Mass-Lumping or not Mass-Lumping for eigenvalue problems

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Abstract

In this paper we analyze the effect of mass-lumping in the linear triangular finite element approximation of second order elliptic eigenvalue problems. We prove that the eigenvalue obtained by using mass-lumping is always below the one obtained with exact integration.

For singular eigenfunctions, as those arising in non convex polygons, we prove that the eigenvalue obtained with mass-lumping is above the exact eigenvalue when the mesh size is small enough. So, we conclude that the use of mass-lumping is convenient in the singular case.

When the eigenfunction is smooth several numerical experiments suggest that the eigenvalue computed with mass-lumping is below the exact one if the mesh is not too coarse.

Key words: Finite elements, eigenvalue problems, mass-lumping.

AMS subject classification: *65N25,65N30.*

1 Introduction

The object of this paper is to analyze the effect of mass-lumping in the piecewise linear finite element approximation of second order elliptic eigenvalue problems in polygonal domains.

Instead of integrating exactly the right-hand side of the weak form of the equation one can use some numerical integration. In [2] it has been proved that, for smooth eigenfunctions, the optimal order of convergence for eigenvalues and eigenfunctions is preserved if the quadrature rule is exact for polynomials of degree one. In particular, one can use the rule based on linear interpolation at the vertices of each triangle which leads to a diagonal matrix on the right-hand side of the generalized discrete eigenvalue problem. This procedure is known as mass-lumping.

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It is known that, for uniform partitions of a domain which is a union of rectangles, the finite element approximation using mass-lumping leads to the standard five point finite difference scheme. For eigenvalue problems these kinds of methods have been widely analyzed (see for example [3, 4, 8, 9]).

When the polygonal domain is not convex, the eigenfunctions are in general non-smooth, i.e., they do not belong to H^2 but only to H^{1+r}, with $r < 1$ depending on the maximum reentrant corner. It is known that, in this case, the order of convergence of finite element approximations is lower than that of the smooth case. For the five point finite difference scheme Forsythe conjectured in [3], based on numerical evidence, that for the singular case the approximate eigenvalue is larger than that of the continuous problem when the mesh-size is small enough.

We will prove that in the singular case, the eigenvalue computed using mass-lumping is larger than the eigenvalue of the continuous problem for small enough mesh-size. In particular the conjecture made by Forsythe is true.

On the other hand, we will prove that the eigenvalue computed with mass-lumping is always below the one obtained with exact integration independently of the smoothness of the eigenfunction.

So, we conclude that in the singular case it is convenient to use mass-lumping, at least for small enough mesh-size. We present several numerical experiments which show that this is true even for coarse meshes which would be a reasonable starting point for an adaptive procedure.

Although when exact integration is used the approximate eigenvalue is always above the exact one, this is not true when mass-lumping is used. For example, in the one dimensional case, when a uniform partition is used, the discrete eigenvalues can be computed explicitly and they are below the eigenvalues of the continuous problem (see [1]). In fact, an easy calculation shows that this is also true in two dimensions when a uniform partition is used in a square domain.

For some particular convex polygonal domain and smooth enough eigenfunction, Forsythe proved in [3] that the discrete eigenvalue approximation obtained by finite differences is below the eigenvalue of the continuous problem when the mesh-size is small enough. We will present several numerical examples which suggest that this is also true for eigenvalues obtained with mass-lumping in general meshes.

In view of the fact that the eigenvalue obtained with exact integration is an upper bound of the exact one it would be very interesting to prove that the mass-lumping procedure gives lower bounds.

2 The Eigenvalue Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Consider the eigenvalue problem:

$$\begin{aligned} Lu(x) &= \lambda u(x) \qquad x \in \Omega, & (2.1)\\ u(x) &= 0 \qquad x \in \partial\Omega, & (2.2) \end{aligned}$$

where

$$Lu(x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_i}u) + b(x)u,$$

with $a_{ij}(x) = a_{ji}(x)$ and $b(x) \geq 0$ are given bounded, real functions on Ω. The operator L is assumed to be uniformly strongly elliptic in Ω, i.e., there is a positive constant γ such that

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^{2} \xi_i^2$$

Let $H = H^1(\Omega)$ and $V = H_0^1(\Omega) \subset H$. We denote by $(\,,\,)$ and $\|\cdot\|_0$ the usual inner product and norm in $L^2(\Omega)$ respectively and by $\|\cdot\|_1$ the norm in $H^1(\Omega)$. The weak formulation of problem (2.1) is given by:

Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$, $u \neq 0$ satisfying

$$\begin{aligned} a(u,v) &= \lambda(u,v) \quad \forall v \in H_0^1(\Omega) \\ \|u\|_0 &= 1 \end{aligned} \tag{2.3}$$

where $a(\cdot,\cdot)$ is a symmetric bilinear form that is continuous on H and coercive on V, i.e., There exist constants M and $\alpha > 0$ such that

$$|a(u,v)| \leq M\|u\|_1\|v\|_1 \quad \forall u,v \in H$$

$$a(v,v) \geq \alpha\|v\|_1^2 \quad \forall v \in V$$

It is well known that the solution of problem (2.3) is given by a sequence of pairs (λ_j, u_j), with positive eigenvalues λ_j diverging to $+\infty$. We assume the eigenvalues to be increasingly ordered: $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$. The associated eigenfunctions satisfy $u_j \in H^{1+r}(\Omega)$, where $r = 1$ if Ω is convex and $r < \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω) otherwise.

In order to approximate the eigenvalue λ and its associated eigenfunction u we consider $\{\mathcal{T}_h\}$ a triangulation of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge. Let h stand for the mesh-size; namely $h = \max_{T\in\mathcal{T}_h} h_T$, with h_T being the diameter of the triangle T. We suppose that the family of triangulations $\mathcal{T}_h$ satisfies the usual shape regularity condition, i.e., there exists a constant $\sigma > 0$ such that $\frac{h_T}{\rho_T} \leq \sigma$, where ρ_T is the diameter of the largest ball contained in T.

We consider the usual finite element space:

$$V_h = \{v_h \in H_0^1(\Omega) : \; v_h|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}$$

($\mathcal{P}_1$ denotes the space of linear polynomials).

Then, the standard finite element approximation problem is the following:

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$, $u_h \neq 0$ such that

$$\begin{aligned} a(u_h,v_h) &= \lambda_h(u_h,v_h) \quad \forall v_h \in V_h \\ \|u_h\|_0 &= 1 \end{aligned} \tag{2.4}$$

Another possible discretization is obtained by using quadrature rule on the right-hand side of (2.4). A usual approach, known as "mass-lumping", leads to the following approximation problem:

Find $\lambda_h^{ml} \in \mathbb{R}$ and $u_h^{ml} \in V_h$, $u_h^{ml} \neq 0$ such that

$$\begin{aligned} a(u_h^{ml}, v_h) &= \lambda_h^{ml} \int_\Omega I_h(u_h^{ml} v_h) \quad \forall v_h \in V_h, \\ \|u_h^{ml}\|_0 &= 1 \end{aligned} \tag{2.5}$$

where I_h denotes the piecewise linear interpolation on the vertices of the triangulation $\mathcal{T}_h$.

Remark 2.1 *For simplicity, we assume that the left-hand sides of (2.4) and (2.5) can be exactly integrated. However, it is not difficult to see that all our results hold in more general cases if numerical quadratures of appropriate degrees of precision according to the results of [2] are used.*

The two problems above reduce to generalized eigenvalue problems involving positive definite symmetric matrices. They attain a finite number of eigenpairs $(\lambda_{j,h}, u_{j,h})$ and $(\lambda_{j,h}^{ml}, u_{j,h})$, $1 \leq j \leq N_h = \dim V_h$, respectively, with positive eigenvalues which we assume increasingly ordered: $\lambda_{1,h} \leq \cdots \leq \lambda_{N_h,h}$ and $\lambda_{1,h}^{ml} \leq \cdots \leq \lambda_{N_h,h}^{ml}$.

Our first goal is to show that the eigenvalue obtained by mass-lumping is always below the one obtained by the standard finite element approximation, i.e., $\lambda_{h,j}^{ml} \leq \lambda_{h,j}$, $1 \leq j \leq N_h = \dim V_h$.

We introduce the following notation: Let $\mathcal{E}_\mathrm{I}$ be the set of all interior edges of the mesh (i.e., edges of triangles not lying on $\partial\Omega$). For each interior edge $\ell \in \mathcal{E}_\mathrm{I}$ we denote by $p_1(\ell)$ and $p_2(\ell)$ the endpoints of the edge ℓ and $T_{1,\ell}$ and $T_{2,\ell}$ the two triangles sharing this edge and set $\Omega_l = T_{1,\ell} \cup T_{2,\ell}$. Then we have

Lemma 2.1 *For any $v_h \in V_h$,*

$$\int_\Omega \left(I_h(v_h^2) - v_h^2 \right) = \frac{1}{12} \sum_{\ell \in \mathcal{E}_\mathrm{I}} \left((v_h(p_1(\ell)) - v_h(p_2(\ell)) \right)^2 |\Omega_\ell|$$

in particular

$$\int_\Omega I_h(v_h^2) \geq \int_\Omega v_h^2$$

Proof. Since v_h is a piecewise linear function and $v_h \mid_{\partial\Omega} = 0$ we observe that

$$\int_\Omega v_h^2 = \sum_{\ell \in \mathcal{E}_\mathrm{I}} v_h^2(m_\ell) \frac{|\Omega_\ell|}{3}$$

where m_ℓ denote the midpoint of the edge ℓ. From $v_h(m_\ell) = \frac{v_h(p_1(\ell)) + v_h(p_2(\ell))}{2}$ we have that

$$\begin{aligned} \sum_{\ell \in \mathcal{E}_\mathrm{I}} v_h^2(m_\ell) \frac{|\Omega_\ell|}{3} &= \sum_{\ell \in \mathcal{E}_\mathrm{I}} \left(\frac{v_h(p_1(\ell)) + v_h(p_2(\ell))}{2} \right)^2 \frac{|\Omega_\ell|}{3} \\ &= \sum_{\ell \in \mathcal{E}_\mathrm{I}} \left(\frac{v_h(p_1(\ell))^2 + v_h(p_2(\ell))^2}{2} \right) \frac{|\Omega_\ell|}{3} - \sum_{\ell \in \mathcal{E}_\mathrm{I}} \left(\frac{v_h(p_1(\ell)) - v_h(p_2(\ell))}{2} \right)^2 \frac{|\Omega_\ell|}{3} \end{aligned} \tag{2.6}$$

It is easy to see that

$$\int_\Omega I_h(v_h^2) = \sum_{\ell\in\mathcal{E}_{\rm I}} \left(\frac{v_h(p_1(\ell))^2 + v_h(p_2(\ell))^2}{2}\right) \frac{|\Omega_\ell|}{3}$$

and therefore the Lemma holds.$\square$

As a consequence of the previous Lemma we have that

Corollary 2.1 *For any $v_h \in V_h$ there exists a constant C such that*

$$0 \le \int_\Omega \left(I_h(v_h^2) - v_h^2\right) \le Ch^2 \|\nabla v_h\|_0^2$$

Proof. Using Lemma 2.1 and the fact that $\frac{\partial v_h}{\partial \ell} = \frac{v_h(p_2(\ell)) - v_h(p_1(\ell))}{|\ell|}$ we have that

$$\begin{aligned}
\int_\Omega \left(I_h(v_h^2) - v_h^2\right) &= \frac{1}{12}\sum_{\ell\in\mathcal{E}_{\rm I}} \left((v_h(p_1(\ell)) - v_h(p_2(\ell))\right)^2 |\Omega_\ell| \\
&= \frac{1}{12}\sum_{\ell\in\mathcal{E}_{\rm I}} \left(\frac{\partial v_h}{\partial \ell}\right)^2 |\ell|^2 |\Omega_\ell|
\end{aligned}$$

The proof concludes by observing that for any triangle $T \in \mathcal{T}_h$, $\left(\frac{\partial v_h}{\partial \ell}\right)^2 \le |\nabla v_h|^2$. $\square$

Theorem 2.1 *Let $\lambda_{h,j}$ and $\lambda_{h,j}^{ml}$, $1 \le j \le N_h$ be the eigenvalues of problems (2.4) and (2.5) respectively. Then*

$$\lambda_{h,j}^{ml} \le \lambda_{h,j} \quad 1 \le j \le N_h \tag{2.7}$$

Proof. It is known that the eigenvalues can be characterized using the minimum-maximum principle (see for example [1]), i.e, for any j, $1 \le j \le N_h$ we have that

$$\lambda_{h,j} = \min_{V_{h,j}} \max_{v_h\in V_{h,j}} \frac{a(v_h, v_h)}{\int_\Omega v_h^2} \tag{2.8}$$

and

$$\lambda_{h,j}^{ml} = \min_{V_{h,j}} \max_{v_h\in V_{h,j}} \frac{a(v_h, v_h)}{\int_\Omega I_h(v_h^2)} \tag{2.9}$$

where $V_{h,j}$ denote any subspace of V_h of dimension j.

In view of Lemma 2.1 we have that

$$\frac{a(v_h, v_h)}{\int_\Omega I_h(v_h^2)} \le \frac{a(v_h, v_h)}{\int_\Omega v_h^2} \qquad \forall v_h \in V_{h,j} \tag{2.10}$$

So, for any $V_{h,j}$

$$\max_{v_h\in V_{h,j}} \frac{a(v_h, v_h)}{\int_\Omega I_h(v_h^2)} \le \max_{v_h\in V_{h,j}} \frac{a(v_h, v_h)}{\int_\Omega v_h^2}$$

and consequently

$$\lambda_{h,j}^{ml} \le \lambda_{h,j} \qquad 1 \le j \le N_h.\square$$

The next lemma gives an expression for the difference between λ_j and the mass-lumping approximation. Although, the result is a particular case of Lemma 5.1 of [2] we include it for the sake of completeness.

Lemma 2.2 *Let $(\lambda_{h,j}^{ml}, u_j^{ml})$ and (λ_j, u_j) be the solutions of problems (2.3) and (2.5) respectively. Then we have that*

$$\lambda_{h,j}^{ml} - \lambda_j = a(u_{h,j}^{ml} - u_j, u_{h,j}^{ml} - u_j) - \lambda_j \|u_{h,j}^{ml} - u_j\|_0^2 - \lambda_{h,j}^{ml} \int_\Omega \left(I_h((u_{h,j}^{ml})^2) - (u_{h,j}^{ml})^2 \right) \tag{2.11}$$

Proof.

$$\begin{aligned}
\lambda_{h,j}^{ml} + \lambda_j &= \lambda_{h,j}^{ml} \|u_{h,j}^{ml}\|_0^2 + \lambda_j \|u_j\|_0^2 \\
&= \lambda_{h,j}^{ml} \int_\Omega I_h((u_{h,j}^{ml})^2) + \lambda_j \|u_j\|_0^2 + \lambda_{h,j}^{ml} \left(\|u_{h,j}^{ml}\|_0^2 - \int_\Omega I_h((u_{h,j}^{ml})^2) \right) \\
&= a(u_{h,j}^{ml}, u_{h,j}^{ml}) + a(u_j, u_j) + \lambda_{h,j}^{ml} \left(\|u_{h,j}^{ml}\|_0^2 - \int_\Omega I_h((u_{h,j}^{ml})^2) \right) \\
&= a(u_{h,j}^{ml} - u_j, u_{h,j}^{ml} - u_j) + 2a(u_{h,j}^{ml}, u_j) + \lambda_{h,j}^{ml} \int_\Omega \left((u_{h,j}^{ml})^2 - I_h((u_{h,j}^{ml})^2) \right) \\
&= a(u_{h,j}^{ml} - u_j, u_{h,j}^{ml} - u_j) + 2\lambda_j - \lambda_j \|u_{h,j}^{ml} - u_j\|_0^2 + \lambda_{h,j}^{ml} \int_\Omega \left((u_{h,j}^{ml})^2 - I_h((u_{h,j}^{ml})^2) \right)
\end{aligned}$$

So, (2.11) holds. □

It is known that, when Ω is not convex, the eigenfunctions of problem (2.3) are singular in many cases, i.e., they belong only to the space $H^{1+r}(\Omega)$, with $r < \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω).

Error estimates for eigenvalue problems considering numerical integration have been obtained in [2]. Although they do not state explicitly the results for non-smooth eigenfunctions it can be seen using their arguments that

$$\begin{aligned}
\|u_{h,j}^{ml} - u_j\|_0 &\leq Ch^{2r} \\
\|\nabla(u_{h,j}^{ml} - u_j)\|_0 &\leq Ch^r
\end{aligned} \tag{2.12}$$

with an appropriate choice of sign for the normalize discrete eigenfunction.

Now, our goal is to show that in the singular case the eigenvalue given by "mass- lumping" approximation (2.5) is better than the eigenvalue approximation given by the standard finite element (2.4) for h small enough.

Corollary 2.2 *Let $\lambda_{h,j}^{ml}$ and λ_j be the eigenvalues of problems (2.3) and (2.5) respectively. If there exists a constant c such that $\|\nabla(u_{h,j}^{ml} - u_j)\|_0 \geq ch^r$, with $r < 1$ then for h small enough*

$$\lambda_j \leq \lambda_{h,j}^{ml} \tag{2.13}$$

Proof. From Lemma (2.2) we know that

$$\lambda_{h,j}^{ml} - \lambda_j = a(u_{h,j}^{ml} - u_j, u_{h,j}^{ml} - u_j) - \lambda_j \|u_{h,j}^{ml} - u_j\|_0^2 - \lambda_{h,j}^{ml} \int_\Omega \left(I_h((u_{h,j}^{ml})^2) - (u_{h,j}^{ml})^2 \right)$$

Since a is coercive, we have that

$$\lambda_{h,j}^{ml} - \lambda_j \geq \alpha \|u_{h,j}^{ml} - u_j\|_1^2 - \lambda_j \|u_{h,j}^{ml} - u_j\|_0^2 - \lambda_{h,j}^{ml} \int_\Omega \left(I_h((u_{h,j}^{ml})^2) - (u_{h,j}^{ml})^2 \right) \tag{2.14}$$

From our hypothesis, the first term on the right-hand side of (2.14) is greater than a constant times h^{2r} and, in view of (2.12) and Corollary 2.1, the second and third terms are of higher order (h^{4r} and h^2 respectively). Therefore, if h is small enough, the sign of $\lambda_{h,j}^{ml} - \lambda_j$ is given by the first term on (2.14) so, we conclude the proof. □

3 Numerical Examples

The object of this section is to compare the eigenvalue approximation obtained with and without mass-lumping in several examples.

We consider the following problem

$$\begin{aligned} -\triangle u &= \lambda u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

with different domains Ω.

First we consider the case of an L-domain. For this domain, it is known that the first eigenfunction is singular.

In Figure 1 we show the first mesh that we use. The subsequent meshes are obtained by uniform refinement (each triangle is divided into four congruent triangles).

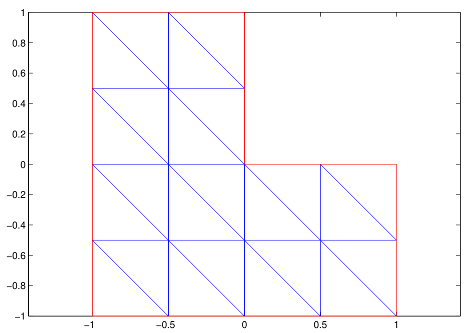

Figure 1: Initial mesh for the L-domain

In the next table we present the numerical approximation of the corresponding eigenvalue.

number of nodes	$\lambda_{h,1}$	$\lambda_{h,1}^{ml}$
21	13.199179221542	9.071796769724
65	10.573955451157	9.641425460959
225	9.916549032001	9.693162213551
833	9.728372729312	9.673506476037
3201	9.66981732232	9.65620182015

Table 1

The exact value has to be less than the last value of $\lambda_{h,1}$ so, we observe that the values obtained with mass-lumping are much better than those obtained with exact integration even for very coarse meshes.

In our next two examples we take Ω as non-convex polygons which are approximations of different levels to the fractal Koch domain. Also in these cases it is known that the first eigenfunctions are singular [5, 6]. In Figure 2 and Figure 3 we show the first meshes for the two examples. As before the subsequent meshes are obtained by uniform refinement.

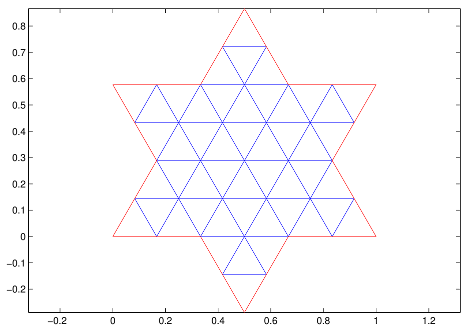

Figure 2: Initial mesh for level 1 approximation of the Koch domain

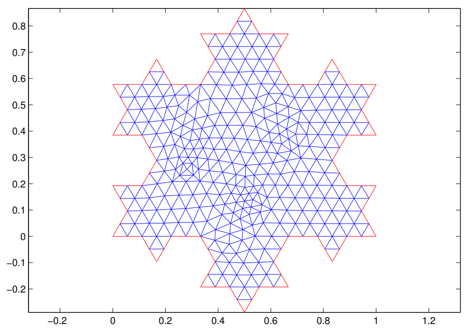

Figure 3: Initial mesh for level 2 approximation of the Koch domain

In Tables 2 and 3 we present the numerical approximation of the first eigenvalues for the domains of Figures 2 and 3 respectively.

number of nodes	$\lambda_{h,1}$	$\lambda_{h,1}^{ml}$
37	46.993282224519	40.401005031470
121	42.121650466929	40.635844194708
433	40.796435658176	40.438418441151

Table 2

number of nodes	$\lambda_{h,1}$	$\lambda_{h,1}^{ml}$
329	40.94016461357	40.34117804088
1217	40.17948566684	40.03394074483

Table 3

Again, we observe that it is convenient to use mass-lumping.

We end the paper by giving some numerical examples for the case of smooth eigenfunctions. In this case, the eigenvalue computed with mass-lumping may be below or above the exact one. In fact, for a uniform mesh in a square domain the eigenvalues of the discrete problem can be obtained explicitly and they are below the exact ones. On the other hand, if one take a mesh of the square with only one interior node close to a corner (see Figure 4) the eigenvalue obtained with mass-lumping is larger than the exact one.

However, the experiments that we show below, as well as other with several different meshes, suggest that in the smooth case the approximate eigenvalue is below the exact one if the mesh is not too coarse.

In all our examples the exact eigenvalues are known explicitly.

In Figure 4 and 5 we present the first and last meshes for the first example in the square domain. In Figure 6 we present the first mesh for the second example in the square domain. In this example the subsequent meshes are obtained by uniform refinement.

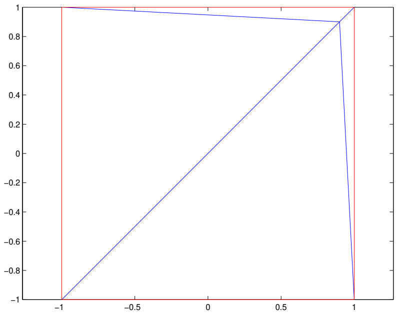

Figure 4

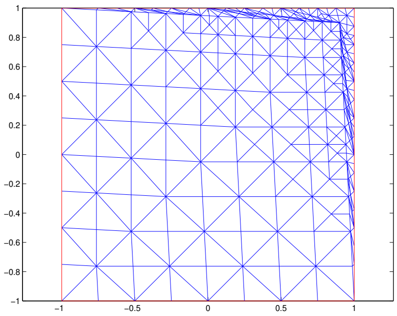

Figure 5

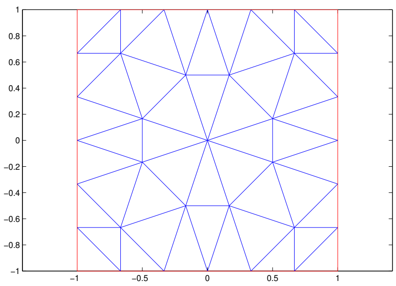

Figure 6

The next tables show the approximation using mass-lumping for the two families of meshes in a square domain. We recall that the exact value is $\lambda_1 = \pi^2/2 = 4.93480220054468....$

number of nodes	$\lambda_{h,1}^{ml}$(initial mesh in Figure 4)
5	15.789473684211
8	3.123922607067
14	3.091168190991
26	3.903152296554
55	4.646900604880
116	4.742942128240
259	4.888070813057

Table 4

number of nodes	$\lambda_{h,1}^{ml}$(initial mesh in Figure 6)
37	4.442736170666
121	4.810061215139
433	4.903574330061

Table 5

In the following two examples the domain is an equilateral triangle. In Figures 7 and 8 we present the first meshes, the subsequent meshes are obtained by uniform refinement. In Tables 6 and 7 we show the discrete eigenvalues obtained for this two cases. The exact value is $\lambda_1 = \frac{16\pi^2}{3} = 52.63789014....$ (see [7])

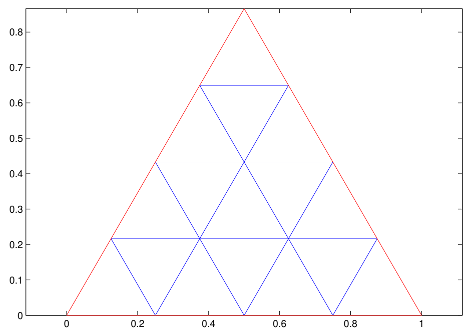

Figure 7

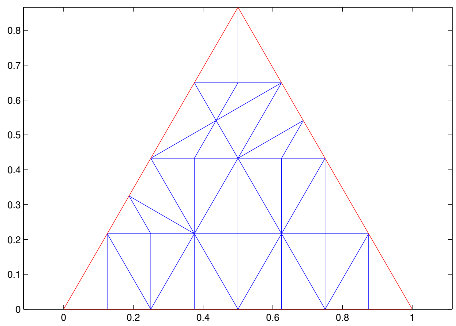

Figure 8

number of nodes	$\lambda_{h,1}^{ml}$(initial mesh in Figure 7)
15	42.666666666667
45	49.987109344163
153	51.964905805628
561	52.468994312245

Table 6

number of nodes	$\lambda_{h,1}^{ml}$(initial mesh in Figure 8)
28	46.839659383781
91	51.318366941074
325	52.331156996197

Table 7

Finally we take as Ω a circle of radius 1. Figure 9 and 10 show two different initial meshes that we have used. The corresponding results are given in Tables 8 and 9. For this case the exact value is $\lambda_1 = 5.78318596294679....$

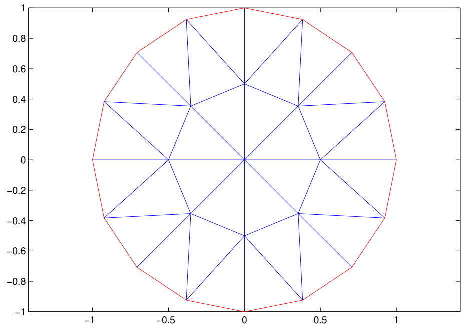

Figure 9

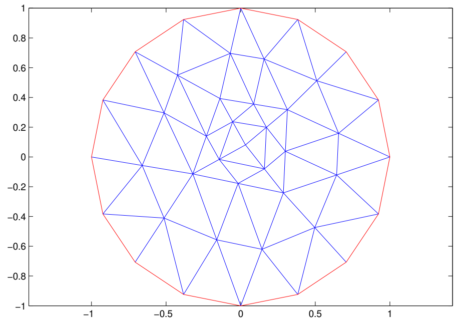

Figure 10

number of nodes	$\lambda_{h,1}^{ml}$(initial mesh in Figure 9)
25	4.86961861262
81	5.52128687409
289	5.71540330139
1089	5.76609636891

Table 8

number of nodes	$\lambda_{h,1}^{ml}$(initial mesh in Figure 10)
41	5.469108031446
145	5.698898965742
545	5.761760229781

Table 9

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