A remark on spurious eigenvalues in a square

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Abstract

We study the finite element approximation of the eigensolutions of a second order problem in a square arising in fluid-structure interaction. We analyze the schemes described in [5] and show that, when a uniform mesh is used, the first two methods produce spurious eigenmodes of different nature, while the third one provides eigensolutions converging to the continuous ones.

1 Introduction

Let Ω be the square with side π and let us consider the eigenvalue problem:

$$\begin{aligned}&\text{find } (\underline{\sigma},\lambda) \text{ in } H_0(\mathrm{div};\Omega)\times\mathbb{R} \text{ such that } \underline{\sigma}\neq 0 \text{ and}\\ &(\mathrm{div}\,\underline{\sigma},\mathrm{div}\,\underline{\tau})=\lambda(\underline{\sigma},\underline{\tau})\quad \forall\underline{\tau}\in H_0(\mathrm{div};\Omega)\end{aligned}\tag{1}$$

where

$$H_0(\mathrm{div};\Omega)=\{\underline{\tau}\in[L^2(\Omega)]^2:\ \mathrm{div}\,\underline{\tau}\in L^2(\Omega),\ \underline{\tau}\cdot\underline{n}=0 \text{ on } \partial\Omega\}.\tag{2}$$

Since the bilinear forms in (1) are symmetric, the eigenvalues are real numbers.

Let Σ_h be a sequence of finite dimensional subspaces of $H_0(\mathrm{div};\Omega)$, then the discretization of problem (1) reads:

$$\begin{aligned}&\text{find } (\underline{\sigma}_h,\lambda_h) \text{ in } \Sigma_h\times\mathbb{R} \text{ such that } \underline{\sigma}_h\neq 0 \text{ and}\\ &(\mathrm{div}\,\underline{\sigma}_h,\mathrm{div}\,\underline{\tau}_h)=\lambda_h(\underline{\sigma}_h,\underline{\tau}_h)\quad \forall\underline{\tau}_h\in \Sigma_h.\end{aligned}\tag{3}$$

It is clear that in problem (1) the zero frequency $\lambda=0$ is associated to the infinitely many eigenfunctions which satisfy the condition $\mathrm{div}\,\underline{\sigma}=0$. Therefore the corresponding modes are purely circulating: $\underline{\sigma}=\mathrm{curl}\,\psi$ for some $\psi\in H^1_0(\Omega)$. On the other hand, the eigenvalues corresponding to eigensolutions with non vanishing divergence are strictly positive, moreover the strong formulation of (1) enforces the following constraint $\mathrm{rot}\,\underline{\sigma}=0$.

Since Ω is a square, by separation of variables, we obtain that the exact eigenpairs with non-zero frequency of problem (1) are given by

$$\begin{aligned}&\lambda^{mn}=m^2+n^2, \text{ for } m,n\in\mathbb{N},\ m+n\neq 0\\ &\underline{\sigma}^{mn}(x,y)=(m\sin(mx)\cos(ny),n\cos(mx)\sin(ny)).\end{aligned}\tag{4}$$

The presence in the spectrum of the zero frequency gives some difficulties in the approximation of the eigenvalue problem; in fact a bad choice of the finite element space Σ_h can produce very poor results. The zero frequency can be badly approximated by positive discrete eigenvalues which are interspersed among the good approximations of the positive ones. Hence the circulation modes may have zero frequencies as well as non-zero frequency in the discretized problem.

Recently, it has been shown that, even if the zero modes are well-approximated, more insidious spurious eigenvalues may pollute the spectrum (see [3, 2]).

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The problem introduced in equation (1) has been object of intensive study in different application fields, as for example fluid-structure interactions or, in a slightly different formulation, electromagnetics. See for instance [1, 5, 3].

In particular Chen and Taylor, in [5], deal with three different discretizations based on piecewise bilinear elements, pointing out that in the first two cases spurious modes appear, while the third one gives good results. Here we show that the spurious modes of the first method are of different nature than those of the second one and prove that the third method gives good approximation of the continuous eigenvalues. These results are obtained by exhibiting the analytical expressions for discrete eigenvalues and eigenvectors and computing their limit as h tends to 0.

In Section 3 we show the relation between problem (1) and the well-known mixed formulation for Laplace equation (see [4, 3]). A deeper analysis of that problem is contained in [2].

2 Approximation schemes: analysis and spurious modes

Here we present some results relative to different finite element methods, all based on the use of the same finite element space. Let us subdivide the square Ω into N^2 squares of side $h = \frac{\pi}{N}$. For each square Q of the decomposition of Ω, let us denote by $Q_1(Q)$, the set of bilinear polynomials on Q. Then we set

$$\Sigma_h = \{\underline{\tau} \in H^1(\Omega) |\ \underline{\tau}_{|Q} \in Q_1(Q)\}. \tag{5}$$

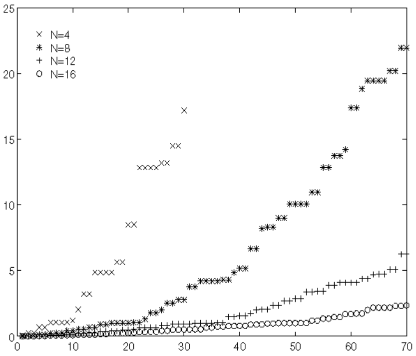

Fig.1 $Q1$ approximation.

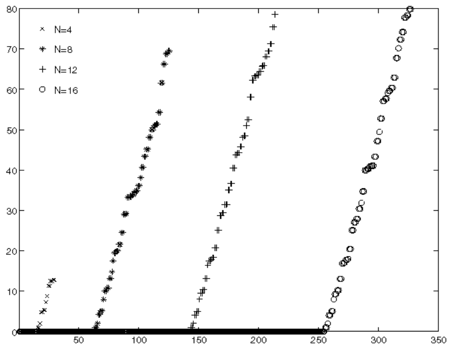

Fig.2 $ProjQ1$ approximation.

In the following we shall refer to this scheme as $Q1$ approximation. In Fig. 1 the first seventy eigenvalues obtained, with different meshes ($N = 4, 8, 12, 16$), are plotted. As we said before, it is impossible to see the difference between the good approximations of the positive eigenvalues and the bad approximations of the zero frequency.

To suppress the circulation modes which have non-zero frequency, one can use reduced integration to compute the stiffness matrix.

Let us introduce the subspace of $L^2(\Omega)$ of piecewise constants:

$$V_h = \{v \in L^2(\Omega) :\ v_{|Q} \in P_0(Q)\}, \tag{6}$$

where $P_0(Q)$ is the set of constant functions on Q. Let $I\!P$ be the L^2 projection operator onto V_h, that is, for $u \in L^2(\Omega)$, $I\!Pu \in V_h$ and $(u - I\!Pu, v) = 0$ for all $v \in V_h$.

Then we consider the following modification of the discrete problem (3) which we shall denote by $ProjQ1$ approximation:

$$\begin{array}{l} \text{find } (\underline{\sigma}_h, \lambda_h) \text{ in } \Sigma_h \times \mathbb{R} \text{ such that } \underline{\sigma}_h \neq 0 \text{ and} \\ (I\!P \operatorname{div} \underline{\sigma}_h, I\!P \operatorname{div} \underline{\tau}_h) = \lambda_h (\underline{\sigma}_h, \underline{\tau}_h) \quad \forall \underline{\tau}_h \in \Sigma_h. \end{array} \tag{7}$$

This discretization of problem (1) gives good results concerning the approximation of the circulation modes, in fact these modes appear all with zero frequency. However, the result is not completely satisfactory because other "spurious" eigenvalues appear in the discrete spectrum as it is shown in Table 1. Between the modes $(0, 4)$ and $(4, 2)$ a spurious mode appears besides the

Table 1: Computed eigenvalues of problem (7)

mode	exact	*ProjQ1* 16×16	*ProjQ1* 24×24	*ProjQ1* 32×32	*ProjQ1* 40×40	*ChT* 32×32
(1,0)	1.00000	1.00322	1.00143	1.00080	1.00051	1.00080
(0,1)	1.00000	1.00322	1.00143	1.00080	1.00051	1.00080
(1,1)	2.00000	1.99997	1.99999	2.00000	2.00000	2.00000
(2,0)	4.00000	4.05166	4.02290	4.01287	4.00823	4.01287
(0,2)	4.00000	4.05166	4.02290	4.01287	4.00823	4.01287
(2,1)	5.00000	5.02876	5.01282	5.00722	5.00462	5.00723
(1,2)	5.00000	5.02876	5.01282	5.00722	5.00462	5.00723
(2,2)	8.00000	7.99785	7.99958	7.99987	7.99995	7.99995
(3,0)	9.00000	9.26313	9.11624	9.06524	9.04171	9.06524
(0,3)	9.00000	9.26313	9.11624	9.06524	9.04171	9.06524
(3,1)	10.0000	10.2066	10.0916	10.0515	10.0329	10.0515
(1,3)	10.0000	10.2066	10.0916	10.0515	10.0329	10.0515
(3,2)	13.0000	13.0736	13.0344	13.0197	13.0127	13.0200
(2,3)	13.0000	13.0736	13.0344	13.0197	13.0127	13.0200
(4,0)	16.0000	16.8382	16.3688	16.2067	16.1320	16.2067
(0,4)	16.0000	16.8382	16.3688	16.2067	16.1320	16.2067
		17.1062				
(4,1)	17.0000	17.7329	17.3234	17.1814	17.1159	17.1816
(1,4)	17.0000	17.7329	17.3234	17.1814	17.1159	17.1816
			17.5951	17.7707	17.8528	
(3,3)	18.0000	17.9749	17.9952	17.9985	17.9994	17.9994
(4,2)	20.0000	20.4515	20.2036	20.1151	20.0738	20.1159
(2,4)	20.0000	20.4515	20.2036	20.1151	20.0738	20.1159
d.o.f.		510	1150	2046	3198	2046
# zeros		255	575	1023	1599	1023

good approximations of the modes $(4,1)$, $(1,4)$ and $(3,3)$. Notice that this spurious mode seems to converge to the value 18, so that the mode $(3,3)$, which is simple, is approximated by two different discrete eigenvalues. In order to have a better understanding of this phenomenon we give an analytical expression of the discrete eigenvalues of problem (7) in terms of m and n.

Lemma 1 *Let λ_h^{mn} and $\underline{\sigma}_h^{mn}=(u^{mn},v^{mn})$ be given as follows:*

$$\lambda_h^{mn}=(4/h^2)\frac{\sin^2(\frac{mh}{2})+\sin^2(\frac{nh}{2})-2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1-(2/3)(\sin^2(\frac{mh}{2})+\sin^2(\frac{nh}{2}))+(4/9)\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})} \tag{8}$$

and

$$\begin{aligned} u^{mn}(x_i,y_j)&=(\tfrac{2}{h})\cos(\tfrac{nh}{2})\sin(\tfrac{mh}{2})\sin(mx_i)\cos(ny_j)\\ v^{mn}(x_i,y_j)&=(\tfrac{2}{h})\sin(\tfrac{nh}{2})\cos(\tfrac{mh}{2})\cos(mx_i)\sin(ny_j)\end{aligned} \tag{9}$$

where (x_i,y_j) for $1\le i,j\le N$ are the mesh points.

Then $(\lambda_h^{mn},\underline{\sigma}_h^{mn})$ for $0\le m,n\le N-1$ and $m+n\neq0$ are eigensolutions of problem (7).

Proof. We give here a brief sketch of the proof. For simplicity we drop the superscript mn and we use the following notation:

$$u_{ij}=u^{mn}(x_i,y_j),\quad v_{ij}=v^{mn}(x_i,y_j),\qquad i,j=1,\dots,N \tag{10}$$

Then the generic rows of the eigenvalue algebraic system associated to (7) read;

$$\begin{aligned} &u_{ij}-\tfrac12(u_{i+1j}-u_{ij+1}+u_{i-1j}-u_{ij-1})-\tfrac14(u_{i+1j+1}+u_{i-1j+1}+u_{i-1j-1}+u_{i+1j-1})\\ &\quad-\tfrac14(v_{i+1j+1}-v_{i-1j+1}+v_{i-1j-1}-v_{i+1j-1})=\lambda_h h^2\left[\tfrac49 u_{ij}\right.\\ &\quad\left.+\tfrac19(u_{i+1j}+u_{ij+1}+u_{i-1j}+u_{ij-1})+\tfrac1{36}(u_{i+1j+1}+u_{i-1j+1}+u_{i-1j-1}+u_{i+1j-1})\right]\end{aligned} \tag{11}$$

$$\begin{aligned} &v_{ij}+\tfrac12(v_{i+1j}-v_{ij+1}+v_{i-1j}-v_{ij-1})-\tfrac14(v_{i+1j+1}+v_{i-1j+1}+v_{i-1j-1}+v_{i+1j-1})\\ &\quad-\tfrac14(u_{i+1j+1}-u_{i-1j+1}+u_{i-1j-1}-u_{i+1j-1})=\lambda_h h^2\left[\tfrac49 v_{ij}\right.\\ &\quad\left.+\tfrac19(v_{i+1j}+v_{ij+1}+v_{i-1j}+v_{ij-1})+\tfrac1{36}(v_{i+1j+1}+v_{i-1j+1}+v_{i-1j-1}+v_{i+1j-1})\right].\end{aligned} \tag{12}$$

Substituting the expressions of the eigenvectors (9) into (11), we get the expression of the eigenvalues (8). Notice that the discrete eigenvectors (9) are essentially the interpolations of the exact eigensolutions multiplied by a properly chosen constant depending on h, m and n. $\square$

The following theorem characterizes all the eigenvalues of problem (7).

Theorem 1 *Problem (7) admits exactly N^2-1 zero eigenvalues. The remaining non vanishing eigenvalues are given by (8).*

Proof. The dimension of Σ_h is $2(N-1)^2+4(N-1)=2(N^2-1)$.

We obtain as many zero eigenvalues as the linear independent elements $\underline{\sigma}_h \in \Sigma_h$ such that $I\!\!P \operatorname{div} \underline{\sigma}_h = 0$. Hence we need to compute the dimension of the kernel

$$K = \{\underline{\sigma}_h \in \Sigma_h : \ (\operatorname{div} \underline{\sigma}_h, v_h) = 0 \ \forall v_h \in V_h\}. \tag{13}$$

Since the dimension of V_h is N^2, the range of the associated operator has dimension at most N^2. On the other hand, if v_h is constant we get that $(\operatorname{div} \underline{\sigma}_h, v_h) = 0$ for every $\underline{\sigma}_h \in \Sigma_h$ due to the boundary conditions. Hence the range is at most N^2-1 and the dimension of K is at least N^2-1.

Since we proved that problem (7) has N^2-1 nonzero eigenvalues, we have that the number of zero eigenvalues is exactly N^2-1 and the lemma is completely proved. $\square$

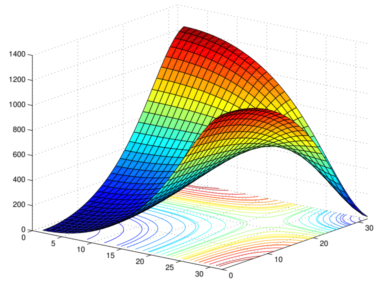

Fig.3 $ProjQ1$ eigenvalues.

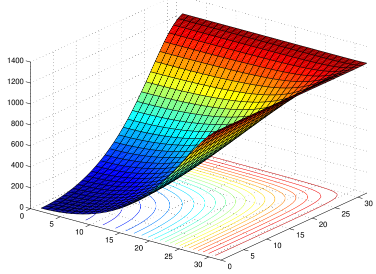

Fig.4 ChT eigenvalues.

Let m and n be fixed, then formulas (8) and (9) give

$$\lim_{h\to 0} \lambda_h^{mn} = m^2+n^2, \quad \lim_{h\to 0} \underline{\sigma}_h^{mn} = \underline{\sigma}^{mn}. \tag{14}$$

On the other hand, the surface where the eigenvalues lie (see Fig. 3) is not increasing as m and n increase. There is a saddle point when $\sin(mh/2) = \sin(nh/2) = \sqrt{3}/2$, that is for $m=n$ close to $2N/3$.

The spurious eigenvalue pointed out in Table 1 corresponds to $\lambda_h^{N-1\,N-1}$. In fact, having in mind the relation between h and N, that is $h=\pi/N$ and using (8), it is easy to evaluate the following limit:

$$\lim_{h\to 0} \lambda_h^{N-1\,N-1} = \lim_{h\to 0} \frac{4}{h^2} \frac{2\cos(h/2)(1-\cos^2(h/2))}{1-(4/3)\cos^2(h/2)+(4/9)\cos^4(h/2)} = 18. \tag{15}$$

Hence two distinct discrete eigenvalues $\lambda_h^{3\,3}$ and $\lambda_h^{N-1\,N-1}$ converge to the value 18. Exploiting the corresponding numerical eigenfunctions one can recognize that only the first one is correct. There are other eigenvalues with spurious multiplicity. Namely, for $i,j\geq 1$ fixed,

$$\lim_{h\to 0} \lambda_h^{N-i\,N-j} = 9\,\lambda^{i\,j} = \lambda^{3i\,3j}. \tag{16}$$

To avoid the spurious eigenvalues generated by the previous method, Chen and Taylor in [5] proposed a new method where also the scalar product in L^2 on the right hand side of (7) is computed by means of a reduced integration.

In order to introduce the Chen and Taylor's scheme, let us denote by $\hat{Q} = (-1,1)\times(-1,1)$ the reference square, by ϕ_i, for $i=1,\ldots,4$, the bilinear basis functions on $\hat{Q}$ such that $\phi_i(a_j)=\delta_{ij}$, a_j being the vertices of $\hat{Q}$. Analyzing the structure of the local stiffness matrix after reduced integration of problem (7), Chen and Taylor observed that it could be obtained simply suppressing in each local basis function the hourglass term xy. Moreover, $I\!P\,\mathrm{div}(xy,0)^t = 0$ and $I\!P\,\mathrm{div}(0,xy)^t = 0$. Hence such terms do not contribute to the under-integrated stiffness matrix. Therefore they decided to drop these hourglass terms also in the computation of the local mass matrix. Denoting by $(\cdot,\cdot)_h$ the scalar product obtained by this procedure the discrete problem (ChT approximation) reads:

$$\begin{array}{l}\text{find } (\underline{\sigma}_h,\lambda_h) \text{ in } \Sigma_h\times\mathbb{R} \text{ such that } \underline{\sigma}_h\neq 0 \text{ and}\\ (I\!P\,\mathrm{div}\,\underline{\sigma}_h, I\!P\,\mathrm{div}\,\underline{\tau}_h) = \lambda_h(\underline{\sigma}_h,\underline{\tau}_h)_h \quad \forall\underline{\tau}_h\in\Sigma_h.\end{array}\tag{17}$$

The eigenvalues corresponding to this approximation with a mesh of 32×32 squares are listed in the last column of Table 1. The numerical results show that no spurious eigenvalue now appears and this is confirmed by the following representation of the non zero eigenvalues of problem (17).

Lemma 2 *Let λ_h^{mn} and $\underline{\sigma}_h^{mn} = (u^{mn}, v^{mn})$ be given as follows:*

$$\lambda_h^{mn} = (4/h^2)\frac{\sin^2(\frac{mh}{2})+\sin^2(\frac{nh}{2})-2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1-(2/3)(\sin^2(\frac{mh}{2})+\sin^2(\frac{nh}{2}))+(1/3)\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}\tag{18}$$

and

$$\begin{array}{l}u^{mn}(x_i,y_j) = (\frac{2}{h})\cos(\frac{nh}{2})\sin(\frac{mh}{2})\sin(mx_i)\cos(ny_j)\\ v^{mn}(x_i,y_j) = (\frac{2}{h})\sin(\frac{nh}{2})\cos(\frac{mh}{2})\cos(mx_i)\sin(ny_j)\end{array}\tag{19}$$

where (x_i,y_j) for $1\le i,j\le N$ are the mesh points.
Then $(\lambda_h^{mn},\underline{\sigma}_h^{mn})$ for $0\le m,n\le N-1$ and $m+n\neq 0$ are eigensolutions of problem (17).

Proof. The proof is the same as that of Lemma 1. The only difference consists in the fact that the matrix in the right hand side is changed, actually the right hand side of a generic row corresponding to a test function with the second component equal to zero is:

$$\begin{array}{l}\lambda_h\frac{h^2}{4}\left[\frac{5}{3}u_{ij}+\frac{1}{2}(u_{i+1j}+u_{ij+1}+u_{i-1j}+u_{ij-1})\right.\\ \qquad\left.+\frac{1}{12}(u_{i+1j+1}+u_{i-1j+1}+u_{i-1j-1}+u_{i+1j-1})\right].\end{array}\tag{20}$$

The same holds for the other rows with v instead of u. □
The same proof as that one of Theorem 1 holds also for the Chen–Taylor element:

Theorem 2 *Problem (17) admits exactly N^2-1 zero eigenvalues. The remaining non vanishing eigenvalues are given by (18).*

The surface corresponding to λ_h^{mn} is plotted in Fig. 4. It is evident that it increases as m and n increase, as it happens for the exact eigenvalues (4). Hence the maximum value is obtained for $\lambda_h^{N-1\,N-1}$ which diverges to $+\infty$ as h goes to zero. Moreover, as in the previous case, if m and n are fixed, then $\lim_{h\to 0}\lambda_h^{mn} = m^2+n^2$ and $\lim_{h\to 0}\underline{\sigma}_h^{mn} = \underline{\sigma}^{mn}$.

This implies the following theorem.

Theorem 3 *Each nonzero eigenvalue of problem (1) with multiplicity ℓ is approximated by exactly ℓ eigenvalues of problem (17) and each compact subset of the resolvent set of (1) does not contain any discrete eigenvalues for h sufficiently small.*

3 Mixed formulation

In this section we recall that the discretizations of problem (1) introduced in Section 2 are related to a mixed finite element approximation of Laplace eigenproblem, see [3]. Hence the above results on the presence of spurious modes can be extended to the approximation of this eigenproblem by means of mixed finite elements, see [2].

Consider the following discrete eigenproblem

$$\text{find } \lambda_h \in \mathbb{R},\ (\psi_h, \chi_h \in \Phi_h \times \Xi_h \setminus \{(0,0)\} \text{ such that}$$
$$\begin{cases} a(\psi_h, \varphi_h) + (\operatorname{div} \varphi_h, \chi_h) = 0 & \forall \varphi_h \in \Phi_h \\ (\operatorname{div} \psi_h, \xi_h) = -\lambda_h(\chi_h, \xi_h) & \forall \xi_h \in \Xi_h, \end{cases} \tag{21}$$

where Φ_h and Ξ_h are finite dimensional subspaces of $H_0(\operatorname{div};\Omega)$ and $L^2(\Omega)_0 = \{\xi \in L^2(\Omega) : \int_\Omega \xi = 0\}$ respectively and $a(\cdot,\cdot)$ is a symmetric continuous bilinear form on $H_0(\operatorname{div};\Omega) \times H_0(\operatorname{div};\Omega)$.

In [3] it has been proved that, choosing $\Phi_h = \Sigma_h$, $\Xi = \operatorname{div} \Sigma_h$, the nonvanishing eigenvalues of problem (3) coincide with the eigenvalues of (21). Moreover the following relations between the corresponding eigenfunctions hold

$$\psi_h = \underline{\sigma}_h \qquad \psi = \nabla_h \chi,$$

where ∇_h is a weak discretization of the gradient operator.

The above discretizations $Q1$, $ProjQ1$ and ChT fit this framework according to the following setting

$$\begin{array}{llll} Q1 & \Phi_h = \Sigma_h & \Xi_h = \operatorname{div} \Sigma_h & a(\cdot,\cdot) = (\cdot,\cdot) \\ ProjQ1 & \Phi_h = \Sigma_h & \Xi_h = V_h & a(\cdot,\cdot) = (\cdot,\cdot) \\ ChT & \Phi_h = \Sigma_h & \Xi_h = V_h & a(\cdot,\cdot) = (\cdot,\cdot)_h. \end{array} \tag{22}$$

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