Finite element analysis of a quadratic eigenvalue problem arising in dissipative acoustics

A. Bermúdez,* R.G. Durán,† R. Rodríguez,‡ and J. Solomin.§

Abstract

A quadratic eigenvalue problem arising in the determination of the vibration modes of an acoustic fluid contained in a cavity with absorbing walls is considered. The problem is shown to be equivalent to the spectral problem for a non-compact operator and a thorough spectral characterization is given.

A numerical discretization based on Raviart-Thomas finite elements to approximate these damped vibration modes is analyzed. The method is proved to be free of spurious modes and to converge with optimal order. Implementation issues and numerical experiments confirming the theoretical results are reported.

Key words. Dissipative acoustics, quadratic eigenvalue problems, finite element spectral approximation

AMS subject classifications. 65N25, 65N30

1 Introduction

This paper concerns the numerical approximation of the damped linear oscillations of an acoustic (i.e., inviscid, compressible, barotropic) fluid contained in a cavity, with absorbing walls able to dissipate acoustic energy.

A large attention has been paid to this kind of problems, particularly in the framework of fluid-structure interactions and mainly related to the goal of decreasing the level of noise in aircraft or cars. Let us mention for instance [18], where a pressure/potential formulation of the fluid is given and numerically solved by finite element and modal reduction methods.

*Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, 15706, Santiago de Compostela, Spain. Partially supported by research project PB97-0508 MEC (Spain). E-mail: *bermudez@zmat.usc.es*.

†Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 - Buenos Aires, Argentina. Partially supported by Universidad de Buenos Aires through grant TX-048 and ANPCyT, through grant PICT 03-00000-00137. Member of CONICET (Argentina). E-mail: *rduran@dm.uba.ar*.

‡Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. Partially supported by FONDECYT through grant 1.990.346 and FONDAP-CONICYT (Chile) through Program A on Numerical Analysis. E-mail: *rodolfo@ing-mat.udec.cl*.

§Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 172, 1900 - La Plata, Argentina. Member of CONICET (Argentina). E-mail: *solo@mate.unlp.edu.ar*.

In fluid-structure interaction problems, since the solid is generally described in terms of displacements, to choose the same variable for the fluid presents advantages because compatibility and equilibrium through the fluid-solid interface satisfy automatically. This approach leads to sparse symmetric matrices and could be in principle applied to the solution of a broad range of problems (in particular non-linear ones, see [1, 23]).

Nevertheless, it is well known that the displacement formulation for the fluid suffers from the presence of zero-frequency spurious circulation modes with no physical meaning. After discretization by standard finite elements, these modes correspond to non-zero frequencies interspersed among those of the physical ones.

Several approaches have been proposed to circumvent this drawback (see [17, 9, 1, 14, 23]). One particularly fruitful has been introduced and analyzed in [2, 5]. It consists of using *Raviart-Thomas* finite elements for the fluid displacements. This method has been proved to be free of spurious modes and to attain optimal order of convergence ([2, 22]). Numerical experiments showing its performance have been reported for 2D and 3D problems in [5] and [3], respectively.

This approach has been recently extended to deal with fluid-structure vibrations with interface damping. In [6], it has been applied to the forced vibration problem of a coupled fluid-solid system with a thin layer of noise damping material separating both media. Optimal order of convergence has been proved in that reference for the Helmholtz-type source problem associated with an external harmonic excitation. Numerical experiments have been reported for 2D and 3D problems in [7] and [4], respectively; the associated non-linear eigenvalue problems have been also numerically solved in both references.

It is well known that when damped systems are subject to a frequency varying periodic excitation, large response peaks are attained for particular values of the external frequency. In most applications it turns out interesting to predict the localization of such peaks. In [7] and [4], it was experimentally shown that the imaginary parts of the obtained eigenvalues practically coincide with the frequencies of the external source for which the response of the damped system is larger. However, no theoretical analysis is provided, neither about a characterization of the solutions of these non-linear eigenvalue problems, nor about the convergence of the proposed numerical method.

In this paper we deal with a simpler problem of this type: the approximation of the damped vibration modes of a fluid filling a rigid cavity with some of its walls covered by a thin layer of a viscoelastic material. As in [6, 7, 4], we model the behavior of the damping material following [18], by conveniently modifying the boundary conditions on the absorbing walls of the fluid domain

In Section 2 we show that these damped vibration modes are solutions of a quadratic eigenvalue problem. We introduce a convenient functional framework to analyze it and prove that this non-linear eigenvalue problem is equivalent to the spectral problem for a non self-adjoint, non-compact bounded operator. In Section 3 we provide a thorough spectral characterization of this operator.

In Section 4 we introduce a finite element discretization of this problem based on Raviart-Thomas elements. We prove approximation and consistency properties as well as further regularity of the generalized eigenfunctions of the continuous problem. Then, by using the spectral approximation theory for non-compact operators from [12], we prove that the proposed numerical scheme converges and is free of spurious modes. In Section

5 we prove optimal order error estimates by conveniently adapting the theory in [13].

Finally, in Section 6, we show how the quadratic eigenvalue problem can be equivalently written to be solved by a standard eigensolver (`eigs` from MATLAB, based on Arnoldi iterations). We also present a numerical experiment showing the good performance of the method and confirming the theoretical results.

2 Model problem

We consider an inviscid compressible barotropic fluid contained in a rigid cavity, with some or all of its walls covered by a thin layer of a viscoelastic material, to absorb part of the acoustic energy of the fluid.

We denote by $\Omega \subset \mathbb{R}^n$ $(n = 2$ or $3)$ the domain occupied by the fluid, which we suppose polyhedral, with boundary $\partial\Omega = \Gamma_{\mathrm{A}} \cup \Gamma_{\mathrm{R}}$. $\Gamma_{\mathrm{A}} = \bigcup_{j=1}^{J} \Gamma_j$, with Γ_j being all the different faces of Ω covered by damping material, is called the "absorbing boundary". Γ_{R} is the union of the remaining faces and we call it the "rigid boundary". We assume that Γ_{A} is not empty. The unit outer normal vector along $\partial\Omega$ is denoted by $\boldsymbol{\nu}$.

We use the following notations for the physical magnitudes: P for the fluid pressure, $\mathbf{U}$ for the displacement field, ρ for the fluid density, and c for the acoustic speed. The dynamic equations for our problem are (see [7, 18]):

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} + \nabla P = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$P = -\rho c^2 \operatorname{div} \mathbf{U} \quad \text{in } \Omega, \tag{2.2}$$

$$P = \left(\alpha\, \mathbf{U} \cdot \boldsymbol{\nu} + \beta\, \frac{\partial \mathbf{U}}{\partial t} \cdot \boldsymbol{\nu} \right) \quad \text{on } \Gamma_{\mathrm{A}}, \tag{2.3}$$

$$\mathbf{U} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{\mathrm{R}}. \tag{2.4}$$

The usual kinematic interface condition for rigid walls, $\mathbf{U} \cdot \boldsymbol{\nu} = 0$, has been relaxed in the absorbing boundary to take into account the effect of the viscoelastic material. Equation (2.3) models this effect: the fluid pressure on the boundary is in equilibrium with the response of the absorbing walls. This response consists of two terms: the first one is proportional to the normal component of the displacements and accounts for the elastic behavior of the material, whereas the second one is proportional to the normal velocity and models the viscous damping.

The *damped vibration modes* of the fluid are complex solutions of equations (2.1)-(2.4) of the form $U(\mathbf{x}, t) = \mathrm{e}^{\lambda t} u(\mathbf{x})$ and $P(\mathbf{x}, t) = \mathrm{e}^{\lambda t} p(\mathbf{x})$. They can be found by solving the following nonlinear eigenvalue problem:

Find $\lambda \in \mathbf{C}$, $\mathbf{u} : \Omega \to \mathbf{C}^n$ *and* $p : \Omega \to \mathbf{C}$, $(\mathbf{u}, p) \neq (\mathbf{0}, 0)$, *such that:*

$$\rho \lambda^2 \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega, \tag{2.5}$$

$$p = -\rho c^2 \operatorname{div} \mathbf{u} \quad \text{in } \Omega, \tag{2.6}$$

$$p = (\alpha + \lambda\beta)\, \mathbf{u} \cdot \boldsymbol{\nu} \quad \text{on } \Gamma_{\mathrm{A}}, \tag{2.7}$$

$$\mathbf{u} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{\mathrm{R}}. \tag{2.8}$$

For each damped vibration mode, $\omega := \operatorname{Im}\lambda$ is the *vibration angular frequency* and $\operatorname{Re}\lambda$ (which is proved below to be negative) the *decay rate*. For a particular angular frequency ω, coefficients α and β in the previous equations are related to the *impedance* Z of the viscoelastic material through the expression:

$$Z := \beta + \frac{\alpha}{\omega}\mathrm{i}.$$

The real and imaginary parts of this impedance can be experimentally measured. For real materials, both coefficients α and β actually depend on ω. However, in a first approach, they can be taken as strictly positive constants. Such assumption turns out realistic when a restricted range of frequencies is considered. See for instance [18], where experimental values of the impedance are reported for a typical acoustic insulating material (glasswool).

A variational formulation of problem (2.5)-(2.8) involving only displacement variables can be easily obtained. Let

$$\mathcal{V} := \left\{ \boldsymbol{\phi} \in \mathrm{H}(\operatorname{div}, \Omega) : \ \boldsymbol{\phi} \cdot \boldsymbol{\nu} \in \mathrm{L}^2(\partial\Omega) \ \text{ and } \ \boldsymbol{\phi} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_{\mathrm{R}} \right\},$$

endowed with its natural norm $\|\boldsymbol{\phi}\|_{\mathcal{V}} := \left(\|\boldsymbol{\phi}\|^2_{\operatorname{div},\Omega} + \|\boldsymbol{\phi}\cdot\boldsymbol{\nu}\|^2_{0,\Gamma_{\mathrm{A}}} \right)^{1/2}$. By integrating by parts (2.5) multiplied by $\bar{\boldsymbol{\phi}} \in \mathcal{V}$ and by using (2.6) and (2.7) we obtain:

Find $\lambda \in \mathbf{C}$ and $\mathbf{u} \in \mathcal{V}$, $\mathbf{u} \neq \mathbf{0}$, such that:

$$\int_\Omega \rho c^2 \operatorname{div}\mathbf{u} \operatorname{div}\bar{\boldsymbol{\phi}} + \int_{\Gamma_{\mathrm{A}}} \alpha\, \mathbf{u}\cdot\boldsymbol{\nu}\, \bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu} + \lambda \int_{\Gamma_{\mathrm{A}}} \beta\, \mathbf{u}\cdot\boldsymbol{\nu}\, \bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu} + \lambda^2 \int_\Omega \rho\, \mathbf{u}\cdot\bar{\boldsymbol{\phi}} = 0 \qquad \forall \boldsymbol{\phi} \in \mathcal{V}. \tag{2.9}$$

This is a quadratic eigenvalue problem.[1] Clearly $\lambda = 0$ is one of its eigenvalues, with corresponding eigenspace

$$\mathcal{K} := \left\{ \mathbf{u} \in \mathcal{V} : \ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega \ \text{ and } \ \mathbf{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega \right\}.$$

The following lemma shows that, for all the other solutions of (2.9), the decay rate is strictly negative. This agrees with what is well-known from the physics viewpoint: the effect of the viscoelastic material is to *damp* the vibrations.

Lemma 2.1 *Let $\lambda \in \mathbf{C}$ and $0 \neq \mathbf{u} \in \mathcal{V}$ be a solution of Problem (2.9). If $\lambda \neq 0$, then* $\operatorname{Re}\lambda < 0$.

Proof: λ is a solution of the algebraic equation $A\lambda^2 + B\lambda + C = 0$ with $A := \int_\Omega \rho|\mathbf{u}|^2 > 0$, $B := \int_{\Gamma_{\mathrm{A}}} \beta|\mathbf{u}\cdot\boldsymbol{\nu}|^2 \geq 0$ and $C := \int_\Omega \rho c^2 |\operatorname{div}\mathbf{u}|^2 + \int_{\Gamma_{\mathrm{A}}} \alpha|\mathbf{u}\cdot\boldsymbol{\nu}|^2 \geq 0$. Then it is easy to check that, for $C > 0$ and $B > 0$, $\operatorname{Re}\lambda < 0$.

[1] An alternative variational formulation in terms of pressure variables can also be obtained:

$$\int_\Omega \nabla p \cdot \nabla \bar{q} + \frac{\lambda^2}{\alpha + \lambda\beta} \int_{\Gamma_{\mathrm{A}}} \rho\, p\bar{q} + \frac{\lambda^2}{c^2} \int_\Omega p\bar{q} = 0 \qquad \forall q \in \mathrm{H}^1(\Omega);$$

however, in this case, the eigenvalue problem turns out to be neither linear nor quadratic.

If $C = 0$, then $B = 0$ and thus $\lambda = 0$. Finally we show that, if $B = 0$, then $C = 0$ too, and thus $\lambda = 0$ again. In fact, if $B = 0$, then $\mathbf{u}\cdot\boldsymbol{\nu} = 0$ on Γ_{A} and hence (2.9) yields

$$\int_\Omega \rho c^2 \operatorname{div}\mathbf{u} \operatorname{div}\bar{\boldsymbol{\phi}} + \lambda^2 \int_\Omega \rho\,\mathbf{u}\cdot\bar{\boldsymbol{\phi}} = 0 \qquad \forall \boldsymbol{\phi}\in\mathcal{V}.$$

Thus, by testing it with different $\boldsymbol{\phi}\in\mathcal{C}^\infty_{\Gamma_{\mathrm{R}}}(\bar{\Omega})^n := \{\boldsymbol{\psi}\in\mathcal{C}^\infty(\bar{\Omega})^n:\ \boldsymbol{\psi} = \mathbf{0} \text{ on } \Gamma_{\mathrm{R}}\}\subset\mathcal{V}$ we have that

$$\begin{aligned} -\rho c^2\nabla(\operatorname{div}\mathbf{u}) &= -\lambda^2\rho\mathbf{u} &&\text{in } \Omega,\\ \operatorname{div}\mathbf{u} &= 0 &&\text{on } \Gamma_{\mathrm{A}}. \end{aligned}$$

Then $q = \operatorname{div}\mathbf{u}\in\mathrm{H}^1(\Omega)$ satisfies

$$\begin{aligned} c^2\Delta q &= \lambda^2 q &&\text{in } \Omega,\\ \frac{\partial q}{\partial\nu} &= 0 &&\text{on } \partial\Omega,\\ q &= 0 &&\text{on } \Gamma_{\mathrm{A}}. \end{aligned}$$

So, if q were not null, then it would be an eigenfunction of the Laplace operator on Ω satisfying simultaneously vanishing Dirichlet and Neumann boundary conditions on Γ_{A}. Since this cannot happen, then $q = 0$ and hence $C = 0$. $\Box$

For the theoretical analysis it is convenient to transform (2.9) into an equivalent linear eigenvalue problem. This is carried out by introducing the new variable $\mathbf{v} := \lambda\mathbf{u}$, as usual in quadratic problems:

Find $\lambda\in\mathbf{C}$ *and* $(\mathbf{u},\mathbf{v})\in\mathcal{V}\times\mathrm{L}^2(\Omega)^n$, $(\mathbf{u},\mathbf{v})\neq(\mathbf{0},\mathbf{0})$, *such that:*

$$\text{(2.10)}\quad \int_\Omega \rho c^2 \operatorname{div}\mathbf{u}\operatorname{div}\bar{\boldsymbol{\phi}} + \int_{\Gamma_{\mathrm{A}}} \alpha\,\mathbf{u}\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu}$$
$$= \lambda\left(-\int_{\Gamma_{\mathrm{A}}}\beta\,\mathbf{u}\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu} - \int_\Omega \rho\,\mathbf{v}\cdot\bar{\boldsymbol{\phi}}\right) \qquad \forall\boldsymbol{\phi}\in\mathcal{V},$$

$$\text{(2.11)}\quad \int_\Omega \rho\,\mathbf{v}\cdot\bar{\boldsymbol{\psi}} = \lambda\int_\Omega\rho\,\mathbf{u}\cdot\bar{\boldsymbol{\psi}} \qquad \forall\boldsymbol{\psi}\in\mathrm{L}^2(\Omega)^n.$$

Remark: In principle, we could have chosen to look for $(\mathbf{u},\mathbf{v})\in\mathcal{V}\times\mathcal{V}$ instead of $\mathcal{V}\times\mathrm{L}^2(\Omega)^n$. However, in such case, in order to prove convergence and error estimates for the numerical scheme introduced below, we would need an estimate similar to (4.24), but in $\mathcal{V}\times\mathcal{V}$ norm (instead of $\mathcal{V}\times\mathrm{L}^2(\Omega)^n$ one), which does not seem to hold. $\Box$

It is clear that $\lambda = 0$ is also an eigenvalue of problem (2.10)-(2.11) with corresponding eigenspace $\widetilde{\mathcal{K}} := \mathcal{K}\times\{\mathbf{0}\}$. Let $\mathcal{G}$ denote the orthogonal complement of $\mathcal{K}$ in $\mathcal{V}$ and $\widetilde{\mathcal{G}} := \mathcal{G}\times\mathrm{L}^2(\Omega)^n$. Notice that (see for instance [15])

$$\mathcal{G} = \left\{\mathbf{u}\in\mathcal{V}:\ \mathbf{u} = \nabla\varphi \text{ for } \varphi\in\mathrm{H}^1(\Omega)\right\}.$$

Now, for $\mathbf{u} = \nabla\varphi\in\mathcal{G}$, φ is a solution of the following Neumann problem:

$$\begin{aligned} \Delta\varphi &= \operatorname{div}\mathbf{u}\in\mathrm{L}^2(\Omega),\\ \frac{\partial\varphi}{\partial\nu} &= \mathbf{u}\cdot\boldsymbol{\nu}\in\mathrm{L}^2(\partial\Omega). \end{aligned}$$

If $\mathbf{u}\cdot\boldsymbol{\nu}\in \mathrm{H}^{1/2}(\Gamma_j)$, for $j=1,\dots,J$, since $\mathbf{u}\cdot\boldsymbol{\nu}=0$ on Γ_{R} because $\mathbf{u}\in\mathcal{V}$, then it is known (see [11, 16]) that there exist $s\in(0,1]$ and $C>0$, both depending only on Ω, such that $\mathbf{u}=\nabla\varphi\in \mathrm{H}^s(\Omega)^n$ and

$$\|\mathbf{u}\|_{s,\Omega}\le C\left(\|\operatorname{div}\mathbf{u}\|_{0,\Omega}+\sum_{j=1}^{J}\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{1/2,\Gamma_j}\right). \tag{2.12}$$

In general, for any $\mathbf{u}\in\mathcal{G}$, since $\mathbf{u}\cdot\boldsymbol{\nu}$ is only in $\mathrm{L}^2(\partial\Omega)$, then $\mathbf{u}=\nabla\varphi\in \mathrm{H}^{\hat{s}}(\Omega)^n$ with $\hat{s}:=\min\{s,\frac{1}{2}\}$ and

$$\|\mathbf{u}\|_{\hat{s},\Omega}\le C\Big(\|\operatorname{div}\mathbf{u}\|_{0,\Omega}+\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{0,\Gamma_{\mathrm{A}}}\Big). \tag{2.13}$$

Here and thereafter C denotes a strictly positive constant not necessarily the same at each occurrence, s denotes a fixed constant in $(0,1]$, which only depends on the geometry of Ω, such that (2.12) holds (for instance, if Ω is convex, $s=1$) and $\hat{s}=\min\{s,\frac{1}{2}\}$.

We introduce some further notation. Let $a:\mathcal{V}\times\mathcal{V}\longrightarrow\mathbf{C}$ be the sesquilinear continuous form defined by

$$a(\mathbf{u},\boldsymbol{\phi}):=\int_\Omega \rho c^2\operatorname{div}\mathbf{u}\operatorname{div}\bar{\boldsymbol{\phi}}+\int_{\Gamma_{\mathrm{A}}}\alpha\,\mathbf{u}\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu}.$$

Let $\widetilde{\mathcal{V}}:=\mathcal{V}\times\mathrm{L}^2(\Omega)^n$ endowed with the product norm and $\tilde{a},\tilde{b}:\widetilde{\mathcal{V}}\times\widetilde{\mathcal{V}}\longrightarrow\mathbf{C}$ the sesquilinear continuous forms defined by

$$\begin{aligned}\tilde{a}\Big((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big) &:= a(\mathbf{u},\boldsymbol{\phi})+\int_\Omega\rho\,\mathbf{v}\cdot\bar{\boldsymbol{\psi}},\\ \tilde{b}\Big((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big) &:= -\int_{\Gamma_{\mathrm{A}}}\beta\,\mathbf{u}\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu}-\int_\Omega\rho\,\mathbf{v}\cdot\bar{\boldsymbol{\phi}}+\int_\Omega\rho\,\mathbf{u}\cdot\bar{\boldsymbol{\psi}}.\end{aligned}$$

The following lemma holds:

Lemma 2.2 *The sesquilinear form $a:\mathcal{G}\times\mathcal{G}\longrightarrow\mathbf{C}$ is $\mathcal{V}$-elliptic and, consequently, $\tilde{a}:\widetilde{\mathcal{G}}\times\widetilde{\mathcal{G}}\longrightarrow\mathbf{C}$ is $\widetilde{\mathcal{V}}$-elliptic.*

PROOF: A priori estimate (2.13) shows that the expression in its right hand side defines a norm on $\mathcal{G}$ equivalent to $\|\cdot\|_{\mathcal{V}}$ and this yields immediately the lemma. □

As a consequence of the ellipticity of $\tilde{a}$ on $\widetilde{\mathcal{G}}$, we are able to introduce the linear bounded operator $A:\widetilde{\mathcal{V}}\longrightarrow\widetilde{\mathcal{V}}$ defined by

$$A(\mathbf{f},\mathbf{g}):=(\mathbf{u},\mathbf{v})\in\widetilde{\mathcal{G}}:\quad \tilde{a}\Big((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big)=\tilde{b}\Big((\mathbf{f},\mathbf{g}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\qquad\forall(\boldsymbol{\phi},\boldsymbol{\psi})\in\widetilde{\mathcal{G}}.$$

Notice that this amounts to

$$\mathbf{v}=\mathbf{f} \tag{2.14}$$

and

$$\mathbf{u}\in\mathcal{G}:\quad a(\mathbf{u},\boldsymbol{\phi})=-\int_{\Gamma_{\mathrm{A}}}\beta\,\mathbf{f}\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu}-\int_\Omega\rho\,\mathbf{g}\cdot\bar{\boldsymbol{\phi}}\qquad\forall\boldsymbol{\phi}\in\mathcal{G}. \tag{2.15}$$

The following lemma shows that the non-zero eigenvalues of A are exactly the inverses of the non-zero eigenvalues of problem (2.10)-(2.11), with the same corresponding eigenfunctions, and consequently of the original quadratic problem (2.9).

Lemma 2.3 *For $\mu \neq 0$, $\big(\mu, (\mathbf{u}, \mathbf{v})\big)$ is an eigenpair of A if and only if $\big(\frac{1}{\mu}, (\mathbf{u}, \mathbf{v})\big)$ is a solution of problem (2.10)-(2.11).*

PROOF: Let $\big(\mu, (\mathbf{u}, \mathbf{v})\big)$ be an eigenpair of A with $\mu \neq 0$. Hence

$$\tilde{a}\Big((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})\Big) = \frac{1}{\mu}\,\tilde{b}\Big((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})\Big) \qquad \forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \widetilde{\mathcal{G}}, \tag{2.16}$$

and then, because of (2.14), $\mathbf{v} = \frac{1}{\mu}\mathbf{u} \in \mathcal{G}$. So, for $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \widetilde{\mathcal{K}}$, $\tilde{b}\Big((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})\Big) = 0$ and $\tilde{a}\Big((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})\Big) = 0$. Thus (2.16) is also valid for $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \widetilde{\mathcal{V}} = \widetilde{\mathcal{G}} \oplus \widetilde{\mathcal{K}}$; that is, (2.10)-(2.11) hold with $\lambda = \frac{1}{\mu}$.

Conversely, let $\big(\frac{1}{\mu}, (\mathbf{u}, \mathbf{v})\big)$ be a solution of problem (2.10)-(2.11). By taking $\boldsymbol{\phi} \in \mathcal{K}$, (2.10) implies that $\int_\Omega \rho\, \mathbf{v} \cdot \bar{\boldsymbol{\phi}} = 0$ and hence $\mathbf{v} \in \mathcal{G}$. On the other hand, (2.11) implies $\mathbf{u} = \mu \mathbf{v} \in \mathcal{G}$. So, $(\mathbf{u}, \mathbf{v}) \in \widetilde{\mathcal{G}}$ and (2.16) holds, showing that $\big(\mu, (\mathbf{u}, \mathbf{v})\big)$ is an eigenpair of A. $\square$

3 Characterization of the spectrum

We will show that the operator A defined in the previous section is not compact. In some simple cases, like a cubic 3D domain Ω with only one of its faces covered by the viscoelastic material, the eigenvalues and eigenfunctions of A can be found solving by separation of variables the non linear spectral problem (2.5)-(2.8) (see [4]). In this case, so-called *overdamped* modes, corresponding to real negative eigenvalues μ, exist and $-\frac{\beta}{\alpha}$ is an accumulation point of them.

In more general problems, the spectrum of the operator, $\sigma(A)$, is not explicitly known. However, we show below that $-\frac{\beta}{\alpha}$ always belongs to its essential spectrum $\sigma_{\mathrm{ess}}(A)$ (i.e., the set consisting of all the limit points of $\sigma(A)$ and the eigenvalues of A having infinite algebraic multiplicity), thus proving the non-compactness of A.

In this section we give a thorough spectral characterization of the operator A by following the general analysis carried out by Krein and Langer in [19]. We do it so, instead of explicitly using the results therein, for the sake of simplicity and completeness. See the remark at the end of this section for a further discussion.

Since $A(\widetilde{\mathcal{V}}) \subset \widetilde{\mathcal{G}}$, then $\sigma(A) = \sigma(A|_{\widetilde{\mathcal{G}}}) \cup \{0\}$. Moreover, since $A(\widetilde{\mathcal{G}}) \subset \mathcal{G} \times \mathcal{G}$, then $\sigma(A) = \sigma(A|_{\mathcal{G}\times\mathcal{G}}) \cup \{0\}$. Because of this, we restrict the analysis throughout this section to the operator $A|_{\mathcal{G}\times\mathcal{G}} : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \times \mathcal{G}$, which, for simplicity, is also denoted by A.

Because of Lemma 2.2, $a(\cdot, \cdot)$ is an inner product on $\mathcal{G}$, inducing on this space a norm equivalent to $\|\cdot\|_{\mathcal{V}}$. Let $A_1, A_2 : \mathcal{G} \longrightarrow \mathcal{G}$ be given by:

$$A_1\mathbf{f} = \mathbf{u}_1 \in \mathcal{G}: \qquad a(\mathbf{u}_1, \boldsymbol{\phi}) = \int_{\Gamma_A} \beta\, \mathbf{f} \cdot \boldsymbol{\nu}\; \bar{\boldsymbol{\phi}} \cdot \boldsymbol{\nu} \qquad \forall \boldsymbol{\phi} \in \mathcal{G}, \tag{3.1}$$

$$A_2\mathbf{g} = \mathbf{u}_2 \in \mathcal{G}: \qquad a(\mathbf{u}_2, \boldsymbol{\phi}) = \int_\Omega \rho\, \mathbf{g} \cdot \bar{\boldsymbol{\phi}} \qquad \forall \boldsymbol{\phi} \in \mathcal{G}. \tag{3.2}$$

Both operators are self-adjoint; A_1 is non-negative and A_2 is positive with respect to $a(\cdot,\cdot)$ (i.e., $a(A_1\mathbf{u},\mathbf{u}) \geq 0$ and $a(A_2\mathbf{u},\mathbf{u}) > 0$, for all $\mathbf{u} \in \mathcal{G}$, $\mathbf{u} \neq \mathbf{0}$). The following lemma shows that A_2 is compact:

Lemma 3.1 *The operator $A_2 : \mathcal{G} \longrightarrow \mathcal{G}$ is compact.*

PROOF: Let $\mathbf{g} \in \mathcal{G}$ and $\mathbf{u}_2 = A_2\mathbf{g}$. Since for $\mathbf{g} \in \mathcal{G}$, (3.2) is also true for $\boldsymbol{\phi} \in \mathcal{K}$, then it holds for all $\boldsymbol{\phi} \in \mathcal{V} = \mathcal{G} \oplus \mathcal{K}$. Therefore, by considering different test functions in $\mathcal{C}^\infty_{\Gamma_{\mathrm{R}}}(\bar{\Omega})^n \subset \mathcal{V}$, we obtain:

$$\begin{aligned} -\rho c^2 \nabla(\operatorname{div} \mathbf{u}_2) &= \rho \mathbf{g} \quad \text{in } \Omega, \\ \rho c^2 \operatorname{div} \mathbf{u}_2 + \alpha\, \mathbf{u}_2 \cdot \boldsymbol{\nu} &= 0 \quad \text{on } \Gamma_j, \quad j = 1, \ldots, J. \end{aligned}$$

Hence $\operatorname{div} \mathbf{u}_2 \in \mathrm{H}^1(\Omega)$ and thus $\mathbf{u}_2 \cdot \boldsymbol{\nu} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{A}})$. Therefore, because of (2.12), $\mathbf{u}_2 \in \mathrm{H}^s(\Omega)^n$ with $s > 0$. So, the lemma follows from the compact inclusions of $\mathrm{H}^1(\Omega)$ in $\mathrm{L}^2(\Omega)$. $\mathrm{H}^s(\Omega)^n$ in $\mathrm{L}^2(\Omega)^n$ and $\mathrm{H}^{1/2}(\Gamma_{\mathrm{A}})$ in $\mathrm{L}^2(\Gamma_{\mathrm{A}})$. $\square$

The operator A can be written as

$$A = \begin{pmatrix} -A_1 & -A_2 \\ I & 0 \end{pmatrix}. \tag{3.3}$$

So, if we define

$$S := \begin{pmatrix} I & 0 \\ 0 & A_2^{1/2} \end{pmatrix}, \qquad U := \begin{pmatrix} -A_1 & -A_2^{1/2} \\ I & 0 \end{pmatrix}, \qquad \text{and} \qquad H := \begin{pmatrix} -A_1 & -A_2^{1/2} \\ A_2^{1/2} & 0 \end{pmatrix},$$

the following relations hold:

$$SA = HS, \qquad A = US, \qquad H = SU, \qquad \text{and} \qquad UH = AU.$$

On the other hand, since A_2 is one-to-one, then so are A, S, U, and H.

The eigenvalues of A and H and their algebraic multiplicities coincide; moreover the corresponding Jordan chains have the same length. In fact, if $\{x_k\}_{k=1}^p$ is a Jordan chain associated with an eigenvalue μ of A, then

$$H(Sx_k) = SAx_k = S(\mu x_k + x_{k-1}) = \mu(Sx_k) + Sx_{k-1}, \quad k = 1, \ldots, p \quad (x_0 = 0).$$

Hence, since S is one-to-one, $\{Sx_k\}_{k=1}^p$ is a Jordan chain of H of the same length. The converse can be shown analogously by using that U is one-to-one and $AU = UH$.

Moreover, the whole spectra of A and H also coincide as shown in the following lemma:

Lemma 3.2 *There holds:*

$$\sigma(A) = \sigma(H).$$

PROOF: Firstly, since A_2 is compact, it is easy to show (see proof of Lemma 3.1) that $A(\mathcal{G} \times \mathcal{G})$ and $H(\mathcal{G} \times \mathcal{G})$ are proper subsets of $\mathcal{G} \times \mathcal{G}$; thus 0 belongs to $\sigma(A)$ and $\sigma(H)$.

Now, let $\mu \notin \sigma(H)$. Since the eigenvalues of A and H coincide, then $(A - \mu I)$ is one-to-one. To prove that it is onto, let $y \in \mathcal{G} \times \mathcal{G}$; since $\mu \notin \sigma(H)$, then there exists $\hat{x} \in \mathcal{G} \times \mathcal{G}$ such that $(H - \mu I)\hat{x} = Sy$. Thus, since $H = SU$, then

$$\hat{x} = \frac{1}{\mu}(H\hat{x} - Sy) = Sx, \qquad \text{for } x = \frac{U\hat{x} - y}{\mu} \in \mathcal{G} \times \mathcal{G}.$$

Hence, since $SA = HS$, we have $S(A-\mu I)x = (H-\mu I)Sx = Sy$, and then $(A-\mu I)x = y$, because S is one-to-one. Therefore $(A - \mu I)$ is onto and thus $\mu \notin \sigma(A)$.

Conversely, for any $\mu \notin \sigma(A)$, the proof that $\mu \notin \sigma(H)$ runs identically, by using that U is one-to-one, $A = US$ and $UH = AU$. □

The operator H can be written as the sum of a self-adjoint operator B and a compact one C. Indeed:

$$H = B + C, \qquad \text{with} \qquad B = \begin{pmatrix} -A_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 0 & -A_2^{1/2} \\ A_2^{1/2} & 0 \end{pmatrix}.$$

Hence, by applying classical Weyl's theorem (see for instance [21]), we have that $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(B)$. The rest of the spectrum $\sigma_{\text{disc}}(H) = \sigma(H) \setminus \sigma_{\text{ess}}(H)$, consists of isolated eigenvalues with finite algebraic multiplicity.

Since $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(-A_1) \cup \{0\}$, then we only need to characterize the essential spectrum of A_1:

Lemma 3.3 *There holds:*

$$\sigma_{\text{ess}}(A_1) = \left\{\frac{\beta}{\alpha}, 0\right\}.$$

PROOF: First we observe that $\frac{\beta}{\alpha}$ is an eigenvalue of A_1 of infinite multiplicity. In fact, it is easy to check that $\mathcal{E} := \{\mathbf{u} \in \mathcal{G} : \operatorname{div} \mathbf{u} = 0\}$ is its associated infinite dimensional eigenspace.

On the other hand, it is easy to see that $\mathcal{F} := \{\mathbf{u} \in \mathcal{G} : \mathbf{u} \cdot \boldsymbol{\nu} \text{ is constant on } \Gamma_{\text{A}}\}$ is the orthogonal complement of $\mathcal{E}$ in $\mathcal{G}$ with respect to $a(\cdot, \cdot)$. So, $A_1(\mathcal{F}) \subset \mathcal{F}$ and $\sigma(A_1) = \{\frac{\beta}{\alpha}\} \cup \sigma(A_1|_{\mathcal{F}})$. Now $A_1\mathbf{u}$ depends only on $\mathbf{u} \cdot \boldsymbol{\nu}$ and so $A_1(\mathcal{F})$ is a one-dimensional space. Therefore, $\sigma(A_1|_{\mathcal{F}})$ consists of $\mu = 0$, which has infinite multiplicity, and a non-zero eigenvalue of multiplicity one. □

Summing up we have the following spectral characterization result:

Theorem 3.1 *The spectrum of A consists of $\sigma_{\text{ess}}(A) = \{-\frac{\beta}{\alpha}, 0\}$ and a set of isolated eigenvalues of finite algebraic multiplicity.*

Remark: The results in this section imply the existence of vibration modes of problem (2.1)-(2.4). To prove those results, we have restricted ourselves to give a spectral characterization of the operator A by following the general theory given by Krein and Langer in [19]. That paper was motivated by the study of oscillations of continua subject to interior damping, instead of boundary damping as in our case. However, the abstract results therein also apply to our problem.

To do this, one should begin with the evolution problem (2.1)-(2.4). By eliminating the pressure and restricting to irrotational displacements, a second order differential equation $T\ddot{\mathbf{u}} + F\dot{\mathbf{u}} + V\mathbf{u} = 0$ is obtained, where T, F and V are (unbounded) linear operators in a closed subspace of $\mathrm{L}^2(\Omega)^n$, namely, the gradients of $\mathrm{H}^1(\Omega)$.

The quadratic spectral problem (2.9), restricted to $\mathcal{G}$, turns out to be a variational form of the spectral problem for the so-called *quadratic pencil* associated to this differential equation: $L(\lambda) := \lambda^2 I + \lambda B + C$, with $B := V^{-1/2} F V^{-1/2}$ and $C := V^{-1/2} T V^{-1/2}$.

In [19] it is shown that $\lambda \neq 0$ is in the spectrum of this quadratic pencil (i.e., $L(\lambda)$ does not have a bounded inverse) if and only if $\mu = \frac{1}{\lambda} \in \sigma(H)$ and that each isolated eigenvalue of H corresponds to an isolated eigenvalue of the pencil (i.e., $L(\lambda)$ is not one-to-one) with the same algebraic multiplicities and corresponding Jordan chains (which are defined in terms of the Jordan chains of the associated operator A).

It is also shown in this reference that each Jordan chain $\{\mathbf{u}_k\}_{k=1}^p$ of the pencil provides p independent elementary solutions of the original evolution problem, namely,

$$\mathbf{U}(\mathbf{x}, t) = e^{\lambda t} \sum_{j=0}^{k-1} \frac{t^j}{j!} \mathbf{u}_{k-j}(\mathbf{x}), \qquad k := 1, ..., p.$$

Therefore, the computation of eigenvalues and eigenfunctions of problem (2.9) is just part of the task of solving the whole spectral problem. Actually, it is the only part we need, in order to know the response peaks whose localization motivates our study. Nevertheless, the analysis in the following section shows that if the quadratic problem had eigenvalues with generalized eigenfunctions, these would be also well approximated by the proposed numerical scheme. □

4 Spectral approximation

In this section we introduce and analyze a finite element method to approximate the solutions of the quadratic eigenvalue problem (2.9). We prove its convergence and that spurious solutions do not arise with this method. Let us remark that such "spurious modes" are usually present, even in the undamped case (see for instance [17]), when standard finite elements are used in a displacement formulation like this.

To avoid them, we use lowest order *Raviart-Thomas* elements (see for instance [8, 20]), which proved to be effective in the undamped case (see [2]). Let $\{\mathcal{T}_h\}$ be a regular family of partitions of Ω in tetrahedra; h stands for the maximum diameter of the elements. Let

$$\mathcal{V}_h := \{\boldsymbol{\phi}_h \in \mathrm{H}(\mathrm{div}, \Omega) : \ \boldsymbol{\phi}_h|_T \in \mathcal{P}_0^n \oplus \mathcal{P}_0 \, \mathbf{x} \ \ \forall T \in \mathcal{T}_h \ \text{ and } \ \boldsymbol{\phi}_h \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_{\mathrm{R}}\} \subset \mathcal{V}$$

($\mathcal{P}_k$ denotes the set of polynomials of degree at most k). Our discrete problem reads:

Find $\lambda_h \in \mathbf{C}$ and $\mathbf{u}_h \in \mathcal{V}_h$, $\mathbf{u}_h \neq \mathbf{0}$, such that:

$$\int_\Omega \rho c^2 \operatorname{div} \mathbf{u}_h \operatorname{div} \bar{\boldsymbol{\phi}}_h + \int_{\Gamma_{\mathrm{A}}} \alpha \, \mathbf{u}_h \cdot \boldsymbol{\nu} \ \bar{\boldsymbol{\phi}}_h \cdot \boldsymbol{\nu} + \lambda_h \int_{\Gamma_{\mathrm{A}}} \beta \, \mathbf{u}_h \cdot \boldsymbol{\nu} \ \bar{\boldsymbol{\phi}}_h \cdot \boldsymbol{\nu} + \lambda_h^2 \int_\Omega \rho \, \mathbf{u}_h \cdot \bar{\boldsymbol{\phi}}_h = 0 \qquad \forall \boldsymbol{\phi}_h \in \mathcal{V}_h. \tag{4.1}$$

We proceed as for the continuous problem; by introducing $\mathbf{v}_h := \lambda_h \mathbf{u}_h$, we transform problem (4.1) into an equivalent linear one:

Find $\lambda_h \in \mathbf{C}$ *and* $(\mathbf{u}_h, \mathbf{v}_h) \in \mathcal{V}_h \times \mathcal{V}_h$, $(\mathbf{u}_h, \mathbf{v}_h) \neq (\mathbf{0}, \mathbf{0})$, *such that:*

$$\int_\Omega \rho c^2 \operatorname{div} \mathbf{u}_h \operatorname{div} \bar{\boldsymbol{\phi}}_h + \int_{\Gamma_{\rm A}} \alpha\, \mathbf{u}_h \cdot \boldsymbol{\nu}\ \bar{\boldsymbol{\phi}}_h \cdot \boldsymbol{\nu} = \lambda_h \left(- \int_{\Gamma_{\rm A}} \beta\, \mathbf{u}_h \cdot \boldsymbol{\nu}\ \bar{\boldsymbol{\phi}}_h \cdot \boldsymbol{\nu} - \int_\Omega \rho\, \mathbf{v}_h \cdot \bar{\boldsymbol{\phi}}_h \right) \qquad \forall \boldsymbol{\phi}_h \in \mathcal{V}_h, \tag{4.2}$$

$$\int_\Omega \rho\, \mathbf{v}_h \cdot \bar{\boldsymbol{\psi}}_h = \lambda_h \int_\Omega \rho\, \mathbf{u}_h \cdot \bar{\boldsymbol{\psi}}_h \qquad \forall \boldsymbol{\psi}_h \in \mathcal{V}_h. \tag{4.3}$$

Clearly $\lambda_h = 0$ is also an eigenvalue of this problem and its associated eigenspace is $\widetilde{\mathcal{K}}_h := \mathcal{K}_h \times \{\mathbf{0}\}$, with $\mathcal{K}_h := \mathcal{K} \cap \mathcal{V}_h$. Let $\mathcal{G}_h$ be the orthogonal complement of $\mathcal{K}_h$ in $\mathcal{V}_h$ and $\widetilde{\mathcal{G}}_h := \mathcal{G}_h \times \mathcal{G}_h \subset \widetilde{\mathcal{V}}$, endowed with the norm of $\widetilde{\mathcal{V}} = \mathcal{V} \times \mathrm{L}^2(\Omega)^n$. Notice that $\mathcal{G}_h \not\subset \mathcal{G}$ and hence $\widetilde{\mathcal{G}}_h \not\subset \widetilde{\mathcal{G}}$. However the sesquilinear form $\tilde{a}$ turns out to be uniformly $\widetilde{\mathcal{V}}$-elliptic on $\widetilde{\mathcal{G}}_h$ as shown below.

The following property of the Helmholtz decomposition of functions in $\mathcal{G}_h$ will be used in the sequel. We recall that $s \in (0, 1]$ and $\hat{s} := \min\{s, \frac{1}{2}\}$ are fixed constants, only depending on Ω, such that estimates (2.12) and (2.13) hold. These constants will be used throughout this section.

Lemma 4.1 *For any* $\mathbf{u}_h \in \mathcal{G}_h$ *we can write*

$$\mathbf{u}_h = \nabla \xi + \boldsymbol{\chi},$$

with $\xi \in \mathrm{H}^1(\Omega)$ *and* $\boldsymbol{\chi} \in \mathcal{K}$ *satisfying*

$$\|\nabla \xi\|_{\hat{s},\Omega} \leq C\Big(\|\operatorname{div} \mathbf{u}_h\|_{0,\Omega} + \|\mathbf{u}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_{\rm A}}\Big), \tag{4.4}$$

$$\|\boldsymbol{\chi}\|_{0,\Omega} \leq C h^{\hat{s}}\Big(\|\operatorname{div} \mathbf{u}_h\|_{0,\Omega} + \|\mathbf{u}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_{\rm A}}\Big), \tag{4.5}$$

Proof: We do not include it here since the arguments are identical to those of decomposition (5.5) in [2] and its corresponding estimates, which are given in the proofs of Theorem 5.4 and Lemma 5.5 of that reference. The only differences are that, in our case, we bound $\|\nabla \xi\|_{\hat{s},\Omega}$ (instead of $\|\nabla \xi\|_{s,\Omega}$) in the first estimate and have $\mathcal{O}(h^{\hat{s}})$ (instead of $\mathcal{O}(h^s)$) in the second one, because we need bounds depending on $\|\mathbf{u}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_{\rm A}}$. □

The first consequence of this lemma is the ellipticity claimed above:

Lemma 4.2 *The sesquilinear form* $a : \mathcal{G}_h \times \mathcal{G}_h \longrightarrow \mathbf{C}$ *is* $\mathcal{V}$*-elliptic, with ellipticity constant not depending on* h*, and, consequently,* $\tilde{a} : \widetilde{\mathcal{G}}_h \times \widetilde{\mathcal{G}}_h \longrightarrow \mathbf{C}$ *is* $\widetilde{\mathcal{V}}$*-elliptic uniformly on* h*.*

Proof: Because of the definition of a it only remains to prove that

$$\|\mathbf{u}_h\|_{0,\Omega}^2 \leq C\Big(\|\operatorname{div} \mathbf{u}_h\|_{0,\Omega}^2 + \|\mathbf{u}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_{\rm A}}^2\Big),$$

with a constant C not depending on h, but this is a direct consequence of (4.4), (4.5) and the fact that $\|\mathbf{u}_h\|_{0,\Omega}^2 = \|\nabla \xi\|_{0,\Omega}^2 + \|\boldsymbol{\chi}\|_{0,\Omega}^2$. □

Now let $A_h : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}$ be defined by $A_h(\mathbf{f}, \mathbf{g}) := (\mathbf{u}_h, \mathbf{v}_h)$, with

$$(\mathbf{u}_h, \mathbf{v}_h) \in \widetilde{\mathcal{G}}_h : \quad \widetilde{a}\Big((\mathbf{u}_h, \mathbf{v}_h), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)\Big) = \widetilde{b}\Big((\mathbf{f}, \mathbf{g}), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)\Big) \qquad \forall (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) \in \widetilde{\mathcal{G}}_h.$$

Because of the previous lemma these operators are well defined and bounded uniformly on h. Notice that $(\mathbf{u}_h, \mathbf{v}_h) = A_h(\mathbf{f}, \mathbf{g})$ if and only if

$$\mathbf{v}_h = P_{\mathcal{G}_h}\mathbf{f}, \tag{4.6}$$

with $P_{\mathcal{G}_h}$ being the L^2-projection onto $\mathcal{G}_h$, and

$$\mathbf{u}_h \in \mathcal{G}_h : \quad a(\mathbf{u}_h, \boldsymbol{\phi}_h) = -\int_{\Gamma_{\mathrm{A}}} \beta\, \mathbf{f} \cdot \boldsymbol{\nu}\; \bar{\boldsymbol{\phi}}_h \cdot \boldsymbol{\nu} - \int_{\Omega} \rho\, \mathbf{g} \cdot \bar{\boldsymbol{\phi}}_h \qquad \forall \boldsymbol{\phi}_h \in \mathcal{G}_h. \tag{4.7}$$

As in the continuous case, the non-zero eigenvalues of A_h are the inverses of the non-zero eigenvalues of problem (4.2)-(4.3), with the same corresponding eigenspaces, and consequently of the discrete quadratic problem (4.1).

Lemma 4.3 *For $\mu_h \neq 0$, $\big(\mu_h, (\mathbf{u}_h, \mathbf{v}_h)\big)$ is an eigenpair of A_h if and only if $\big(\frac{1}{\mu_h}, (\mathbf{u}_h, \mathbf{v}_h)\big)$ is a solution of problem (4.2)-(4.3).*

PROOF: The proof runs almost identically to that of Lemma 2.3, except that, now, we use the orthogonal decomposition $\mathcal{V}_h \times \mathcal{V}_h = \widetilde{\mathcal{G}}_h \oplus (\mathcal{K}_h \times \mathcal{K}_h)$. □

The remainder of this section is devoted to prove that any isolated eigenvalue of A, with algebraic multiplicity m, is approximated by exactly m eigenvalues of A_h (repeated according to their respective algebraic multiplicities) with optimal order of convergence, and similar results for the corresponding invariant subspaces.

Since A is not compact, we will use the theory in [12] to prove convergence of our spectral approximation and non existence of spurious modes, and that in [13] to obtain optimal order error estimates. However we will need to perform a slight modification of the latter in order to encompass our case.

We will make use of some approximation properties for the involved operators and spaces which we prove in the following four lemmas. In all of them we distinguish two different cases:

1. when they apply to functions in the discrete subspace $\widetilde{\mathcal{G}}_h$ (or in the whole space $\widetilde{\mathcal{V}}$);
2. when they apply to functions in an invariant subspace corresponding to an isolated eigenvalue $\mu \in \sigma_{\mathrm{disc}}(A)$.

From now on, let $\mu \in \sigma_{\mathrm{disc}}(A)$ be a fixed isolated eigenvalue with finite algebraic multiplicity m; so, because of Theorem 3.1, $\mu \neq 0$ and $\mu \neq -\frac{\beta}{\alpha}$. Let $E : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}$ be the spectral projector of A relative to μ (see for instance [12] for a precise definition). Then, $E(\widetilde{\mathcal{V}})$ is the invariant subspace of A corresponding to μ (i.e., the span of all its associated generalized eigenfunctions).

Lemma 4.4 *(A priori estimates.)*

1. *Let* $(\mathbf{u},\mathbf{v}) = A(\mathbf{f},\mathbf{g})$ *for* $(\mathbf{f},\mathbf{g}) \in \widetilde{\mathcal{V}}$. *Then* $\mathbf{u} \in \mathrm{H}^{\hat{s}}(\Omega)^n$ *and* $\operatorname{div}\mathbf{u} \in \mathrm{H}^1(\Omega)$, *with*

$$\|\mathbf{u}\|_{\hat{s},\Omega} + \|\operatorname{div}\mathbf{u}\|_{1,\Omega} \le C\|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}. \tag{4.8}$$

2. *Let* $(\mathbf{u},\mathbf{v}) \in E(\widetilde{\mathcal{V}})$. *Then* $\mathbf{u},\mathbf{v} \in \mathcal{G} \cap \mathrm{H}^s(\Omega)^n$, $\operatorname{div}\mathbf{u}$, $\operatorname{div}\mathbf{v} \in \mathrm{H}^{1+s}(\Omega)$ *and* $\mathbf{u}\cdot\boldsymbol{\nu}, \mathbf{v}\cdot\boldsymbol{\nu} \in \mathrm{H}^{1/2+s}(\Gamma_{\mathrm{A}})$, *with*

$$\|\mathbf{u}\|_{s,\Omega} + \|\operatorname{div}\mathbf{u}\|_{1+s,\Omega} + \|\mathbf{u}\cdot\boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} \le C\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}, \tag{4.9}$$

$$\|\mathbf{v}\|_{s,\Omega} + \|\operatorname{div}\mathbf{v}\|_{1+s,\Omega} + \|\mathbf{v}\cdot\boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} \le C\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}. \tag{4.10}$$

Proof: *Case 1.* Consider a Helmholtz decomposition of $\mathbf{g}$ (see for instance [15]):

$$\mathbf{g} = \nabla\xi + \boldsymbol{\chi} \qquad \xi \in \mathrm{H}^1(\Omega), \quad \boldsymbol{\chi} \in \mathcal{K}. \tag{4.11}$$

Then $\|\nabla\xi\|_{0,\Omega} \le \|\mathbf{g}\|_{0,\Omega}$. Because of (2.15), for all $\boldsymbol{\phi} \in \mathcal{G}$ we have

$$a(\mathbf{u},\boldsymbol{\phi}) = -\int_{\Gamma_{\mathrm{A}}} \beta\,\mathbf{f}\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu} - \int_\Omega \rho\,\nabla\xi\cdot\bar{\boldsymbol{\phi}}.$$

Since the equation above is also true for $\boldsymbol{\phi} \in \mathcal{K}$, then it holds for all $\boldsymbol{\phi} \in \mathcal{V}$. Therefore, by considering different test functions in $\mathcal{C}^\infty_{\Gamma_{\mathrm{R}}}(\bar{\Omega})^n \subset \mathcal{V}$, we obtain

$$-\rho c^2\nabla(\operatorname{div}\mathbf{u}) = -\rho\nabla\xi \qquad \text{in } \Omega, \tag{4.12}$$

$$\rho c^2 \operatorname{div}\mathbf{u} + \alpha\,\mathbf{u}\cdot\boldsymbol{\nu} = -\beta\,\mathbf{f}\cdot\boldsymbol{\nu} \qquad \text{on } \Gamma_j, \quad j = 1,\dots,J. \tag{4.13}$$

So, because of (4.12) and the boundedness of A, $\operatorname{div}\mathbf{u} \in \mathrm{H}^1(\Omega)$ with

$$\|\operatorname{div}\mathbf{u}\|_{1,\Omega} \le C\|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}.$$

On the other hand, since $\mathbf{u} \in \mathcal{G}$, because of (2.13) and once more the boundedness of A, $\mathbf{u} = \mathrm{H}^{\hat{s}}(\Omega)^n$ and

$$\|\mathbf{u}\|_{\hat{s},\Omega} \le C\|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}.$$

Case 2. We prove (4.9)-(4.10) for all the generalized eigenfunctions of a Jordan chain of A associated to μ, $\{(\mathbf{u}_k,\mathbf{v}_k)\}_{k=1}^p$. The proof is inductive. Assume $\mathbf{u}_{k-1}$ and $\mathbf{v}_{k-1}$ belongs to $\mathcal{G}$ and satisfy (4.9) and (4.10), respectively; we are going to prove that the same holds for $\mathbf{u}_k$ and $\mathbf{v}_k$. (For $k=1$, i.e., $(\mathbf{u}_k,\mathbf{v}_k)$ an eigenfunction, consider $(\mathbf{u}_0,\mathbf{v}_0) = (\mathbf{0},\mathbf{0})$.)

Since $A(\mathbf{u}_k,\mathbf{v}_k) = \mu(\mathbf{u}_k,\mathbf{v}_k) + (\mathbf{u}_{k-1},\mathbf{v}_{k-1})$, then, because of the boundedness of A,

$$\|(\mathbf{u}_{k-1},\mathbf{v}_{k-1})\|_{\widetilde{\mathcal{V}}} \le C\|(\mathbf{u}_k,\mathbf{v}_k)\|_{\widetilde{\mathcal{V}}}. \tag{4.14}$$

On the other hand, by using (2.14) and (2.15), we have

$$\mu\mathbf{v}_k + \mathbf{v}_{k-1} = \mathbf{u}_k \quad \text{in } \Omega, \tag{4.15}$$

and $\mu\mathbf{u}_k + \mathbf{u}_{k-1} \in \mathcal{G}$ satisfies

$$a(\mu\mathbf{u}_k + \mathbf{u}_{k-1},\boldsymbol{\phi}) = \left(-\int_{\Gamma_{\mathrm{A}}} \beta\,\mathbf{u}_k\cdot\boldsymbol{\nu}\;\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\nu} - \int_\Omega \rho\,\mathbf{v}_k\cdot\bar{\boldsymbol{\phi}}\right) \qquad \forall\boldsymbol{\phi} \in \mathcal{G}.$$

Hence $\mathbf{u}_k, \mathbf{v}_k \in \mathcal{G}$.

So, proceeding as in the previous case, we obtain:

$$-\rho c^2 \nabla\Big(\operatorname{div}(\mu \mathbf{u}_k + \mathbf{u}_{k-1})\Big) = -\rho \mathbf{v}_k \qquad \text{in } \Omega, \tag{4.16}$$

$$\rho c^2 \operatorname{div}(\mu \mathbf{u}_k + \mathbf{u}_{k-1}) + \alpha (\mu \mathbf{u}_k + \mathbf{u}_{k-1}) \cdot \boldsymbol{\nu} = -\beta\, \mathbf{u}_k \cdot \boldsymbol{\nu} \qquad \text{on } \Gamma_j, \quad j = 1, \ldots, J. \tag{4.17}$$

Now, because of (4.16),

$$\|\operatorname{div} \mathbf{u}_k\|_{1,\Omega} \le C \left(\|\operatorname{div} \mathbf{u}_{k-1}\|_{1,\Omega} + \|(\mathbf{u}_k, \mathbf{v}_k)\|_{\widetilde{\mathcal{V}}} \right),$$

and because of (4.17) and the fact that $\mu \neq -\frac{\beta}{\alpha}$,

$$\|\mathbf{u}_k \cdot \boldsymbol{\nu}\|_{1/2,\Gamma_{\mathrm{A}}} \le C \left(\|\operatorname{div} \mathbf{u}_k\|_{1,\Omega} + \|\operatorname{div} \mathbf{u}_{k-1}\|_{1,\Omega} + \|\mathbf{u}_{k-1} \cdot \boldsymbol{\nu}\|_{1/2,\Gamma_{\mathrm{A}}} \right).$$

So, by using (2.12), the last two equations and the fact that $\sum_{j=1}^{J} \|\mathbf{u}_{k-1} \cdot \boldsymbol{\nu}\|_{1/2,\Gamma_j} \le C\, \|\mathbf{u}_{k-1} \cdot \boldsymbol{\nu}\|_{1/2,\Gamma_{\mathrm{A}}}$, we have

$$\|\mathbf{u}_k\|_{s,\Omega} \le C \left(\|\operatorname{div} \mathbf{u}_{k-1}\|_{1,\Omega} + \|\mathbf{u}_{k-1} \cdot \boldsymbol{\nu}\|_{1/2,\Gamma_{\mathrm{A}}} + \|(\mathbf{u}_k, \mathbf{v}_k)\|_{\widetilde{\mathcal{V}}} \right).$$

On the other hand, because of (4.15), (4.16) and (4.17), we obtain, respectively,

$$\begin{aligned}
\|\mathbf{v}_k\|_{s,\Omega} &\le \frac{1}{\mu} \left(\|\mathbf{v}_{k-1}\|_{s,\Omega} + \|\mathbf{u}_k\|_{s,\Omega} \right), \\
\|\operatorname{div} \mathbf{u}_k\|_{1+s,\Omega} &\le C \left(\|\operatorname{div} \mathbf{u}_{k-1}\|_{1+s,\Omega} + \|\mathbf{v}_k\|_{s,\Omega} \right), \\
\|\mathbf{u}_k \cdot \boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} &\le C \left(\|\operatorname{div} \mathbf{u}_k\|_{1+s,\Omega} + \|\operatorname{div} \mathbf{u}_{k-1}\|_{1+s,\Omega} + \|\mathbf{u}_{k-1} \cdot \boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} \right),
\end{aligned}$$

and from (4.15) again

$$\begin{aligned}
\|\operatorname{div} \mathbf{v}_k\|_{1+s,\Omega} &\le \frac{1}{\mu} \left(\|\operatorname{div} \mathbf{v}_{k-1}\|_{1+s,\Omega} + \|\operatorname{div} \mathbf{u}_k\|_{1+s,\Omega} \right), \\
\|\mathbf{v}_k \cdot \boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} &\le \frac{1}{\mu} \left(\|\mathbf{v}_{k-1} \cdot \boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} + \|\mathbf{u}_k \cdot \boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} \right),
\end{aligned}$$

Finally, from the last six equations, the inductive hypothesis that (4.9) and (4.10) hold for $\mathbf{u}_{k-1}$ and $\mathbf{v}_{k-1}$, respectively, and (4.14), we obtain that (4.9) and (4.10) also hold for $\mathbf{u}_k$ and $\mathbf{v}_k$, respectively. So, we conclude the lemma. □

Lemma 4.5 *(Approximation from the discrete spaces.)*

1. *Let* $(\mathbf{u}, \mathbf{v}) = A(\mathbf{f}, \mathbf{g})$ *for* $(\mathbf{f}, \mathbf{g}) \in \widetilde{\mathcal{G}}_h$. *Then there exists* $\widehat{\mathbf{u}}_h \in \mathcal{G}_h$ *such that*

$$\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{\mathcal{V}} \le C h^{\hat{s}} \|(\mathbf{f}, \mathbf{g})\|_{\widetilde{\mathcal{V}}}.$$

2. *Let* $(\mathbf{u}, \mathbf{v}) \in E(\widetilde{\mathcal{V}})$. *Then there exist* $\widehat{\mathbf{u}}_h, \widehat{\mathbf{v}}_h \in \mathcal{G}_h$ *such that*

$$\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{\mathcal{V}} + \|\mathbf{v} - \widehat{\mathbf{v}}_h\|_{\mathcal{V}} \le C h^{s} \|(\mathbf{u}, \mathbf{v})\|_{\widetilde{\mathcal{V}}}.$$

PROOF: *Case 1.* Since $\mathbf{u} \in \mathcal{G}$, let $\mathbf{u}^{\mathrm{I}} \in \mathcal{V}_h$ be the Raviart-Thomas interpolant of $\mathbf{u}$ (notice that $\mathbf{u}^{\mathrm{I}}$ is well defined since $\mathbf{u} \in \mathrm{H}^{\hat{s}}(\Omega)^n$; see for instance [8] for the definition and approximation properties of this interpolant used below). Then

$$\|\mathbf{u}-\mathbf{u}^{\mathrm{I}}\|_{\operatorname{div},\Omega} \leq Ch^{\hat{s}}\left(\|\mathbf{u}\|_{\hat{s},\Omega} + \|\operatorname{div}\mathbf{u}\|_{1,\Omega}\right) \leq Ch^{\hat{s}}\|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}, \tag{4.18}$$

the last inequality because of Lemma 4.4. On the other hand, proceeding as in the proof of the previous Lemma, we are able to prove an equality similar to (4.13) and hence

$$\mathbf{u}\cdot\boldsymbol{\nu} = \frac{1}{\alpha}\left(-\beta\,\mathbf{f}\cdot\boldsymbol{\nu} - \rho c^2 \operatorname{div}\mathbf{u}\right) \quad \text{on } \Gamma_j, \quad j=1,\dots,J.$$

So, taking into account that $\mathbf{u}^{\mathrm{I}}|_{\Gamma_{\mathrm{A}}}\cdot\boldsymbol{\nu} = P(\mathbf{u}|_{\Gamma_{\mathrm{A}}}\cdot\boldsymbol{\nu})$, with P being the L^2 projection onto the piecewise constant functions on Γ_{A}, we have in this case:

$$\begin{aligned}
\|(\mathbf{u}-\mathbf{u}^{\mathrm{I}})\cdot\boldsymbol{\nu}\|_{0,\Gamma_{\mathrm{A}}} &= \frac{1}{\alpha}\left\|\left(-\beta\,\mathbf{f}\cdot\boldsymbol{\nu} - \rho c^2\operatorname{div}\mathbf{u}\right) - P\left(-\beta\,\mathbf{f}\cdot\boldsymbol{\nu} - \rho c^2\operatorname{div}\mathbf{u}\right)\right\|_{0,\Gamma_{\mathrm{A}}}\\
&= \frac{1}{\alpha}\left\|\rho c^2\operatorname{div}\mathbf{u} - P\left(\rho c^2\operatorname{div}\mathbf{u}\right)\right\|_{0,\Gamma_{\mathrm{A}}}\\
&\leq Ch^{1/2}\|\operatorname{div}\mathbf{u}\|_{1,\Omega}\\
&\leq Ch^{\hat{s}}\|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}.
\end{aligned} \tag{4.19}$$

Now, since $\mathcal{V}_h = \mathcal{G}_h \oplus \mathcal{K}_h$, let $\mathbf{u}^{\mathrm{I}} = \hat{\mathbf{u}}_h + \mathbf{u}_{\mathcal{K}_h}$ with $\hat{\mathbf{u}}_h \in \mathcal{G}_h$ and $\mathbf{u}_{\mathcal{K}_h} \in \mathcal{K}_h$. So, since $\mathbf{u} \in \mathcal{G}$ is also $\mathcal{V}$-orthogonal to $\mathbf{u}_{\mathcal{K}_h}$, then

$$\|\hat{\mathbf{u}}_h - \mathbf{u}\|_{\mathcal{V}}^2 \leq \|(\hat{\mathbf{u}}_h-\mathbf{u})\|_{\mathcal{V}}^2 + \|\mathbf{u}_{\mathcal{K}_h}\|_{\mathcal{V}}^2 = \|(\hat{\mathbf{u}}_h-\mathbf{u})+\mathbf{u}_{\mathcal{K}_h}\|_{\mathcal{V}}^2 = \|\mathbf{u}^{\mathrm{I}}-\mathbf{u}\|_{\mathcal{V}}^2,$$

which together with (4.18) and (4.19) allow us to conclude the lemma in this case.

Case 2. Let $\mathbf{u}^{\mathrm{I}}$ be as above. Because of Lemma 4.4 (case 2) we have now:

$$\|\mathbf{u}-\mathbf{u}^{\mathrm{I}}\|_{\operatorname{div},\Omega} \leq Ch^{s}\left(\|\mathbf{u}\|_{s,\Omega} + \|\operatorname{div}\mathbf{u}\|_{1,\Omega}\right) \leq Ch^{s}\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}.$$

Moreover, since $\mathbf{u}\cdot\boldsymbol{\nu} \in \mathrm{H}^{1/2+s}(\Gamma_{\mathrm{A}})$, then

$$\|(\mathbf{u}-\mathbf{u}^{\mathrm{I}})\cdot\boldsymbol{\nu}\|_{0,\Gamma_{\mathrm{A}}} = \|\mathbf{u}\cdot\boldsymbol{\nu} - P(\mathbf{u}\cdot\boldsymbol{\nu})\|_{0,\Gamma_{\mathrm{A}}} \leq Ch^{1/2+s}\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{1/2+s,\Gamma_{\mathrm{A}}} \leq Ch^{s}\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}.$$

So, defining $\hat{\mathbf{u}}_h \in \mathcal{G}_h$ as in the proof of the previous case, we have

$$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_{\mathcal{V}} \leq Ch^{s}\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}.$$

Finally, because of Lemma 4.4 (case 2), $\mathbf{v} \in \mathcal{G}$ and it satisfies estimates identical to those of $\mathbf{u}$. Hence, the same procedure allows defining $\hat{\mathbf{v}}_h$ with analogous approximation properties and we conclude the lemma. □

Lemma 4.6 *(Consistency terms.) Let* $(\mathbf{f},\mathbf{g}) \in \widetilde{\mathcal{V}}$, $(\mathbf{u},\mathbf{v}) = A(\mathbf{f},\mathbf{g})$ *and* $(\mathbf{u}_h,\mathbf{v}_h) = A_h(\mathbf{f},\mathbf{g})$. *There hold:*

1.

$$\sup_{0\neq\boldsymbol{\phi}_h\in\mathcal{G}_h} \frac{a(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_{\mathcal{V}}} \leq Ch^{\hat{s}}\|\mathbf{g}\|_{0,\Omega};$$

2. *if* $(\mathbf{f},\mathbf{g}) \in E(\widetilde{\mathcal{V}})$, *then*

$$a(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\phi}_h) = 0 \qquad \forall \boldsymbol{\phi}_h \in \mathcal{G}_h.$$

PROOF: Let $\boldsymbol{\phi}_h \in \mathcal{G}_h$; by Lemma 4.1 we have

$$\boldsymbol{\phi}_h = \nabla\xi + \boldsymbol{\chi} \qquad \xi \in \mathrm{H}^1(\Omega), \quad \boldsymbol{\chi} \in \mathcal{K}, \tag{4.20}$$

and

$$\|\boldsymbol{\chi}\|_{0,\Omega} \le C h^{\hat{s}} \|\boldsymbol{\phi}_h\|_{\mathcal{V}}. \tag{4.21}$$

On the other hand, because of the definition of a and (2.15),

$$a(\mathbf{u}, \boldsymbol{\phi}_h) = a(\mathbf{u}, \nabla\xi) = -\int_{\Gamma_\mathrm{A}} \beta\, \mathbf{f}\cdot\boldsymbol{\nu}\, \nabla\bar{\xi}\cdot\boldsymbol{\nu} - \int_\Omega \rho\, \mathbf{g}\cdot\nabla\bar{\xi}.$$

So, from this, (4.7) and (4.20), we have

$$a(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\phi}_h) = \int_{\Gamma_\mathrm{A}} \beta\, \mathbf{f}\cdot\boldsymbol{\nu}\, \bar{\boldsymbol{\chi}}\cdot\boldsymbol{\nu} + \int_\Omega \rho\, \mathbf{g}\cdot\bar{\boldsymbol{\chi}} = \int_\Omega \rho\, \mathbf{g}\cdot\bar{\boldsymbol{\chi}}. \tag{4.22}$$

Case 1. It is a direct consequence of (4.22) and (4.21).

Case 2. Since for $(\mathbf{f},\mathbf{g}) \in E(\widetilde{\mathcal{V}})$, $\mathbf{g} \in \mathcal{G}$, and $\mathcal{K}$ is L^2-orthogonal to $\mathcal{G}$, (4.22) yields the lemma. □

Lemma 4.7 *(Convergence.) Let* $(\mathbf{f},\mathbf{g}) \in \widetilde{\mathcal{V}}$, $(\mathbf{u},\mathbf{v}) = A(\mathbf{f},\mathbf{g})$ *and* $(\mathbf{u}_h,\mathbf{v}_h) = A_h(\mathbf{f},\mathbf{g})$. *There hold:*

1. *if* $(\mathbf{f},\mathbf{g}) \in \widetilde{\mathcal{G}}_h$, *then* $\mathbf{v} = \mathbf{v}_h$ *and*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{V}} \le C h^{\hat{s}} \|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}; \tag{4.23}$$

2. *if* $(\mathbf{f},\mathbf{g}) \in E(\widetilde{\mathcal{V}})$ *then*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{V}} + \|\mathbf{v}-\mathbf{v}_h\|_{0,\Omega} \le C h^{s} \|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}. \tag{4.24}$$

PROOF: Since $\mathbf{u}$ and $\mathbf{u}_h$ are given by (2.15) and (4.7), respectively, and $\widetilde{\mathcal{G}}_h \not\subset \widetilde{\mathcal{G}}$, we apply classical Second Strang's Lemma (see for instance [10]), which in our case reads:

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{V}} \le C \left[\inf_{\widehat{\mathbf{u}}_h \in \mathcal{G}_h} \|\mathbf{u}-\widehat{\mathbf{u}}_h\|_{\mathcal{V}} + \sup_{0 \ne \boldsymbol{\phi}_h \in \mathcal{G}_h} \frac{a\,(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_{\mathcal{V}}} \right]. \tag{4.25}$$

Case 1. Due to Lemmas 4.5 and 4.6 (case 1), estimate (4.25) yields (4.23). On the other hand, for $\mathbf{f} \in \mathcal{G}_h$, because of (2.14) and (4.6),

$$\mathbf{v} = \mathbf{f} = P_{\mathcal{G}_h}\mathbf{f} = \mathbf{v}_h.$$

Case 2. Due to Lemmas 4.5 and 4.6 (case 2), estimate (4.25) yields now

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{V}} \le C h^{s} \|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}} \le C h^{s} \|(\mathbf{f},\mathbf{g})\|_{\widetilde{\mathcal{V}}}.$$

On the other hand, because of (2.14) and (4.6), $\mathbf{v} = \mathbf{f}$ and $\mathbf{v}_h = P_{\mathcal{G}_h}\mathbf{f}$. Thus, since $P_{\mathcal{G}_h}$ is the L^2-projection onto $\mathcal{G}_h$, by applying Lemma 4.5 (case 2), we have

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega} \leq \|\mathbf{v} - \hat{\mathbf{v}}_h\|_{0,\Omega} \leq Ch^s \|(\mathbf{u}, \mathbf{v})\|_{\widetilde{\mathcal{V}}} \leq Ch^s \|(\mathbf{f}, \mathbf{g})\|_{\widetilde{\mathcal{V}}},$$

thus concluding the lemma. □

Now, we can prove the following properties, *P1* and *P2*, which are all what we need to apply the spectral approximation theory in [12] to the operators A and $A_h|_{\widetilde{\mathcal{G}}_h} : \widetilde{\mathcal{G}}_h \longrightarrow \widetilde{\mathcal{G}}_h$.

Lemma 4.8 *P1: There holds:*

$$\lim_{h\to 0} \left\| (A - A_h)|_{\widetilde{\mathcal{G}}_h} \right\|_{\widetilde{\mathcal{V}}} = 0.$$

P2: For all $(\mathbf{u}, \mathbf{v}) \in E(\widetilde{\mathcal{V}})$*, there holds:*

$$\lim_{h\to 0} \left[\inf_{(\mathbf{u}_h, \mathbf{v}_h) \in \widetilde{\mathcal{G}}_h} \|(\mathbf{u}, \mathbf{v}) - (\mathbf{u}_h, \mathbf{v}_h)\|_{\widetilde{\mathcal{V}}} \right] = 0.$$

PROOF: *P1* is a direct consequence of case 1 of Lemma 4.7. *P2* is a consequence of the fact that, for all $(\mathbf{u}, \mathbf{v}) \in E(\widetilde{\mathcal{V}})$, there exists $(\mathbf{f}, \mathbf{g}) \in E(\widetilde{\mathcal{V}})$ such that $(\mathbf{u}, \mathbf{v}) = A(\mathbf{f}, \mathbf{g})$, and case 2 of the same lemma. □

The following theorem, proved in [12], shows that our method does not introduce spurious modes:

Theorem 4.1 *Let* $K \subset \mathbf{C}$ *be a compact set not intersecting* $\sigma(A)$*. There exists* $h_0 > 0$ *such that, if* $h \leq h_0$*, then* K *does not intersect* $\sigma(A_h)$*.*

PROOF: This is a direct consequence of *P1*, as it is shown in Theorem 1 of [12]. □

Let $D \subset \mathbf{C}$ be a closed disk centered at μ, such that $0 \notin D$ and $D \cap \sigma(A) = \{\mu\}$. Let $\mu_{1h}, \ldots, \mu_{m(h)h}$ be the eigenvalues of A_h contained in D (repeated according to their algebraic multiplicities). Under assumptions *P1* and *P2*,[2] it is proved in [12] that $m(h) = m$, for h small enough, and that $\lim_{h\to 0} \mu_{kh} = \mu$, for $k = 1, \ldots, m$.

5 Error estimates

The aim of this section is to prove error estimates for the finite element method introduced above. In principle, the theory in [22] about non conforming approximation for non compact operators could be adapted to attain this goal (we notice that our problem does not fit exactly in its framework, because the sesquilinear form $\tilde{a}$ is not elliptic on the whole $\widetilde{\mathcal{V}}$). However, this theory would provide estimates depending on $\left\| (A - A_h)|_{\widetilde{\mathcal{G}}_h} \right\|_{\widetilde{\mathcal{V}}}$, which in our case are $\mathcal{O}(h^{\hat{s}})$, and then not optimal.

[2] Actually, *P2* is assumed in [12] for all $(\mathbf{u}, \mathbf{v}) \in \widetilde{\mathcal{V}}$; however, this property is only used in the proof of Theorem 3 of that reference, and for $(\mathbf{u}, \mathbf{v}) \in E(\widetilde{\mathcal{V}})$.

So, instead of applying the theory in [22], we proceed as in that reference: we prove optimal error estimates by performing minor modifications of the theory in [13], so that it can be applied to a non conforming method like ours. Furthermore, the ellipticity of $\widetilde{a}$ when restricted to $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}_h$ (Lemmas 2.2 and 4.2, respectively) will suffice for these proofs.

Let A_* and $A_{*h} : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}$ be defined by $A_*(\mathbf{f},\mathbf{g}) := (\mathbf{u}_*,\mathbf{v}_*)$ and $A_{*h}(\mathbf{f},\mathbf{g}) := (\mathbf{u}_{*h},\mathbf{v}_{*h})$, with $(\mathbf{u}_*,\mathbf{v}_*)$ and $(\mathbf{u}_{*h},\mathbf{v}_{*h})$ satisfying:

$$\begin{aligned} (\mathbf{u}_*,\mathbf{v}_*) \in \widetilde{\mathcal{G}}: &\quad \widetilde{a}\Big((\boldsymbol{\phi},\boldsymbol{\psi}),(\mathbf{u}_*,\mathbf{v}_*)\Big) = \widetilde{b}\Big((\boldsymbol{\phi},\boldsymbol{\psi}),(\mathbf{f},\mathbf{g})\Big) \qquad \forall(\boldsymbol{\phi},\boldsymbol{\psi}) \in \widetilde{\mathcal{G}},\\ (\mathbf{u}_{*h},\mathbf{v}_{*h}) \in \widetilde{\mathcal{G}}_h: &\quad \widetilde{a}\Big((\boldsymbol{\phi}_h,\boldsymbol{\psi}_h),(\mathbf{u}_{*h},\mathbf{v}_{*h})\Big) = \widetilde{b}\Big((\boldsymbol{\phi}_h,\boldsymbol{\psi}_h),(\mathbf{f},\mathbf{g})\Big) \qquad \forall(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) \in \widetilde{\mathcal{G}}_h. \end{aligned}$$

Then A_* and A_{*h} are the adjoints of A and A_h, respectively, with respect to the sesquilinear form $\widetilde{a}$. It is simple to show (see for instance [13]) that $\bar{\mu}$ is an eigenvalue of A_* with the same multiplicity m as that of μ. Analogously, for h small enough, $\bar{\mu}_{1h},\ldots,\bar{\mu}_{mh}$ are the eigenvalues of A_{*h} belonging to $\bar{D} := \{z \in \mathbf{C} : \bar{z} \in D\}$.

Straightforward computations show that, in our case,

$$(\mathbf{u}_*,-\mathbf{v}_*) = A(\mathbf{f},-\mathbf{g}) \qquad \text{and} \qquad (\mathbf{u}_{*h},-\mathbf{v}_{*h}) = A_h(\mathbf{f},-\mathbf{g}). \tag{5.1}$$

Furthermore,

$$(\mathbf{f},\mathbf{g}) \in E(\widetilde{\mathcal{V}}) \qquad \text{if and only if} \qquad (\bar{\mathbf{f}},-\bar{\mathbf{g}}) \in E_*(\widetilde{\mathcal{V}}), \tag{5.2}$$

where E_* is the spectral projector of A_* associated to $\bar{\mu}$. Thus, results similar to those in Lemma 4.7 hold for A_* and A_{*h}, and, consequently,

$$\lim_{h\to 0} \left\| (A_* - A_{*h})|_{\widetilde{\mathcal{G}}_h} \right\|_{\widetilde{\mathcal{V}}} = 0$$

and

$$\left\| (A_* - A_{*h})|_{E(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} \le Ch^s.$$

Let $\Pi_h : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}$ be the projector with range $\widetilde{\mathcal{G}}_h = \mathcal{G}_h \times \mathcal{G}_h$ defined by

$$\widetilde{a}\Big((\mathbf{u},\mathbf{v}) - \Pi_h(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h)\Big) = 0 \qquad \forall(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) \in \widetilde{\mathcal{G}}_h.$$

Notice that Π_h is bounded uniformly on h, because of the continuity of $\widetilde{a}$ and Lemma 4.2. On the other hand, since $\widetilde{a}$ is symmetric, Π_h is self-adjoint with respect to $\widetilde{a}$.

For conforming methods $A_h = \Pi_h A$. This is for instance assumed in the spectral approximation theory in [13] and used in the proofs therein. In our case A_h and $\Pi_h A$ do not coincide on its whole domain $\widetilde{\mathcal{V}}$, but in the following lemma we show that they coincide on $E(\widetilde{\mathcal{V}})$ and an analogous result for the corresponding adjoints.

Lemma 5.1 *There holds:*

$$A_h|_{E(\widetilde{\mathcal{V}})} = \Pi_h A|_{E(\widetilde{\mathcal{V}})} \qquad \textit{and} \qquad A_{*h}|_{E_*(\widetilde{\mathcal{V}})} = \Pi_h A_*|_{E_*(\widetilde{\mathcal{V}})}.$$

PROOF: Let $(\mathbf{f},\mathbf{g}) \in E(\widetilde{\mathcal{V}})$, $(\mathbf{u},\mathbf{v}) = A(\mathbf{f},\mathbf{g})$ and $(\mathbf{u}_h,\mathbf{v}_h) = A_h(\mathbf{f},\mathbf{g})$. Then, because of (2.14) and (4.6), $\mathbf{v}_h = P_{\mathcal{G}_h}\mathbf{f} = P_{\mathcal{G}_h}\mathbf{v}$. So, because of Lemma 4.6 (case 2) and the fact that $P_{\mathcal{G}_h}$ is the L^2 projection onto $\mathcal{G}_h$, we have

$$\widetilde{a}\Big((\mathbf{u},\mathbf{v})-(\mathbf{u}_h,\mathbf{v}_h),(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h)\Big) = a(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\phi}_h)+\int_\Omega \rho\,(\mathbf{v}-\mathbf{v}_h)\cdot\boldsymbol{\psi}_h = 0 \qquad \forall(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h)\in\widetilde{\mathcal{G}}_h.$$

That is, $A_h(\mathbf{f},\mathbf{g}) = \Pi_h A(\mathbf{f},\mathbf{g})$.

The second equality is an immediate consequence of the first one, (5.1), (5.2) and the definition of $\widetilde{a}$. □

As a consequence of the previous lemma the following estimates hold:

Lemma 5.2 *There holds:*

$$\left\|(I-\Pi_h)\,|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} \le Ch^s \qquad \textit{and} \qquad \left\|(I-\Pi_h)\,|_{E_*(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} \le Ch^s.$$

PROOF: For $(\mathbf{u},\mathbf{v}) \in E(\widetilde{\mathcal{V}})$, $(A-\mu I)^m(\mathbf{u},\mathbf{v}) = 0$, and hence $(\mathbf{u},\mathbf{v}) = A(\mathbf{z},\mathbf{w})$ with $(\mathbf{z},\mathbf{w}) = \sum_{k=0}^{m-1} c_k A^k(\mathbf{u},\mathbf{v}) \in E(\widetilde{\mathcal{V}})$, where the coefficients c_k only depend on k, m and $\mu \neq 0$. Then, as a consequence of the previous lemma and case 2 of Lemma 4.7, we have

$$\begin{aligned}
\left\|(I-\Pi_h)\,|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} &= \sup_{(\mathbf{0},\mathbf{0})\neq(\mathbf{u},\mathbf{v})\in E(\widetilde{\mathcal{V}})} \frac{\|(\mathbf{u},\mathbf{v})-\Pi_h(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}}{\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}} \\
&= \sup_{(\mathbf{0},\mathbf{0})\neq(\mathbf{u},\mathbf{v})\in E(\widetilde{\mathcal{V}})} \frac{\|A(\mathbf{z},\mathbf{w})-\Pi_h A(\mathbf{z},\mathbf{w})\|_{\widetilde{\mathcal{V}}}}{\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}} \\
&= \sup_{(\mathbf{0},\mathbf{0})\neq(\mathbf{u},\mathbf{v})\in E(\widetilde{\mathcal{V}})} \frac{\|A(\mathbf{z},\mathbf{w})-A_h(\mathbf{z},\mathbf{w})\|_{\widetilde{\mathcal{V}}}}{\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}} \\
&\le Ch^s \sup_{(\mathbf{0},\mathbf{0})\neq(\mathbf{u},\mathbf{v})\in E(\widetilde{\mathcal{V}})} \frac{\|(\mathbf{z},\mathbf{w})\|_{\widetilde{\mathcal{V}}}}{\|(\mathbf{u},\mathbf{v})\|_{\widetilde{\mathcal{V}}}} \le Ch^s.
\end{aligned}$$

The proof of the other estimate runs identically. □

Let $B_h := A_h\Pi_h : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}$. Then $\sigma(A_h) = \sigma(B_h)$ and, for any non-zero eigenvalue, the corresponding invariant subspaces coincide. Let $B_{*h} := A_{*h}\Pi_h : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}$. Notice that B_{*h} is actually the adjoint of B_h with respect to $\widetilde{a}$.

Let F_h be the spectral projector of B_h, relative to the eigenvalues $\mu_{1h},\ldots,\mu_{mh}$. It can be proved as in Lemma 1 of [13] that there exists $h_0 > 0$ such that $\|(zI-B_h)^{-1}\|$ is uniformly bounded for $h \le h_0$ and $z \in \partial D$; consequently, for h small enough, the spectral projectors F_h are uniformly bounded with respect to h. Let F_{*h} be the spectral projector of B_{*h} relative to $\bar{\mu}_{1h},\ldots,\bar{\mu}_{mh}$, respectively. The following estimates hold:

Lemma 5.3 *There exist constants $h_0 > 0$ and $C > 0$ such that, for $h \le h_0$,*

$$\left\|(E-F_h)|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} \le C\left\|(A-B_h)|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} \le Ch^s, \tag{5.3}$$

and

$$\left\| (E_* - F_{*h})|_{E_*(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} \leq C \left\| (A_* - B_{*h})|_{E_*(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} \leq Ch^s. \tag{5.4}$$

PROOF: The proof of the left hand inequality in (5.3) is identical to that of the corresponding one in Lemma 3 of [13].

Regarding the right hand inequality, we have

$$\begin{aligned} \left\| (A - B_h)|_{E(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} &\leq \left\| (A - A_h)|_{E(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} + \left\| A_h (I - \Pi_h)|_{E(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} \\ &\leq \left\| (A - A_h)|_{E(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} + \| A_h \|_{\widetilde{\mathcal{V}}} \left\| (I - \Pi_h)|_{E(\widetilde{\mathcal{V}})} \right\|_{\widetilde{\mathcal{V}}} \leq Ch^s, \end{aligned}$$

the latter because of Lemmas 4.7 and 5.2, and the fact that $\|A_h\|_{\widetilde{\mathcal{V}}}$ is uniformly bounded with respect to h.

Analogous proofs are valid for the inequalities in (5.4). □

Now we are able to prove optimal order error estimates for the approximate invariant subspace of A_h, which, as stated above, coincides with $F_h(\widetilde{\mathcal{V}})$. We recall the definition of the gap $\widehat{\delta}$ between two closed subspaces, $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{Z}}$, of $\widetilde{\mathcal{V}}$:

$$\widehat{\delta}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) := \max \left\{ \delta(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}), \delta(\widetilde{\mathcal{Z}}, \widetilde{\mathcal{Y}}) \right\},$$

with

$$\delta(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) := \sup_{\substack{(\phi,\psi)\in\widetilde{\mathcal{Y}} \\ \|(\phi,\psi)\|_{\mathcal{V}}=1}} \left[\inf_{(\widehat{\phi},\widehat{\psi})\in\widetilde{\mathcal{Z}}} \|(\boldsymbol{\phi}, \boldsymbol{\psi}) - (\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}})\|_{\mathcal{V}} \right].$$

Theorem 5.1 *There exist constants $h_0 > 0$ and $C > 0$ such that, for $h \leq h_0$,*

$$\widehat{\delta}\left(F_h(\widetilde{\mathcal{V}}), E(\widetilde{\mathcal{V}})\right) \leq Ch^s.$$

PROOF: It relies on Lemma 5.3 and is essentially identical to that of Theorem 1 in [13]. □

Let $\Lambda_h := F_h|_{E(\widetilde{\mathcal{V}})} : E(\widetilde{\mathcal{V}}) \longrightarrow F_h(\widetilde{\mathcal{V}})$. The arguments in the proof of Theorem 1 in [13] show that, for h small enough, Λ_h^{-1} exists and $\left\|\Lambda_h^{-1}\right\|_{\widetilde{\mathcal{V}}}$ is uniformly bounded with respect to h.

Let $\widehat{A} := A|_{E(\widetilde{\mathcal{V}})} : E(\widetilde{\mathcal{V}}) \longrightarrow E(\widetilde{\mathcal{V}})$ and $\widehat{B}_h := \Lambda_h^{-1} B_h \Lambda_h : E(\widetilde{\mathcal{V}}) \longrightarrow E(\widetilde{\mathcal{V}})$. The operator $\widehat{A}$ has a unique eigenvalue μ of algebraic multiplicity m; the operator $\widehat{B}_h$ has the eigenvalues $\mu_{1h}, \ldots, \mu_{mh}$.

The following lemma is the key to prove error estimates for the approximate eigenvalues. Its proof is a slight modification of that of Theorem 2.2 in [22], taking care of the fact that the sesquilinear form $\widetilde{a}$ is not elliptic on the whole space $\widetilde{\mathcal{V}}$. We include the complete proof for the sake of completeness.

Lemma 5.4 *There exist constants $h_0 > 0$ and $C > 0$ such that, for $h \leq h_0$,*

$$\left\| (\widehat{A} - \widehat{B}_h) \right\|_{\widetilde{\mathcal{V}}} \leq Ch^{2s}.$$

PROOF: Let $(\mathbf{u},\mathbf{v}) \in E(\widetilde{\mathcal{V}})$ with $\|(\mathbf{u},\mathbf{v})\| = 1$. Since $\widetilde{a}$ is $\widetilde{\mathcal{V}}$-elliptic on $\widetilde{\mathcal{G}} \supset E(\widetilde{\mathcal{V}})$ (Lemma 2.2), then

$$\begin{aligned}
\left\|(\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v})\right\|_{\widetilde{\mathcal{V}}}^2 &\leq C\,\widetilde{a}\Big((\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}),(\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v})\Big)\\
&\leq C \sup_{\substack{(\phi,\psi)\in\widetilde{\mathcal{G}}\\ \|(\phi,\psi)\|_{\widetilde{\mathcal{V}}}=1}} \widetilde{a}\Big((\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\left\|(\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v})\right\|_{\widetilde{\mathcal{V}}}.
\end{aligned}$$

On the other hand, since $(\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}) \in E(\widetilde{\mathcal{V}})$ and E_* is the adjoint of E with respect to $\widetilde{a}$ (the latter because A_* is the adjoint of A with respect to $\widetilde{a}$), then

$$\widetilde{a}\Big((\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big) = \widetilde{a}\Big((\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}),E_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big).$$

Hence

$$\begin{aligned}
\left\|(\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v})\right\|_{\widetilde{\mathcal{V}}} &\leq C \sup_{\substack{(\phi,\psi)\in\widetilde{\mathcal{G}}\\ \|(\phi,\psi)\|_{\widetilde{\mathcal{V}}}=1}} \widetilde{a}\Big((\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}),E_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\
&\leq C\,\|E_*\|_{\widetilde{\mathcal{V}}} \sup_{\substack{(\phi,\psi)\in E_*(\widetilde{\mathcal{V}})\\ \|(\phi,\psi)\|_{\widetilde{\mathcal{V}}}=1}} \widetilde{a}\Big((\widehat{A}-\widehat{B}_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big).
\end{aligned} \tag{5.5}$$

By using that $\left(\Lambda_h^{-1}F_h - I\right)A|_{E(\widetilde{\mathcal{V}})} = 0$ and that B_h commutes with its spectral projector F_h, it is straightforward to prove that

$$\widehat{A}-\widehat{B}_h = (A-B_h)\,|_{E(\widetilde{\mathcal{V}})} + \left(\Lambda_h^{-1}F_h - I\right)(A-B_h)\,|_{E(\widetilde{\mathcal{V}})}. \tag{5.6}$$

Let $(\boldsymbol{\phi},\boldsymbol{\psi}) \in E_*(\widetilde{\mathcal{V}})$ with $\|(\boldsymbol{\phi},\boldsymbol{\psi})\| = 1$. Since $F_h\left(\Lambda_h^{-1}F_h - I\right) = 0$ and F_{*h} is the adjoint of F_h with respect to $\widetilde{a}$ (the latter because B_{*h} is the adjoint of B_h with respect to $\widetilde{a}$), we have

$$\begin{aligned}
&\left|\widetilde{a}\Big((\Lambda_h^{-1}F_h - I)(A-B_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\right|\\
&\qquad = \left|\widetilde{a}\Big((\Lambda_h^{-1}F_h - I)(A-B_h)(\mathbf{u},\mathbf{v}),(I-F_{*h})(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\right|\\
&\qquad \leq C\left\|\Lambda_h^{-1}F_h - I\right\|_{\widetilde{\mathcal{V}}}\left\|(A-B_h)|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}}\left\|(I-F_{*h})|_{E_*(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}}\\
&\qquad \leq Ch^{2s},
\end{aligned} \tag{5.7}$$

where we have used Lemma 5.3 and the fact that F_h and Λ_h^{-1} are bounded independently of h.

On the other hand,

$$\begin{aligned}
\widetilde{a}\Big((A-B_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big) &= \widetilde{a}\Big((A-B_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\
&\quad + \widetilde{a}\Big((A-B_h)(\mathbf{u},\mathbf{v}),(I-\Pi_h)(\boldsymbol{\phi},\boldsymbol{\psi})\Big).
\end{aligned} \tag{5.8}$$

The second term in the right hand side above is bounded by using Lemmas 5.2 and 5.3 as follows:

$$\left|\widetilde{a}\Big((A-B_h)(\mathbf{u},\mathbf{v}),(I-\Pi_h)(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\right| \leq C\left\|(A-B_h)|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}}\left\|(I-\Pi_h)|_{E_*(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} \leq Ch^{2s}, \tag{5.9}$$

whereas for the first term we have:

$$\begin{aligned}\widetilde{a}\Big((A-B_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big) &= \widetilde{a}\Big((A-A_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &\quad + \widetilde{a}\Big((A_h-B_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &= \widetilde{a}\Big((A_h-B_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big),\end{aligned} \tag{5.10}$$

the latter since $\widetilde{a}\Big((A-A_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big)$ vanishes, because of Lemma 4.6 (case 2) and (4.6).

Now,

$$\begin{aligned}\widetilde{a}\Big((A_h-B_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big) &= \widetilde{a}\Big(A_h(I-\Pi_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &= \widetilde{b}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &= \widetilde{b}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &\quad - \widetilde{b}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),(I-\Pi_h)(\boldsymbol{\phi},\boldsymbol{\psi})\Big).\end{aligned} \tag{5.11}$$

The second term in the right hand side above is bounded by using Lemma 5.2 as follows:

$$\left|\widetilde{b}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),(I-\Pi_h)(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\right| \leq C\left\|(I-\Pi_h)|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}}\left\|(I-\Pi_h)|_{E_*(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}} \leq Ch^{2s}, \tag{5.12}$$

whereas for the first term we have:

$$\begin{aligned}\widetilde{b}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big) &= \widetilde{a}\Big((\mathbf{u},\mathbf{v}),A_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big) - \widetilde{b}\Big(\Pi_h(\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &= \widetilde{a}\Big((\mathbf{u},\mathbf{v}),A_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big) - \widetilde{a}\Big(\Pi_h(\mathbf{u},\mathbf{v}),A_{*h}(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &= \widetilde{a}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),A_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &\quad + \widetilde{a}\Big(\Pi_h(\mathbf{u},\mathbf{v}),(A_*-A_{*h})(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\\ &= \widetilde{a}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),A_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big),\end{aligned} \tag{5.13}$$

the latter since $\widetilde{a}\Big(\Pi_h(\mathbf{u},\mathbf{v}),(A_*-A_{*h})(\boldsymbol{\phi},\boldsymbol{\psi})\Big)$ vanishes, because of Lemma 4.6 (case 2), (4.6), (5.1), and the definition of $\widetilde{a}$.

Finally, by using again Lemma 5.2, we have

$$\begin{aligned}\left|\widetilde{a}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),A_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\right| &= \left|\widetilde{a}\Big((I-\Pi_h)(\mathbf{u},\mathbf{v}),(I-\Pi_h)A_*(\boldsymbol{\phi},\boldsymbol{\psi})\Big)\right|\\ &\leq C\left\|(I-\Pi_h)|_{E(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}}\left\|(I-\Pi_h)|_{E_*(\widetilde{\mathcal{V}})}\right\|_{\widetilde{\mathcal{V}}}\|A_*\|_{\widetilde{\mathcal{V}}}\\ &\leq Ch^{2s}.\end{aligned} \tag{5.14}$$

Now the lemma is a consequence of (5.5) to (5.14). □

As a conclusion of the previous lemma, we obtain optimal order error estimates for the approximate eigenvalues:

Theorem 5.2 *There exist constants $h_0 > 0$ and $C > 0$ such that, for $h \le h_0$,*

$$\begin{aligned} \left| \mu - \frac{1}{m} \sum_{k=1}^{m} \mu_{kh} \right| &\le Ch^{2s}, \\ \left| \frac{1}{\mu} - \frac{1}{m} \sum_{k=1}^{m} \frac{1}{\mu_{kh}} \right| &\le Ch^{2s}, \\ \max_{k=1,\dots,m} |\mu - \mu_{kh}| &\le Ch^{2s/p}, \end{aligned}$$

where p is the ascent of the eigenvalue μ of A, (i.e., the length of the longest Jordan chain of A associated with μ).

Proof: It is essentially identical to that of Theorem 3 in [13]. □

6 Numerical results

To end this paper we present some numerical experiments to show the performance of this method and how the quadratic eigenvalue problem (4.1) can be efficiently solved by using a standard eigensolver, namely, `eigs` from Matlab version 5.3. This code is based on the classical *Arnoldi* iteration and allows solving a generalized linear eigenvalue problem with an arbitrary matrix on the left hand side, but a hermitian and positive definite one on the right hand side.

Let $\{\boldsymbol{\phi}_j\}_{j=1}^N$ be a nodal basis of $\mathcal{V}_h$. For $\mathbf{u}_h \in \mathcal{V}_h$, we write $\mathbf{u}_h = \sum_{j=1}^N u_j \boldsymbol{\phi}_j$ in terms of this basis and denote $u = (u_1, \dots, u_N)$ the vector of its nodal components. Let $K := (K_{ij})$, $B := (B_{ij})$ and $M := (M_{ij})$, with

$$K_{ij} := \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{\phi}_i \operatorname{div} \bar{\boldsymbol{\phi}}_j, \quad B_{ij} := \int_{\Gamma_A} \boldsymbol{\phi}_i \cdot \boldsymbol{\nu}\, \bar{\boldsymbol{\phi}}_j \cdot \boldsymbol{\nu}, \quad M_{ij} := \int_\Omega \rho\, \boldsymbol{\phi}_i \cdot \bar{\boldsymbol{\phi}}_j, \quad i, j = 1, \dots, N.$$

Problem (4.1) can be written in the following matrix form:

$$Ku + (\alpha + \lambda_h \beta) Bu + \lambda_h^2 Mu = 0. \tag{6.1}$$

K and M are the stiffness and mass matrices of the fluid, respectively, whereas B is used to take into account the effect of the absorbing wall. The three matrices are hermitian; furthermore, K and B are semipositive and M positive definite.

The matrix form of the equivalent linear problem (4.2)-(4.3), used for the theoretical analysis, is

$$\begin{pmatrix} K + \alpha B & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda_h \begin{pmatrix} -\beta B & -M \\ M & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $v = \lambda_h u$. The matrices in the right and left hand sides are singular and indefinite, respectively. Thus this generalized eigenvalue problem cannot be directly solved with Matlab eigensolver `eigs`. Instead we will transform problem (6.1) into other suitable equivalent linear one.

For $\lambda_h \neq 0$, let $\mu_h := \frac{1}{\lambda_h}$; then problem (6.1) is equivalent to

$$Mu + \mu_h \beta B u + \mu_h^2 (K + \alpha B) u = 0.$$

Now, by introducing $w = \mu_h u$, we rewrite this problem in the following way

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \mu_h \begin{pmatrix} -\beta B & -(K+\alpha B) \\ M & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} -\beta B & -(K+\alpha B) \\ M & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \lambda_h \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}. \tag{6.2}$$

This last problem is equivalent to (4.1), except for $\lambda_h = 0$, and the matrix in its right hand side is hermitian and positive definite. Hence, it can be safely solved by means of the eigensolver `eigs`.

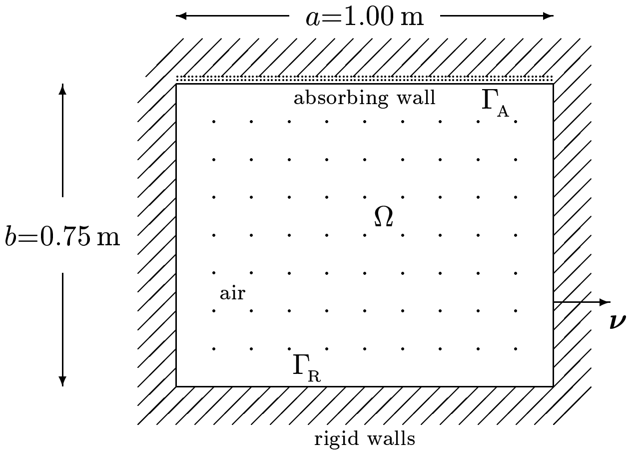

Figure 1: Fluid in a cavity with one absorbing wall.

We apply our method to compute the vibration modes of a 2D rectangular rigid cavity filled with air and with one absorbing wall as that shown in Figure 1. For such simple geometry, problem (2.9) can be solved by separation of variables (see [7] for details). Apart from $\lambda = 0$, its solutions are

$$\lambda_j \in \mathbf{C}, \qquad \mathbf{u}_j(x,y) = \nabla \left(\cos \frac{j\pi x}{a} \cosh \eta_j y \right), \qquad j = 0, 1, 2, \ldots$$

with λ_j and $\eta_j \in \mathbf{C}$ being solutions of the so-called *dispersion equations:*

$$\eta_j^2 = \frac{\lambda_j^2}{c^2} + \frac{\pi^2}{a^2} j^2, \tag{6.3}$$

$$\eta_j \tanh \eta_j b = -\frac{\rho \lambda_j^2}{(\alpha + \lambda_j \beta)}. \tag{6.4}$$

Notice that all the eigenfunctions are oscillatory in the horizontal direction, the integer parameter j in these equations being the number of corresponding half-waves. In general, the system above has an infinite number of complex solutions for each $j \geq 0$. Furthermore, at least for j large enough, it also has real solutions with λ_j negative, corresponding to the overdamped modes quoted in Section 3. These real negative λ_j only have one accumulation point at $-\frac{\alpha}{\beta}$ (see [7]).

We have taken the geometrical data given in Figure 1 and the following physical data: $\rho = 1\,\mathrm{kg/m^3}$, $c = 340\,\mathrm{m/s}$, $\alpha = 5 \times 10^4\,\mathrm{N/m^3}$ and $\beta = 200\,\mathrm{N\,s/m^3}$. The two latter correspond to an ideal very viscous insulating material and were chosen for overdamped modes to occur for j not too large.

We have computed both, overdamped and normally damped vibration modes, the latter being the eigenvalues with non-null imaginary part. We have used uniform meshes obtained by successively refining that in Figure 2. We denote by N the refinement level ($N = 1$ for that in Figure 2, $N = 2$ for the one obtained by halving the meshsize h, etc.).

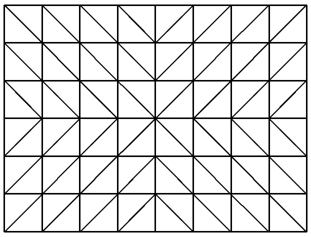

Figure 2: Initial mesh: $N = 1$.

Table 1 shows the eigenvalues of the discrete problem (6.2) with lowest positive vibration frequencies: $0 < f := \frac{\omega}{2\pi} < 600\,\mathrm{Hz}$ (recall $\omega := \operatorname{Im}\lambda$). Those with smallest decay rate (i.e., $-\operatorname{Re}\lambda$ small) correspond to the largest response peaks of the damped cavity, which are the magnitudes of interest in most applications.

Table 1: Damped acoustic modes for a rectangular rigid cavity with an absorbing wall.

$N = 2$ (604 d.o.f)	$N = 4$ (2360 d.o.f)	$N = 8$ (9328 d.o.f)	'exact'	order
$-317.98 + 267.76\,\mathrm{i}$	$-320.02 + 267.68\,\mathrm{i}$	$-320.54 + 267.66\,\mathrm{i}$	$-320.71 + 267.65\,\mathrm{i}$	1.99
$-259.50 + 813.05\,\mathrm{i}$	$-259.28 + 813.23\,\mathrm{i}$	$-259.21 + 813.27\,\mathrm{i}$	$-259.21 + 813.29\,\mathrm{i}$	2.34
$-90.32 + 1281.55\,\mathrm{i}$	$-90.04 + 1281.40\,\mathrm{i}$	$-89.98 + 1281.36\,\mathrm{i}$	$-89.95 + 1281.35\,\mathrm{i}$	2.00
$-299.64 + 2176.41\,\mathrm{i}$	$-297.81 + 2179.97\,\mathrm{i}$	$-297.36 + 2180.85\,\mathrm{i}$	$-297.21 + 2181.15\,\mathrm{i}$	2.00
$-27.81 + 2247.92\,\mathrm{i}$	$-27.47 + 2249.79\,\mathrm{i}$	$-27.39 + 2250.25\,\mathrm{i}$	$-27.37 + 2250.41\,\mathrm{i}$	2.01
$-237.61 + 2408.21\,\mathrm{i}$	$-236.93 + 2408.97\,\mathrm{i}$	$-236.76 + 2409.15\,\mathrm{i}$	$-236.70 + 2409.21\,\mathrm{i}$	2.02
$-144.34 + 3029.85\,\mathrm{i}$	$-143.47 + 3025.26\,\mathrm{i}$	$-143.24 + 3024.08\,\mathrm{i}$	$-143.16 + 3023.68\,\mathrm{i}$	1.98
$-13.36 + 3269.81\,\mathrm{i}$	$-12.84 + 3279.02\,\mathrm{i}$	$-12.73 + 3281.31\,\mathrm{i}$	$-12.69 + 3282.07\,\mathrm{i}$	2.01
$-309.77 + 3566.92\,\mathrm{i}$	$-304.30 + 3583.08\,\mathrm{i}$	$-303.02 + 3587.10\,\mathrm{i}$	$-302.60 + 3588.43\,\mathrm{i}$	2.01
$-281.62 + 3723.38\,\mathrm{i}$	$-276.89 + 3734.25\,\mathrm{i}$	$-275.78 + 3736.92\,\mathrm{i}$	$-275.41 + 3737.81\,\mathrm{i}$	2.01

The table includes the eigenvalues computed with three different meshes (the corresponding number of degrees of freedom (d.o.f) are also given) and the 'exact' ones computed by solving the non linear system (6.3)-(6.4). An excellent agreement between 'exact' and computed values can be observed, even for the coarsest mesh, showing the effectiveness of our method. Finally the table shows the computed orders of convergence which compares very well with the theoretically predicted one (since the cavity is convex, $s = 1$ in (2.12), and hence Theorem 5.2 yields a quadratic order of convergence for the eigenvalues).

Table 2 shows some real eigenvalues of the discrete problem (6.2) which correspond to overdamped modes. The corresponding 'exact' eigenvalues and computed orders of convergence are also shown in the table.

Table 2: Overdamped modes for a rectangular rigid cavity with an absorbing wall.

j	$N = 2$	$N = 4$	$N = 8$	'exact'	order
2	-347.04	-343.11	-342.15	-341.83	2.02
3	-300.35	-297.39	-296.66	-296.41	2.00
4	-285.58	-282.49	-281.71	-281.45	1.99
5	-278.36	-274.99	-274.13	-273.84	1.98
6	-274.16	-270.49	-269.53	-269.21	1.97
...					
2	-1026.37	-1069.12	-1080.41	-1084.23	1.96
3	-1721.86	-1854.08	-1892.02	-1905.19	1.90
4	-2295.75	-2576.38	-2664.69	-2696.36	1.83
5	-2766.17	-3249.14	-3417.56	-3480.48	1.75
6	-3156.62	-3876.48	-4153.53	-4261.77	1.68
...					

For the physical parameters considered in this example, the system of equations (6.3)-(6.4) (and consequently the continuous problem) has real solutions if and only if $j \geq 2$. In that case, for each j, the system has two different pairs of solutions, one with $-\frac{2\alpha}{\beta} < \lambda_j < -\frac{\alpha}{\beta}$ and the other with $\lambda_j < -\frac{2\alpha}{\beta}$. Thus two sequences of eigenvalues are obtained, the former converges to $-\frac{\alpha}{\beta}$ and the latter diverges to $-\infty$ (see [7]).

Once more a very good agreement between 'exact' and computed values can be observed and the orders of convergence are close to the theoretical one. However, the quality of the approximation deteriorates for larger numbers of half-waves j, particularly for the second sequence.

7 Acknowledgement

This work was initiated during a stay of the last three authors at the Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, which was partially supported by Programa de Cooperación Científica con Iberoamérica, Ministerio de Educación y Ciencia, Spain.

References

[1] K.J. Bathe, C. Nitikitpaiboon, and X. Wang, A mixed displacement-based finite element formulation for acoustic fluid-structure interaction. *Comp. Struct.*, **56** (1995) 225-237.

[2] A. Bermúdez, R. Durán, M.A. Muschietti, R. Rodríguez, and J. Solomin, Finite element vibration analysis of fluid-solid systems without spurious modes, *SIAM J. Numer. Anal.*, **32** (1995) 1280-1295.

[3] A. Bermúdez, L. Hervella-Nieto, and R. Rodríguez, Finite element computation of three-dimensional elastoacustic vibrations. *J. Sound Vibr.*, **219** (1999) 277-304.

[4] A. Bermúdez, L. Hervella-Nieto, and R. Rodríguez, Finite element computation of the vibrations of a plate-fluid system with interface damping, *Technical Report* 98-25, Universidad de Concepción, 1998.

[5] A. Bermúdez and R. Rodríguez, Finite element computation of the vibration modes of a fluid-solid system, *Comp. Methods Appl. Mech. Engng.* **119** (1994) 355-370.

[6] A. Bermúdez and R. Rodríguez, Numerical computation of elastoacoustic vibrations with interface damping, in *Équations aux Dérivées Partielles et Applications*, Gauthier-Villars, 1998, pp. 165-187.

[7] A. Bermúdez and R. Rodríguez, Modeling and numerical solution of elastoacoustic vibrations with interface damping, *Int. J. Numer. Methods Engng.* **46** (1999) 1763-1779.

[8] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.

[9] H.C. Chen and R.L. Taylor, Vibration analysis of fluid-solid systems using a finite element displacement formulation. *Int. J. Numer. Methods Engng.*, **29** (1990) 683-698.

[10] P.G. Ciarlet, Basic error estimates for elliptic problems, in *Handbook of Numerical Analysis, Vol. II*, P.G. Ciarlet and J.L. Lions, eds., North Holland, 1991.

[11] M. Dauge, Elliptic boundary value problems on corner domains: smoothness and asymptotics of solutions, *Lecture Notes in Mathematics*, **1341**, Springer-Verlag, 1988.

[12] J. Descloux, N. Nassif, and J. Rappaz, On spectral approximation. Part 1: The problem of convergence., *R.A.I.R.O. Anal. Numér.*, **12** (1978) 97-112.

[13] J. Descloux, N. Nassif, and J. Rappaz, *On spectral approximation. Part 2: Error estimates for the Galerkin method*, *R.A.I.R.O. Anal. Numér.*, **12** (1978) 113-119.

[14] L. Gastaldi, Mixed finite element methods in fluid structure systems. *Numer. Math.*, **74** (1996) 153-176.

[15] V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, 1986.

[16] P. Grisvard, *Elliptic Problems for Non-smooth Domains*, Pitman, 1985.

[17] M. Hamdi, Y. Ouset, and G. Verchery, A displacement method for the analysis of vibrations of coupled fluid-structure systems, *Int. J. Numer. Methods Engng.*, **13** (1978) 139-150.

[18] V. Kehr-Kandille and R. Ohayon, Elastoacoustic damped vibrations. Finite element and modal reduction methods, in *New Advances in Computational Structural Mechanics*, O.C. Zienkiewicz and P. Ladevèze, eds., Elsevier, 1992, pp. 321-334.

[19] M.G. Krein and H. Langer, On some mathematical principles in the linear theory of damped oscillations of continua I. *Integral Equations and Operator Theory*, **1/3** (1978) 364-399.

[20] P.A. Raviart and J.M. Thomas, A mixed finite element method for second order elliptic problems, in *Mathematical Aspects of Finite Element Methods*, Lecture Notes in Mathematics **606**, Springer-Verlag, 1977, pp. 292-315.

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV Analysis of operators*, Academic Press, 1978.

[22] R. Rodríguez and J. Solomin, The order of convergence of eigenfrequencies in finite element approximations of fluid-structure interaction problems. *Math. Comp.*, **65** (1996) 1463–1475.

[23] X. Wang and K.J. Bathe, Displacement/pressure based mixed finite element formulations for acoustic fluid-structure interaction problems. *Int. J. Numer. Methods Engng.*, **40** (1997) 2001-2017.