(r,z)| \\
&\quad \le Cz \int_{-1}^1 \frac{(1-t^2)^k}{(1-2rt+r^2+z^2)^{\frac{n}{2}+1}} dt \\
&\quad \le Cz \left(\int_{-1}^1 \frac{(1-t^2)^m}{(1-2rt+r^2+z^2)^{\frac{n+2}{2}}} dt\right)^{\frac{1}{2}} \left(\int_{-1}^1 \frac{(1-t^2)^{m+1}}{(1-2rt+r^2+z^2)^{\frac{n+2}{2}}} dt\right)^{\frac{1}{2}}
\end{aligned}$$

and, since now $\frac{n+2}{2} \in \mathbb{N}$, we deduce from the previous case that

$$\begin{aligned}
\left|\frac{d}{dz} I(r,z)\right| &\le Cz[(1-r^2)+z^2]^{\frac{-(n+2)+2m+3}{2}} \\
&= Cz[(1-r)^2+z^2]^{-\frac{3}{2}} \\
&\le Cz[|1-r|+z]^{-3}
\end{aligned}$$

Therefore,

$$I(r,z)=\int_0^z \frac{d}{dt}I(r,t)\,dt \le Cz[|1-r|+s]^{-2}|_0^z \le Cz[|1-r|+z]^{-2}$$

which implies

$$\|g_1\|_{L^s(\frac{dr}{r})} \sim \frac{1}{z^{1-\frac{1}{s}}}.$$

It remains to check the case $k=-\frac{1}{2}$ (i.e., $n=2$). To this end, we write

$$I(r,z)=\underbrace{\int_{-1}^0 \frac{(1-t^2)^{-\frac{1}{2}}}{(1-2at+a^2+z^2)}\,dt}_{(i)}+\underbrace{\int_0^1 \frac{(1-t^2)^{-\frac{1}{2}}}{(1-2at+a^2+z^2)}\,dt}_{(ii)}$$

Clearly,

$$(i)\le \int_{-1}^0 \frac{dt}{(1+t)^{\frac{1}{2}}}=2$$

while

$$\begin{aligned}(ii) &\le \int_0^1 \frac{(1-t)^{-\frac{1}{2}}}{(1-2at+a^2+z^2)}\,dt\\ &= -2\int_0^1 \frac{\frac{d}{dt}[(1-t)^{\frac{1}{2}}]}{1-2at+a^2+z^2}\,dt\\ &\le 4a\int_0^1 \frac{(1-t^2)^{\frac{1}{2}}}{(1-2at+a^2+z^2)^2}\,dt\end{aligned}$$

and the last integral can be bounded as before (notice that it corresponds to the case $n=4$).

We are now able to see that (5.4) holds. Indeed, by our previous calculations, we need to bound

$$\int_0^1 \left(\frac{1}{z^{1-\frac{1}{s}}(1+z^2)^{\frac{\alpha}{2}}}+\frac{1}{(1+z^2)^{\frac{n+\alpha}{2}}}\right)^{p'} dz$$
$$+\int_1^\infty \left(\frac{1}{z^n(1+z^2)^{\frac{\alpha}{2}}}+\frac{1}{z^{\beta+\frac{n}{q'}}(1+z^2)^{\frac{\alpha}{2}}}\right)^{p'} dz$$

When $z\to 0$, the integrability condition is $p'(1-\frac{1}{s})<1$, which holds because of (1.21) and (5.3). When $z\to\infty$, since we are assuming that $\beta<\frac{n}{q}$, there holds that $n>\beta+\frac{n}{q'}$, whence the integralibity condition is $p'(\beta+\frac{n}{q'}+\alpha)>1$, that is, $\alpha+\beta>\frac{1}{p'}-\frac{n}{q'}$. But, by (1.18) this condition is equivalent to $\frac{n}{p'}>0$, which trivially holds. This concludes the proof of the theorem.

6. Proof of Theorem 1.2

As in the case of the Caffarelli-Kohn-Nirenberg interpolation inequality, if we simply apply Theorem 1.3 to $|\nabla f|$ we obtain (1.15) provided that

$$1\le p\le q<\infty \tag{6.1}$$

$$\frac{n}{q}-\frac{n+1}{p}=\alpha+\beta-1 \tag{6.2}$$

and

$$-\frac{n}{q'} < \beta < \frac{n}{q}. \tag{6.3}$$

Notice that this last condition is equivalent to $-\frac{n+1}{p} + 1 < \alpha < \frac{n+1}{p'}$ because of (6.2).

To prove Theorem 1.2 we need to see that condition $\alpha < \frac{n+1}{p'}$ is unnecessary for inequality (1.15) to hold. Indeed, with a similar argument as that used for Theorem 1.1, we will prove that if the inequality holds for α and β then it also holds for $\alpha+1$ and $\beta - 1$ provided that $\alpha p \neq -1$.

To see this, consider $f(x)|x|$ (strictly speaking, we would need to replace $|x|$ by a regularized distance, to guarantee that the product is in C_0^∞). Then,

$$\begin{aligned}\|f(x,0)|x|^{-\beta+1}\|_{L^q(\mathbb{R}^n)} &\leq C\||(y,z)|^{\alpha}\nabla(|(y,z)|f(y,z))\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\\ &\leq C\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\\ &\quad + \||(y,z)|^{\alpha} f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\end{aligned}$$

Therefore, it suffices to see that

$$\||(y,z)|^{\alpha} f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} \leq C\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}.$$

To this end, consider

$$\begin{aligned}&\||(y,z)|^{\alpha} f(y,z)\|^p_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\\ &\qquad= \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |(y,z)|^{p\alpha}|f(y,z)|^p\,dy\,dz\\ &\qquad\leq C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |\nabla|(y,z)|^{p\alpha+1}||f(y,z)|^p\,dy\,dz\\ &\qquad\leq C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |(y,z)|^{p\alpha+1}|\nabla|f(y,z)|^p|\,dy\,dz\\ &\qquad\leq C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |(y,z)|^{p\alpha+1}|f(y,z)|^{p-1}|\nabla f(y,z)|\,dy\,dz\\ &\qquad= C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |(y,z)|^{\alpha(p-1)}|f(y,z)|^{p-1}|(y,z)|^{\alpha+1}|\nabla f(y,z)|\,dy\,dz\end{aligned}$$

Applying Hölder's inequality we see that

$$\||(y,z)|^{\alpha} f(y,z)\|_p^p \leq C\||(y,z)|^{\alpha} f(y,z)\|_p^{\frac{p}{p'}}\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_p$$

and it follows immediately that

$$\||(y,z)|^{\alpha} f(y,z)\|_p \leq C\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_p$$

as we wanted to see.

Iterating the same argument we see that if inequality (1.15) holds for α and β, then it holds for $\alpha + k$ and $\beta - k$ with $k \in \mathbb{N}_0$. Therefore, to see that condition $\alpha < \frac{n+1}{p'}$ is uneccessary, it suffices to see that any $\alpha \geq \frac{n+1}{p'}$ can be written as $(\alpha - k) + k$, with $-\frac{n+1}{p} + 1 < \alpha - k < \frac{n+1}{p'}$ and $(\alpha - k)p \neq -1$.

But, $\frac{n+1}{p'} - \left(-\frac{n+1}{p} + 1\right) = n$, and therefore k can be chosen as above, except when $n = 1$ and $\alpha = \frac{n+1}{p'} = \frac{2}{p'}$ (that is, $\beta = -\frac{1}{q'}$) that cannot happen because for $n = 1$, $\alpha > \frac{2}{p'}$ (because of (6.3) and (6.2)). This completes the proof of the theorem.

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