An adaptive time step procedure for a parabolic problem with blow-up

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Abstract. In this paper we introduce and analyze a fully discrete approximation for a parabolic problem with a non linear boundary condition which implies that the solutions blow up in finite time. We use standard linear elements with mass lumping for the space variable. For the time discretization we write the problem in an equivalent form which is obtained by introducing an appropriate time re-scaling and then, we use explicit Runge-Kutta methods for this equivalent problem.

In order to motivate our procedure we present it first in the case of a simple ordinary differential equation and show how the blow up time is approximated in this case.

We obtain necessary and sufficient conditions for the blow-up of the numerical solution and prove that the numerical blow-up time converges to the continuous one. We also study, for the explicit Euler approximation, the localization of blow-up points for the numerical scheme.

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1 Introduction.

For many differential equations or systems the solutions can become unbounded in finite time, a phenomenon that is known as blow up. Typical examples where this happens are parabolic problems involving nonlinear reaction terms. These problems have been widely analyzed from a mathematical point of view (see for example [7] and the references therein) but, only few results concerning the numerical approximation of them can be found in the literature (we refer for example to [1, 2, 3, 4, 5]). Indeed, even for an elementary ordinary differential equation $y' = f(y)$ having blow up, the usual analysis to obtain error estimates and adaptive step procedures do not apply because they are based on regularity assumptions which are not satisfied in this case.

An important question is how to approximate the blow up time. For the elementary ordinary differential equation $y' = y^m$ with $m > 1$, a result in this direction has been given in [9] where the Euler method with a fix time step is analyzed.

Another interesting problem is to say whether the numerical solution behaves similar to the exact one, for example, concerning the localization of blow up.

In this paper we study the behavior of solutions of a numerical approximation of the following model parabolic problem involving a nonlinear flux boundary condition.

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$$
\begin{cases}
u_t(x,t) = u_{xx}(x,t) & \text{in } (0,1)\times[0,T),\\
u_x(1,t) = f(u(1,t)) & \text{on } [0,T),\\
u_x(0,t) = 0 & \text{on } [0,T),\\
u(x,0) = u_0(x)\geq 0 & \text{on } [0,1].
\end{cases}
\tag{1.1}
$$

where f is increasing, convex, positive and smooth and, we assume compatibility between the initial and boundary data in order to have that, before the blow up time, the solution is as regular as needed in our arguments.

The behavior of the exact solutions of this problem has been analyzed in several papers (see for example [6, 8, 10]). In particular, it is known that if $\int^{\infty} 1/f < +\infty$ then, the blow up occurs in the sense that there exists a time T such that $\lim_{t\to T} u(1,t) = +\infty$ but the blow up set is localized at the boundary ($u(x,t)$ remains bounded for every $x\neq 1$) (see [8]).

In [5] a semidiscretization in space of this problem was analyzed. The space discretization leads in a usual way to a system of ordinary differential equations. A necessary and sufficient condition on the f for blow up of that system was obtained in that paper. Also, it was proven in [5] that the blow up time of the semidiscrete problem converges to that of the continuous one when the mesh size goes to zero.

The object of this work is to introduce and analyze totally discrete approximations of problem (1.1). As in [5] we use a standard piecewise linear finite element method with mass lumping on a uniform mesh for the space variable which, as is well known, coincides with the classical central finite difference second order scheme. This semidiscritezation reads as follows

$$
\begin{aligned}
&\hat{U}' = \tfrac{1}{h^2}A\hat{U} + \tfrac{2}{h}e_N f(\hat{U}_N)\\
&\hat{U}(0) = u_0^I
\end{aligned}
\tag{1.2}
$$

where $\hat{U} = \hat{U}(t) = (\hat{U}_1,...,\hat{U}_N)^t$, e_N is the N-canonical column vector, $u_0^I = (u(x_0,0),...,u(x_N,0))^t$, $x_j = jh$, for $0\leq j\leq N$, and

$$
A = \begin{pmatrix}
-2 & 2 & 0 & . & . & . & . & 0\\
1 & -2 & 1 & . & . & 0 & . & 0\\
0 & . & . & . & . & . & . & 0\\
. & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & .\\
0 & . & . & . & . & 1 & -2 & 1\\
0 & . & . & . & . & . & 2 & -2
\end{pmatrix}
\tag{1.3}
$$

From [5], we know that (1.2) has blowing-up solutions if and only if $\int^{\infty}\frac{1}{f(s)}ds < \infty$.

Our time discretization of (1.2) is based on the use of standard explicit methods combined with an appropriate re-scaling. In the case of the lowest order method, the method corresponds to a simple adaptive time step procedure.

In Section 2 we explain our time discretization procedure by applying it to an elementary ordinary differential equation. In Section 3 we introduce a general discretization of (1.2) and prove several properties which will be useful for our subsequent analysis. In Section 4 we prove convergence for the general scheme when the solution is smooth, i.e., before the blow up time. In Section 5 we give a necessary and sufficient condition on f for blow up of the discrete solutions. The condition is $\int^{+\infty} 1/f < +\infty$ which is the same as in the semidiscrete case but is slightly different than that of the continuous problem which is $\int^{+\infty} 1/ff' < +\infty$, ([10]). In Section 6 we prove that the blow up time of the discrete problem converges to that of the continuous one. Finally, Section 7 deals with the rate and localization of blow up in the discrete problem. For the sake of simplicity we restrict the analysis in that section to the Euler method.

2 Adaptive procedure for an ordinary differential equation.

In this section we motivate the adaptive procedure by applying it to an ordinary differential equation with blowing up solutions. We show that with this procedure the discrete approximation also blows up in finite time (in a sense to be precise below) and the numerical blow up time converges to the continuous one.

Let $\hat{y}(t)$ be the solution of the ordinary differential equation

$$\left\{ \begin{array}{l} \hat{y}'(t) = f(\hat{y}(t)) \\ \hat{y}(0) = y_0 > 0 \end{array} \right. \tag{2.1}$$

where f is a positive, increasing and smooth function.

Assume that $\int^{+\infty} 1/f < +\infty$ then, $\hat{y}(t)$ blows up in finite time T and we can compute exactly the blow-up time. Indeed we have

$$T = \int_{y_0}^{+\infty} \frac{1}{f(s)} ds$$

Let us first consider the Euler method. In this case the o.d.e. (2.1) is aproximated by

$$\left\{ \begin{array}{r} \hat{y}^{j+1} = \hat{y}^j + \Delta t^j f(\hat{y}^j) \\ \hat{y}^0 = y_0 \end{array} \right.$$

If we fix $\lambda > 0$ and select the time step Δt^j such that

$$\Delta t^j f(\hat{y}^j) = \lambda$$

we have that

$$\hat{y}^{j+1} = \hat{y}^j + \lambda$$

and hence,

$$\hat{y}^j = y_0 + j\lambda$$

therefore,

$$\Delta t^j = \frac{\lambda}{f(\hat{y}^j)} = \frac{\lambda}{f(y_0 + j\lambda)}$$

From this we can obtain that the numerical scheme also blows up in the sense that $\hat{y}^j \to \infty$ while $\sum_j \Delta t^j < +\infty$. Also a precise estimate of the numerical blow up time can be given. Indeed, we have

$$\sum_{j=0}^{\infty} \Delta t^j = \Delta t^0 + \sum_{j=1}^{\infty} \frac{\lambda}{f(y_0 + j\lambda)} \leq \frac{\lambda}{f(y_0)} + \int_0^{+\infty} \frac{\lambda}{f(y_0 + s\lambda)} ds$$

$$= \frac{\lambda}{f(y_0)} + \int_{y_0}^{+\infty} \frac{1}{f(s)} ds = \frac{\lambda}{f(y_0)} + T.$$

Therefore, $\sum_j \Delta t^j < +\infty$ and calling $T_\lambda = \sum_j \Delta t^j$ we have

$$T_\lambda - T \leq \frac{\lambda}{f(y_0)}$$

On the other hand, since y is convex, it is easy to see that the numerical solution is always below the exact one and hence

$$T \leq T_\lambda$$

So,

$$|T - T_\lambda| \leq \frac{\lambda}{f(y_0)}$$

In order to generalize this adaptive procedure to methods of any order we observe that, introducing a new variable τ, the o.d.e. (2.1) can be equivalently written as the system

$$\begin{cases} \frac{dt}{d\tau} = \frac{1}{f(y(t))} & t(0) = 0 \\ \frac{dy}{d\tau} = 1 & y(0) = y_0 \end{cases} \tag{2.2}$$

where $y(\tau) = \hat{y}(t)$.

Now, if we apply the Euler method with a fixed step λ to this system we recover the method introduced above.

On the other hand, we can apply any method to the system (2.2). For example, we can use any Runge-Kutta method with a fixed step λ. Clearly, the second equation of (2.2) will be integrated exactly while the different discretizations of the first one will give different ways for advancing the time t.

For example, the second order Heun method gives

$$\begin{cases} t^{j+1} = t^j + \frac{\lambda}{2}(\frac{1}{f(y^j)} + \frac{1}{f(y^j+\lambda)}) \\ t^0 = 0 \end{cases}$$

while the half step second order Runge-Kutta yields,

$$\begin{cases} t^{j+1} = t^j + \lambda \frac{1}{f(y^j + \frac{\lambda}{2})} \\ t^0 = 0 \end{cases}$$

and the standard fourth order Runge-Kutta method gives,

$$\begin{cases} t^{j+1} = t^j + \frac{\lambda}{6}(\frac{1}{f(y^j)} + \frac{4}{f(y^j+\frac{\lambda}{2})} + \frac{1}{f(y^j+\lambda)}) \\ t^0 = 0 \end{cases}$$

As we have mentioned above, for the Euler method, this procedure corresponds to a simple adaptive procedure applied to the original equation.

Note that all the above methods can be written in the form

$$\begin{cases} t^{j+1} = t^j + \lambda \sum_k c_k \frac{1}{f(\xi_k^j)} \\ t^0 = 0 \end{cases} \tag{2.3}$$

where $\sum c_k = 1$, $0 \leq c_k \leq 1$ and $y_j \leq \xi_k^j \leq y_{j+1}$ $\forall k$.

For any of these methods we can see (as we did for the Euler method) that

$$T_\lambda = \sum_{j=0}^{\infty} \Delta t^j$$

approximates T. This can be seen using the expression (2.3). Indeed, since f is increasing we have

$$\frac{1}{f(y^j)} \geq \sum c_k \frac{1}{f(\xi_k^j)}$$

and then $T_\lambda \leq T_\lambda^{euler}$. On the other hand

$$\frac{1}{f(y^{j+1})} \leq \sum c_k \frac{1}{f(\xi_k^j)}$$

and so $T_\lambda \geq T_\lambda^{euler} - \frac{\lambda}{f(y_0)}$. Therefore,

$$|T - T_\lambda| \le \frac{2\lambda}{f(y_0)} \tag{2.4}$$

Of course, the last estimate is not sharp. One expects that methods of higher order yield better approximations of the blow up time and in fact, it is not difficult to see that (2.4) can be improved under appropriate assumptions on f. For example, for the second order Heun method we have

$$T - T_\lambda = \sum_{j=0}^{\infty} \left\{ \int_{y^j}^{y^j+\lambda} \frac{1}{f(s)}\, ds - \frac{\lambda}{2}\left(\frac{1}{f(y^j)} + \frac{1}{f(y^j+\lambda)} \right) \right\}$$

and so $T - T_\lambda$ can be bounded by using the error formula for the trapezoidal rule applied to the funcftion $\frac{1}{f}$. Consequently, if the function $\frac{1}{f}$ has derivatives up to the second order and $D^2(\frac{1}{f})$ is monotone decreasing an integrable at infinity it is easy to see that

$$|T - T_\lambda| \le C\lambda^2 \int_{y^{j-1}}^{\infty} D^2\left(\frac{1}{f(s)}\right)\, ds$$

In the same way one can check that a general Runge-Kutta method of order k_0 will approximate the integral of the function $\frac{1}{f}$ by an integration rule of order k_0 and therefore it can be proved that

$$|T - T_\lambda| \le C\lambda^{k_0} C \int_{y^{j-1}}^{\infty} D^{k_0}\left(\frac{1}{f(s)}\right)\, ds$$

if the function $\frac{1}{f}$ has derivatives up to the order k_0 and $D^{k_0}(\frac{1}{f})$ is monotone decreasing an integrable at infinity.

3 Adaptive procedure for the Partial Differential Equation

Using the idea described in the previous section we give now an equivalent form of problem (1.2). In order to do that, we introduce a time rescaling given by

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{1}{f(\hat{U}_N(t))} \\ t(0) &= 0 \end{aligned} \tag{3.1}$$

Observe that, since f is positive, $t(\tau)$ defines a one to one function.

Now, in terms of τ, i.e. calling $\hat{U}(t(\tau)) = U(\tau) = (U_1, \cdots, U_N)^t$, the system (1.2) becomes

$$U' = F(U) \tag{3.2}$$

where

$$F(U) = \frac{1}{f(U_N)} \frac{1}{h^2} AU + \frac{2}{h} e_N$$

As in the o.d.e. case analyzed in the previous section, we can approximate (3.2) by any Runge-Kutta method with a fixed step λ. Also in this case, it is not difficult to see that, when the Euler method is applied, the rescaling corresponds to a simple time step adaptive procedure.

In the rest of this section we are going to show some properties of a general Runge-Kutta method applied to problem (3.2). We will always consider positive initial data U^0. Any explicit Runge-Kutta method with constant step λ can be written as

$$U^{j+1} = U^j + \lambda \sum_{s=1}^{k} \gamma_s F_s \tag{3.3}$$

for some nonnegative γ_s such that $\sum_{s=1}^{k} \gamma_s = 1$, where $F_1 = F(U^j)$, and F_s is defined in terms of F_{s-1} by

$$F_s = F(U^j + \lambda\theta_{s-1}F_{s-1})$$

for some θ_{s-1} satisfying $0 \le \theta_{s-1} \le 1$. Then, it is not difficult to see that there exist polynomials P and Q such that $P(0) = 1$ and $Q(0) = 1$ wich can be written as

$$P(x) = a_0^j + a_1^j(1+x) + \cdots + a_k^j(1+x)^k$$

and,

$$Q(x) = b_0^j + b_1^j(1+x) + \cdots + b_{k-1}^j(1+x)^{k-1}$$

with $a_i^j \ge 0$ and $b_i^j \ge 0$, such that,

$$U^{j+1} = P(\mu A)U^j + \frac{2\lambda}{h}Q(\mu A)e_N \tag{3.4}$$

where $\mu = \delta\frac{\lambda}{h^2}k$ with $\delta = 2\max\{\delta_i\}$ (where δ_i are some evaluations of $1/f$ which actually depends on j and will be precised below).

Let us consider, for example, the second order Heun method. In this case $\delta_1 = 1/f(U_N^j)$ and $\delta_2 = 1/f(\overline{U}_N^j)$, with $\overline{U}_N^j = U_N^j + \lambda F(U^j)_N$ and the iteration (3.3) is given by (3.4) with $P(x) = a_0^j + a_1^j(1+x) + a_2^j(1+x)^2$ and $Q(x) = b_0^j + b_1^j(1+x)$ with the coefficients defined in the following way

$$\begin{aligned} a_0^j &= 1 - \frac{1}{4}(\frac{\delta_1}{\delta} + \frac{\delta_2}{\delta}) + \frac{1}{8}\frac{\delta_1\delta_2}{\delta^2}, \\ a_1^j &= \frac{1}{4}(\frac{\delta_1}{\delta} + \frac{\delta_2}{\delta}) - \frac{1}{4}\frac{\delta_1\delta_2}{\delta^2}, \\ a_2^j &= \frac{1}{8}\frac{\delta_1\delta_2}{\delta^2}, \\ b_0^j &= 1 - \frac{1}{4}\frac{\delta_2}{\delta}, \end{aligned}$$

and,

$$b_1^j = \frac{1}{4}\frac{\delta_2}{\delta}.$$

One can easily check that a_k^j and b_k^j are positive.

Now we indicate the proof of (3.4) for the general iteration (3.3). The technical, but not difficult, details are left to the reader. Observe that

$$\lambda F_1 = \delta_1\frac{\lambda}{h^2}AU^j + 2\frac{\lambda}{h}e_N \tag{3.5}$$

with

$$\delta_1 = \frac{1}{f(U_N^j)}. \tag{3.6}$$

Also, we have

$$\lambda F_2 = \delta_2\frac{\lambda}{h^2}AU^j + \theta_1\delta_1\delta_2(\frac{\lambda}{h^2})^2A^2U^j + 2\frac{\lambda}{h}e_N + 2\theta_1\delta_2\frac{\lambda^2}{h^3}Ae_N \tag{3.7}$$

with

$$\delta_2 = \frac{1}{f(U_N^j + \lambda\theta_1(F_1)_N)}. \tag{3.8}$$

Inductively, it is easy to see that λF_s has the form

$$\lambda F_s = G_s^j + H_s^j \tag{3.9}$$

where

$$\begin{aligned} G_s^j &= \delta_s \frac{\lambda}{h^2} A U^j + \theta_{s-1}\delta_{s-1}\delta_s (\frac{\lambda}{h^2})^2 A^2 U^j + \cdots \\ &+ \theta_1 \cdots \theta_{s-1}\delta_1 \cdots \delta_{s-1}\delta_s (\frac{\lambda}{h^2})^s A^s U^j \end{aligned} \tag{3.10}$$

and,

$$H_s^j = 2\frac{\lambda}{h} e_N + 2\delta_s\theta_{s-1}\frac{\lambda^2}{h^3} A e_N + \cdots + 2\theta_1 \cdots \theta_{s-1}\delta_2 \cdots \delta_s \frac{\lambda^{s+1}}{h^{2s+1}} A^{s-1} e_N. \tag{3.11}$$

The numbers δ_i represents intermediate evaluations of $1/f$, indeed

$$\delta_s = \frac{1}{f(U_N^j + \lambda\theta_{s-1}(F_{s-1})_N)}. \tag{3.12}$$

We remark that, in fact, we should write δ_i^j instead of δ_i but, we have dropped the j to simplify the notation.

Then, using (3.9), (3.10), (3.11) and the fact that $\delta = 2\max\{\delta_i\}$ and $\mu = \delta\frac{\lambda}{h^2}k$, one can show that, for $s = 1, \cdots, k$,

$$U^j + \lambda F_s = P_s(\mu A) U^j + \frac{2\lambda}{h} Q_s(\mu A) e_N$$

with

$$P_s(x) = 1 + \alpha_1^j x + \frac{\alpha_2^j}{2!} x^2 + \cdots + \frac{\alpha_s^j}{s!} x^s \tag{3.13}$$

and,

$$Q_s(x) = 1 + \beta_1^j x + \frac{\beta_2^j}{2!} x^2 + \cdots + \frac{\beta_{s-1}^j}{(s-1)!} x^{(s-1)}$$

where the coefficients satisfy the following relations.

$$0 < \alpha_{l+1}^j \leq \frac{\alpha_l^j}{2} \leq \frac{1}{2} \tag{3.14}$$

and,

$$0 < \beta_{l+1}^j \leq \frac{\beta_l^j}{2} \leq \frac{1}{2} = \beta_1^j \tag{3.15}$$

Now, it is not difficult to show that, in view of the inequalities (3.14) and (3.15), P_s and Q_s have positive coefficients when they are written in powers of $(1+x)$.

Consequently, (3.4) follows from the fact that $\sum_{s=1}^k \gamma_s = 1$.

By using the general expression (3.4) we can show some properties which will be useful in our subsequent analysis.

Lemma 3.1 *For the matrix A defined in (1.3) and a polynomial $P(x) = a_0 + a_1(1+x) + \cdots + a_k(1+x)^k = 1 + \alpha_1 x + \cdots + \frac{\alpha_k}{k!} x^k$ such that $a_j \geq 0$, $\alpha_j \geq 0$ and $\alpha_{j+1} \leq \frac{\alpha_j}{2}$ the following properties hold,*

1. *For $0 \leq \mu < 1/2$ the matrix $P(\mu A)$ has nonnegative entries.*

2. *For $0 \le \mu < 1/2$ and any vector U then , any component W_s of the vector $W = P(\mu A)U$ verifies*

$$W_s = \sum_{l=1}^{N} \omega_l U_l \tag{3.16}$$

for some nonnegative numbers ω_l, such that $\sum_{l=1}^{N} \omega_l = 1$.

3. *For any $s = 1, \cdots, N-1$, the coefficient ω_{s+1} in (3.16) satisfies $\omega_{s+1} \ge \frac{\alpha_1}{2}\mu$.*

4. *If $0 \le \mu < 1/3$ then, for any positive and increasing vector U (i.e. $0 \le U_i \le U_{i+1}$), the vector $W = P(\mu A)U$ is also positive and increasing.*

Proof: If $0 \le \mu < 1/2$ then, $I + \mu A$ has nonnegative entries and so 1) follows from the fact that $a_j \ge 0$. Now, since the sum of the elements of any row of A is zero, the same is true for any power of A. Therefore, the sum of the elements of any row of the matrix $P(\mu A)$ is equal to one and so, 2) is a consequence of 1) and the fact that $P(0) = 1$. Using again that the coefficients of $I + \mu A$ are positive, it follows that ω_{s+1} is greater than or equal to the $(s, s+1)$ component of $a_1(I + \mu A)$ which is bounded from below by $a_1\mu$. On the other hand, using that $\alpha_{j+1} \le \alpha_j/2$, it is easy to see that $a_1 = P'(-1) \ge \alpha_1/2$ and so, 3) holds.

Finally, since $a_j \ge 0$, to prove 4) it is enough to show that $W = (I + \mu A)U$ is positive and increasing. To show this we define $Z_i = W_{i+1} - W_i$ and $Y_i = U_{i+1} - U_i$, and we want to prove that $Z_i \ge 0$ whenever $Y_i \ge 0$. Observe that Z_i verifies

$$\begin{aligned} Z_i - Y_i &= (W_{i+1} - U_{i+1}) - (W_i - U_i) \\ &= \mu(U_{i+2} - 2U_{i+1} + U_i - U_{i+1} + 2U_i - U_{i-1}) \\ &= \mu(Y_{i+1} - 2Y_i + Y_{i-1}) \end{aligned}$$

for every $i = 1, \cdots, N-1$.

On the other hand, for $i = 1$ we have,

$$\begin{aligned} Z_1 - Y_1 &= \mu(U_3 - 2U_2 + U_1 - 2U_2 + 2U_1) \\ &= \mu(Y_2 - 3Y_1) \end{aligned}$$

and for $i = N-1$,

$$\begin{aligned} Z_{N-1} - Y_{N-1} &= (W_N - U_N) - (W_{N-1} + U_{N-1}) \\ &= \mu(U_{N-1} - U_N - U_N + 2U_{N-1} - U_{N-2}) \\ &= \mu(Y_{N-2} - 3Y_{N-1}). \end{aligned}$$

Therefore, the result follows by observing that $\mu < 1/3$. □

Remark 3.1 *The polynomials P and Q in (3.4) satisfies the hypotheses of Lemma 3.1. In particular, observe that the inequalities for the coefficients α_j follows from (3.14) and (3.15).*

In view of this remark we have the following immediate consequence of the previous lemma.

Lemma 3.2 *If U^0 is positive and increasing and $\mu < 1/3$ then, the iteration defined by (3.3) satisfies $U_i^j > 0$ and $U_i^j \le U_{i+1}^j$, $\forall\, i, j$ (i.e., the numerical solution is positive and increasing in x).*

Remark 3.2 *Observe that the restriction $\mu = \delta \frac{\lambda}{h^2} k < 1/3$ is a natural one because we are using explicit methods.*

As mentioned above, we will always consider positive U^0. In particular, in view of Lemma 3.1, the numerical solution U^j will be positive for all j. Another consequence of Lemma 3.1 is the following lemma giving some auxiliary results which will be used below.

Lemma 3.3 *Let $\{U^j\}_{j\in\mathbb{N}}$ be the sequence given by (3.3) for a given initial U^0. If $\mu < 1/2$. Then,*

1. *For any j, and any $s = 1, \cdots, N$ there exist nonnegative numbers ω_l and r_s (which depend on j) such that $\sum_{l=1}^{N} \omega_l = 1$, $r_s = 0$ for $1 \leq s \leq N-k$, and*

$$U_s^{j+1} = \sum_{l=1}^{N} \omega_l U_l^j + r_s \tag{3.17}$$

2. *The minimum of U^j (i.e. $min_{1\leq s\leq N}\{U_s^j\}$) is non decreasing with j.*

3. *If U^j remains bounded by above, i.e., $U_l^j \leq M$ for a constant M, then, for any $1 \leq s \leq N-1$, ω_{s+1} in equation (3.17) verifies $\omega_{s+1} > c$ for a $c > 0$ depending only on M and $\frac{\lambda}{h^2}$.*

Proof: Applying 2) of Lemma 3.1 for the polynomial P defined in (3.4) we have that

$$[P(\mu A)U^j]_s = \sum_{l=1}^{N} \omega_l U_l^j$$

and using 1) of Lemma 3.1 for Q and the fact that the components of e_N are nonnegative we get

$$r_s = [Q(\mu A)e_N]_s \geq 0$$

therefore, (3.17) follows from (3.4).

Observing that $Ae_N = -2e_N + e_{N-1}$, one easily gets that the vector $Q(\mu A)e_N$ belongs to the subspace spanned by $e_N, \cdots, e_{N-k+1}$, and this implies that $r_s = 0$ for $N-k \leq s \leq N$ which concludes the proof of 1). Now, 2) is an immediate consequence of 1).

Using 3) of Lemma 3.1 we have that $\omega_{s+1} > \frac{\alpha_1}{2}\mu$ where α_1 is the coefficient multiplying x in the expression of P in powers of x obtained from (3.13). Therefore, recalling that $\mu = \delta\frac{\lambda}{h^2}k$, 3) will be true if we prove that δ and α_1 are bounded from below by a positive constant which depends only on M and $\frac{\lambda}{h^2}$.

Now, in view of 2) we have that

$$min_{1\leq l\leq N}\{[U^j + \lambda\theta_s F_s]_l\} \geq min_{1\leq l\leq N}\{U_l^j\} \geq min_{1\leq l\leq N}\{U_l^0\}$$

So, looking at (3.12) and using that f is an increasing function we see that $\delta = 2max\{\delta_s\} \leq C$ for a positive constant C depending only on U^0.

On the other hand, looking at (3.9), (3.10) and (3.11), it is easy to see that $[U^j + \lambda\theta_s F_s]_N$ is bounded by a constant which depends only on M and $\frac{\lambda}{h^2}$. Therefore, $\delta_s > c$ for some positive c depending only on M and $\frac{\lambda}{h^2}$.

Finally, looking at the argument showing the existence of P and Q in (3.4), it is not difficult to see that the boundedness from below of α_1 follows from the fact that $c < \delta_s \leq C$.□

The following corollary is a sort of "quasi" maximum principle for the discrete problem. We will show later that the true maximum principle holds for j large enough.

Corollary 3.1 *Let $\{U^j\}_{j\in\mathbb{N}}$ be the sequence given by (3.3) for a given initial U^0. If $\mu < 1/2$ then, for a given J, if we call $M = max_{\{0\leq s\leq N, 0\leq j\leq J\}}\{U_s^j\}$ we have,*

$$M = max\{max_{\{N-k+1\leq s\leq N, 0\leq j\leq J\}}\{U_s^j\}, max_{0\leq s\leq N}\{U_s^0\}\}$$

Proof: It is an immediate consequence of part 1) of the previous lemma. □

Our next goal is to prove a theorem which says that the discrete solutions goes to infinity. In order to do that, we first give two lemmas. The first one is a comparison principle for super and subsolutions of the discrete problem while the second one, proves that the discrete solution goes to infinity when the initial datum is increasing. We will use the following definiton.

Definition 3.1 *For the general method given by (3.4) we say that W is a supersolution (resp.: subsolution) of the numerical scheme if it verifies (3.4), for polynomials P and Q corresponding to a given initial U^0, with an inequality $\geq$ (resp.: $\leq$) instead of an equality.*

It is important to remark that in this definition the initial U^0 is fixed. So, the polynomials P and Q in (3.4) depend on j and on U^0. Therefore, it has to be clear that we are not saying that W is a supersolution (or subsolution) of the original nonlinear problem.

Lemma 3.4 *Assume that $\mu < 1/2$.*

1. *If $\underline{W}$ is a subsolution and $\overline{W}$ a supersolution of (3.4) such that $\underline{W}_i^0 < \overline{W}_i^0$ for $i = 1, \cdots, N$ then, $\underline{W}_i^j \leq \overline{W}_i^j$, $\forall i, j$.*
2. *If $\underline{W}$ and $\overline{W}$ are solutions of (3.4), with P and Q corresponding to a given initial U^0, then, $\max_i\{\overline{W}_i^{j+1} - \underline{W}_i^{j+1}\} \leq \max_i\{\overline{W}_i^0 - \underline{W}_i^0\}$*

Proof: In the first case the difference $Z^j = \overline{W}^j - \underline{W}^j$ satisfies

$$Z^{j+1} \geq P(\mu A)Z^j$$

then, 1) follows by using 1) of Lemma 3.3 and recalling that $Z^0 \geq 0$.

On the other hand, if both are solutions, the difference Z^j satisfies

$$Z^{j+1} = P(\mu A)Z^j$$

and so, 1) follows also from 1) of Lemma 3.3.□

Lemma 3.5 *Let U^0 be increasing and $\mu < 1/8$. Then, the iteration defined by (3.4) satisfies $\lim_{j\to\infty} U_N^j = \infty$.*

Proof: Since $\mu < 1/3$ it follows from Lemma 3.2 that U^j is increasing (i.e. $U_{s+1}^j \geq U_s^j$). Hence $U_1^j = \min_{1\leq s\leq N}\{U_s^j\}$ and therefore, from 2) of Lemma 3.3, we have that U_1^j is non decreasing with j. Using (3.17) for $s = 1$ we have

$$U_1^{j+1} = \sum_l \omega_l U_l^j + r_1 \tag{3.18}$$

with $r_1 \geq 0$.

We now argue by contradiction. If the the sequence U_N^j is bounded, the same is true for U_1^j. In particular this implies that there exists α such that $U_1^j \to \alpha$ when $j \to \infty$. On the other hand, since $U_l^j \geq U_1^j$ for $l = 3, \cdots, N$ and $\sum_l \omega_l = 1$ we have

$$U_1^{j+1} \geq (1-\omega_2)U_1^j + \omega_2 U_2^j$$

and so,

$$\frac{U_1^{j+1} - U_1^j}{\omega_2} + U_1^j \geq U_2^j$$

but, we know from 3) of Lemma 3.3 that there exists $c > 0$, independent of j, such that $\omega_2 > c$ and therefore,

$$\frac{U_1^{j+1} - U_1^j}{c} + U_1^j \geq U_2^j \geq U_1^j$$

and consequently, $U_2^j \to \alpha$ when $j \to \infty$.

Proceeding by induction we get that the limit of each U_s^j is α, but this contradicts the equation for U_N^j because there exists a constant c such that $r_N > C > 0$. Indeed, since the infinite norm of A verifies $\|A\|_\infty = 4$, we have that $\rho(\mu A)_\infty < 1/2$. Then, since e_N is unitary, the existence of the constant $c > 0$ depending only on λ and h, but not on j, such that $r_N > c$, follows from (3.15) recalling that $r_N = [Q(\mu A)e_N]_N$.□

Theorem 3.1 *Let $\{U^j\}_{j\in\mathbb{N}}$ be the sequence given by (3.3) for a given initial U^0. If $\mu < 1/8$ then,*

1. $U_N^j \to \infty$.
2. *There exist C_1 and C_2 depending only on U^0 (in particular, independent of h and λ) such that*

$$C_1 max_{1\le s\le N}\{U_s^j\} \le U_N^j \le C_2 max_{1\le s\le N}\{U_s^j\}$$

3. *There exists K, depending on h but independent of λ such that, for $U_N^j > K$, the following inequalities hold*

$$U_N^j + \frac{\lambda}{h} \le U_N^{j+1} \le U_N^j + \frac{3\lambda}{h} \tag{3.19}$$

$$U_s^{j+1} \le U_s^j + \frac{\lambda}{2h} \qquad for \qquad 1 \le s \le N-1. \tag{3.20}$$

and,

$$\frac{1}{f(U_N^j + \frac{3\lambda}{h})} \le \delta_s \le \frac{1}{f(U_N^j)} \tag{3.21}$$

and also,

$$U_N^j = max_{1\le s\le N}\{U_s^j\} \tag{3.22}$$

In particular, in view of 1), there exists j_0 depending on h and λ such that, for $j \ge j_0$, (3.19), (3.20), (3.21) and (3.22) hold.

Proof: Given $U^0 > 0$ we construct increasing $\underline{U}^0$ and $\overline{U}^0$ such that $\underline{U}^0 < U^0 < \overline{U}^0$. Let $\underline{U}^j$ and $\overline{U}^j$ the vectors obtained by the iteration (3.4) with initials $\underline{U}^0$ and $\overline{U}^0$ where the polynomials P and Q are those corresponding to the initial datum U^0.

Suppose now that U_N^j remains bounded then, by 1) of Lemma 3.4, the same is true for $\underline{U}^0$.

Now, applying the maximum principle given by 2) of Lemma 3.4 to the difference $\overline{U}^j - \underline{U}^j$ we get that $max_s\{\overline{U}_s^j - \underline{U}_s^j\}$ is reached at $j = 0$. Hence, calling $M = max_s\{\overline{U}_s^0 - \underline{U}_s^0\}$ we have

$$\overline{U}_N^j \le M + \underline{U}_N^j \tag{3.23}$$

and therefore, $\overline{U}_N^j$ also remains bounded. So, since $\overline{U}^j$ is increasing, all its components remain bounded.

Then, using again 1) of Lemma 3.4 we obtain the boundedness of U^j.

Now, with the same argument used in the proof of Lemma 3.5, we can show that $\overline{U}_N^j \to \infty$ (recall that ω_l and r_s in equation (3.18) depend only on the polynomials P and Q which depends only on U^j) which is a contradiction. Therefore 1) holds.

Given j, let s_0 (which may depend on j) be such that the maximum of U_s^j is reached at $U_{s_0}^j$. Then, using (3.23) and that $\overline{U}^j$ is increasing, we obtain

$$U_{s_0}^j = max_{1\le s\le N}\{U_s^j\} \le max_{1\le s\le N}\{\overline{U}_s^j\} \le \overline{U}_N^j \le \underline{U}_N^j + M \le U_N^j + M$$

and so, 2) holds.

Now, since f is superlinear (i.e., $w/f(w) \to 0$ when $w \to 0$), it follows from 2) and the expressions of λF_1 and δ_1 given by (3.5) and (3.6) that

$$|\lambda F_1 - 2\frac{\lambda}{h}e_N| < \varepsilon \tag{3.24}$$

whenever $U_N^j > K$ for some K which depends on ε, U^0 and h but is independent of λ. In the same way, using now (3.24), (3.7) and (3.8) we obtain

$$|\lambda F_2 - 2\frac{\lambda}{h}e_N| < \varepsilon$$

for $U_N^j > K$, with K as above. Analogously, we can show that

$$|\lambda F_s - 2\frac{\lambda}{h}e_N| < \varepsilon$$

again for U_N^j as above.

Then, we can choose K, depending only on U^0 and h, such that, the first $N-1$ components of λF_s are smaller than $\frac{\lambda}{2h}$ while the last one is between $\frac{\lambda}{h}$ and $\frac{3\lambda}{h}$ whenever $U_N^j > K$. Therefore (3.19) and (3.20) follows from (3.3).

To prove (3.21) we proceed in the same way and use now (3.12) and the fact that $0 \le \theta_s \le 1$.

Finally, (3.22) follows immediately from (3.19) and (3.20). □

4 Convergence

In this section we prove the convergence of the general Runge-Kutta method (3.3) for intervals of time in which the solution is smooth. If T is the blow-up time and $T_1 < T$ we show that the error in the interval $[0, T_1]$ converges to zero when λ and h converge to zero. As usual, these results hold under appropriate regularity assumptions on the solution which are known to hold when the initial data is smooth and compatible.

It is not difficult to see that it is enough to prove the error estimates for a problem of the form (1.1) with f replaced by a globally Lipschitz function $\tilde{f}$ such that

$$f(v) = \tilde{f}(v) \quad \text{for} \quad v \le \|u\|_{L^\infty([0,1]\times[0,T_1])} + 1$$

In fact, by uniqueness, u satisfies the equation (1.1) with $\tilde{f}$ instead of f. Moreover, once the uniform convergence is proved for the problem with $\tilde{f}$, it follows that, for h and λ small enough, $f(U_N^j) = \tilde{f}(U_N^j)$.

It is also important to note that $\frac{1}{f}$ is globally Lipschitz in an interval of the form $[a, \infty)$ with $a > 0$. On the other hand it follows from (3.17) that the evaluations of f in (3.12) are restricted to the interval $[a, \infty)$ with $a = \min_s\{U_s^0\}$. Therefore, we can take $\tilde{f}$ such that $1/\tilde{f}$ is globally Lipschitz, strictly positive and bounded in $[0, \infty]$.

Therefore, in the rest of this section we will work with $\tilde{f}$. However, in order to simplify notation, all the matrices and polynomials associated with the Runge-Kutta methods are denoted as in previous sections although they are constructed using $\tilde{f}$ instead of f.

Throughout this section we will assume the stability restriction $\frac{\lambda}{h^2} \le C_{est}$ needed for the results of the previous section. We remark that the constant C_{est} can be taken depending only on k and U_0. Indeed, the restriction needed is $\frac{\lambda}{h^2} < \frac{1}{8k\delta}$ but, δ is bounded by a constant which depends only on U^0.

Let us first give some lemmas. We will call u^j the vector with components $u_i^j = u(x_i, t(\tau^j))$ where u is the exact solution of (1.1).

Lemma 4.1 *If u has bounded second derivatives in the x variable we have that, for a given l, the components of the vector $V = \frac{1}{h^2}A^l u^j$ satisfy*

$$|V_s| \leq C \qquad if \qquad 1 \leq s \leq N-l$$

and,

$$|V_s| \leq \frac{C}{h} \qquad if \qquad N-l < s \leq N$$

where C is a constant which depends only on u.

Proof: We proceed by induction in l. For $l=1$ we observe that, for $2 \leq s \leq N-1$, V_s is a second centered difference and so, $|V_s|$ can be bounded by a constant depending only on the second derivatives of u. If $s=1$ we use that $u_x(0,t)=0$ and hence,

$$\frac{u_2^j - u_1^j}{h^2} = \frac{u(h,t(\tau^j)) - u(0,t(\tau^j))}{h^2} = u_{xx}(\theta, t(\tau^j))$$

with $0 < \theta < h$. Finally for $s = N$

$$\frac{u_{N-1}^j - u_N^j}{h^2} = -\frac{u(1,t(\tau^j)) - u(1-h,t(\tau^j))}{h^2} = \frac{u_x(\xi, t(\tau^j))}{h}$$

with $1-h < \xi < 1$, and so, the case $l=1$ is proved. For $l > 1$, we write

$$\frac{1}{h^2} A^l u^j = A^{l-1}(\frac{1}{h^2} A u^j)$$

and the result follows by observing that $A^{l-1} e_N$ belongs to the subspace spanned by $\{e_{N-l}, \cdots, e_N\}$. □

Our proof of convergence will be based on comparison arguments. In order to do that we introduce the following function which will be used to construct a supersolution. Let $L > 0$ be a constant (to be chosen below) and define

$$w_L(x) = e^{Lx} - Lx$$

and W_L the vector with components $W_{L,i} = w_L(x_i)$ for $i = 1, \cdots, N$.

We will need the following elementary result on the behavior of w_L.

Lemma 4.2 *Let C_1 be an arbitrary positive constant. Then, there exist $h_0 > 0$ and L, depending only on C_1, such that for any $0 < h < h_0$*

$$\frac{w_L'(1-h)}{w_L(1)} \geq C_1$$

Proof: A direct computation gives

$$c(h,L) := \frac{w_L'(1-h)}{w_L(1)} = \frac{L(e^{L(1-h)} - 1)}{e^L - L}$$

and it is easy to see that

$$lim_{h \to 0} c(h, 1/h) = \infty$$

So, we can take h_0 such that $c(h_0, 1/h_0) \geq C_1$. Then, calling $L = 1/h_0$ we have that $c(h,L) \geq C_1$ for any $h < h_0$, since $C(h,L)$ is decreasing with h. □

For a given Runge-Kutta method with k evaluations as in (3.3) let P be the polynomial arising in the expression (3.4) of the iteration. Let us introduce the matrix $B_k = \frac{1}{\lambda}(P(\mu A) - I)$ (for the sake of clarity we write now explicitly the dependence on k since it plays an important role in the following arguments).

For example, for the Euler method we have $B_1 = \frac{1}{f(U_N^j)} \frac{1}{h^2} A$. We can prove, by the same argument used in Lemma 4.1 that

$$|(B_1 W_L)_s| \leq C_2 \qquad \text{if} \qquad 1 \leq s \leq N-1$$

with C_2 depending on L.

On the other hand, from Lemma 4.2, we have that, for a fixed $C_1 > 0$, there exist h_0 and L depending only on C_1 such that

$$(B_1 W_L)_N \leq -\frac{C_1 W_{L,N}}{h}$$

for any $h < h_0$.

In fact,

$$(B_1 W_L)_N = -\frac{w_L'(\xi)}{h}$$

with $1 - h < \xi < 1$. And using that w_L' is increasing and Lemma 4.2 we obtain

$$(B_1 W_L)_N \leq -\frac{w_L'(1-h)}{h} \leq -\frac{C_1 w_L(1)}{h} = -\frac{C_1 W_{L,N}}{h}$$

In a similar way it can be proved using (3.14) that, for a general Runge-Kutta method of the form (3.3), the following lemma holds. Since the ideas are the same as those used above we leave the details to the reader.

Lemma 4.3 *If $\frac{\lambda}{h^2} \leq C_{est}$ then, for any k and any $C_1 > 0$ there exist h_0, L and C_2 depending on C_1 and on the stability constant C_{est} such that, for $h < h_0$,*

$$|(B_k W_L)_s| \leq C_2 \qquad \textit{if} \qquad 1 \leq s \leq N-k$$

and,

$$(B_k W_L)_s \leq -\frac{C_1 W_{L,N}}{h} \qquad \textit{if} \qquad N-k < s \leq N$$

We can now prove the convergence of the numerical approximation. Moreover, the following theorem gives error estimates in terms of the discretization parameters h and λ and the order of the Runge-Kutta method used. We will give the proof in detail for the Euler and Heun methods and we will indicate how the same ideas can be applied for the general method. We will denote with $\mathbf{1}$ the vector with all its components equal to 1 and, as usual, vector inequalities have to be understood component by component.

Theorem 4.1 *Assume that the solution u of (1.1) has bounded derivatives up to the order k_0+1 in t and $2(k_0+1)$ in x for $t \in [0, T_1)$. Let U^j be the solution of (3.1) and (3.2) obtained by a Runge-Kutta method of order k_0 of the form given in (3.3). If the stability condition $\frac{\lambda}{h^2} \leq C_{est}$ is satisfied then, there exists a constant $C = C(u, T_1)$ such that, for h small enough,*

$$|u(x_s, t(\tau^j)) - U_s^j| \leq C(h^2 + \lambda^{k_0})$$

where $\tau^j = \lambda j$

Proof: Let us consider first the Euler method. For $u \in C^{4,2}$ and $u_i^j = u(x_i, t(\tau^j))$ we have

$$\frac{u^{j+1} - u^j}{\lambda} = \frac{1}{\tilde{f}(u_N^j)} \frac{1}{h^2} A u^j + \frac{2}{h} e_N + O(h^2 + \lambda) \tag{4.1}$$

and so, the error $E_i^j = U_i^j - u_i^j$ satisfies

$$\frac{E^{j+1}-E^j}{\lambda} = \frac{1}{\tilde{f}(U_N^j)}\frac{1}{h^2}AE^j + (\frac{1}{\tilde{f}(U_N^j)} - \frac{1}{\tilde{f}(u_N^j)})\frac{1}{h^2}Au^j + O(h^2+\lambda)$$

Therefore,

$$\frac{E^{j+1}-E^j}{\lambda} \leq \frac{1}{\tilde{f}(U_N^j)}\frac{1}{h^2}AE^j + C|E_N^j||\frac{1}{h^2}Au^j| + C(h^2+\lambda) \tag{4.2}$$

where we have used that $\frac{1}{f}$ is Lipschitz. Then, from Lemma 4.1 we obtain

$$\frac{E^{j+1}-E^j}{\lambda} \leq B_1(E^j) + C|E_N^j|(\mathbf{1} + \frac{C}{h}e_N) + C(h^2+\lambda) \tag{4.3}$$

In the same way one can check that an analogous inequality holds for $-E^j$, namely,

$$\frac{(-E^{j+1})-(-E^j)}{\lambda} \leq B_1(-E^j) + C|E_N^j|(\mathbf{1} + \frac{C}{h}e_N) + C(h^2+\lambda) \tag{4.4}$$

where we can take the same constant C than in (4.3).

On the other hand we have $|E_i^0| \leq Ch^2$, where once again we can take the same C.

Let us now define the function $\Phi(x,\tau) = D_1 e^{D_2\tau} w_L(x)(h^2+\lambda)$. Using Lemma 4.3, it is easy to see that we can choose L and h_0 depending on C and afterwards D_1 and D_2 , depending on C and L but independent of h and λ such that, for $h < h_0$ the vectors W^j defined as $W_i^j = \Phi(x_i, \tau^j)$ satisfy

$$\frac{W^{j+1}-W^j}{\lambda} \geq B_1(W^j) + CW_N^j(\mathbf{1} + \frac{C}{h}e_N) + C(h^2+\lambda) \tag{4.5}$$

and,

$$W_i^0 > Ch^2.$$

Then, calling $Z^j = W^j - |E^j|$, we have that $Z^0 > 0$ and so, subtracting (4.3) from (4.5) and using the same argument as in Lemma 3.4 we obtain that $W^1 \geq E^1$. In the same way, using now (4.4) we obtain that $W^1 \geq -E^1$. Therefore, $W^1 \geq |E^1|$. Clearly, we can proceed inductively to prove that

$$|E_i^j| \leq W_i^j$$

and consequently,

$$|E_i^j| \leq D_1(e^L - L)e^{D_2 T_1}(h^2+\lambda) := C(u, T_1)(h^2+\lambda)$$

which concludes the proof for the Euler method.

Consider now the second order Heun method. In this case we can write

$$\begin{aligned} &\tfrac{U^{j+1}-U^j}{\lambda} = \tfrac{1}{2}(\tfrac{1}{\tilde{f}(U_N^j)} + \tfrac{1}{\tilde{f}(\overline{U}_N^j)})\tfrac{1}{h^2}AU^j + \tfrac{1}{\tilde{f}(U_N^j)\tilde{f}(\overline{U}_N^j)}\tfrac{1}{2}\tfrac{\lambda}{h^4}A^2U^j \\ &+\tfrac{2}{h}e_N + \tfrac{\lambda}{h^3}\tfrac{1}{\tilde{f}(\overline{U}_N^j)}Ae_N \end{aligned} \tag{4.6}$$

with

$$\overline{U}_N^j = U_N^j + \frac{1}{\tilde{f}(U_N^j)}\frac{\lambda}{h^2}(u_{N-1}^j - u_N^j) + \frac{\lambda}{h}.$$

Now, observe that for u smooth enough, $u_i^j = u(x_i, t(\tau^j))$ satisfy,

$$\begin{aligned} &\tfrac{u^{j+1}-u^j}{\lambda} = \tfrac{1}{2}(\tfrac{1}{\tilde{f}(u_N^j)} + \tfrac{1}{\tilde{f}(\overline{u}_N^j)})\tfrac{1}{h^2}Au^j + \tfrac{1}{\tilde{f}(u_N^j)\tilde{f}(\overline{u}_N^j)}\tfrac{1}{2}\tfrac{\lambda}{h^4}A^2u^j \\ &+\tfrac{2}{h}e_N + \tfrac{\lambda}{h^3}\tfrac{1}{\tilde{f}(\overline{u}_N^j)}Ae_N + C(h^2+\lambda^2) \end{aligned} \tag{4.7}$$

with

$$\overline{u}_N^j = u_N^j + \frac{1}{\tilde{f}(u_N^j)} \frac{\lambda}{h^2}(u_{N-1}^j - u_N^j) + \frac{\lambda}{h}.$$

Hence, $E_i^j = U_i^j - u_i^j$ satisfy

$$\begin{array}{l} \frac{E^{j+1}-E^j}{\lambda} = \frac{1}{2}(\frac{1}{\tilde{f}(U_N^j)} + \frac{1}{\tilde{f}(\overline{U}_N^j)})\frac{1}{h^2}AE^j + \\ + \frac{1}{\tilde{f}(U_N^j)\tilde{f}(\overline{U}_N^j)}\frac{1}{2!}\frac{\lambda}{h^4}A^2E^j + \\ + \left(\frac{1}{2}(\frac{1}{\tilde{f}(u_N^j)} + \frac{1}{\tilde{f}(\overline{u}_N^j)}) - \frac{1}{2}(\frac{1}{\tilde{f}(U_N^j)} + \frac{1}{\tilde{f}(\overline{U}_N^j)})\right)\frac{1}{h^2}Au^j + \\ + \left(\frac{1}{\tilde{f}(U_N^j)\tilde{f}(\overline{U}_N^j)} - \frac{1}{\tilde{f}(u_N^j)\tilde{f}(\overline{u}_N^j)}\right)\frac{1}{2!}\frac{\lambda}{h^4}A^2u^j + \\ +(\frac{1}{\tilde{f}(\overline{U}_N^j)} - \frac{1}{\tilde{f}(\overline{u}_N^j)})\frac{\lambda}{h^3}Ae_N + C(h^2+\lambda^2) \end{array} \tag{4.8}$$

Let us observe that, since $1/f$ is Lipschitz, we obtain by using the stability condition that,

$$|\overline{U}_N^j - \overline{u}_N^j| \le \max\{|U_N^j - u_N^j|, |U_{N-1}^j - u_{N-1}^j|\}$$

and so, writing $|E^j| = \max_i |E_i^j|$ and using again that $\frac{1}{\tilde{f}}$ is Lipschitz, we have

$$\begin{array}{l} \frac{E^{j+1}-E^j}{\lambda} \le \frac{1}{2}(\frac{1}{\tilde{f}(U_N^j)} + \frac{1}{\tilde{f}(\overline{U}_N^j)})\frac{1}{h^2}AE^j + \\ + \frac{1}{\tilde{f}(U_N^j)\tilde{f}(\overline{U}_N^j)}\frac{1}{2!}\frac{\lambda}{h^4}A^2E^j + \\ +C|E^j|\frac{1}{h^2}Au^j + \\ +C|E^j|(\frac{\lambda}{h^4})A^2u^j + \\ +C|E^j|\frac{\lambda}{h^3}Ae_N + C(h^2+\lambda^2) \end{array} \tag{4.9}$$

Then, from Lemma 4.1 and the fact that Ae_N belongs to the subspace spanned by $\{e_{N-1}, e_N\}$, it follows that

$$\frac{E^{j+1} - E^j}{\lambda} \le B_2E^j + C|E^j|(\mathbf{1} + \frac{1}{h}e_N + \frac{1}{h}e_{N-1}) + C(h^2+\lambda^2) \tag{4.10}$$

As in the Euler case, we have an analogous estimate for $-E^j$. Taking again $\Phi(x,\tau)$ it follows from Lemma 4.3 that we can choose L, h_0, D_1 and D_2, depending on C but independent of λ and h such that, for $h < h_0$, $W_i^j = \Phi(x_i, t_j)$ satisfy

$$\frac{W^{j+1} - W^j}{\lambda} \ge B_2(W^j) + CW_N^j(\mathbf{1} + \frac{1}{h}e_N + \frac{1}{h}e_{N-1}) + C(h^2+\lambda^2) \tag{4.11}$$

Hence, using again a comparison argument we obtain by induction,

$$|E^j| \le W^j \le C(u, T_1)(h^2+\lambda^2).$$

The proof for the general method follows the same lines as the cases described above. Indeed, for a method of the form (3.3) of order k_0, the idea is to show that the error vectors E^j satisfy

$$\frac{E^{j+1} - E^j}{\lambda} \le B_k(E^j) + C|E^j|(\mathbf{1} + \frac{1}{h}e_N + \cdots + \frac{1}{h}e_{N-k+1}) + C(h^2+\lambda^{k_0}) \tag{4.12}$$

As in the previous cases, an analogous estimate holds for $-E^j$. These inequalities are obtained by estimating the consistency error and using Lemma 4.1.

Once more defining $W_i^j = \Phi(x_i, t_j)$ we can choose the constants L, h_0, D_1 and D_2 such that, for $h < h_0$,

$$\frac{W^{j+1} - W^j}{\lambda} \ge B_k(W^j) + CW_N^j(\mathbf{1} + \frac{1}{h}e_N + \cdots + \frac{1}{h}e_{N-k+1}) + C(h^2+\lambda^{k_0}) \tag{4.13}$$

and the proof concludes by comparison.□

5 Blow-up of the numerical solution

In this section we obtain a necessary and sufficient condition for the existence of blow-up of the discrete solution.

We have shown in Section 3 that, for h and λ fixed, at least one component of the discrete solution U^j goes to infinity when $j \to \infty$. To obtain this result we have worked with (3.2) without taking into account the discretization of (3.1). This was possible because the system (3.2) does not depend explicitly on t.

In order to analize now the behavior of the "true" time variable we have to consider also the numerical integration of the equation (3.1). A general Runge-Kutta method with k evaluations applied to (3.1) gives

$$\Delta t^j = t^{j+1} - t^j = \lambda \sum_{s=1}^{k} \gamma_s \delta_s \tag{5.1}$$

with $\gamma_s \geq 0$ such that $\sum_0^k \gamma_s = 1$ and δ_s given by (3.12). Note that, as in previous sections, we have dropped the dependence on j to simplify the notation.

For example, for the Euler method we have

$$\Delta t^j = \frac{\lambda}{f(U_N^j)}$$

The following theorem gives a necessary and sufficient condition for the existence of a finite value $T_{h,\lambda}$ such that

$$t^j \to T_{h,\lambda}$$

If such value exists we say that the discrete solution blows up and that $T_{h,\lambda}$ is the discrete blow-up time.

The blow-up condition obtained here is the same that appears in the semidiscrete case analyzed in ([5]) but it is slightly different from the one of the continuous problem which is $\int^{+\infty} 1/ff' < +\infty$, ([10]).

Theorem 5.1 *If the stability condition $\frac{\lambda}{h^2} \leq C_{est}$ then, the solution of (3.2) and (3.1) with positive initial U^0 obtained by a general Runge-Kutta method blows up if and only if $\int^{+\infty} 1/f < +\infty$.*

Proof: Clearly, t^j will have a finite limit if and only if $\sum_{m=0}^{\infty} \Delta t^m < \infty$.

From 3) of Theorem 3.1 we know that there exists j_0 such that, for $j \geq j_0$, we have

$$\frac{1}{f(U_N^j + \frac{3\lambda}{h})} \leq \delta_s \leq \frac{1}{f(U_N^j)}$$

and so, since $\gamma_s \geq 0$ and $\sum_0^k \gamma_s = 1$, it follows from (5.1) that, for $j \geq j_0$

$$\frac{\lambda}{f(U_N^j + \frac{3\lambda}{h})} \leq \Delta t^j \leq \frac{\lambda}{f(U_N^j)} \tag{5.2}$$

On the other hand, from (3.19), we obtain by induction that, for any $m \geq 0$

$$U_N^{j_0} + m\frac{\lambda}{h} \leq U_N^{j_0+m} \leq U_N^{j_0} + m\frac{3\lambda}{h}$$

Therefore, it follows from (5.2) and the fact that f is increasing, that

$$\frac{\lambda}{f(U_N^{j_0} + m\frac{3\lambda}{h})} \leq \Delta t^{j_0+m} \leq \frac{\lambda}{f(U_N^{j_0} + m\frac{\lambda}{h})}$$

Then,

$$\sum_{m=1}^{\infty} \frac{\lambda}{f(U_N^{j_0} + m\frac{3\lambda}{h})} \leq \sum_{m=1}^{\infty} \Delta t^{j_0+m} \leq \sum_{m=1}^{\infty} \frac{\lambda}{f(U_N^{j_0} + m\frac{\lambda}{h})}$$

and using again that f is increasing it is easy to see that this last inequality implies

$$h \int_{U^{j_0}_{N+\frac{3\lambda}{h}}}^{\infty} \frac{1}{f(s)} ds \le \sum_{m=1}^{\infty} \Delta t^{j_0+m} \le h \int_{U_N^{j_0}}^{\infty} \frac{1}{f(s)} ds$$

which concludes the proof.□

6 Convergence of the blow-up time.

We know from [5] that, provided that $(u_0)_{xx} \ge \alpha > 0$, the numerical blow-up time T_h for the semidiscretization in space given by (1.2) converges to the continuous blow-up time T when the space discretization parameter h goes to zero.

Since (1.2) is equivalent to (3.2) through the change of variables (3.1), we can conclude that the former has blow-up if and only if $lim_{\tau\to\infty} t(\tau) < \infty$, moreover we have

$$lim_{\tau\to\infty} t(\tau) \nearrow T_h \tag{6.1}$$

Let us call $U_N(\tau)$ the value of the semidiscretization at $x_N = 1$.

Theorem 6.1 *Assume that $(u_0)_{xx} \ge \alpha > 0$ and that $\int^{+\infty} 1/f < +\infty$. Let T be the blow-up time of the continuous problem (1.1) and $T_{h,\lambda}$ the blow-up time of the discretization obtained by a Runge-Kutta method of the form (3.3). Then,*

$$\lim_{h\to 0} \lim_{\lambda\to 0} T_{h,\lambda} = T.$$

Proof: Since we know that $T_h \to T$ when $h \to 0$, we have to show that, for a fixed h,

$$\lim_{\lambda\to 0} T_{h,\lambda} = T_h$$

Let us recall that

$$T_{h,\lambda} = lim_{j\to\infty} t^j = \sum_{j=0}^{\infty} \Delta t^j$$

From 3) of Theorem 3.1 we know that, there exists a constant K depending on h, but independent of λ, such that if $U_N^j > K$ then,

$$U_N^{j+1} \ge U_N^j + \frac{\lambda}{h} \tag{6.2}$$

and,

$$\delta_s \le \frac{1}{f(U_N^j)}$$

Therefore, in view of (5.1) we have for $U_N^j > K$ that

$$\Delta t^j = t^{j+1} - t^j \le \frac{\lambda}{f(U_N^j)}$$

and so by using (6.2) we obtain

$$\Delta t^{j+m} \le \frac{\lambda}{f(U_N^j + m\frac{\lambda}{h})} \tag{6.3}$$

Now, given $\varepsilon > 0$ we can choose K, depending also on ε such that

$$\int_K^{+\infty} \frac{1}{f(s)} ds \leq \frac{\varepsilon}{3} \tag{6.4}$$

On the other hand, in view of (6.1), we can take $\overline{\tau}$ depending on ε and h such that

$$T_h - t(\overline{\tau}) \leq \frac{\varepsilon}{3} \tag{6.5}$$

and,

$$U_N(\overline{\tau}) \geq 2K \tag{6.6}$$

Now, up to the time $t(\overline{\tau})$ the problem is regular (i.e., we are below the blow-up time) and so, for h fixed, it follows from the standard convergence results for Runge-Kutta methods that, for λ small enough depending on h and $\overline{\tau}$ (and so, on h and ε) that

$$|t(\overline{\tau}) - t^{j_0+1}| = |t(\overline{\tau}) - \sum_{j=0}^{j_0} \Delta t_j| < \frac{\varepsilon}{3} \tag{6.7}$$

for j_0 such that $j_0\lambda \leq \overline{\tau} < (j_0+1)\lambda$.

On the other hand, using again the convergence of the scheme below the blow-up time we have, in view of (6.6), that for λ small enough depending on ε and h,

$$U_N^{j_0} > K$$

and so, from (6.2), $U_N^j > K$ for every $j \geq j_0$.

Then, from (6.3) and (6.4) we have

$$\sum_{j=j_0+1}^{\infty} \Delta t^j \leq h \int_K^{\infty} \frac{1}{f(s)} ds \leq \frac{\varepsilon}{3} \tag{6.8}$$

and the proof concludes by writing

$$|T_h - T_{h,\lambda}| \leq |T_h - t(\overline{\tau})| + |t(\overline{\tau}) - t^{j_0+1}| + |t^{j_0+1} - T_{h,\lambda}|$$

and using (6.5), (6.7) and (6.8). □

7 Localization of blow-up points.

In Section 5 we have shown that, under the same condition needed for the blow up of the semidiscrete problem, the discrete solution blows up at least in one point. The object of this section is to analize the asymptotic behavior and the localization of blow-up for the discrete solution for h and λ fixed. In order to simplify the exposition we will restrict our analysis to the Euler method.

For the continuous problem it is known that the blow-up set is localized at the boundary provided that $\int^{+\infty} 1/f < +\infty$ (i. e., $u(x,t)$ remains bounded for every $x \neq 1$, see [8]). We will show that the behavior of the discrete solution is different. Indeed, the number of nodes such that the numerical solution go to infinity can be greater than one and is determined by f. However, for $f(s) = s^p$ with $p > 1$ the blow up is essentially local in the sense that the number of nodes in which the numerical solution blows up is independent of h.

We will use the notation $a(j) \sim b(j)$ to indicate that there exist positive constants c, C, independent of j, such that $ca(j) \leq b(j) \leq Ca(j)$.

To state our theorem we introduce first some notation. We set

$$F_1(j) = \sum_{i=1}^{j} \frac{i}{f(i)}$$

and for any $k \geq 1$,

$$F_{k+1}(j) = \sum_{i=1}^{j} \frac{F_k(i)}{f(i)}$$

Theorem 7.1 *Assume that U^0 is increasing, that f satisfies $\int^{+\infty} 1/f < +\infty$ (in particular, the discrete solution blows up), and that for every $b > 0$ there exists positive constants c_1, c_2 and s_0 depending on b such that*

$$c_1 \leq \frac{f(bs)}{f(s)} \leq c_2 \qquad \forall\ s \geq s_0 \tag{7.1}$$

If U_i^j is the solution of (3.2) obtained by the Euler method with the discretization parameters satisfying the stability condition $\frac{\lambda}{h^2} \leq C_{est}$ then,

$$U_N^j \sim j \qquad \text{and} \qquad \Delta t^j \sim \frac{1}{f(j)}$$

U_{N-1}^j goes to infinity if and only if $\lim_{j\to+\infty} F_1(j) = +\infty$ and,

$$U_{N-1}^j \sim F_1(j).$$

In general, U_{N-k}^j goes to infinity if and only if $\lim_{j\to+\infty} F_k(j) = +\infty$ and moreover,

$$U_{N-k}^j \sim F_k(j).$$

We remark that the theorem states that the discrete solution may go to infinity in more than one node but, the asymptotic behavior proven shows that $\frac{U_i^j}{U_{i+1}^j} \to 0$ when $j \to \infty$ (this will be shown in the proof of the theorem).

An interesting example is the case $f(s) = s^p$, $(p > 1)$. For this problem the theorem says that the number of blow-up points is finite and depends on the power p but is independent of h. This gives a sort of "numerical localization" of the blow-up set near $x = 1$ when $h \to 0$. Indeed, for this case we have the following,

Corollary 7.1 *Let $f(s) = s^p$ $(p > 1)$ then if $\frac{m+2}{m+1} < p \leq \frac{m+1}{m}$, the discrete solution of (3.2) obtained by the Euler method satisfies*

$$U_{N-l}^j \to_{j\to\infty} +\infty, \qquad l = 1, \cdots, m$$
$$\text{but} \qquad U_{N-(m+1)}^j \qquad \text{is bounded.}$$

Proof: The property (7.1) is obvious in this case. Then the result follows by analyzing the behavior of the functions F_k which can be easily done using that $F_1(j) \sim \int_1^j s^{1-p}\, ds$ and analogous equivalences for all the F_k.□

Another interesting example is given by the function $f(s) = s(\ln s)^p$, $(p > 1)$. In this case, it follows from the theorem that all the U_i^j go to infinity as $j \to \infty$. Indeed, the condition (7.1) is easily verified and it is not difficult to see that in this case $\lim_{j\to\infty} F_k(j) = \infty$ for all k. For example,

$$F_1(j) = \sum_{i=1}^{j} \frac{1}{(ln\, i)^p} \geq \frac{j}{(ln\, j)^p}$$

hence $F_1(j) \to \infty$ and,

$$F_2(j) = \sum_{i=1}^{j} \frac{F_1(i)}{f(i)} \geq \sum_{i=1}^{j} \frac{1}{(ln\, i)^{2p}} \geq \frac{j}{(ln\, j)^{2p}}$$

and therefore $F_2(j) \to \infty$ and we can proceed in the same way to show that $F_k(j) \to \infty$ for all k.

So, in this example, the numerical solution blows up at every point in $[0, 1]$. However, when $p > 2$, it is known that the continuous solution blows up only at $x = 1$.

We remark once again that, in this section, we consider positive solutions of (2.2) with h and λ fixed. Then, we denote with C a positive constant that may depend on h and λ and may vary from one line to another, but is independent of j.

Proof of Theorem 7.1 From (3.19) we know that

$$U_N^j \sim j$$

so, using (7.1) we obtain

$$\Delta t^j = \frac{\lambda}{f(U_N^j)} \sim \frac{1}{f(j)}$$

Now, U_{N-1}^j satisfies

$$U_{N-1}^{j+1} = U_{N-1}^j + \Delta t^j \frac{1}{h^2}(U_N^j - 2U_{N-1}^j + U_{N-2}^j)$$

So, using Lemma 3.2 we obtain

$$U_{N-1}^{j+1} \leq U_{N-1}^j + C\Delta t^j U_N^j$$

and then,

$$U_{N-1}^j - U_{N-1}^1 \leq C \sum_{i=1}^{j-1} \Delta t^i U_N^i \sim C \sum_{i=1}^{j} \frac{i}{f(i)} = CF_1(j)$$

Therefore, if $F_1(j)$ remains bounded, so does U_{N-1}^j. But if $F_1(j) \to \infty$ we have that

$$\frac{U_{N-1}^j}{U_N^j} \to 0 \qquad j \to \infty$$

Indeed, since $U_{N-1}^j \leq CF_1(j)$ it follows that

$$\frac{U_{N-1}^j}{U_N^j} \leq C\frac{F_1(j)}{j}$$

but, $\frac{F_1(j)}{j} \to 0$. In fact, this follows from the fact that f is superlinear (i.e., $\frac{j}{f(j)} \to 0$) and the elementary result saying that if a sequence a_j converges to zero then, $\frac{1}{j}\sum_{i=1}^{j} a_i$ also converges to zero.

Hence, U_{N-1}^j verifies

$$U_{N-1}^{j+1} - U_{N-1}^j = \Delta t^j \frac{1}{h^2}(U_N^j - 2U_{N-1}^j + U_{N-2}^j) \sim \Delta t^j U_N^j \sim \frac{j}{f(j)}$$

So, we can conclude that $U_{N-1}^{j} \to \infty$ and that

$$U_{N-1}^{j} \sim \sum_{i=1}^{j} \frac{i}{f(i)} = F_1(j)$$

Now we look at U_{N-2}. We have that

$$U_{N-2}^{j+1} = U_{N-2}^{j} + \Delta t^j \frac{1}{h^2}(U_{N-1}^{j} - 2U_{N-2}^{j} + U_{N-3}^{j}) \leq U_{N-2}^{j} + C\Delta t^j U_{N-1}^{j}$$

Hence,

$$U_{N-2}^{j} - U_{N-2}^{1} \leq C \sum_{i=1}^{j} \frac{U_{N-1}^{i}}{f(U_N^i)} \sim F_2(j)$$

If $F_2(j)$ is bounded then, the same is true for U_{N-2}^{j}, but if $F(j) \to \infty$ we have that

$$\frac{U_{N-2}^{j}}{U_{N-1}^{j}} \to 0, \qquad j \to \infty.$$

Indeed, this follows from the estimate

$$\frac{U_{N-2}^{j}}{U_{N-1}^{j}} \leq C\frac{F_2(j)}{F_1(j)}$$

and the elementary result, known as Stolz theorem, which says that if a_j and b_j are two positive sequences such that a_j/b_j converges to zero and $\sum b_j$ is a divergent series then, $\frac{\sum_{i=1}^{j} a_i}{\sum_{i=1}^{j} b_i}$ also converges to zero.

Then,

$$\begin{aligned} U_{N-2}^{j+1} - U_{N-2}^{j} &= \Delta t^j \frac{1}{h^2}(U_{N-1}^{j} - 2U_{N-2}^{j} + U_{N-3}^{j}) \\ &\sim \Delta t^j U_{N-1}^{j} \sim \frac{1}{f(j)} \end{aligned}$$

and this implies that

$$U_{N-2}^{j} \sim F_2(j)$$

We can repeat this procedure as long as $\lim_{j\to+\infty} F_k(j) = +\infty$ concluding the proof of the theorem. □

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