A POSTERIORI ERROR ANALYSIS FOR NONCONFORMING APPROXIMATION OF MULTIPLE EIGENVALUES

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Abstract. In this paper we study an a posteriori error indicator introduced in E. Dari, R.G. Durán, C. Padra, *Appl. Numer. Math.*, 2012, for the approximation of Laplace eigenvalue problem with Crouzeix–Raviart non-conforming finite elements. In particular, we show that the estimator is robust also in presence of eigenvalues of multiplicity greater than one. Some numerical examples confirm the theory and discuss the convergence of an adaptive algorithm when dealing with multiple eigenvalues.

1. Introduction

Although the a posteriori error analysis for eigenvalue problems arising from partial differential equations is a mature field of research, some intriguing questions remain open when discussing the convergence of an adaptive scheme for the approximation of eigenvalues with multiplicity greater than one.

In this paper we consider the approximation of Laplace eigenvalue by standard Crouzeix–Raviart finite elements (see [3] and, for instance, [2]). In [5] an a posteriori error indicator has been proposed for this problem and its efficiency and reliability have been proved. The analysis of [5] showed that the indicator is equivalent to the energy norm of the error in the eigenfunctions (up to higher order terms) and that it provides an upper bound for the *first* eigenvalue (up to higher order terms). In this paper we are mainly interested in the case when an eigenvalue may have multiplicity greater than one. This topic has been the object of little research and only very recently people started investigating the issues originating from the presence of multiple eigenvalues (see, in particular, [12, 4, 7]).

The presented results contain a theoretical part, included in Sections 3 and 4, and some numerical experiments reported in Section 5.

In Section 3 we study the error estimates for the eigenfunctions and, recalling the results of [5], we show that the results extend in a natural way to the case of multiple eigenvalues. In Section 4, using some special tools adapted from [10], we extend the estimates for the eigenvalues to the general case of multiplicity $q \geq 1$. One of the main difficulties comes from the fact that, when using non-conforming finite elements, the min-max lemma is no longer valid and there is no reason why the discrete eigenvalues should be upper bounds of the corresponding continuous ones. In our analysis we study separately the cases when an eigenvalue is approximated by q discrete eigenvalues by above or by below. Our analysis does not apply to the case when a continuous eigenvalue corresponds to discrete eigenvalues which can approximate it simultaneously from above of from below. It should however be noted that in most situations Crouzeix-Raviart element provides lower bound: this has been proved for singular eigenspaces (see [1] and [5]). Known examples of discrete eigenvalues which provide approximation from above are rare and computed on very coarse meshes.

The numerical results shown in Section 5 confirm the theory and aim at investigating the behavior of an adaptive procedure based on the studied indicator in case of multiple eigenvalues. As expected, it turns out that a correct procedure should take into account all discrete eigenfunctions approximating the same eigenspace (see [12]). One of the main issues raised by this investigation is that in general it is not known a priori (besides very particular situations like the one considered in our tests) the multiplicity of an eigenvalue of the continuous problem and it is not obvious to detect which discrete values correspond to it. This phenomenon requires further investigation and will be the object of future study.

2. Setting of the problem

Let $\Omega\subset\mathbb{R}^d$, $d=2,3$ be a polygonal or polyhedral Lipschitz domain, we consider the Laplacian eigenproblem: find $\lambda\in\mathbb{R}$ and $u\in H^1_0(\Omega)$ with $u\ne0$ such that

$$a(u,v)=\lambda(u,v)\quad\forall v\in H^1_0(\Omega),\tag{1}$$

where

$$a(u,v)=\int_\Omega\nabla u\nabla v\,dx\qquad(u,v)=\int_\Omega uv\,dx.$$

It is well known that the eigenvalues of the problem above form an increasing sequence tending to infinity:

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_i\le\cdots\tag{2}$$

We denote by u_i an eigenfunction associated to the eigenvalue λ_i; it is well known that the eigenfunctions can be chosen such that the following properties are satisfied:

$$\begin{aligned}&(u_i,u_i)=1 &&(u_i,u_j)=0\quad\text{if } i\ne j\\ &a(u_i,u_i)=\lambda_i &&a(u_i,u_j)=0\quad\text{if } i\ne j.\end{aligned}\tag{3}$$

Let us introduce the Crouzeix–Raviart non conforming finite element space we shall work with (see [3]). We consider a regular family of decompositions of Ω into closed triangles or tetrahedra. Let h_K denote the diameter of the element K and $h=\max_{K\in\mathcal{T}}h_K$. The set of all faces F of elements in $\mathcal{T}_h$ is denoted by $\mathcal{F}_h$. For any internal face F let K and K' be two elements such that $K\cap K'=F$, we denote by $[v]_F$ the jump across F for $v\in L^2(K\cup K')$. For a face $F\subset\partial\Omega$ we set $[v]_F=v$. Then we define

$$V_h^{nc}=\{v\in L^2(\Omega):\ v|_K\in\mathcal{P}_1(K)\ \forall K\in\mathcal{T}_h\text{ and }\int_F[v]_F=0\ \forall F\in\mathcal{F}_h\}.$$

We introduce the following discrete bilinear form defined on $V_h^{nc}\times V_h^{nc}$

$$a_h(u,v)=\sum_{K\in\mathcal{T}_h}\nabla u\nabla v\,dx=\int_\Omega\nabla_hu\nabla_hv\,dx\qquad\forall u,v\in V_h^{nc}$$

where

$$\nabla_hu|_T=\nabla(u|_T).$$

Let us recall some standard notation. We set $\|\cdot\|_0^2=(\cdot,\cdot)$, the L^2-norm, and

$$\begin{aligned}&\|u\|_1^2=a(u,u)=\|\nabla u\|_0^2 &&\forall u\in H^1_0(\Omega)\\ &\|u\|_h^2=a_h(u,u)=\|\nabla_hu\|_0^2 &&\forall u\in V_h^{nc}\end{aligned}\tag{4}$$

Notice that thanks to the Poincaré inequality and to its discrete version for non conforming elements (see [8]) both $\|\cdot\|_1$ and $\|\cdot\|_h$ are norms on $H^1_0(\Omega)$ and V_h^{nc}, respectively.

Let $\tilde V = H^1_0(\Omega) + V_h^{nc}$, that is any element $\tilde u$ of $\tilde V$ can be written as the sum $\tilde u = u + u_h$ with $u \in H^1_0(\Omega)$ and $u_h \in V_h^{nc}$. We have that $\|\cdot\|_h$ is a norm in $\tilde V$ and that in the case of $u \in H^1_0(\Omega)$ it holds $\|u\|_h = \|u\|_1$.

Then the discrete eigenproblem reads: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h^{nc}$ with $u_h \neq 0$ such that

$$a_h(u_h, v) = \lambda_h (u_h, v) \qquad \forall v \in V_h^{nc}. \tag{5}$$

Problem (5) admits exactly $N_h = \dim(V_h^{nc})$ positive eigenvalues with

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{N_h,h}. \tag{6}$$

Moreover, we denote by $u_{i,h}$ a discrete eigenfunction associated to the eigenvalue $\lambda_{i,h}$ with the following properties:

$$\begin{aligned} &(u_{i,h}, u_{i,h}) = 1 && (u_{i,h}, u_{j,h}) = 0 \quad \text{if } i \neq j \\ &a_h(u_{i,h}, u_{i,h}) = \lambda_{i,h} && a_h(u_{i,h}, u_{j,h}) = 0 \quad \text{if } i \neq j. \end{aligned} \tag{7}$$

We indicate with $E_{i,\dots,j} \subset H^1_0(\Omega)$ (resp. $E_{i,\dots,j,h} \subset V_h^{nc}$) the span of the eigenvectors $\{u_i, \dots, u_j\}$ (resp. $\{u_{i,h}, \dots, u_{j,h}\}$) and $P_{i,\dots,j}$ (resp. $P_{i,\dots,j,h}$) the elliptic projection onto $E_{i,\dots,j}$ (resp. $E_{i,\dots,j,h}$), that is

$$\begin{aligned} &\text{for } u \in H^1_0(\Omega),\ P_{i,\dots,j} u \in E_{i,\dots,j} \ \text{ s.t. } a(u - P_{i,\dots,j} u, v) = 0 \quad \forall v \in E_{i,\dots,j} \\ &\text{for } u \in \tilde V,\ P_{i,\dots,j,h} u \in E_{i,\dots,j,h} \ \text{ s.t. } a_h(u - P_{i,\dots,j,h} u, v) = 0 \quad \forall v \in E_{i,\dots,j,h}. \end{aligned} \tag{8}$$

In our a posteriori error analysis we shall also make use of the space of conforming piecewise linear elements

$$V_h^c = \{ v \in H^1_0(\Omega) : v|_K \in \mathcal{P}_1(K)\ \forall K \in \mathcal{T}_h \}.$$

The conforming discretization of the eigenvalue problem under consideration reads: find $\lambda_h^c \in \mathbb{R}$ and $u_h^c \in V_h^c$ with $u_h^c \neq 0$ such that

$$a(u_h^c, v) = \lambda_h^c (u_h^c, v) \qquad v \in V_h^c. \tag{9}$$

Problem (9) admits $N_h^c = \dim(V_h^c)$ positive eigenvalues

$$0 < \lambda_{1,h}^c \le \lambda_{2,h}^c \le \cdots \le \lambda_{N_h^c,h}^c. \tag{10}$$

As in the case of non conforming discretization we denote by $u_{i,h}^c$ the eigenfunction associated to the eigenvalue $\lambda_{i,h}^c$ such that $(u_{i,h}^c, u_{i,h}^c) = 1$ with the following orthogonality properties:

$$\begin{aligned} &(u_{i,h}^c, u_{i,h}^c) = 1 && (u_{i,h}^c, u_{j,h}^c) = 0 \quad \text{if } i \neq j \\ &a_h(u_{i,h}^c, u_{i,h}^c) = \lambda_{i,h}^c && a_h(u_{i,h}^c, u_{j,h}^c) = 0 \quad \text{if } i \neq j. \end{aligned} \tag{11}$$

Notice that $V_h^c = H^1_0(\Omega) \cap V_h^{nc}$, hence $N_h^c < N_h$ and $\lambda_{i.h} \le \lambda_{i,h}^c$ for $i = 1, \dots, N_h^c$ because of the min max characterization.

Let P_h^c be the elliptic projection from $\tilde V$ onto V_h^c, that is: for all $u \in \tilde V$, $P_h^c u \in V_h^c$ such that

$$a(P_h^c u, v) = a_h(u, v) \quad \forall v \in V_h^c. \tag{12}$$

Similarly to the nonconforming approximation, we denote by $E^c_{i,\dots,j,h} \subset V^c_h$ the span of the eigenvectors $\{u^c_{i,h},\dots,u^c_{j,h}\}$ and by $P^c_{i,\dots,j,h}$ the elliptic projection onto $E^c_{i,\dots,j,h}$, that is: for all $u \in \tilde{V}$, $P^c_{i,\dots,j,h}u \in E^c_{i,\dots,j,h}$ such that

$$a_h(u - P^c_{i,\dots,j,h}u, v) = 0 \qquad \forall v \in E^c_{i,\dots,j,h}. \tag{13}$$

We shall make use of the Rayleigh quotient associated to the eigenvalue problem (1)

$$\mathcal{R}(w) = \frac{a(w,w)}{(w,w)} \quad \forall w \in H^1_0(\Omega) \setminus \{0\} \tag{14}$$

and of the analogous quotient associated to the nonconforming discretization

$$\mathcal{R}_h(w) = \frac{a_h(w,w)}{(w,w)} \quad \forall w \in V^{nc}_h \setminus \{0\} \tag{15}$$

In case of multiple eigenvalues we shall need to estimate the distance between eigenspaces associated to them and to their discrete counterpart. Let E and F be two subspaces of $\tilde{V}$, then the distance between them is defined as

$$\delta_h(E,F) = \sup_{\substack{u \in E\\ \|u\|_h=1}} \inf_{v\in F} \|u-v\|_h.$$

For nonzero functions u and v, if $E = \mathrm{span}\{u\}$, we write $\delta(u,F)$ instead of $\delta(E,F)$ and if $E = \mathrm{span}\{u\}$ and $F = \mathrm{span}\{v\}$, we write $\delta(u,v)$ for $\delta(E,F)$. We have $0 \le \delta_h(E,F) \le 1$ and $\delta_h(E,F) = 0$ if and only if $E \subseteq F$. If $\dim E = \dim F < \infty$ then $\delta_h(E,F) = \delta_h(F,E)$. If P and Q are the orthogonal projections onto E and F, respectively, then $\delta_h(E,F)$ equals the largest singular value of the operator $(I-Q)P$ and

$$\delta_h(E,F) = \|(I-Q)P\|_{\mathcal{L}(\tilde{V})}. \tag{16}$$

See for example [11] for these results and the characterization of the distance between subspaces.

3. Error estimates for the eigenfunctions

In this section we introduce the error indicators and present the a posteriori error estimates for the eigenfunctions.

First of all let us recall some properties of the Couzeix–Raviart space. Given $w \in H^1_0(\Omega)$, we denote by $w_I \in V^{nc}_h$ its edge/face average interpolant such that

$$\int_F w_I = \int_F w \qquad \forall F \in \mathcal{F}_h. \tag{17}$$

It is well known that $\nabla_h w_I$ is the L^2-projection of ∇w onto the piecewise constant vector fields and that the following estimates hold true

$$\begin{aligned} &\|\nabla_h w_I\|_0 \le \|\nabla w\|_0,\\ &\|w - w_I\|_{L^2(K)} \le C_1 h_K \|\nabla w\|_{L^2(K)}. \end{aligned} \tag{18}$$

Our error indicators make use of the following conforming postprocessing for the elements in V^{nc}_h. To any element $v \in V^{nc}_h$ we associate an element $\tilde{v} \in V^c_h$ obtained by averaging the value of v at the vertices of the triangulation $\mathcal{T}_h$. Namely, following [5], for each internal vertex P we consider all elements $K_i \in \mathcal{T}_h$ for $i = 1,\dots,M$

which share the vertex P and define

$$\tilde{v}(P) = \sum_{i=1}^{M} w_i v|_{K_i}(P), \tag{19}$$

where w_i are suitable weights such that $\sum_{i=1}^{M} w_i = 1$.

Lemma 1. *The following estimates hold true with constants C independent of h*

$$\begin{aligned} &\|\tilde{v}\|_0 \le C\|v\|_0 \\ &\|\nabla\tilde{v}\|_0 \le C\|\nabla_h v\|_0 \\ &\|\tilde{v} - v\|_0 \le Ch\|\nabla_h(\tilde{v} - v)\|_0. \end{aligned} \tag{20}$$

Proof. Let P be a vertex of the mesh. It is proved in [5, Th. 5.2], that for all $w \in H_0^1(\Omega)$

$$|\tilde{v}(P) - v|_K(P)| \le \frac{C}{h_K^{d/2-1}}\|\nabla_h(v - w)\|_{L^2(\Omega_P)}, \tag{21}$$

where Ω_P is the union of elements K containing P.

For $w = 0$, using an inverse estimate, we also have

$$|\tilde{v}(P) - v|_K(P)| \le \frac{C}{h_K^{d/2}}\|v\|_{L^2(\Omega_P)},$$

Then, for an element K we write, using the standard notation N_i for nodal basis functions,

$$\tilde{v} - v = \sum_{i=1}^{d+1} \left(\tilde{v}(P_i) - v|_K(P_i)\right) N_i$$

and

$$\nabla(\tilde{v} - v) = \sum_{i=1}^{d+1} \left(\tilde{v}(P_i) - v|_K(P_i)\right) \nabla N_i$$

and therefore, if $\tilde{K}$ is the union of neighbors of K, using the above estimates and standard estimates for the basis functions N_i, we obtain

$$\|\tilde{v} - v\|_{0,K} \le C\|v\|_{0,\tilde{K}}$$

and

$$\|\nabla(\tilde{v} - v)\|_{0,K} \le C\|\nabla_h v\|_{0,\tilde{K}}.$$

Then the triangle inequality yields the first two estimates in (20). The last one can also be easily obtained from (21) taking into account that $\|N_i\|_0 \le Ch_K^{d/2}$ and choosing $w = \tilde{v}$.

$\square$

We define the local and global error estimators as follows

$$\begin{aligned} \mu_{i,T}^2 &= \|\nabla\tilde{u}_{i,h} - \nabla_h u_{i,h}\|_{L^2(T)}^2, \qquad & \mu_i^2 &= \sum_T \mu_{i,T}^2 \\ \eta_{i,T}^2 &= h_T^2\|\lambda_{i,h} u_{i,h}\|_{L^2(T)}^2, & \eta_i^2 &= \sum_T \eta_{i,T}^2. \end{aligned} \tag{22}$$

The following theorem gives the error estimates for the eigenfunctions in term of the above error indicators.

Theorem 2. *Let λ_i be an eigenvalue of (1) with multiplicity $q \geq 1$ (that is $\lambda_i = \cdots = \lambda_{i+q-1}$) and let $E_{i,\dots,i+q-1}$ be the associated eigenspace. Assume that $\lambda_{j,h}$ is a discrete eigenvalue of (5) converging to λ_i and that $E_{j,h}$ is the associated eigenspace ($j = i, \dots, i+q-1$). Then*

$$\delta(E_{j,h}, E_{i,\dots,i+q-1}) \leq \delta(E_{j,h}, H_0^1(\Omega)) + C_1 \frac{\eta_j}{\lambda_{j,h}} + h.o.t. \tag{23}$$

More precisely, we have

$$|h.o.t.| \leq \frac{C_\Omega}{\lambda_{j,h}} \left((\lambda_i - \lambda_{j,h}) + (\lambda_i \lambda_{j,h})^{1/2} \inf_{v \in E_{i,\dots,i+q-1}} \|v - u_{j,h}\|_{L^2(\Omega)} \right),$$

where C_1 is the constant in (18) and C_Ω is the Poincaré constant.

Proof. The proof is based on that of [5, Th. 3.2]. Here we make more precise the case of multiple eigenvalues.

Let us fix $j = i, \dots, i+q-1$ and let us consider the eigensolution $(\lambda_{j,h}, u_{j,h})$ of (5); we recall that $\|u_{j,h}\|_0 = 1$. Let $u_j(h) \in E_{i,\dots,i+q-1}$ be such that $\|u_j(h)\|_0 = 1$ and

$$\|u_j(h) - u_{j,h}\|_h = \inf_{v \in E_{i,\dots,i+q-1}} \|v - u_{j,h}\|_h. \tag{24}$$

We observe that $u_j(h)$ is an eigenfunction associated to the multiple eigenvalue λ_i, hence it satisfies (1). Then applying the same argument as in the proof of [5, Th. 3.2], we have that

$$\begin{aligned} \|u_j(h) - u_{j,h}\|_h \leq \inf_{v \in H_0^1(\Omega)} \|v - u_{j,h}\|_h + C_1 \eta + C_\Omega \Big((\lambda_i - \lambda_{j,h}) \\ + (\lambda_i \lambda_{j,h})^{1/2} \|u_j(h) - u_{j,h}\|_0 \Big). \end{aligned} \tag{25}$$

From the definition of the gap we have to estimate

$$\delta(E_{j,h}, E_{i,\dots,i+q-1}) = \sup_{\substack{u \in E_{j,h} \\ \|u\|_h = 1}} \inf_{v \in E_{i,\dots,i+q-1}} \|u - v\|_h = \inf_{v \in E_{i,\dots,i+q-1}} \left\| \frac{u_{j,h}}{\|u_{j,h}\|_h} - v \right\|_h,$$

since $E_{j,h}$ is generated by $u_{j,h}$. With a simple computation, using the above estimate for the eigenfunction $u_{j,h}$ and the fact that $\|u_{j,h}\|_h = \lambda_{j,h}$, we obtain the desired bound. □

It remains to estimate the gap between $E_{j,h}$ and $H_0^1(\Omega)$ in terms of our indicators.

Lemma 3. *Under the same assumptions as in Theorem 2, the following estimate holds true:*

$$\delta(E_{j,h}, H_0^1(\Omega)) \leq \frac{1}{\lambda_{j,h}} \mu_j. \tag{26}$$

Proof. By definition we have

$$\begin{aligned} \delta(E_{j,h}, H_0^1(\Omega)) &= \sup_{\substack{u \in E_{j,h} \\ \|u\|_h = 1}} \inf_{v \in H_0^1(\Omega)} \|u - v\|_h \\ &= \frac{1}{\lambda_{j,h}} \inf_{v \in H_0^1(\Omega)} \|u_{j,h} - v\|_h \\ &\leq \frac{1}{\lambda_{j,h}} \|u_{j,h} - \tilde{u}_{j,h}\|_h = \frac{1}{\lambda_{j,h}} \mu_j. \end{aligned}$$

□

For the efficiency of these error estimators we refer to [5] where the following local bounds from below of the error are proved.

Theorem 4. *Let λ_i be an eigenvalue of (1) with multiplicity $q \geq 1$ and let $\lambda_{j,h}$ be a discrete eigenvalue converging to λ_i ($j = i, \dots, i+q-1$). Let $u_j(h) \in E_{i,\dots,i+q-1}$ be such that (24) holds true. Then there exist constants C depending only on the regularity of the elements such that for all elements $K \in \mathcal{T}_h$ it holds*

$$\mu_K \leq C\|\nabla_h(u_j(h) - u_{j,h})\|_{L^2(K^*)}$$
$$\eta_K \leq C\|\nabla_h(u_j(h) - u_{j,h})\|_{L^2(K)} + h.o.t.,$$

where K^* is the union of all the elements in $\mathcal{T}_h$ sharing a vertex with K and

$$h.o.t. = h_K\|\lambda_i u_j(h) - \lambda_{j,h}u_{j,h}\|_{L^2(K)}.$$

4. Error estimates for the eigenvalues

In this section we prove error estimates for the eigenvalues using the a posteriori error indicators introduced in (22). In the case of conforming approximation of the eigenvalue problem (1) it is well known that each discrete eigenvalue is greater than or equal to the corresponding continuous one. In the case of nonconforming discretization this is not true in general. In [1, 5] it is proved that, for singular eigenfunctions, the Crouzeix-Raviart approximation provides asymptotic lower bounds of the corresponding eigenvalue. For this reason, in our analysis we consider separately the cases where a multiple eigenvalue is approximated by below or by above. More precisely, given a multiple eigenvalue λ_i of multiplicity $q \geq 1$, we assume that either $\lambda_{j,h} \leq \lambda_i$ or $\lambda_i \leq \lambda_{j,h}$ for all $j = i, \dots, i+q-1$.

Let us consider first the case when the eigenvalues are approximated from below.

The first theorem gives an estimate of the relative error for the eigenvalues in terms of the norm of the distance of the discrete eigenspace from the subspace of conforming finite elements orthogonal to the span of the first i conforming eigenfunctions.

Theorem 5. *Let λ_i be an eigenvalue with multiplicity q so that*

$$\lambda_{i-1} < \lambda_i = \dots = \lambda_{i+q-1} < \lambda_{i+q},$$

and let $\lambda_{i,h} \leq \dots \leq \lambda_{i+q-1,h}$ be the q discrete eigenvalues converging to λ_i. We assume that $\lambda_{j,h} \leq \lambda_i$ for $j = i, \dots, i+q-1$. Then

$$\frac{\lambda_i - \lambda_{j,h}}{\lambda_i} \leq \|(I - P_h^c + P_{1,\dots,i-1,h}^c)P_{i,\dots,j,h}\|^2_{\mathcal{L}(\tilde{V})}. \tag{27}$$

Proof. We observe that the discretization by conforming finite elements produces q discrete eigenvalues converging to λ_i and that it holds $\lambda_i \leq \lambda_{j,h}^c$ for $j = i, \dots, i+q-1$.

Let us fix j with $i \leq j \leq i+q-1$, then by assumption we have

$$\lambda_{j,h} \leq \lambda_i \leq \lambda_{j,h}^c.$$

The operators $I - P_h^c + P_{1,\dots,i-1,h}^c$ and $P_{i,\dots,j,h}$ are orthogonal projections with respect to the norm $\|\cdot\|_h$ of $\tilde{V}$. Therefore $\|(I - P_h^c + P_{1,\dots,i-1,h}^c)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})} \leq 1$ (see [9, Th. 6.34, p. 56]). If $\|(I - P_h^c + P_{1,\dots,i-1,h}^c)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})} = 1$ then the bound (27) is obviously true, since $\lambda_{j,h} \leq \lambda_i$. Hence we assume that

$$\|(I - P_h^c + P_{1,\dots,i-1,h}^c)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})} < 1.$$

Thanks to [9, Th. 3.6, Chap. I] this inequality implies that

$$\dim((P_h^c - P_{1,\dots,i-1,h}^c)E_{i,\dots,j,h}) = \dim(E_{i,\dots,j,h}) = j - i + 1.$$

We choose $\overline{u} \in (P_h^c - P_{1,\dots,i-1,h}^c)E_{i,\dots,j,h} \subset V_h^c$ such that $\|\overline{u}\|_h = \|\overline{u}\| = 1$ and

$$\mathcal{R}(\overline{u}) = \max_{\substack{w\in(P_h^c-P_{1,\dots,i-1,h}^c)E_{i,\dots,j,h}\\ w\neq 0}} \mathcal{R}(w),$$

where $\mathcal{R}(w)$ is the Rayleigh quotient defined in (14).

Let us consider the following orthogonal decomposition of $\overline{u}$ in $\tilde{V}$:

$$\overline{u} = u + v \quad \text{with } u \in E_{1,\dots,j,h} \text{ and } v \in (E_{1,\dots,j,h})^\perp,$$

that is $a_h(w, v) = 0$ for all $w \in E_{1,\dots,j,h}$. Notice that since $u \in V_h^{nc}$ also $v \in V_h^{nc}$. We have that

$$\begin{aligned}
\|v\|_h &= \delta_h(\overline{u}, E_{1,\dots,j,h}) && \text{by definition of } v\\
&\le \delta_h((P_h^c - P_{1,\dots,i-1,h}^c)E_{i,\dots,j,h}, E_{1,\dots,j,h}) && \overline{u} \in (P_h^c - P_{1,\dots,i-1,h}^c)E_{i,\dots,j,h}\\
&\le \delta_h((P_h^c - P_{1,\dots,i-1,h}^c)E_{i,\dots,j,h}, E_{i,\dots,j,h}) && \text{the inf is taken on a smaller subset}\\
&= \|(I - P_h^c + P_{1,\dots,i-1,h}^c)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})} && \text{this is a characterization of the gap.}
\end{aligned}$$

We now prove that

$$0 \le \frac{\lambda_{j,h}^c - \lambda_{j,h}}{\lambda_{j,h}^c} \le \|v\|_h$$
$$0 \le \frac{\lambda_i - \lambda_{j,h}}{\lambda_i} \le \|v\|_h.$$

We observe that the first inequality implies the second one.
By definition of $\overline{u}$ and the min-max principle for the eigenvalues we have that

$$\lambda_{j,h}^c \le \mathcal{R}(\overline{u}).$$

Moreover, since $u \in E_{1,\dots,j,h}$, we have that $u = \sum_{s=1}^j \alpha_s u_{s,h}$ and

$$\mathcal{R}_h(u) = \frac{a_h(u,u)}{(u,u)} = \frac{\sum_{s=1}^j \alpha_s^2 a_h(u_{s,h}, u_{s,h})}{\sum_{s=1}^j \alpha_s^2 (u_{s,h}, u_{s,h})} = \frac{\sum_{s=1}^j \alpha_s^2 \lambda_{s,h}}{\sum_{s=1}^j \alpha_s^2} \le \lambda_{j,h}.$$

In conclusion, the following inequalities hold true ($i \le j \le i + q - 1$):

$$\mathcal{R}_h(u) \le \lambda_{j,h} \le \lambda_j \le \lambda_{j,h}^c \le \mathcal{R}(\overline{u}),$$

and the rest of the proof is based on a bound for $1/\mathcal{R}_h(u) - 1/\mathcal{R}(\overline{u})$.

Since $v \in (E_{1,\dots,j,h})^\perp$ we have that $a_h(u, v) = 0$. We want to show that also $(u, v) = 0$. Since we know that $v \in V_h^{nc}$ and that $E_{1,\dots,j,h}$ is invariant with respect to T_h, we have also $a_h(T_h u, v) = 0$. Hence

$$0 = a_h(T_h u, v) = (u, v),$$

due to the definition of T_h.

We now compute

$$
\begin{aligned}
\frac{1}{\mathcal{R}_h(u)} - \frac{1}{\mathcal{R}(\overline{u})} &= \frac{(u,u)}{a_h(u,u)} - \frac{(u,u)+(v,v)}{a_h(u,u)+a_h(v,v)} \\
&= \frac{a_h(u,u)(u,u)+a_h(v,v)(u,u)-a_h(u,u)(u,u)-a_(u,u)(v,v)}{a_h(u,u)\left(a_h(u,u)+a_h(v,v)\right)} \\
&= \frac{1}{a_h(u,u)+a_h(v,v)}\left(\frac{a_h(v,v)}{a_h(u,u)}\left((u,u)+(v,v)\right) - \left(\frac{a_h(v,v)}{a_h(u,u)}+1\right)(v,v)\right) \\
&= \left(\frac{1}{\mathcal{R}(\overline{u})} - \frac{1}{\mathcal{R}_h(v)}\right)\frac{a_h(v,v)}{a_h(u,u)} \\
&\le \frac{1}{\mathcal{R}(\overline{u})}\frac{a_h(v,v)}{a_h(u,u)} \le \frac{1}{\lambda_j}\frac{a_h(v,v)}{a_h(u,u)}.
\end{aligned}
$$

We get

$$
\frac{1}{\lambda_{jh}} - \frac{1}{\lambda_j} \le \frac{1}{\mathcal{R}_h(u)} - \frac{1}{\mathcal{R}(\overline{u})} \le \frac{1}{\lambda_j}\frac{a_h(v,v)}{a_h(u,u)}
$$

from which we obtain

$$
\frac{\lambda_j}{\lambda_{j,h}} \le 1 + \frac{a_h(v,v)}{a_h(u,u)} = \frac{a_(u,u)+a_h(v,v)}{a_h(u,u)} = \frac{1}{a_h(u,u)}
$$

and then

$$
\frac{\lambda_j - \lambda_{j,h}}{\lambda_j} = 1 - \frac{\lambda_{j,h}}{\lambda_j} \le 1 - a_h(u,u) = a_h(v,v) = \|v\|_h^2.
$$

We can obtain also

$$
\frac{\lambda_{j,h}^c - \lambda_{j,h}}{\lambda_{j,h}^c} \le a_h(v,v) = \|v\|_h^2,
$$

by using the following inequality

$$
\frac{1}{\lambda_{j,h}} - \frac{1}{\lambda_{j,h}^c} \le \frac{1}{\mathcal{R}_h(u)} - \frac{1}{\mathcal{R}(\overline{u})} \le \frac{1}{\lambda_{j,h}^c}\frac{a_h(v,v)}{a_h(u,u)}.
$$

□

We now want to estimate the right hand side of (27) in terms of P_h^c and $P_{i,\dots,j,h}$ only.

First of all, we observe that

$$
\|(I - P_h^c + P_{1,\dots,i-1,h}^c)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})}^2 = \|(I-P_h^c)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})}^2 + \|P_{1,\dots,i-1,h}^c P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})}^2;
$$

hence it remains to estimate the second term, which represents the projection of the nonconforming invariant subspace associated to the eigenvalues numbered from i to j onto the subspace of conforming invariant subspace generated by the first $i-1$ eigenvalues.

Proposition 6. *Let λ_i be an eigenvalue with multiplicity q, so that*

$$
\lambda_{i-1} < \lambda_i = \dots = \lambda_{i+q-1} < \lambda_{i+q},
$$

and let $\lambda_{i,h} \le \dots \le \lambda_{i+q-1,h}$ be the q discrete eigenvalues converging to λ_i. We assume that $\lambda_{j,h} \le \lambda_i$ for $j = i,\dots,i+q-1$ and that

$$
\lambda_{i-1,h}^c < \lambda_{i,h}.
$$

If the positive quantity $\lambda_{i,h}-\lambda^c_{i-1,h}$ is large enough, then we have:
(28)
$$\|P^c_{1,\dots,i-1,h}P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde V)} \le \frac{\|(I-P^c_h)T_hP^c_{1,\dots,i-1,h}\|_{\mathcal{L}(\tilde V)}}{d-\delta}\|(I-P^c_h)P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde V)}$$
where
$$d-\delta=\frac{2\lambda_{i,h}-\lambda_{i+q-1,h}-\lambda^c_{i-1,h}}{\lambda_{i,h}\lambda^c_{i-1,h}}$$
and T_h is the resolvent operator associated to problem (5).

Proof. Using the same notation as in [10, Th. 4.2], we introduce the following operators:
$$P=P_{i,\dots,j,h},\quad \tilde R=P^c_{1,\dots,i-1,h},$$
so that P is the elliptic projection onto the invariant nonconforming subspace $E_{i,\dots,j,h}$, and $\tilde R$ is the elliptic projection onto the invariant conforming subspace $E^c_{1,\dots,i-1,h}$, that is
$$\begin{aligned}&\tilde R=P^c_{1,\dots,i-1,h}:\tilde V\to E^c_{1,\dots,i-1,h}\\&a(\tilde Rw,v)=a_h(w,v)\quad\forall v\in E^c_{1,\dots,i-1,h}.\end{aligned}$$
We observe that $\|\tilde Rw\|_1\le\|w\|_h$

Moreover, the spectrum of $(\tilde RT_h\tilde R)|_{\mathrm{Im}\tilde R}$ is equal to $\{1/\lambda^c_{1,h},\dots,1/\lambda^c_{i-1,h}\}$. Indeed, $\mathrm{Im}\tilde R$ is the span of $\{u^c_{1,h},\dots,u^c_{i-1,h}\}$ and by definition $T_hu^c_{k,h}$ $(1\le k\le i-1)$ belongs to V^{nc}_h and is given by
$$a_h(T_hu^c_{k,h},v)=(u^c_{k,h},v)\quad\forall v\in V^{nc}_h.$$
Hence, $\tilde RT_hu^c_{k,h}$ belongs to $E^c_{1,\dots,i-1,h}$ and satisfies
$$a(\tilde RT_hu^c_{k,h},v)=a_h(T_hu^c_{k,h},v)=(u^c_{k,h},v)\quad\forall v\in E^c_{1,\dots,i-1,h}.$$
It follows that
$$\tilde RT_hu^c_{k,h}=\frac{1}{\lambda^c_{k,h}}u^c_{k,h},$$
so that $1/\lambda^c_{k,h}$ $(1\le k\le i-1)$ coincides with the spectrum of $(\tilde RT_h\tilde R)|_{\mathrm{Im}\tilde R}$ (there cannot be other eigenvalues, since the dimension of $\mathrm{Im}\tilde R$ is equal to $i-1$).

Since the spectrum of $(\tilde RT_h\tilde R)|_{\mathrm{Im}\tilde R}$ does not contain the eigenvalues $\mu_{j,h}=1/\lambda_{j,h}$ for $j=i,\dots,i+q-1$, the operator $\tilde R(T_h-\mu_{j,h})\tilde R$ has a bounded inverse and
$$d\|\tilde RP\|_{\mathcal{L}(\tilde V)}\le\|\tilde R(T_h-\mu_{j,h})\tilde RP\|_{\mathcal{L}(\tilde V)},$$
where
$$d=\min_{k=1,\dots,i-1}|\mu^c_{k,h}-\mu_{j,h}|=|\mu^c_{i-1,h}-\mu_{j,h}|\le|\mu^c_{i-1,h}-\mu_{i,h}|=\frac{\lambda_{i,h}-\lambda^c_{i-1,h}}{\lambda_{i,h}\lambda^c_{i-1,h}}.$$
We have that
$$\tilde R(T_h-\mu_{j,h})\tilde RP=\tilde R(T_h-\mu_{j,h})P^c_hP. \tag{29}$$
Namely, since $\tilde R\tilde RP=\tilde RP^c_hP$, it is enough to show that $\tilde RT_h\tilde RP=\tilde RT_hP^c_hP$. We have that $T^c_h=P^c_hT_h$, which implies that $\tilde RT_h\tilde RP=\tilde RT^c_h\tilde RP$ and that $\tilde RT_hP^c_hP=\tilde RT^c_hP^c_hP$. Hence, we only have to show that $\tilde RT^c_h\tilde RP=\tilde RT^c_hP^c_hP$. Indeed, it holds $\tilde RT^c_h\tilde R=\tilde RT^c_hP^c_h$. In order to show this result, let's take $f\in V^{nc}_h$, then

$P_h^c f = \sum_{i=1}^{\dim V_h^c} \alpha_i u_{i,h}^c$ and $\tilde{R} f = \sum_{i\in I} \alpha_i u_{i,h}^c$, where I is the finite set of indices corresponding to the range of $\tilde{R}$. The equality (29) is then easily obtained by comparing $\tilde{R} T_h^c \tilde{R} f$ and $\tilde{R} T_h^c P_h^c f$ and taking into account that $u_{i,h}^c$ are eigenfunctions of T_h^c.

From (29) we obtain

$$\begin{aligned}\tilde{R}(T_h - \mu_{j,h}) P_h^c P &= \tilde{R} T_h P_h^c P - \tilde{R} T_h P + \tilde{R} T_h P - \tilde{R} \mu_{j,h} P_h^c P \\ &= -\tilde{R} T_h (I - P_h^c) P + \tilde{R}(T_h - \mu_{j,h}) P + \mu_{j,h} \tilde{R}(I - P_h^c) P.\end{aligned}$$

The last term is equal to zero since $\tilde{R} = \tilde{R} P_h^c$. Hence

$$\begin{aligned}d\|\tilde{R}P\|_{\mathcal{L}(\tilde{V})} &\le \|\tilde{R} T_h (I - P_h^c) P\|_{\mathcal{L}(\tilde{V})} + \|\tilde{R}(T_h - \mu_{j,h}) P\|_{\mathcal{L}(\tilde{V})} \\ &\le \|\tilde{R} T_h (I - P_h^c)\|_{\mathcal{L}(\tilde{V})} \|(I - P_h^c) P\|_{\mathcal{L}(\tilde{V})} + \|\tilde{R}P\|_{\mathcal{L}(\tilde{V})} \|P(T_h - \mu_{j,h}) P\|_{\mathcal{L}(\tilde{V})} \\ &\le \|(I - P_h^c) T_h \tilde{R}\|_{\mathcal{L}(\tilde{V})} \|(I - P_h^c) P\|_{\mathcal{L}(\tilde{V})} + \delta \|\tilde{R}P\|_{\mathcal{L}(\tilde{V})},\end{aligned}$$

where δ is given by the following inequality

$$\|P(T_h - \mu_{j,h})P\|_{\mathcal{L}(\tilde{V})} \le \delta.$$

Since P is the elliptic projection onto the invariant nonconforming subspace $E_{i,\dots,j,h}$ we have that

$$\delta \le |\mu_{i+q-1,h} - \mu_{i,h}| = \frac{1}{\lambda_{i,h}} - \frac{1}{\lambda_{i+q-1,h}} = \frac{\lambda_{i+q-1,h} - \lambda_{i,h}}{\lambda_{i,h}\lambda_{i+q-1,h}} \le \frac{\lambda_{i+q-1,h} - \lambda_{i,h}}{\lambda_{i,h}\lambda_{i-1,h}^c}.$$

For h small enough, $d - \delta > 0$ and we conclude that

$$\|\tilde{R}P\|_{\mathcal{L}(\tilde{V})} \le \frac{\|(I - P_h^c) T_h \tilde{R}\|_{\mathcal{L}(\tilde{V})}}{d - \delta} \|(I - P_h^c) P\|_{\mathcal{L}(\tilde{V})}.$$

□

Combining the results of Theorem 5 and of Proposition 6 we have the following result

$$\frac{\lambda_i - \lambda_{j,h}}{\lambda_i} \le \left(1 + \frac{\|(I - P_h^c) T_h P_{1,\dots,i-1,h}^c\|_{\mathcal{L}(\tilde{V})}}{d - \delta}\right) \|(I - P_h^c) P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})}^2, \tag{30}$$

from which we deduce the following a posteriori estimate involving the indicators introduced in (22)

Theorem 7. *Let us assume the same hypotheses as in Theorem 5 and Proposition 6. Then, for h small enough, we have*

$$\frac{\lambda_i - \lambda_{j,h}}{\lambda_i} \le C\delta^2(E_{i,\dots,j,h}, V_h^c) \le C \sum_{k=i}^{j} \frac{1}{\lambda_{k,h}^2} \mu_k^2.$$

Proof. The quotient within the parentheses in (30) tends to zero as h tends to zero, hence it is bounded. On the other hand, from (16) we have that

$$\|(I - P_h^c) P_{i,\dots,j,h}\|_{\mathcal{L}(\tilde{V})}^2 = \delta^2(E_{i,\dots,j,h}, V_h^c).$$

Thanks to (7), $u_{k,h}$ for $k = i, \dots, j$ form an orthogonal basis for $E_{i,\dots,j,h}$, so that (see, e.g. [10, Cor. 2.2])

$$\delta^2(E_{i,\dots,j,h}, V_h^c) \le \sum_{k=i}^{j} \delta^2(E_{k,h}, V_h^c).$$

Applying Lemma 3 we arrive to the desired estimate. □

Let us now consider the case of discrete nonconforming eigenvalues approximating the continuous ones from above. We estimate first the distance between an eigenvalue λ_i of multiplicity q and the average of the discrete eigenvalues $\lambda_{k,h}$ for $k = i, \dots, j$ (here $j = i, \dots, i + q - 1$).

Lemma 8. *Let λ_i be an eigenvalue with multiplicity q, so that*

$$\lambda_{i-1} < \lambda_i = \cdots = \lambda_{i+q-1} < \lambda_{i+q},$$

and let $\lambda_{i,h} \le \cdots \le \lambda_{i+q-1,h}$ be q discrete eigenvalues converging to λ_i. We assume that $\lambda_{j,h} \ge \lambda_i$ for $j = i, \dots, i + q - 1$, then for all $j = i, \dots, i + q - 1$

$$\frac{1}{J}\sum_{k=i}^{j}\lambda_{k,h} - \lambda_i \le \frac{1}{J}\sum_{k=i}^{j}\left(6\mu_k^2 + 4C_1\eta_k^2 + 4|h.o.t.|^2\right) + |h.o.t.|_2 \tag{31}$$

where

$$|h.o.t.|_2 = \frac{C}{J}\sum_{k=i}^{j}\sum_{m=1}^{k-i} h^{2m}\sum_{l=i}^{k-m} C(\mu_k + \mu_l)^2\lambda_{l,h},$$

$J = j - i + 1$ and the higher order terms $|h.o.t.|$ are defined in Th. 2.

Proof. The proof is divided into two parts. We start by estimating the error for the first discrete eigenvalue $\lambda_{i,h}$ converging to λ_i, next we shall deal with the general case.

First case. Let λ_i with $i \ge 1$ be a multiple eigenvalue with multiplicity $q \ge 1$.

It holds that $\lambda_i \le \lambda_{i,h} \le \lambda_{i,h}^c$. The first inequality holds by assumption and the second one is due to the min-max principle for the eigenvalues and the fact that $V_h^c \subset V_h^{nc}$. Let $u_{i,h} \in V_h$ be the eigensolution associated with $\lambda_{i,h}$ and $u_i(h) \in E_{i,\dots,i+q-1}$ be such that $\|u_i(h)\|_0 = 1$ and satisfies (24) and (25), then, for all $v \in (P_h^c - P_{1,\dots,i-1,h}^c)E_{i,h}$ with $\|v\|_0 = 1$, we have

$$\lambda_i + \lambda_{i,h} \le \|\nabla u_i(h)\|_0^2 + \mathcal{R}(v) = \|\nabla u_i(h)\|_0^2 + \|\nabla v\|_0^2.$$

Hence

$$\begin{aligned}
\lambda_i + \lambda_{i,h} &\le \|\nabla u_i(h)\|_0^2 + \|\nabla v\|_0^2 \\
&= \|\nabla(u_i(h) - v)\|_0^2 + 2\int_\Omega \nabla u_i(h)\nabla v\,dx \\
&= \|\nabla(u_i(h) - v)\|_0^2 + 2\lambda_i\int_\Omega u_i(h)v\,dx \\
&= \|\nabla(u_i(h) - v)\|_0^2 - \lambda_i\|u_i(h) - v\|_0^2 + 2\lambda_i
\end{aligned}$$

and

$$\lambda_{i,h} - \lambda_i \le \|\nabla(u_i(h) - v)\|_0^2 \le 2\|\nabla_h(u_i(h) - u_{i,h})\|_0^2 + 2\|\nabla_h(u_{i,h} - v)\|_0^2.$$

The first term on the right hand side can be estimated with (25). Since v is arbitary in $(P_h^c - P_{1,\dots,i-1,h}^c)E_{i,h}$, we can set $v = \tilde{u}_{i,h}$ and we obtain

$$\lambda_{i,h} - \lambda_i \le 4\mu_i^2 + C_1\eta_i^2 + |h.o.t.|^2.$$

Second case. Due to the min-max characterization for the eigenvalues we have that for $k = i, \dots, j$

$$\lambda_i \le \lambda_{k,h} \le \lambda_{k,h}^c.$$

Let us take J arbitrary elements $v_1, \dots, v_J$ of V_h^c belonging to $(E_{1,\dots,i-1,h}^c)^\perp$ which are orthogonal to each other, that is $(v_l, v_m) = 0$ for $l \neq m$.

We show that

$$\sum_{k=i}^{j} \lambda_{k,h}^c \leq \sum_{l=1}^{J} \mathcal{R}(v_l), \tag{32}$$

where $\mathcal{R}(v_l)$ denotes the Rayleigh quotient (see (14)). For simplicity, we show the result under the normalization $\|v_l\|_0 = 1$ ($l = 1, \dots, J$), so that $\mathcal{R}(v_l) = a(v_l, v_l)$ (this does not limit the generality of our proof). Let us write v_l using the following representation:

$$v_l = \sum_{k=i}^{j} \alpha_{lk} u_{k,h}^c + w_l.$$

It turns out that

$$\mathcal{R}(v_l) = a(v_l, v_l) \geq \sum_{k=i}^{j} \alpha_{lk}^2 \lambda_{k,h}^c + \left(1 - \sum_{k=i}^{j} \alpha_{lk}^2\right) \lambda_{j+1,h}^c,$$

so that

$$\sum_{l=1}^{J} \mathcal{R}(v_l) \geq \sum_{k=i}^{j} \lambda_{k,h}^c + \sum_{k=i}^{j} (\lambda_{j+1,h}^c - \lambda_{k,h}^c) \left(1 - \sum_{l=1}^{J} \alpha_{lk}^2\right).$$

Hence, the estimate (32) is proved if we can show that

$$\sum_{l=1}^{J} \alpha_{lk}^2 \leq 1 \quad \forall k.$$

This follows from the orthogonality of the v_l's. Indeed, for all k,

$$\sum_{l=1}^{J} \alpha_{lk}^2 = \Big(\sum_{l=1}^{J} \alpha_{lk} v_l, u_{k,h}^c \Big) \leq \Big\| \sum_{l=1}^{J} \alpha_{lk} v_l \Big\|_0 = \sqrt{\sum_{l=1}^{J} \alpha_{lk}^2}.$$

As in the proof of the first case, for $k = i, \dots, j$, let $u_k(h) \in E_{i,\dots,i+q-1}$ with $\|u_k(h)\|_0 = 1$ be such that (25) holds true. Then we have

$$\begin{aligned} J\lambda_i + \sum_{k=i}^{j} \lambda_{k,h} &\leq \sum_{k=i}^{j} \left(\|\nabla u_k(h)\|_0^2 + \|\nabla v_{k-i+1}\|_0^2 \right) \\ &= \sum_{k=i}^{j} \left(\|\nabla (u_k(h) - v_{k-i+1})\|_0^2 - \lambda_i \|u_k(h) - v_{k-i+1}\|_0^2 \right) + 2J\lambda_i \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{J} \sum_{k=i}^{j} \lambda_{k,h} - \lambda_i &\leq \frac{1}{J} \sum_{k=i}^{j} \|\nabla (u_k(h) - v_{k-i+1})\|_0^2 \\ &\leq \frac{2}{J} \sum_{k=i}^{j} \left(\|\nabla_h (u_k(h) - u_{k,h})\|_0^2 + \|\nabla_h (u_{k,h} - v_{k-i+1})\|_0^2 \right) \end{aligned} \tag{33}$$

The first term in the sum appearing in the last line of (33) can be bounded by (25), while for the remaining term we need to define v_l, $l = 1, \dots, J$.

For $k = i, \dots, j$ let $\tilde{u}_{k,h} \in V_h^c$ be obtained by averaging the values of $u_{k,h}$ at the vertices of the triangulation as described in (19). Then we choose

$$v_1 = \tilde{u}_{i,h}$$
$$v_l = \tilde{u}_{i+l-1,h} - \sum_{m=1}^{l-1} \frac{(\tilde{u}_{i+l-1,h}, v_m)}{\|v_m\|_0^2} v_m, \qquad l = 2, \dots, J. \tag{34}$$

By construction, we have that for h small enough

$$\frac{1}{2} \le \|v_l\|_0 \le C. \tag{35}$$

Let's start with $j = i$ in the last line of (33). By definition of the estimator (22) we have

$$\|\nabla_h (u_{i,h} - v_1)\|_0^2 = \mu_i^2. \tag{36}$$

Moreover, from Lemma 1

$$\|\nabla v_1\|_0 = \|\nabla \tilde{u}_{i,h}\|_0 \le C \|\nabla_h u_{i,h}\|_0 = \lambda_{i,h}^{1/2}. \tag{37}$$

For $k = i+1, \dots, j$, set $l = k - i + 1$, so that we have

$$\|\nabla_h (u_{k,h} - v_l)\|_0^2 \le \|\nabla_h (u_{k,h} - \tilde{u}_{k,h})\|_0^2 + \sum_{m=1}^{l-1} \frac{|(\tilde{u}_{k,h}, v_m)|^2}{\|v_m\|_0^4} \|\nabla v_m\|_0^2. \tag{38}$$

We detail the cases when $k = i+1$ and $k = i+2$; the general situation should then be clear.

If $k = i+1$, then $l = 2$, so that the last term in (38) can be estimated using (35) and (20) as follows:

$$\begin{aligned} |(\tilde{u}_{i+1,h}, v_1)| &\le |(\tilde{u}_{i+1,h} - u_{i+1,h}, v_1)| + |(u_{i+1,h}, v_1 - u_{i,h})| \\ &\le \|\tilde{u}_{i+1,h} - u_{i+1,h}\|_0 \|v_1\|_0 + \|u_{i+1,h}\|_0 \|\tilde{u}_{i,h} - u_{i,h}\|_0 \\ &\le Ch(\mu_{i+1} + \mu_i). \end{aligned} \tag{39}$$

Hence we obtain

$$\begin{aligned} \|\nabla_h (u_{i+1,h} - v_2)\|_0^2 &\le \|\nabla_h (u_{i+1,h} - \tilde{u}_{i+1,h})\|_0^2 + Ch^2(\mu_{i+1}^2 + \mu_i^2)\lambda_{i,h} \\ &= \mu_{i+1}^2 + Ch^2(\mu_{i+1}^2 + \mu_i^2)\lambda_{i,h} \end{aligned}$$

and

$$\|\nabla v_2\|_0 \le \|\nabla \tilde{u}_{i+1,h}\|_0 + \frac{|(\tilde{u}_{i+1,h}, v_1)|}{\|v_1\|_0^2} \|\nabla v_1\|_0 \le C\lambda_{i+1,h}^{1/2} + Ch(\mu_{i+1} + \mu_i)\lambda_{i,h}^{1/2}. \tag{40}$$

If $k = i+2$, then $l = 3$ and we have two terms in the sum on the right hand side of (38), that is $(\tilde{u}_{i+2,h}, v_1)$ and $(\tilde{u}_{i+2,h}, v_2)$. For the first term, working as in (39), we obtain

$$\begin{aligned} |(\tilde{u}_{i+2,h}, v_1)| &\le C\|\tilde{u}_{i+2,h} - u_{i+2,h}\|_0 \|v_1\|_0 + \|u_{i+2,h}\|_0 \|\tilde{u}_{i,h} - u_{i,h}\|_0 \\ &\le Ch(\mu_{i+2} + \mu_i). \end{aligned} \tag{41}$$

Next, from the definition of v_1 and v_2 we have using also (20) and (39)

$$\begin{aligned} |(\tilde{u}_{i+2,h}, v_2)| &\le |(\tilde{u}_{i+2,h} - u_{i+2,h}, v_2)| + |(u_{i+2,h}, \tilde{u}_{i+1,h} - u_{i+1,h})| \\ &\quad + \frac{|(\tilde{u}_{i+1,h}, v_1)|}{\|v_1\|_0^2} |(u_{i+2,h}, \tilde{u}_{i,h} - u_{i,h})| \\ &\le Ch(\mu_{i+2} + \mu_{i+1}) + Ch^2(\mu_{i+1} + \mu_i)\mu_i. \end{aligned} \tag{42}$$

Therefore, putting things together, we get

$$\|\nabla_h(u_{i+2,h}-v_3)\|_0^2 \le \mu_{i+2}^2 + Ch^2\sum_{l=i}^{i+1}(\mu_{i+2}+\mu_l)^2\lambda_{l,h} + Ch^4(\mu_{i+1}+\mu_i)^2\mu_i^2\lambda_{i+1,h}.$$

From this estimate it is straightforward to obtain the estimate of $\|\nabla v_3\|_0$, and so on. □

It is now easy to obtain the desired a posteriori error estimate for the eigenvalues approaching the continuous one from above.

Theorem 9. *Under the same hypotheses as in Lemma 8 the following bound holds true for $j = i, \dots, i+q-1$*

$$\frac{\lambda_{j,h}-\lambda_i}{\lambda_i} \le \frac{1}{\lambda_i}\left(\frac{1}{J}\sum_{k=i}^{j}\left(6\mu_k^2 + 4C_1\eta_k^2 + 4|h.o.t.|^2\right) + |h.o.t.|_2 + \frac{1}{J}\sum_{k=i}^{j-1}(\lambda_{j,h}-\lambda_{k,h})\right).$$

Proof. The proof is straighforward since

$$\lambda_{j,h}-\lambda_i = \frac{1}{J}\sum_{k=i}^{j-1}(\lambda_{j,h}-\lambda_{k,h}) + \frac{1}{J}\sum_{k=i}^{j}\lambda_{k,h}-\lambda_i,$$

which combined with (31) gives the result.

□

5. Numerical results

In this section we present some numerical results obtained applying the estimator discussed in this paper.

The aim of our tests is not to show that the reliability and the efficiency of the error indicator, since results in this direction have been already reported in [5]. The emphasis of our computations will be put on the approximation of multiple eigenvalues. There are few papers concerning adaptivity for the numerical approximation of multiple eigenvalues, among those we recall in particular [12], [4], and [7]. A consequence of the results contained in these two papers is, in particular, that an adaptive algorithm aiming at a refinement procedure for the approximation of a multiple eigenvalue, needs to take into account indicators belonging to all the discrete eigenfunctions approximating the continuous eigenspace.

In all our numerical tests the discrete eigenvalues are lower bounds for the continuous ones (as it is mostly often the case with Crouzeix–Raviart element). Moreover, we focus our attention on eigenvalues of multiplicity two. We consider an adaptive algorithm based on standard Dörfler marking strategy [6]. In case of double eigenvalues, we choose either to mark elements according to the error indicator based on one of the two discrete corresponding eigenfunctions, or to the sum of the indicators of the two eigenfunctions. From the discussion of the results we are going to present, it will be clear that the results of [12], [4], and [7] are confirmed. A more comprehensive and detailed study of the numerical tests will be included in a forthcoming paper.

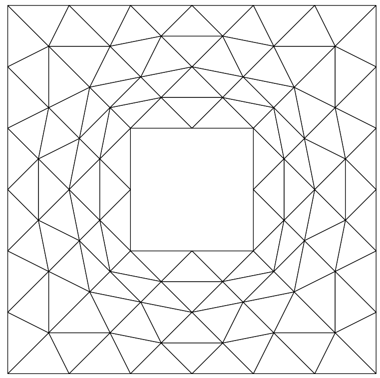

FIGURE 1. The initial mesh for the square ring test case

5.1. **The square ring domain.** Let Ω be the domain obtaining by subtracting the square $(1/3, 2/3)^2$ from the square $(0, 1)^2$. This is the same domain considered in [12, Sect. 2].

It turns out that λ_2 is an eigenvalue with multiplicity two. More precisely, the two-dimensional eigenspace corresponding to $\lambda_2 = \lambda_3$ can be generated by two singular eigenfunctions: each of them has a singularity at two opposite reentrant corners of Ω.

We start by using a refinement strategy based on the discrete eigenfunction $u_{2,h}$ (notice that, due to the fact that the discrete eigenvalues are lower bounds, this corresponds to the worstly approximating eigenvalue). The initial non-structured mesh is shown in Figure 1. The plot of the discrete eigenvalues is reported in Figure 2.

It is clear that the phenomenon already explained in [12] is present. The two discrete eigenvalues apparently change their position as the mesh is refined. For example, on the first mesh the two values are equal since the mesh is symmetric. After the first refinement they separate and, as effect of the second refinement, the approximation provided by $\lambda_{2,h}$ improves, so that after one more refinement the role of $\lambda_{2,h}$ and of $\lambda_{3,h}$ is exchanged. This behavior is confirmed by the plots of the corresponding eigenfunctions which is reported in Figure 3. It is interesting to look at the underlying mesh: in the second plot, for instance, it can be seen that the mesh has been refined in the top left and bottom right region, according to the eigenfunction computed on the initial mesh; however the eigenfunction corresponding to the second mesh calls for a refinement in the other two corners.

In order to better highlight this phenomenon, in Figure 4 we report a sequence of meshes obtained after eight level of refinements. It is clear that the method refines a region close to all four reentrant corners even if we are constructing our indicator only according to the second discrete eigenfunction. Moreover, the refinement strategy is not optimal since at each step only two of the four involved regions are considered.

As a second test, we consider an adaptive strategy driven by the approximation of the eigenfunction $u_{3,h}$ corresponding to the third eigenvalue. A plot of the values

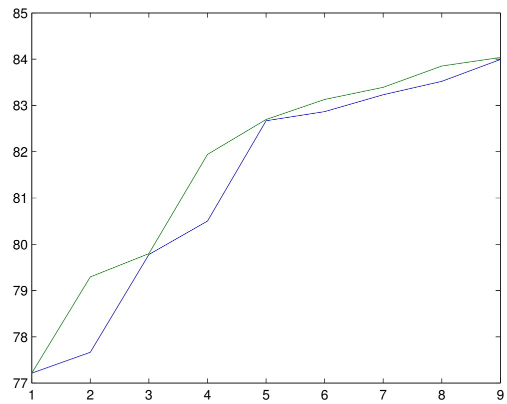

FIGURE 2. The values of the discrete eigenvalues $\lambda_{2,h}$ and $\lambda_{3,h}$ on the square ring for different refinement levels (non structured initial mesh, refined based on $u_{2,h}$)

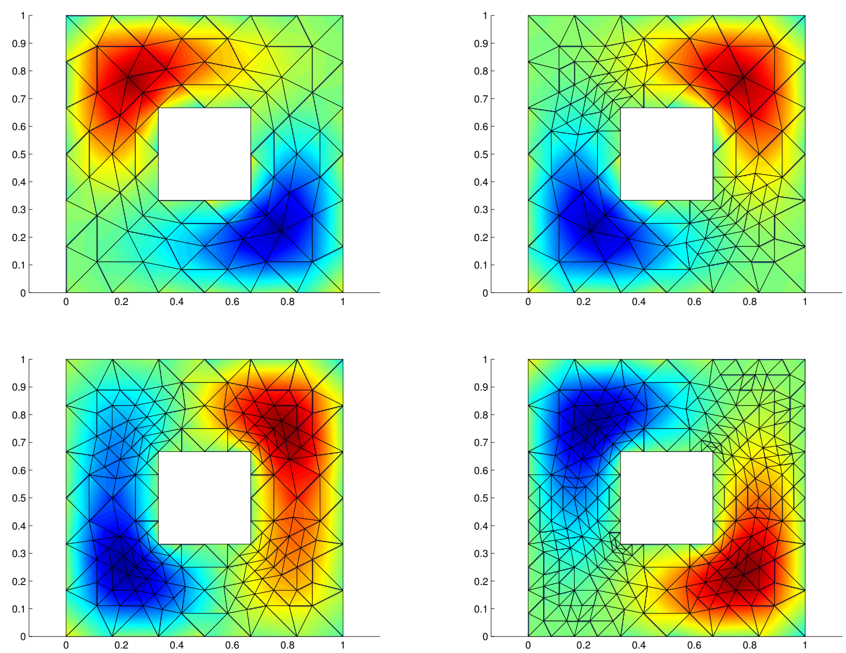

FIGURE 3. The eigenfunctions corresponding to $\lambda_{2,h}$ on the square ring for the first four refinement levels (non structured initial mesh, refined based on $u_{2,h}$)

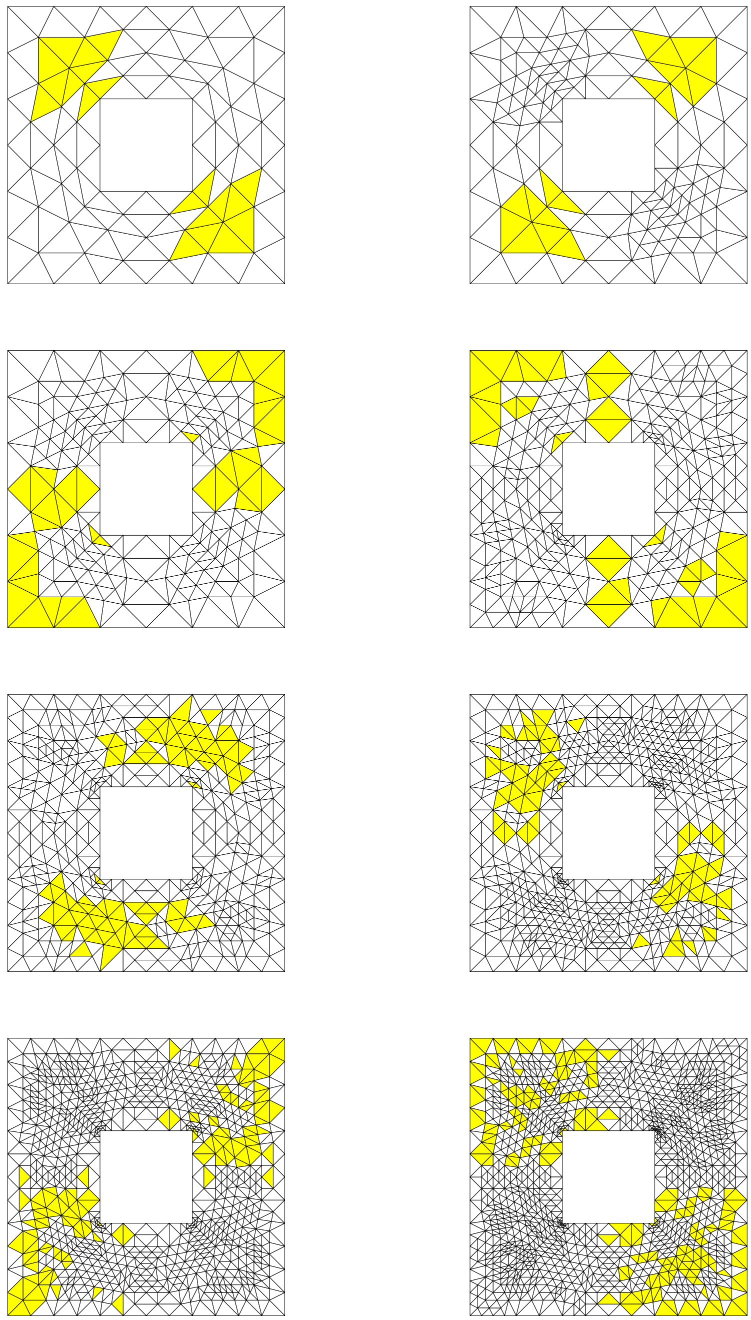

FIGURE 4. Eight level of refinements of the square ring domain: indicator based on $u_{2,h}$

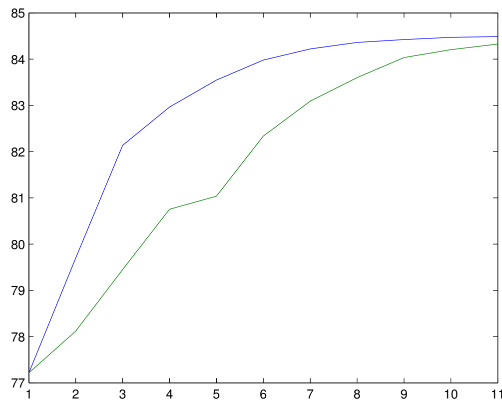

FIGURE 5. The values of the discrete eigenvalues $\lambda_{2,h}$ and $\lambda_{3,h}$ on the square ring for different refinement levels (non structured initial mesh, refined based on $u_{3,h}$)

of $\lambda_{2,h}$ and $\lambda_{3,h}$ is reported in Figure 5. It is clear that in this case the adaptive procedure is effective in pushing the convergence of $\lambda_{3,h}$, while $\lambda_{2,h}$ is converging more slowly.

The corresponding meshes are reported in Figure 6, where it can be seen that the refinements is always performed in a neighborhood of the same two corners.

The eigenfunctions on the first four meshes are plotted in Figure 7: it can be seen that they always have singularities about the top right and bottom left reentrant corners.

To conclude this section, we repeat the same computation with an error indicator based on both singular eigenfunctions $u_{2,h}$ and $u_{3,h}$. The plot of the eigenvalues is reported in Figure 8 where it can be observed that now the two discrete values approximating the double eigenvalue $\lambda_2 = \lambda_3$ are almost superimposed.

For completeness, we report in Figure 9 the eigenfunctions corresponding to $\lambda_{2,h}$ and $\lambda_{3,h}$ after three refinements and in Figure 10 the sequence of mesh obtained after eight level of refinements.

The convergence history of the adaptive algorithm is shown in Figure 11. It can be seen that the procedure is performing optimally with respect to the degrees of freedom.

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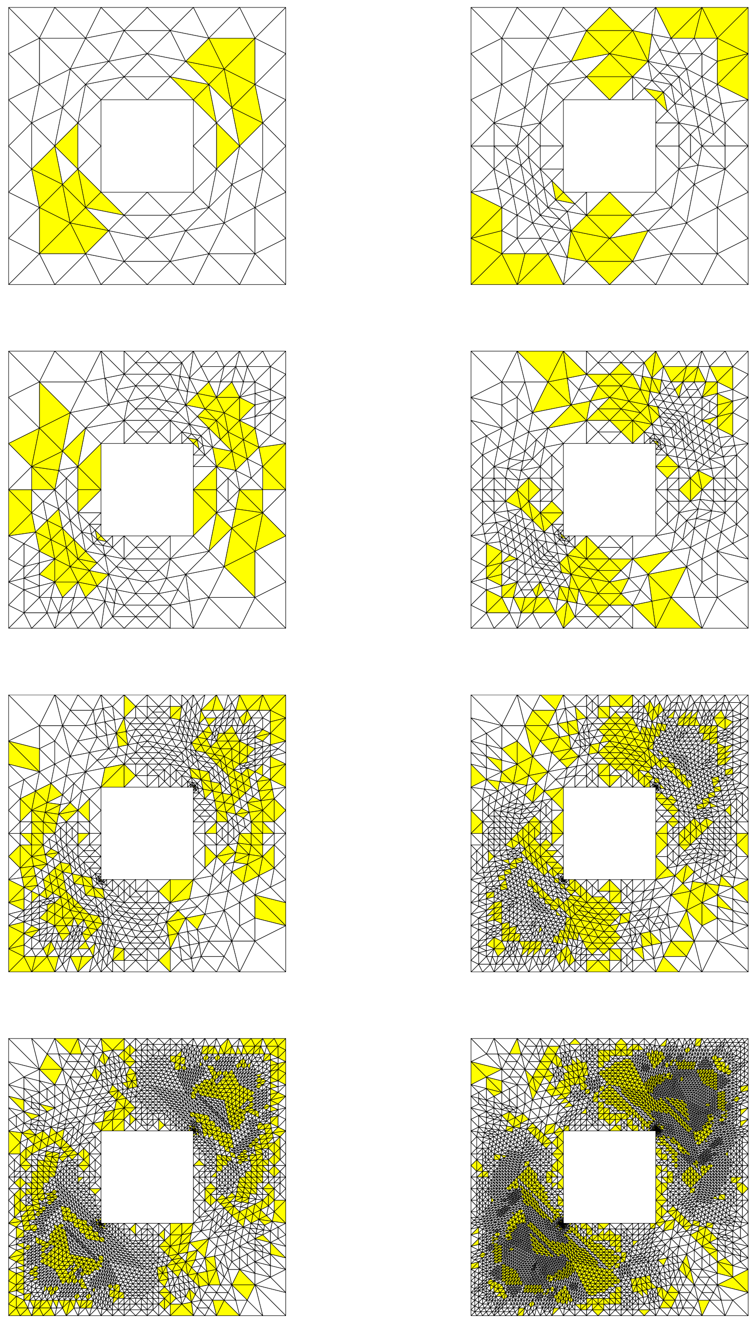

FIGURE 6. Eight level of refinements of the square ring domain: indicator based on $u_{3,h}$

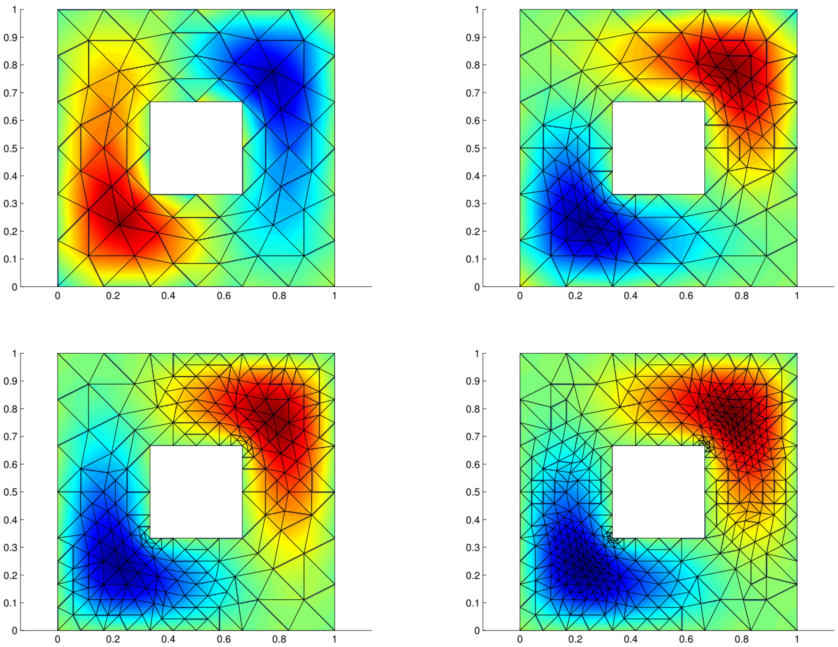

FIGURE 7. The eigenfunctions corresponding to $\lambda_{3,h}$ on the square ring for the first four refinement levels (non structured initial mesh, refined based on $u_{3,h}$)

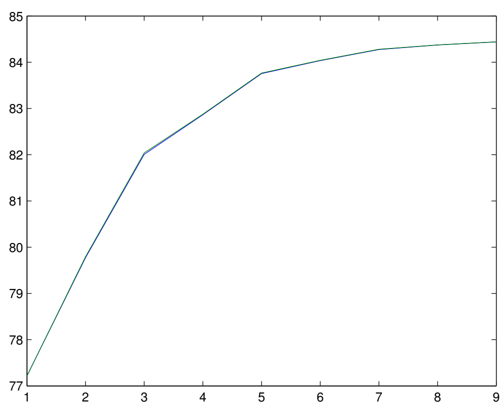

FIGURE 8. The values of the discrete eigenvalues $\lambda_{2,h}$ and $\lambda_{3,h}$ on the square ring for different refinement levels (non structured initial mesh, refined based on both $u_{2,h}$ and $u_{3,h}$)

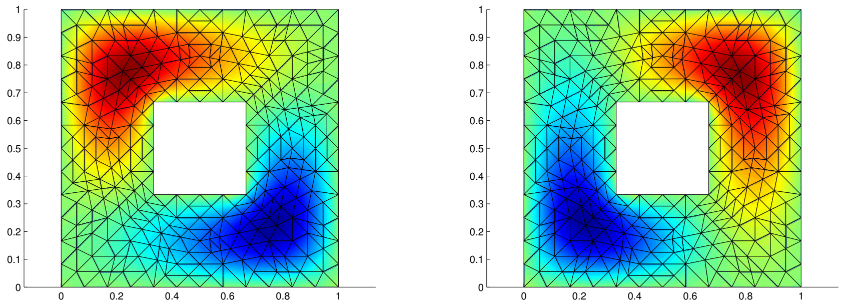

FIGURE 9. Eigenfunctions corresponding to $\lambda_{2,h}$ and $\lambda_{3,h}$ after three levels of refinements: indicator based on $u_{2,h}$ and $u_{3,h}$

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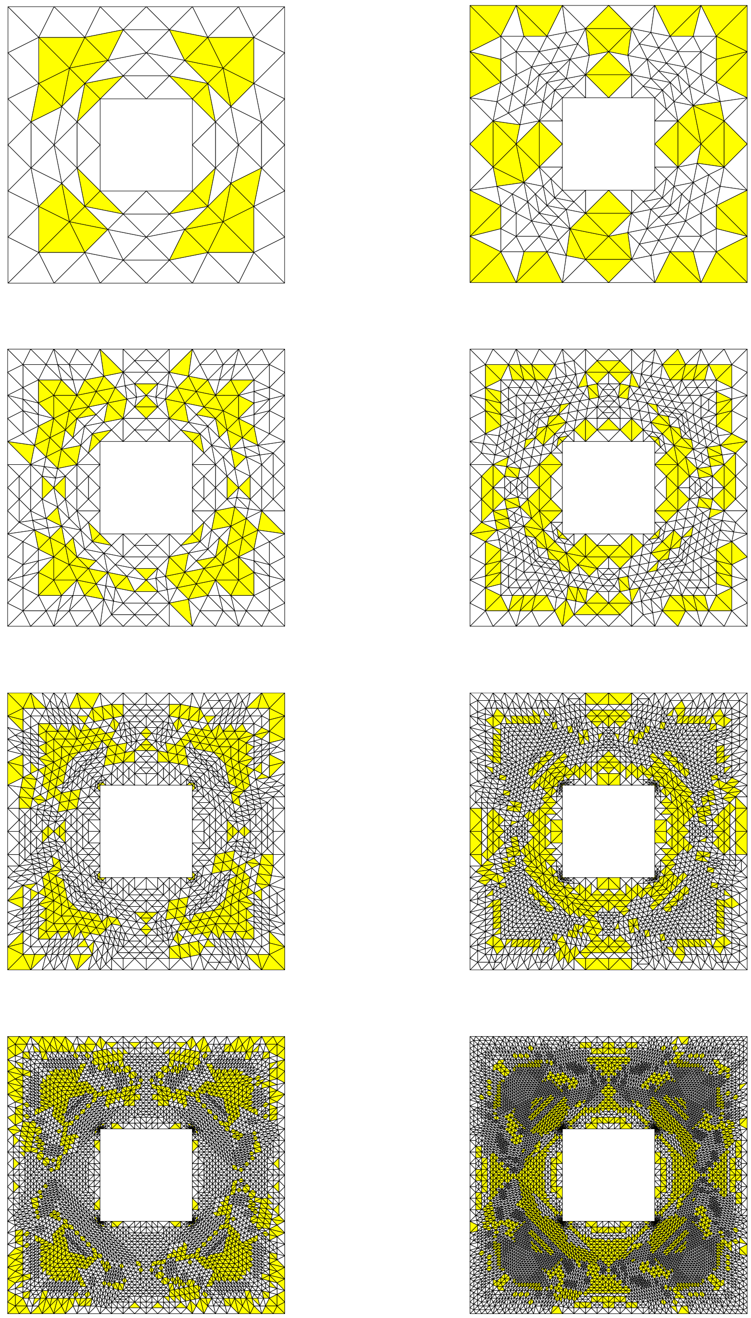
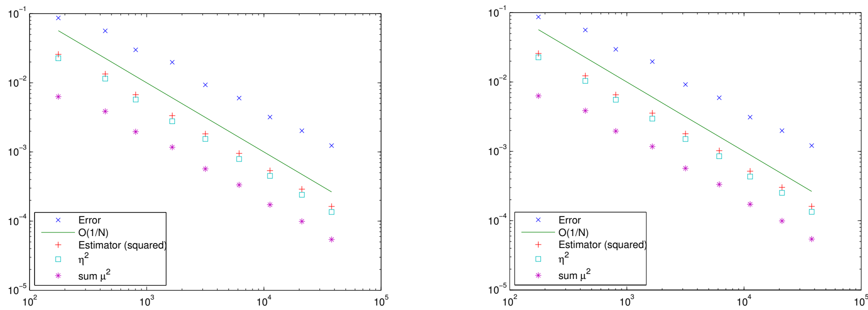


FIGURE 10. Eight level of refinements of the square ring domain: indicator based on $u_{2,h}$ and $u_{3,h}$



(a) $\lambda_{2,h}$ (b) $\lambda_{3,h}$

FIGURE 11. Convergence history of the adaptive procedure: indicator based on $u_{2,h}$ and $u_{3,h}$