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Radial Solutions for Hamiltonian Elliptic Systems With Weights

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Abstract

We prove the existence of infinitely many radial solutions for elliptic systems in $\mathbb{R}^n$ with power weights. A key tool for the proof will be a weighted imbedding theorem for fractional-order Sobolev spaces, that could be of independent interest.
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1 Introduction

In this paper, we study the existence of non-trivial, radially symmetric solutions of the following Hamiltonian elliptic system in $\mathbb{R}^n$:

$$\begin{cases} -\Delta u + u &= |x|^a |v|^{p-2} v \\ -\Delta v + v &= |x|^b |u|^{q-2} u \end{cases} \tag{1.1}$$

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More precisely, we will prove the following theorem:

Theorem 1.1 *Assume that the following conditions hold:*

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1 \tag{1.2}$$

$$0 < a < \frac{(n-1)(p-2)}{2}, \quad 0 < b < \frac{(n-1)(q-2)}{2} \tag{1.3}$$

$$\frac{n+a}{p} + \frac{n+b}{q} > n-2 \tag{1.4}$$

and

$$q < \frac{2(n+b)}{n-4}, \quad p < \frac{2(n+a)}{n-4} \quad \text{if } n \geq 5. \tag{1.5}$$

Then, (1.1) admits infinitely many radially symmetric weak-solutions (see Definition 4.1 below).

In order to explain the significance of our result, let us briefly review some of the related results in the literature. The related scalar equation

$$\begin{cases} -\Delta u & = & |x|^a u^{p-1} & \text{in} & \Omega \\ u > 0 & & & \text{in} & \Omega \\ u = 0 & & & \text{on} & \partial\Omega \end{cases} \tag{1.6}$$

with Ω being the unit ball in $\mathbb{R}^n$, $a > 0$ and $p > 2$ is known in the literature as the Hénon equation, since M. Hénon introduced it as a model of spherically symmetric stellar clusters [7]. It is well known that the presence of the weight $|x|^a$ modifies the global homogeneity of the equation, and shifts up the threshold between existence and non existence given by the application of the Pohozaev Identity. Indeed, W. M. Ni proved in [10] the existence of a solution of (1.6) if $2 < p < 2^* + \frac{2a}{n-2}$ (here $2^* = \frac{2n}{n-2}$ denotes the usual Sobolev critical exponent). The solutions found by Ni are radial and arise via application of the Mountain Pass Theorem in the space of radial functions. Problems in the whole space have also been considered. For instance, for the related equation

$$-\Delta u + |x|^a u = |x|^b u^{p-1}, \quad u \in H^1(\mathbb{R}^n) \tag{1.7}$$

P. Sintzoff [14] proved the existence of infinitely many radial solutions in the case

$$n \geq 3, \ \ p > 1, \ \ 2 < p < 2^* + \frac{2a}{n-2}, \ \ 2b - \left(1 + \frac{p}{2}\right) a < (n-1)(p-2). \tag{1.8}$$

Elliptic systems like (1.1) have also been extensively studied. An important feature of this system is its Hamiltonian structure, that allows us to find weak solutions using the methods of critical point theory. More precisely, its solutions are given by the critical points of a functional in a suitable functional space (see

(4.2) below). We refer the reader to the excellent survey [4] for more details on this subject.

A closely related problem is the system

$$\begin{cases} -\Delta u &= |x|^a |v|^{p-2} v \quad \text{in} \quad \Omega \\ -\Delta v &= |x|^b |u|^{q-2} u \quad \text{in} \quad \Omega \end{cases} \tag{1.9}$$

with Dirichlet boundary conditions ($u = v = 0$ in $\partial\Omega$), where $\Omega \subset \mathbb{R}^n$ is a bounded domain. It is known that in the unweighted case $a = b = 0$, (1.9) admits infinitely many non-trivial solutions if

$$\begin{gathered} p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1 \quad \text{(superlinearity)} \\ \frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{n} \quad \text{(critical hyperbola)} \\ q < \frac{2n}{n-4}, \quad p < \frac{2n}{n-4} \quad \text{if } n \geq 5. \end{gathered} \tag{1.10}$$

For this result, see the paper [1] by T. Bartsch, D. G. de Figueiredo and the previous works [5] and [8]. In [1] the existence of infinitely many radial solutions for the unweighted system in $\mathbb{R}^n$ is also proved (i.e. problem (1.1) with $a = b = 0$).

In view of the known results for the Henón equation, one would expect that the presence of weights should also cause a shift in the critical hyperbola (1.10), and, indeed, in [6] D. G. de Figueiredo, I. Peral and J. Rossi extended these results to problem (1.9) with non-trivial weights in an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$, with $0 \in \Omega$, obtaining existence of many strong solutions and at least one positive solution under conditions (1.2), (1.4) and (1.5), namely,

$$\frac{1}{p} + \frac{1}{q} < 1, \quad \frac{n+a}{p} + \frac{n+b}{q} > n - 2$$

and

$$q < \frac{2(n+b)}{n-4}, \quad p < \frac{2(n+a)}{n-4} \quad \text{if } n \geq 5.$$

(Note that condition (1.3) is not needed in the case of a bounded domain).

Essentially, our aim is to complement the above results by proving existence of infinitely many radially symmetric solutions of the weighted system (1.1) in the whole of $\mathbb{R}^n$ under appropriate restrictions on the weights.

A key role in our proof will be played by a weighted imbedding theorem for radial functions (Theorem 2.1) that we believe could also be of independent interest. The proof of this imbedding relies on some $L^p - L^q$ estimates for the so-called Fourier-Bessel (or Hankel) transform proved by L. De Carli in [2], and is given in Section 2.

It is worth noting that in the case of a bounded domain, the corresponding imbedding theorem (Proposition 2.1 of [6]) can be obtained by applying the classical imbedding theorem combined with Hölder's inequality. However, for an unbounded domain this approach is not possible since the power weights $|x|^r$ are not inegrable. On the other hand, our imbedding theorem has some extra restrictions (see (2.2) below) that do not appear in the bounded case (and that is why we have conditions (1.3) in Theorem 1.1).

The compactness of our imbedding will also be of fundamental importance (since it will allow us to prove a suitable form of the Palais-Smale compactness condition, see Lemma 4.1). The idea of obtaining better imbedding properties (in particular, compactness) by restricting to subespaces of radially symetric functions goes back to the works of W. Strauss [15] and W. Rother [12], and was futher generalized in different directions by P. L. Lions [9] and W. Sickel and L. Skrzypczak [16].

The other important ingredient of our proof of Theorem 1.1 is an abstract minimax theorem from [1] that we recall as Theorem 3.1 in order to make this paper self-contained. Finally, in Section 4 we complete the proof of Theorem 1.1 by checking that all the conditions of Theorem 3.1 hold. Notice that we need to use this abstract minimax theorem instead of the much simpler one used in the bounded domain case (Theorem 3.1 in [6]) since the proof of Lemma 3.2 in that paper (which is analogous to Lemma 4.3 in our paper) essentially depends on the fact that the Laplacian has discrete spectrum in a bounded domain (which is not the case for the linear part of (1.1) in $\mathbb{R}^n$).

Finally, to complete the picture, let us observe that the same methods of [6] could be used to obtain radially symmetric solutions of (1.9) if Ω is a ball, under the same hypotheses of Theorem 1.1 in that paper (i.e. without the restrictions (1.3)), by working in a subspace of radially symmetric functions $H^s_{rad}(\mathbb{R}^n) \times H^t_{rad}(\mathbb{R}^n)$, and then using the principle of Symmetric Criticality (see [11]).

As a final remark, we point out that, for the sake of simplicity, we concentrate on this paper on the model problem (1.1), though it is clear that the same techniques could be used to handle more general Hamiltonian elliptic systems of the form:

$$\left\{ \begin{array}{rcl} -\Delta u + u & = & H_v(|x|, u, v) \\ -\Delta v + v & = & H_u(|x|, u, v) \end{array} \right. \tag{1.11}$$

in $\mathbb{R}^n$ under suitable hypotheses on the Hamiltonian function H (analogous to those in [1]).

2 An imbedding theorem for fractional order Sobolev spaces with weights

Using an idea that goes back to [5], we shall work in the fractional Sobolev spaces defined, as usual, by:

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\omega|^2)^s |\hat{u}(\omega)|^2 \, d\omega < +\infty \right\}$$

were $\hat{u}$ denotes the Fourier transform of u. This is a Hilbert space with the inner product given by:

$$\langle u, v\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\omega|^2)^s \hat{u}(\omega)\overline{\hat{v}(\omega)}\, d\omega$$

For the proof of Theorem 1.1, we shall need the following weighted imbedding theorem for $H^s_{rad}(\mathbb{R}^n)$, the subspace of radially symmetric functions of $H^s(\mathbb{R}^n)$[1]:

Theorem 2.1 *Let* $0 < s < \frac{n}{2}$, $2 < q < 2^*_c = \frac{2(n+c)}{n-2s}$. *Then, we have the compact imbedding*

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c dx) \tag{2.1}$$

provided that

$$-2s < c < \frac{(n-1)(q-2)}{2}. \tag{2.2}$$

Remark 2.1 The case $s = 1$ was proved by W. Rother [13] (this theorem was used by P. Sintzoff in [14] for problem (1.7)). The case $c = 0$ gives the classical Sobolev imbedding (in the case of radial functions):

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \text{ for } 2 < q < \frac{2n}{n-2s}.$$

In that case, the compactness of the imbedding is given by the following theorem of P. L. Lions [9]:

Theorem 2.2 *Assume that* $0 < s < \frac{n}{2}$ *and* $2 < q < \frac{2n}{n-2s}$. *Then the imbedding*

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$$

is compact.

The proof of Theorem (2.1) is based on an $L^p - L^q$ estimate for the so-called Fourier-Bessel (or Hankel) transform due to L. De Carli [2]. In this section, we shall therefore follow the notations in that paper, that we recall here for sake of completeness:

For given parameters α, ν, μ De Carli introduced the operator

$$L^\alpha_{\nu,\mu} f(y) = y^\mu \int_0^\infty (xy)^\nu f(x) J_\alpha(xy)\, dx$$

where J_α denotes the Bessel function of order α. A particular case of this operator is the Fourier-Bessel transform:

$$\tilde{\mathcal{H}}_\alpha f(x) = L^\alpha_{\alpha+1,-2\alpha-1} f(x) \tag{2.3}$$

[1]Notice that troughout this section p and q are not the exponents in (1.1).

The importance of this operator for our purposes is due to the fact that it provides an expresion for the Fourier transform of a radial function $u(x) = u_0(|x|)$:

$$\hat{u}(|\omega|) = (2\pi)^{\frac{n}{2}} \tilde{H}_{\frac{n}{2}-1}(u_0)(|\omega|). \tag{2.4}$$

Morover, we recall that we have the inversion formula:

$$\tilde{\mathcal{H}}_\alpha(\tilde{\mathcal{H}}_\alpha(u))(x) = u(x) \text{ (equation (2.4) from [2])} \tag{2.5}$$

Now, we state De Carli's theorem (Theorem 1.1 in [2]):

Theorem 2.3 *$L^\alpha_{\nu,\mu}$ is a bounded operator from $L^p(0,\infty)$ to $L^q(0,\infty)$ whenever $\alpha \geq -\frac{1}{2}$, $1 \leq p \leq q \leq \infty$, if and only if*

$$\mu = \frac{1}{p'} - \frac{1}{q} \;\; and \; -\alpha - \frac{1}{p'} < \nu \leq \frac{1}{2} - max\left(\frac{1}{p'} - \frac{1}{q}, 0\right).$$

Finally, we observe that the De Carli operators $L^\alpha_{\nu,\mu}$ enjoy two invariance properties that will be useful in obtaining weighted estimates (and that are immediate from their definition):

$$y^e L^\alpha_{\nu,\mu}(f)(y) = L^\alpha_{\nu,\mu+e}(f)(y) \tag{2.6}$$

$$L^\alpha_{\nu,\mu}(f) = L^\alpha_{\nu-\sigma,\mu+\sigma}(x^\sigma f). \tag{2.7}$$

Now we are ready to give a proof of Theorem 2.1:

Proof. Let $u(x) = u_0(|x|) \in H^s_{rad}(\mathbb{R}^n)$. Using polar coordinates we have that:

$$\left(\int_{\mathbb{R}^n} |x|^c |u|^q \, dx\right)^{\frac{1}{q}} = C\left(\int_0^\infty r^{c+n-1}|u_0(r)|^q \, dr\right)^{\frac{1}{q}}.$$

Thanks to the inversion formula (2.5) for the Fourier-Bessel transform of order $\alpha = \frac{n}{2} - 1$ (which is just the usual Fourier inversion formula for radial functions) we obtain:

$$\left(\int_{\mathbb{R}^n} |x|^c |u|^q \, dx\right)^{\frac{1}{q}} = C\left(\int_0^\infty r^{c+n-1}|\tilde{\mathcal{H}}_\alpha(\tilde{\mathcal{H}}_\alpha(u_0))(r)|^q \, dr\right)^{\frac{1}{q}}$$

$$= C\left(\int_0^\infty r^{c+n-1}|L^\alpha_{\alpha+1,-2\alpha-1}(\tilde{\mathcal{H}}_\alpha(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \text{ (using (2.3))}$$

$$= C\left(\int_0^\infty |L^\alpha_{\alpha+1,-2\alpha-1+\frac{c+n-1}{q}}(\tilde{\mathcal{H}}_\alpha(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \text{ (using (2.6))}$$

$$= C\left(\int_0^\infty |L^\alpha_{\alpha+1-\sigma,-2\alpha-1+\frac{c+n-1}{q}+\sigma}(r^\sigma \tilde{\mathcal{H}}_\alpha(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \text{ (using (2.7))}$$

where the value of the parameter σ can be chosen to fulfill our needs.

Indeed, now we apply Theorem 2.3 with the following choice of parametes

$$p = \frac{nq}{nq-n-c}, \quad \sigma = \frac{n-1}{p}, \quad \alpha = \frac{n}{2}-1, \quad \nu = \alpha+1-\sigma, \quad \mu = -2\alpha-1+\frac{c+n-1}{q}+\sigma$$

Since it easy to see that, under the hypotheses of our theorem, all the restrictions of Theorem 2.3 are fulfilled, we get the bound:

$$\left(\int_{\mathbb{R}^n} |x|^c |u|^q \, dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty |r^\sigma \tilde{\mathcal{H}}_\alpha(u_0)(r)|^p \, dr\right)^{\frac{1}{p}}$$

$$= C\left(\int_0^\infty (1+r^2)^{\frac{sp}{2}}(1+r^2)^{-\frac{sp}{2}} |\tilde{\mathcal{H}}_\alpha(u_0)(r)|^p \, r^{n-1} \, dr\right)^{\frac{1}{p}}$$

and, using Hölder's inequality with exponent $\frac{2}{p}$,

$$\leq C\left(\int_0^\infty (1+r^2)^s |\tilde{\mathcal{H}}_\alpha(u_0)(r)|^2 r^{n-1} \, dr\right)^{\frac{1}{2}} \left(\int_0^\infty (1+r^2)^{-\frac{sp}{2-p}} r^{n-1} \, dr\right)^{\frac{2-p}{2p}}$$

$$\leq C\|u\|_{H^s},$$

where in the last inequality we have used (2.4) and the fact that, under the restrictions of our theorem,

$$\int_0^\infty (1+r^2)^{-\frac{sp}{2-p}} \, r^{n-1} \, dr < +\infty \quad \left(\text{recall that } \frac{2n}{2s+n} < p\right).$$

It remains to prove that the imbedding $H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c dx)$ is compact. It is enough to show that if $u_n \to 0$ weakly in $H^s_{rad}(\mathbb{R}^n)$, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^n, |x|^c \, dx)$. Since

$$2 < q < 2^*_c = \frac{2(n+c)}{n-2s}$$

by hypothesis, it is possible to choose r and $\tilde{q}$ so that $2 < r < q < \tilde{q} < 2^*_c$. We write $q = \theta r + (1-\theta)\tilde{q}$ with $\theta \in (0,1)$ and, using Hölder's inequality, we have that

$$\int_{\mathbb{R}^n} |x|^c |u_n|^q \, dx \leq \left(\int_{\mathbb{R}^n} |u_n|^r \, dx\right)^\theta \left(\int_{\mathbb{R}^n} |x|^{\tilde{c}} |u_n|^{\tilde{q}} \, dx\right)^{1-\theta} \tag{2.8}$$

where $\tilde{c} = \frac{c}{1-\theta}$. By choosing r close enough to 2 (hence making θ small), we can fulfill the conditions

$$\tilde{q} < \frac{2(n+\tilde{c})}{n-2s}, \quad -2s < \tilde{c} < \frac{(n-1)(\tilde{q}-2)}{2}.$$

Therefore, by the imbedding that we have already established:

$$\left(\int_{\mathbb{R}^n} |x|^{\tilde{c}} |u_n|^{\tilde{q}} \, dx\right)^{1/\tilde{q}} \leq C\|u_n\|_{H^s} \leq C'.$$

Since the imbedding $H^s_{rad}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ is compact by Lions theorem, we have that $u_n \to 0$ in $L^r(\mathbb{R}^n)$. From (2.8) we conclude that $u_n \to 0$ strongly in $L^q(\mathbb{R}^n, |x|^c\, dx)$, which shows that the imbedding in our theorem is also compact. This concludes the proof of Theorem 2.1.

3 An abstract critical point theorem

In order to prove Theorem 1.1 we will use an abstract critical point result from [1]. For the reader's convenience, we will try to keep the notation from that paper. We start by recalling the specific form of the Palais-Smale-Cerami compactness condition used in [1]:

Definition 3.1 We consider a Hilbert space E and a functional $\Phi \in C^1(E, \mathbb{R})$. Given a sequence $(X_n)_{n\in\mathbb{N}}$ of finite dimensional subspaces of X, with $X_n \subset X_{n+1}$ and $\overline{\bigcup X_n} = E$, the functional Φ is said to satisfy *condition* $(PS)^{\mathcal{F}}_c$ *at level* c if every sequence $(z_j)_{j\in\mathbb{N}}$ with $z_j \in X_{n_j}$, $n_j \to +\infty$ and such that

$$\Phi(z_j) \to c \quad \text{and} \quad (1 + \|z_j\|)(\Phi|_{X_{n_j}})'(z_j) \to 0$$

(a so-called $(PS)^{\mathcal{F}}_c$ sequence) has a subsequence which converges to a critical point of Φ.

Theorem 3.1 (Fountain Theorem, Theorem 2.2 from [1]) *Assume that the Hilbert space E splits as a direct sum $E = E^+ \oplus E^-$, and that $E^\pm_1 \subset E^\pm_2 \subset \ldots \subset E^\pm_n \subset$ are strictly increasing sequences of finite dimensional subspaces such that $\overline{\bigcup_{n=1}^\infty E^\pm_n} = E^\pm$ and let $E_n = E^+_n \oplus E^-_n$. Furthermore, assume that the functional Φ satisfies the following assumptions:*

(Φ_1) $\Phi \in C^1(E, \mathbb{R})$ *and satisfies* $(PS)^{\mathcal{F}}_c$ *with respect to* $\mathcal{F} = (E_n)_{n\in\mathbb{N}}$ *and every* $c > 0$.

(Φ_2') *There exists a sequence* $r_k > 0$ $(k \in \mathbb{N})$ *such that*

$$b_k = \inf\{\Phi(z) : z \in E^+,\ z \perp E_{k-1}\ \|z\| = r_k\}$$

satisfy $b_k \to +\infty$.

(Φ_3') *There exist a sequence of isomorphisms* $T_k : E \to E$ $(k \in \mathbb{N})$ *with* $T_k(E_n) = E_n$ *for all k and n, and there exists a sequence* $R_k > 0$ $(k \in \mathbb{N})$ *such that, for* $z = z^+ + z^- \in E^+_k \oplus E^-$ *with* $\max(\|z^+\|, \|z^-\|) = R$ *one has*

$$\|T_k\| > r_k \quad \textit{and} \quad \Phi(T_k z) < 0.$$

(Φ_4') $d_k = \sup\{\Phi(T_k(z^+ + z^-)) : z^+ \in E^+_k,\ z^- \in E^-,\ \|z^+\|, \|z^-\| \le R_k\} < +\infty$.

(Φ_5) Φ *is even, i.e.* $\Phi(-z) = \Phi(z)\ \forall\, z \in E$.

Then Φ has an unbounded sequence of critical values.

In our application, we will also use Remark 2.2 from [1], that we state here as a lemma for the sake of completeness:

Lemma 3.1 *Let E be a Hilbert space, and $E_1 \subset E_2 \subset E_3 \subset \dots$ be a sequence of finite dimensional subspaces of E such that $E = \overline{\bigcup_{n=1}^{\infty} E_n}$. Assume that we have a compact imbedding $E \subset X$, where X is a Banach space.*

Let $\Phi \in C^1(E, R)$ be a functional of the form $\Phi = P - \Psi$ where

$$P(z) \geq \alpha \|z\|_E^p \; \forall \, z \in E$$

and

$$|\Psi(x)| \leq \beta(1 + \|z\|_X^q) \; \forall \, z \in E$$

where α, β and $q > p$ are positive constants. Then, there exist $r_k > 0$ $(k \in \mathbb{N})$ such that

$$b_k = \inf\{\Phi(z) : z \in E^+, \; z \perp E_{k-1} \; \|z\| = r_k\} \to +\infty$$

i.e. condition (Φ_2') in theorem 3.1 holds.

4 Proof of the main Theorem

4.1 The Functional Setting

Using conditions (1.2), (1.4) and (1.5) we may choose s, t such that $0 < s, t < \frac{n}{2}$, $s + t = 2$ and

$$2 < p < \frac{2(n+a)}{n-2t}, \quad 2 < q < \frac{2(n+b)}{n-2s}.$$

From Theorem 2.1 we then have the compact imbeddings

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^b dx), \quad H^t_{rad}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, |x|^a dx). \tag{4.1}$$

The natural functional associated to (1.11) is given by:

$$\Phi(u, v) = \int_{\mathbb{R}^n} A^s u \cdot A^t v - \int_{\mathbb{R}^n} H(x, u, v) \tag{4.2}$$

in the subspace $E = H^s_{rad}(\mathbb{R}^n) \times H^t_{rad}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \times H^t(\mathbb{R}^n)$, with the pseudo-differential operator $A^s u = (-\Delta + I)^{s/2}$ given in terms of the Fourier transform:

$$\widehat{A^s u}(\omega) = (1 + |\omega|^2)^{s/2} \hat{u}(\omega),$$

and where H is the Hamiltonian:

$$H(x, u, v) = \frac{|x|^b |u|^q}{q} + \frac{|x|^a |v|^p}{p}.$$

The imbeddings (4.1) imply that Φ is well defined in E and $\Phi \in C^1(E, \mathbb{R})$ (see the appendix of [12]).

Definition 4.1 We say that $z = (u, v) \in H^s_{rad} \times H^t_{rad}$ is an (s,t)-*weak solution* of the system (1.1) if z is a critical point of the functional (4.2).

Remark 4.1 The functional Φ is not well-defined in $H^s(\mathbb{R}^n) \times H^t(\mathbb{R}^n)$ because the imbedding (4.1) is not valid in general for non-radial functions. However, the functional is well-defined in $\tilde{E} = (H^s(\mathbb{R}^n) \cap L^q(\mathbb{R}^n, |x|^b dx)) \times (H^t(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, |x|^a dx))$. Moreover, since Φ is invariant with respect to radial symmetries, the critical points of Φ in E are also critical points in $\tilde{E}$ thanks to the Symmetric Criticality Principle (see Theorem A 5.4 of [11]).

Next, consider the bilinear form $B : E \times E \to \mathbb{R}$ given by:

$$B[z, \eta] := \int (A^s u A^t \phi + A^s \psi A^t v) \ \text{ where } \ z = (u, v), \ \eta = (\psi, \phi).$$

Asociated with B we have the quadratic form

$$Q(z) = \frac{1}{2} B(z, z) = \int A^s u A^t v.$$

It is well-known that the operator $L : E \to E$ defined by $\langle Lz, \eta \rangle = B[z, \eta]$ has exactly two eigenvalues $+1$ and -1 and that the corresponding eigenspaces are given by

$$E^+ = \{(u, A^{-t} A^s u) : u \in E^s\}, \qquad E^- = \{(u, -A^{-t} A^s u) : u \in E^s\}.$$

Then, we have that

$$\Phi(z) = \frac{1}{2} \langle Lz, z \rangle - \Psi(z) \tag{4.3}$$

where:

$$\Psi(z) = \int H(x, u, v).$$

We now define the sequence of finite dimensional subspaces that we need to apply Theorem 3.1. For this purpose, choose an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of $H^s_{rad}(\mathbb{R}^n)$. By density, we can choose $e_j \in \mathcal{S}(\mathbb{R}^n)$ (the Schwarz class). Then $f_j = A^{-t} A^s e_j$ form an orthonormal basis of $H^t_{rad}(\mathbb{R}^n)$, $f_j \in \mathcal{S}(\mathbb{R}^n)$, and we may define the following finite dimensional subspaces:

$$E^s_n = \langle e_j : j = 1 \ldots n \rangle \subset H^s_{rad}(\mathbb{R}^n)$$

$$E^t_n = \langle f_j : j = 1 \ldots n \rangle \subset H^t_{rad}(\mathbb{R}^n)$$

$$E_n = E^s_n \oplus E^t_n.$$

4.2 The Palais-Smale condition

In what follows, we will prove that the functional Φ satisfies conditions $(\Phi_1), (\Phi_2') - (\Phi_4'), (\Phi_5)$ in Theorem 3.1. We begin by checking the compactness condition $(PS)_c^{\mathcal{F}}$:

Lemma 4.1 *Condition* (Φ_1) *holds.*

Proof. Using the imbedding in Theorem 2.1, it follows from standard arguments (see for example [12]) that Φ is well defined, and morover $\Phi \in C^1(E, \mathbb{R})$. It remains to show that Φ satisfies the $(PS)_c^{\mathcal{F}}$ condition. Assume that we have a sequence $z_j \in E_{n_j}$ such that $\Phi(z_j) \to c, (1 + \|z_j\|)(\Phi|_{E_{n_j}})'(z_j) \to 0$.

We observe that since the functional Φ has the form (4.3) where $L : E \to E$ is a linear Fredholm operator of index zero and $\nabla\Psi : E \to E$ is completely continuous (due to the compactness of the imbeddings (4.1)), then by Remark 2.1 of [1], it is enough to prove that z_j is bounded.

Since $(\Phi|_{E_{n_j}})'(z_j) \to 0$, in particular we have that

$$|\Phi'(z_j)(w)| \leq C\|w\|_E \text{ for all } w \in E_{n_j}. \tag{4.4}$$

If $z_j = (u_j, v_j)$, taking $w_j = \frac{pq}{p+q}\left(\frac{1}{p}u_j, \frac{1}{q}v_j\right)$, we have that

$$\begin{aligned} C(1 + \|w_j\|_E) &\geq \Phi(z_j) - \Phi'(z_j)(w_j) \\ &= \int A^s u_j A^t v_j - \int H(x, u_j, v_j) \\ &- \left[\frac{q}{p+q}\int A^s u_j A^t v_j + \frac{p}{p+q}\int A^s u_j A^t v_j\right. \\ &\left.- \frac{q}{p+q}\int H_u(x, u_j, v_j)u_j - \frac{p}{p+q}\int H_v(x, u_j, v_j)v_j\right] \\ &= \left(\frac{pq}{p+q} - 1\right)\int H(x, u_j, v_j). \end{aligned}$$

Using (1.2) and Theorem 2.1 we obtain

$$\int H(u_j, v_j, x)\,dx = \int \frac{|x|^b|u_j|^q}{q} + \frac{|x|^a|v_j|^p}{p} \leq C(1 + \|u_j\|_{H^s} + \|v_j\|_{H^t}). \tag{4.5}$$

Now, considering $w = (\psi, 0), \psi \in E^s_{n_j} \subset H^s_{rad}(\mathbb{R}^n)$ in (4.4)

$$\begin{aligned} |Q'(z_j)(w)| = \left|\int A^s\psi A^t v_j\right| &\leq \int |H_u(u_j, v_j, x)\psi|\,dx + C\|\psi\|_{H^s} \\ &= \int |x|^b|u_j|^{q-2}u_j\psi + C\|\psi\|_{H^s} \\ &\leq \|u_j\|^{q-1}_{L^q(|x|^b)}\|\psi\|_{L^q(|x|^b)} + C\|\psi\|_{H^s} \end{aligned}$$

and using Theorem 2.1 we conclude that

$$\left| \int A^s \psi A^t v_j \right| \leq C \left(\|u_j\|_{L^q(|x|^b)}^{q-1} + 1 \right) \|\psi\|_{H^s}.$$

Using a duality argument (and the fact that $\int A^s \psi A^t v_j = 0 \ \forall \psi \in (E_{n_j}^s)^\perp$), this implies that

$$\|v_j\|_{H^t} \leq C \left(\|u_j\|_{L^q(|x|^b)}^{q-1} + 1 \right). \tag{4.6}$$

Similarly, taking $w = (0, \psi), \psi \in E_{n_j}^t$ in (4.4), we obtain

$$|Q'(z_j)(w)| = \left| \int A^s u_j A^t \psi \right| \leq C \left(\|v\|_{L^p(|x|^a)}^{p-1} + 1 \right) \|\psi\|_{H^t}$$

hence,

$$\|u_j\|_{H^s} \leq C \left(\|v_j\|_{L^p(|x|^a)}^{p-1} + 1 \right). \tag{4.7}$$

Therefore, replacing (4.6) and (4.7) into (4.5), we obtain

$$\left(\frac{1}{C} \|u_j\|_{H^s} - 1 \right)^{q/(q-1)} + \left(\frac{1}{C} \|v_j\|_{H^t} - 1 \right)^{p/(p-1)} \leq C \left(1 + \|u_j\|_{H^s} + \|v_j\|_{H^t} \right).$$

Since $p, q > 1$, we conclude that z_j is bounded in E, as we have claimed. It follows that Φ satisfies the $(PS)_c^{\mathcal{F}}$ condition.

Lemma 4.2 *Condition* $(\Phi_{2'})$ *holds.*

Proof. We follow the proof of Lemma 3.2 of [1]. We apply Lemma 3.1 in E^+, with

$$P(z) := Q(z) = \int A^s u A^t v, \quad \Psi(z) := \int H(x, u, v) \, dx.$$

Since $z \in E^+$,

$$Q(z) = \int A^s u A^t (A^{-t} A^s u) \, dx = \int |A^s u|^2 \, dx = \|u\|_{H^s}^2 = \frac{1}{2} \|z\|_E^2.$$

Using the imbeddings (4.1), we have that

$$\left| \int H(x, u, v) \right| \leq C \left(\|u\|_{H^s}^q + \|v\|_{H^t}^p \right) \leq C \left(\|z\|_E^q + \|z\|_E^p \right) \leq \|z\|_E^{\max(p,q)}.$$

Thus, we have $\Phi = P - \psi$ with $P(z) \geq \frac{1}{2} \|z\|_E^2$ and $|\psi(z)| = |\int H(x, u, v)| \leq C(1 + \|z\|_E^{\max(p,q)})$, with $\max(p, q) > 2$. Therefore, by Lemma 3.1, condition $(\Phi_{2'})$ holds.

4.3 The Geometry of the Functional Φ

In the next two lemmas, we check the requiered conditions on the geometry of the functional Φ:

Lemma 4.3 *Condition* $(\Phi_3\prime)$ *holds.*

Proof. We follow the proof of Lemma 5.1 of [1]. We want to prove that there exist isomorphisms $T_k : E \to E$ $(k \in \mathbb{N})$ such that $T_k(E_n) = E_n$ for all k, n and that there exist $R_k > 0 (k \in \mathbb{N})$ such that, if $z = z^+ + z^- \in E_k^+ \oplus E_k^-$ with $R_k = \max(\|z^+\|, \|z^-\|)$, then $\|T_k z\| > r_k$ and $\phi(T_k z) < 0$ (r_k being the same as that in condition $(\Phi_2\prime)$).

We want to see that there exists λ_k such that the above condition holds with $T_k = T_{\lambda_k}$ and $R_k = \lambda_k$, where

$$T_{\lambda_k}(u, v) = (\lambda_k^\mu u, \lambda_k^\nu v) \ \text{ with } \ \mu = \frac{m-q}{q}, \ \ \nu = \frac{m-p}{p}, \ \ m > \max(p, q).$$

Clearly, $T_k : E \to E$ is isomorphism for all k. Moreover, $T_{\lambda_k} E_n = E_n$ for all k and, for all $\lambda > 0$, we have that

$$\int_{\mathbb{R}^n} H(x, T_\lambda z) \geq C \left(\lambda^{\mu q} \int_{\mathbb{R}^n} |u|^q |x|^b + \lambda^{\nu p} \int_{\mathbb{R}^n} |v|^p |x|^a \right). \tag{4.8}$$

For $z = z^+ + z^- \in E_k^+ \oplus E_k^-$, let $z^- = z_1^- + z_2^-$ with $z_1^- \in E_k^-$ and $z_2^- \perp E_k^-$, and let $\bar{z} = z^+ + z_1^-$. If $z = (u, v)$, we extend these definitions to u and have that $\bar{u} = u^+ + u_1^-$ and, therefore, $u_2^- \perp \bar{u}$ in L^2. Then,

$$\|\bar{u}\|_{L^2}^2 = |\langle \bar{u}, \bar{u} \rangle_{L^2}| = |\langle \bar{u} + u_2^-, \bar{u} \rangle_{L^2}| = |\langle u, \bar{u} \rangle_{L^2}|$$

$$= \int_{\mathbb{R}^n} |u||\bar{u}||x|^{b/q}|x|^{-b/q} \leq \|u\|_{L^q(|x|^b)} \|\bar{u}\|_{L^{q'}(|x|^{-b/(q-1)})}.$$

But, since $\bar{u} \in E_s^k \subset \mathcal{S}(\mathbb{R}^n)$, we have that

$$\|\bar{u}\|_{L^{q'}(|x|^{-b/(q-1)})} = \left(\int_{\mathbb{R}^n} |\bar{u}|^{q'} |x|^{-b/(q-1)} \right)^{1/q'} < +\infty \text{ since } b < n(q-1)$$

and, thanks to the equivalence of the norms $\|\bar{u}\|_{L^2}$ and $\|\bar{u}\|_{L^{q'}(|x|^{-b/(q-1)})}$ (in the finite dimensional subespace E_k^s), we obtain

$$\|u\|_{L^q(|x|^b)} \geq \gamma_k \|\bar{u}\|_{H^s} \ \forall\, u \in E_k^s$$

for some $\gamma_k > 0$. Similarly there exists $\overline{\gamma}_k > 0$ such that

$$\|v\|_{L^p(|x|^a)} \geq \overline{\gamma}_k \|\bar{v}\|_{H^t} \ \forall\, v \in E_k^t.$$

It then follows from (4.8) that

$$\int_{\mathbb{R}^n} H(x, T_\lambda z) \geq C \left(\lambda^{\mu q} \ \gamma_k^q \|\overline{u}\|_{H^s}^q + \lambda^{\nu p} \ \overline{\gamma}_k^p \|\overline{v}\|_{H^t}^p \right)$$

and (as in lemma 4.2 of [1]) we get a lower bound of the form:

$$\int_{\mathbb{R}^n} H(x, T_\lambda z) \geq c \min \left\{ \frac{1}{2^q} \lambda^{\mu q} \gamma_k^q \lambda^q, \frac{1}{2^p} \lambda^{\nu p} \overline{\gamma}_k^p \lambda^p \right\} \geq \sigma_k \lambda^m$$

provided that $\|z^+\|_E = \lambda$.

On the other hand,

$$Q(T_\lambda z) = \lambda^{\nu+\mu}(\|z^+\|_E^2 - \|z^-\|_E^2) \leq \lambda^{\nu+\mu+2}$$

for $\|z^+\|_E = \lambda$. As a consequence, we have that

$$\Phi(T_\lambda z) \leq \lambda^{\nu+\mu+2} - \sigma_k \lambda^m.$$

Since $m > \nu + \mu + 2$, it follows that there is a $\lambda_0(k)$ such that $T_{\lambda_k}(z) < 0$ if $\lambda_k > \lambda_0(k)$. Also we have that

$$\|T_\lambda z\|_E \geq \lambda^{\min(\nu,\mu)} \|z\|_E^2$$

which implies that

$$\|T_\lambda z\|_E \geq \lambda_k^{\min(\mu,\nu)+2} \text{ for } \max(\|z^+\|_E, \|z^-\|_E) = \lambda_k$$

Therefore, it is possible to select $\lambda_k > 0$ such that

$$\Phi(T_{\lambda_k} z) \leq 0 \quad \text{and} \quad \|T_{\lambda_k} z\|_E \geq r_k$$

for any given r_k.

Finally, we observe that condition (Φ_5) holds trivially. Therefore, all the conditions of Theorem 3.1 are fulfilled, and hence the proof of Theorem 1.1 is complete.

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