WEIGHTED POINCARÉ AND KORN INEQUALITIES FOR HÖLDER α DOMAINS

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Abstract. It is known that the classic Korn inequality is not valid for Hölder α domains. In this paper we prove a family of weaker inequalities for this kind of domains, replacing the standard L^p-norms by weighted norms where the weights are powers of the distance to the boundary.

In order to obtain these results we prove first some weighted Poincaré inequalities and then, generalizing an argument of Kondratiev and Oleinik, we show that weighted Korn inequalities can be derived from them.

The Poincaré type inequalities proved here improve previously known results.

We show by means of examples that our results are optimal.

1. Introduction

The Korn inequality is a fundamental tool in the analysis of the linear elasticity equations and has been the object of many papers since the works of Korn [14, 15].

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. For a displacement field $u \in W^{1,p}(\Omega)^n$ we denote with $\varepsilon(u)$ the linear part of the strain tensor, namely,

$$\varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$$

The classic Korn inequality states that

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|\varepsilon(u)\|_{L^2(\Omega)}. \tag{1.1}$$

To obtain this inequality appropriate conditions on the field u have to be imposed in order to remove the non constant infinitesimal rigid motions (i.e., fields u such that the right hand side vanishes while the left one does not). The two conditions considered by Korn were $u = 0$ on $\partial\Omega$ (usually called *first case*), and $\int_\Omega \operatorname{rot} u = 0$ (*second case*). These two cases correspond to essential and natural boundary conditions for the elasticity equations respectively. It is known that (1.1) can be derived in both cases (as well as in the more general case corresponding to mixed type boundary conditions), by using compactness arguments, from the following inequality (which we state in the more general case of L^p, $1 < p < \infty$):

$$\|\nabla u\|_{L^p(\Omega)} \leq C\{\|u\|_{L^p(\Omega)} + \|\varepsilon(u)\|_{L^p(\Omega)}\} \tag{1.2}$$

for all $u \in W^{1,p}(\Omega)^n$.

After the works of Korn a lot of different arguments to prove the inequality in its different forms have been developed by several authors, see for example [7, 8, 13, 17, 18, 5], the books [2, 12] and their references, and the survey article [11].

For the first case it is known that the inequality (1.1) is valid for any domain (see for example [12]). However, the situation is quite different in the second case or for the general inequality (1.2). This inequality has been proved for bounded Lipschitz domains (see for

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example [12, 17]) and more recently for the more general class of bounded extension domains of P. Jones [5]. On the other hand, it is known that (1.2) is not valid for an arbitrary bounded domain. Indeed, counter-examples showing that the inequality does not hold true for domains with external cusps has been given in [9, 19]. Also, in the old paper [7], Friedrichs gave a very nice counter-example for an inequality for complex analytic functions which can be derived from (1.1) in the second case.

In view of the counter-examples mentioned above it is natural to ask whether a weaker inequality similar to (1.2) can be obtained for domains with external cusps. To give an answer to this question we consider in this paper weighted norms where the weights are powers of the distance to the boundary. We prove, for example, the following generalized Korn inequality:

Let Ω be a Hölder α domain (i.e., $\partial\Omega$ is locally the graph of a Hölder α function in an appropriate coordinate system), $0 < \alpha \leq 1$, and, for $x \in \Omega$, denote with $d(x)$ the distance of x to $\partial\Omega$ then, for $1 < p < \infty$, there exists a constant C which depends only on Ω and p such that

$$\|d^{1-\alpha}\nabla u\|_{L^p(\Omega)} \leq C\left\{\|\varepsilon(u)\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right\} \tag{1.3}$$

Observe that in particular, when Ω is a Lipschitz domain (i.e., α=1), we recover (1.2).

In fact we obtain a more general inequality where part or all the weight can be put on the right hand side, see Theorem 3.1, but we present here this particular case to simplify notation in the introduction. Also we show by an example that our result is optimal in the sense that an analogous inequality with a lower power of d is not true.

Our proof of the generalized Korn inequality is based on the arguments introduced by Kondratiev and Oleinik in [13] to prove the classic inequality. Indeed, by a generalization of their method we show that weighted Korn inequalities can be derived from some appropriate weighted Poincaré inequalities. For example, (1.3) is a consequence of

$$\|d^{1-\alpha}f\|_{L^p(\Omega)} \leq C\|d\nabla f\|_{L^p(\Omega)} \tag{1.4}$$

for functions f satisfying some vanishing weighted average condition.

As we will see, the arguments to derive Korn from Poincaré inequalities applies for arbitrary bounded domains. Therefore, our problem is reduced to obtain weighted Poincaré inequalities for Hölder α domains. Estimates of this kind were obtained in [1] by using results on weighted Sobolev spaces given in the book [16]. However, the Poincaré estimates obtained in this way are not optimal. Indeed, using Theorem 19.7 of [16, page 272] and reproducing the arguments of [1], one can only obtain an estimate like (1.4) but with a power of the distance on the left hand side higher than $1-\alpha$. So, in order to obtain the optimal weighted Poincaré inequalities needed for our purposes, we generalize the "conning argument" introduced in [3] which allows one to obtain new Poincaré inequalities from known ones in higher dimensions.

The rest of the paper is organized as follows: Section 2 deals with the weighted Poincaré inequalities. Although, as mentioned above, our motivation for these estimates are the generalized Korn inequalities, we believe that they are of interest in themselves, in particular they improve previously known results and moreover they are optimal. In Section 3, we derive the weighted Korn inequalities, we show that they are optimal and finally, we show how the results on compactness given in [19] can be derived from our inequalities by using imbedding theorems for weighted Sobolev spaces proved in [16].

2. Weighted Poincaré inequalities

In this section we prove some weighted Poincaré inequalities for Hölder α domains. Our proof is based on the arguments introduced in [3].

We begin by recalling an equivalent characterization of Hölder α domains. Given α such that $0 < \alpha \leq 1$ we set $\gamma = 1/\alpha$.

Definition 2.1. *A set $C \subset \mathbb{R}^m$ is an α-cusp, if there exist an $h > 0$ and some neighborhood of the origin $S_C \subset \mathbb{R}^{m-1}$ such that, in some orthogonal coordinate system $(x_1, \cdots, x_m)$,*

$$C = \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : 0 < x_m < h, x_m^{-\gamma} x' \in S_C\}$$

In some cases, we will work also with an analogous definition but choosing another variable x_j in place of x_m.

Remark 2.1. *It can be seen that a bounded open set $A \subset \mathbb{R}^m$ is a Hölder α domain if and only if for any $x_0 \in \partial A$ there exists a neighborhood U of x_0 such that $x + C \subset A$ for all $x \in U \cap \overline{A}$. For the Lipschitz case (i. e. $\alpha = 1$) this is proved for example in* [10]*. It is not difficult to see that similar arguments apply for $0 < \alpha < 1$.*

Given a subset $A \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$ we denote with $d_A(x)$ the distance from x to the boundary of A.

For $A \subset \mathbb{R}^n$, $k \in \mathbb{N}$, and $0 \leq t \leq 1$ we introduce as in [3],

$$A^{k,t} = \{(x, y) \in A \times \mathbb{R}^k : |y| < d_A(x)^t\}$$

For simplicity we assume that our domain Ω has diameter less than one (we can scale the original domain in order to satisfy this requirement).

Lemma 2.1. *If $(x, y) \in \Omega^{k,t}$ then $d_{\Omega^{k,t}}(x, y) \leq d_\Omega(x)$.*

Proof. Given $y \in \mathbb{R}^k$, let $\Omega_y = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in \Omega\}$. If $t = 0$ we have, for all $(x, y) \in \Omega^{k,0}$

$$d_\Omega(x) = d_{\Omega_y}(x, y) \geq d_{\Omega^{k,0}}(x, y).$$

On the other hand, since the diameter of Ω is less than one, we have that for $0 \leq t \leq 1$,

$$\Omega^{k,t} \subset \Omega^{k,0}$$

and therefore, for $(x, y) \in \Omega^{k,t}$,

$$d_{\Omega^{k,t}}(x, y) \leq d_{\Omega^{k,0}}(x, y) \leq d_\Omega(x)$$

(since $d_{\Omega^{k,0}}$ is a cilynder of section $d_\Omega(x)$) as we wanted to prove. □

The next lemma allows us to apply for Hölder α domains the ideas introduced in [3]. Since the proof is rather technical, we give all the details only for the two dimensional case. However, it is not difficult to see that the arguments can be extended to higher dimensions.

Lemma 2.2. *If $\Omega \subset \mathbb{R}^n$ is a Hölder α domain then $\Omega^{k,t} \subset \mathbb{R}^{n+k}$ is also Hölder α*

Proof. We need to construct for any point of $\partial\Omega^{k,t}$ a neighborhood and an α-cusp in such a way that the translations quoted in Remark 2.1 are contained in $\Omega^{k,t}$. In order to do that we will decompose $\partial\Omega^{k,t}$ in two parts: a middle one, consisting in a thin strip containing the set $(\partial\Omega, 0) := \{(x, 0) : x \in \partial\Omega\} \subset \partial\Omega^{k,t}$ and its complement. We divide the proof in three steps, the first two steps deal with the middle part of the boundary. In the last step we prove that the complement of the middle part is in fact smoother, showing that it is locally the graph of a Lipchitz function.

1) Let $x_0 \in \partial\Omega$. Since Ω is Hölder α there exists a neighborhood $U \subset \mathbb{R}^n$ of x_0 and an α-cusp C such that $x + C \subset \Omega$ for all $x \in U \cap \Omega$. The aim of this part is to prove the following

Claim 1: There exists an α-cusp $D \subset \mathbb{R}^{n+k}$ such that $(x, 0) + D \subset \Omega^{k,t}$ for all $x \in U \cap \Omega$.

If

$$C = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < x_n < h, x_n^{-\gamma} x' \in S_C\},$$

we define

$$D = \{(x', x_n, x'') \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^k : 0 < x_n < \frac{h}{3}, x_n^{-\gamma}(x', x'') \in S_D\}$$

where

$$S_D = \{(x', x'') \in \mathbb{R}^{n-1} \times \mathbb{R}^k : x' \in S_C, |x''| < d_C^t((\frac{h}{3})^\gamma x', \frac{h}{3})\}.$$

Observe that it is enough to show

$$D \subset C^{k,t}, \tag{2.5}$$

indeed, this inclusion gives

$$x + D \subset x + C^{k,t} = (x + C)^{k,t} \subset \Omega^{k,t}$$

as it is stated in *Claim 1*.

In order to show (2.5), let us consider $(x', a, x'') \in D$. Then, by definition, $0 \le a \le \frac{h}{3}$ and $a^{-\gamma}(x', x'') \in S_D$, so $a^{-\gamma}x' \in S_C$ and hence

$$(x', a) \in C. \tag{2.6}$$

On the other hand, from the definitions of D and S_D, we know that $|a^{-\gamma}x''| \le d_C^t((\frac{h}{3})^\gamma a^{-\gamma}x', \frac{h}{3})$, that is $|x''| \le a^\gamma d_C^t((\frac{h}{3})^\gamma a^{-\gamma}x', \frac{h}{3})$, and then we have to prove

$$a^\gamma d_C^t\left(\left(\frac{h}{3}\right)^\gamma a^{-\gamma}x', \frac{h}{3}\right) \le d_C^t(x', a). \tag{2.7}$$

Observe that it is enough to consider the case $t = 1$, since $a < h < 1$. To simplify notation, we prove this inequality for $n = 2$ (as mentioned above, similar arguments apply in the general case). Then we can assume that $S_C = [-b, b]$ $(b > 0)$ and x_0 is the origin. Let

$$f(x_1) = a^\gamma d_C\left(\left(\frac{h}{3}\right)^\gamma a^{-\gamma}x_1, \frac{h}{3}\right) - d_C(x_1, a), \qquad 0 \le x_1 \le ba^\gamma.$$

The function d_C is differentiable in $C - \{x_1 = 0\}$ and so f is differentiable in $\{x_1 \ne 0\}$. We have

$$f'(x_1) = \left(\frac{h}{3}\right)^\gamma \frac{\partial d_C}{\partial x_1}\left(\left(\frac{h}{3}\right)^\gamma a^{-\gamma}x_1, \frac{h}{3}\right) - \frac{\partial d_C}{\partial x_1}(x_1, a)$$

Now, let $(\eta, \xi) \in C$ with $0 \le \xi \le h$ (and then $0 \le \eta \le b\xi^\gamma$) and let $0 < \hat{\beta} < \frac{\pi}{2}$ be the acute angle between the axis x_1 and the line L passing through (η, ξ) which is orthogonal to the graph of $x_2 = b^{-\alpha}x_1^\alpha$ (see Figure 1).

Since $0 \le \xi \le \frac{h}{3}$, the distance $d_C(\eta, \xi)$ is realized along the line L, and hence

$$\frac{\partial d_C}{\partial L} = -1 \qquad \frac{\partial d_C}{\partial L^\perp} = 0$$

where ∂L is understood as the outward direction along L and $L^\perp$ stands for the orthogonal line to L. It follows

$$\frac{\partial d_C}{\partial x_1} = -\cos\hat{\beta}.$$

If $\hat{\beta}_a$ and $\hat{\beta}_{\frac{h}{3}}$ are the acute angles corresponding to the points (x_1, a) and $((\frac{h}{3})^\gamma a^{-\gamma}x_1, \frac{h}{3})$ respectively, it is easy to see that $\hat{\beta}_{\frac{h}{3}} > \hat{\beta}_a$ and then

$$f'(x_1) = -\left(\frac{h}{3}\right)^\gamma \cos\hat{\beta}_{\frac{h}{3}} + \cos\hat{\beta}_a \ge 0$$

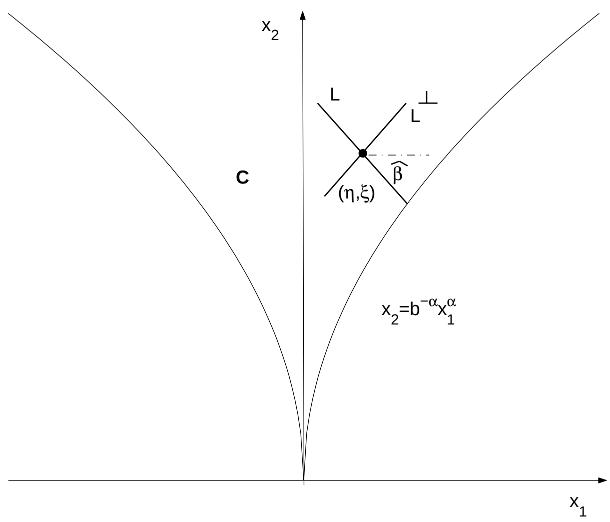

FIGURE 1

since $\frac{h}{3} \leq 1$. Besides $f(ba^\gamma) = 0$, and then $f(x_1) < 0$ for $0 \leq x_1 \leq ba^\gamma$, so inequality (2.7) holds for all $(x_1, a) \in C$. Hence (2.5) is true and the first step is proved.

2) Now we prove the following

Claim 2: Given $x_0 \in \partial\Omega$ *there exists a neighborhood* $V \subset \mathbb{R}^{n+k}$ *of* $(x_0, 0)$ *and a* α*-cusp* D *such that* $(x, y) + D \subset \Omega^{k,t}$ *for all* $(x, y) \in V \cap \Omega^{k,t}$.

Given $x_0 \in \partial\Omega$ let U and C be as in the previous step and define $V = U \times \mathbb{R}^k$. For $0 < t < 1$ we can modify, if necessary, U and C in such a way that

$$\operatorname{diam}(U) < \frac{1}{2} t^{\frac{1}{1-t}} \qquad \text{and} \qquad \operatorname{diam}(C^{k,1}) < \frac{1}{2} t^{\frac{1}{1-t}}. \tag{2.8}$$

By step 1 we know that $C^{k,1}$ contains an α-cusp D, hence, in order to prove *Claim 2* we will show that

$$(v, w) + C^{k,1} \subset \Omega^{k,t} \tag{2.9}$$

for all $(v, w) \in V \cap \Omega^{k,t}$. For such a (v, w) let $(x, y) \in (v, w) + C^{k,1}$ with $v_n - x_n \leq \frac{h}{3}$ and $y = w + \tilde{y}$. Then $|\tilde{y}| \leq d_{v+C}(x)$, and therefore $|y| = |\tilde{y} + w|$ verifies

$$|y| \leq d_\Omega^t(v) + d_{v+C}(x) \leq (d_\Omega(v) + d_{v+C}(x))^t. \tag{2.10}$$

In the case $0 < t < 1$ the last inequality follows from the conditions (2.8). Let $\tilde{x} \in \partial\Omega$ such that $d_\Omega(x) = \operatorname{dist}(x, \tilde{x})$ (see Figure 2) and let $\hat{x} = \overline{x\tilde{x}} \cap \partial C$. Finally, let $\tilde{v}$ such that $|v - \tilde{v}| = d_\Omega(v)$ and $\overline{x\tilde{x}}$ is parallel to $\overline{v\tilde{v}}$. Then, it follows that

$$d_\Omega(x) = |x - \tilde{x}| = |x - \hat{x}| + |\hat{x} - \tilde{x}| \geq d_{v+C}(x) + |v - \tilde{v}| = d_{v+C}(x) + d_\Omega(v). \tag{2.11}$$

From (2.10) and (2.11) we have $|y| \leq d_\Omega(x)^t$ and then (2.9) holds.

3) By step 2, for each $x \in \partial\Omega$ there exists a neighborhood $U_x \subset \mathbb{R}^{n+k}$ of $(x, 0)$ such that $\partial\Omega^{k,t} \cap U_x$ is the graph of a Hölder α function. We have that $(\partial\Omega, 0) \subset \cup_{x\in\partial\Omega} U_x$ and then we can extract $\{x_i\}_{i=1}^r$ such that $(\partial\Omega, 0) \subset \cup_{i=1}^r U_{x_i}$. On the other hand there exists $\varepsilon > 0$ such that $\{(x, y) \in \Omega^{k,t} : d_\Omega(x) < 2\varepsilon\} \subset \cup_{i=1}^r U_{x_i}$. In order to conclude the proof of the Lemma we will show

Claim 3: The set $\{(x, y) \in \partial\Omega^{k,t} : d_\Omega(x) > \varepsilon\}$ *is locally the graph of a Lipschitz function.*

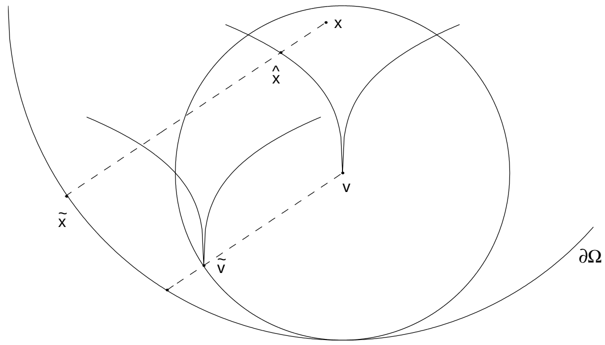

FIGURE 2

Let $(x, y) \in \partial\Omega^{k,t}$ with $d_\Omega(x) > \varepsilon$. Since $y_1^2 + \ldots + y_k^2 = d(x)^{2t}$ we can suppose that $|y_1|, \ldots, |y_{k-1}| < d_\Omega(x)^t$ and $y_k^2 \geq \frac{d_\Omega(x)^{2t}}{k}$. Then $y_1^2 + \ldots + y_{k-1}^2 = d_\Omega(x)^{2t} - y_k^2 \leq (1 - \frac{1}{k}) d_\Omega(x)^{2t}$. Define

$$D = \left\{ (\eta, \xi') \in \mathbb{R}^n \times \mathbb{R}^{k-1} : |x - \eta| < \frac{d_\Omega(x)}{2}, \xi_1^2 + \ldots + \xi_{k-1}^2 < \left(1 - \frac{1}{2k}\right) d_\Omega(\eta)^{2t} \right\}.$$

Thus, $D \subset \mathbb{R}^n \times \mathbb{R}^{k-1}$ is a neighborhood of (x, y'), where $y = (y', y_k)$. Then we consider the function $f : D \to \mathbb{R}$ defined by

$$\xi_k = f(\eta, \xi') = \sqrt{d(\eta)^{2t} - \xi_1^2 - \ldots - \xi_{k-1}^2}.$$

It can be seen that, f is a Lipschitz function. Indeed, in view of

$$d_\Omega(\eta)^{2t} - |\xi'|^2 \geq \frac{1}{2k} d(\eta)^{2t} \geq \frac{1}{2k} \varepsilon^{2t}$$

for all $(\eta, \xi') \in D$, if Λ_1 is such that $|\sqrt{a} - \sqrt{b}| \leq \Lambda_1 |a - b|$ when $a, b \geq \frac{1}{2k}\varepsilon^{2t}$, we have

$$\begin{aligned} f(\eta, \xi') - f(\alpha, \beta') &= \sqrt{d_\Omega(\eta)^{2t} - |\xi'|^2} - \sqrt{d_\Omega(\alpha)^{2t} - |\beta'|^2} \\ &\leq \Lambda_1 \left| (d_\Omega(\eta)^{2t} - |\xi'|^2) - (d_\Omega(\alpha)^{2t} - |\beta'|^2) \right| \\ &= \Lambda_1 \left| (d_\Omega(\eta)^{2t} - d_\Omega(\alpha)^{2t}) + (|\beta'|^2 - |\xi'|^2) \right| \end{aligned}$$

and now, if Λ_2 is such that $|a^{2t} - b^{2t}| \leq \Lambda_2 |a - b|$ for all $a, b > \frac{\varepsilon}{2}$ and $|a^2 - b^2| \leq \Lambda_2 |a - b|$ for all $a, b < \operatorname{diam} \Omega$, we get, recalling that d_Ω is Lipschitz with constant 1,

$$\begin{aligned} f(\eta, \xi') - f(\alpha, \beta') &\leq \Lambda_1 \Lambda_2 \left(|d_\Omega(\eta) - d_\Omega(\alpha)| + \big| |\beta'| - |\xi'| \big| \right) \\ &\leq \Lambda \left(|\eta - \alpha| + |\beta' - \xi'| \right) \end{aligned}$$

with $\Lambda = \Lambda_1 \Lambda_2$, as we wanted to prove. □

We can now prove the weighted Poincaré inequalities for Hölder α domains. Our results generalize the inequality obtained in [1] which corresponds to the case $\beta = 1$ in the next theorem. To simplify notation we drop the subindex from the distance when the domain is Ω and write $d = d_\Omega$.

Theorem 2.1. *Let Ω be a Hölder α domain, B a ball contained in Ω, and $\phi \in C^\infty(\Omega)$ such that supp $\phi \subset B$ and $\int_B \phi \, dx \neq 0$. If $f \in W^{1,p}(\Omega)$ satisfies $\int_B f\phi \, dx = 0$ then*

$$\|d^{1-\beta} f\|_{L^p(\Omega)} \leq C \|d^{\alpha-\beta+1} \nabla f\|_{L^p(\Omega)}$$

for $\alpha \leq \beta \leq 1$

Proof. Let $\omega_0 = \mathrm{dist}\,(\mathrm{supp}\,\phi, \partial\Omega)^t$ and

$$\psi : \Omega^{k,t} \to \mathbb{R} \qquad \psi(x) = \phi(x)\Pi_{i=1}^{k}\rho(y_i)$$

where $\rho \in C^\infty[-\omega_0, \omega_0]$ with $\mathrm{supp}\,\rho \subset\subset (-\omega_0, \omega_0)$ and $\int \rho \neq 0$. Then $\psi \in C^\infty(\Omega^{k,t})$ and $\int \psi \neq 0$.

Consider the function $F : \Omega^{k,t} \to \mathbb{R}$ defined by $F(x, y) = f(x)$. Then we have

$$\int_{\Omega^{k,t}} F\psi = \int_B f(x)\phi(x)dx \left\{ \int_{-\omega_0}^{\omega_0} \rho(t)dt \right\}^k = 0.$$

By Lemma 2.2 we know that $\Omega^{k,t}$ is Hölder α and then, from the generalized Poincaré inequality proved in [1], it follows that there exists a constant C which depends only on $\Omega^{k,t}$ such that

$$\|F\|_{L^p(\Omega^{k,t})} \leq C\|d^\alpha_{\Omega^{k,t}} \nabla F\|_{L^p(\Omega)}.$$

But,

$$\|F\|^p_{L^p(\Omega^{k,t})} = c_k \int_\Omega f(x)^p d(x)^{tk} = c_k \|d^{\frac{tk}{p}} f\|^p_{L^p(\Omega)}$$

and, using Lemma 2.1, if $1 \leq i \leq n$,

$$\left\| d^\alpha_{\Omega^{k,t}} \frac{\partial F}{\partial x_i} \right\|^p_{L^p(\Omega)} \leq \int_\Omega \frac{\partial f}{\partial x_i} d(x)^{tk+p\alpha} = \|d^{\frac{tk}{p}+\alpha} \frac{\partial f}{\partial x_i}\|^p_{L^p(\Omega)},$$

while if $1 \leq i \leq k$,

$$\left\| d^\alpha_{\Omega^{k,t}} \frac{\partial F}{\partial y_i} \right\|_{L^p(\Omega)} = 0.$$

Therefore, we obtain

$$\|d^{\frac{tk}{p}} f\|_{L^p(\Omega)} \leq C\|d^{\frac{tk}{p}+\alpha} \nabla f\|_{L^p(\Omega)}.$$

The proof concludes by choosing for example $k = [p(1-\beta)] + 1$ and $t = \frac{p(1-\beta)}{[p(1-\beta)]+1}$. $\square$

The following example shows that an estimate of the form

$$\|d^\beta f\|_{L^p(\Omega)} \leq C\|d^\delta \nabla f\|_{L^p(\Omega)} \tag{2.12}$$

is not valid if $\delta - \beta > \alpha$. Therefore, the result obtained in the previous theorem is optimal.

Given $0 < \alpha < 1$ we call $\gamma = 1/\alpha$. Let Ω be the α-cusp defined by

$$\Omega = \{(x_1, x_2) : 0 < x_1 < 1, -x_1^\gamma < x_2 < x_1^\gamma\}.$$

Consider the function

$$f(x_1, x_2) = x_1^{-\nu} - k$$

for some $\nu > 0$ to be chosen below and a constant k such that $\int_\Omega f\phi = 0$ for some ϕ satisfying the hypotheses of Theorem 2.1.

We can easily check that the function $d(x_1, x_2)$ verifies

$$d(x_1, x_2) \sim x_1^\gamma - |x_2|$$

Then, for any $\beta > 0$ and $\delta > 0$ we have

$$\int_\Omega |f|^p d^{\beta p} \sim \int_0^1 x_1^{\frac{\beta p+1}{\alpha} - \nu p} dx_1 \qquad \text{and} \qquad \int_\Omega |\nabla f|^p d^{\delta p} \sim \int_0^1 x_1^{\frac{\delta p+1}{\alpha} - (\nu+1)p} dx_1.$$

Then, if $\delta - \beta > \alpha$, we can choose ν such that

$$\frac{\beta}{\alpha}+\frac{1}{p\alpha}+\frac{1}{p}\le\nu<\frac{\delta}{\alpha}+\frac{1}{p\alpha}+\frac{1}{p}-1$$

and therefore, for such ν we have

$$\int_\Omega |f|^p d^{\beta p}=\infty \qquad \text{and} \qquad \int_\Omega |\nabla f|^p d^{\delta p}<\infty.$$

So, it follows that inequality (2.12) can not be true.

3. Weighted Korn inequalities

In this section we prove the weighted Korn inequalities for Hölder α domains. With this goal we generalize the method introduced in [13] to prove the classic Korn inequality in the Lipschitz case. In this way we show that weighted Korn inequalities can be derived from some appropriate weighted Poincaré inequalities. It is important to remark that no assumption on the domain Ω, other than that it is bounded, is needed for this derivation.

The following lemma was proved in [4]. Since the proof is short we reproduce it here for the sake of completeness. Note that no restriction on the domain is needed for this proof. For the particular case $p=2$ and $\mu=0$ a different argument was given in [13]. As in the previous section $d(x)$ denotes the distance from x to the boundary of Ω.

Lemma 3.1. *Let $\Omega\subset\mathbb{R}^n$ be an arbitrary bounded domain and $1\le p<\infty$. If f is a harmonic function in Ω then*

$$\|d^{1-\mu}\nabla f\|_{L^p(\Omega)}\le C\|d^{-\mu}f\|_{L^p(\Omega)}$$

for all $\mu\in\mathbb{R}$.

Proof. Given $x\in\Omega$, let $B(x,R)\subset\Omega$ be the ball with center at x and radius R. Since f is harmonic in Ω it satisfies the following inequality (see for example [6]),

$$|\nabla f(x)|^p\le\frac{C}{R^{n+p}}\int_{B(x,R)}|f(y)|^p dy$$

Now, given $x\in\Omega$, let us take $R=d(x)/2$ in this inequality. Then we have

$$\int_\Omega |\nabla f(x)|^p d(x)^{p(1-\mu)}dx\le C\int_\Omega d(x)^{-n-p\mu}\left(\int_{B(x,d(x)/2)}|f(y)|^p dy\right)dx$$

But, since $|d(x)-d(y)|\le|x-y|$, we have that $\frac{d(x)}{2}\le d(y)\le\frac{3}{2}d(x)$ whenever $|x-y|<\frac{d(x)}{2}$. Therefore, we can change the order of integration and replace $d(x)$ by $d(y)$ to obtain

$$\int_\Omega |\nabla f(x)|^p d(x)^{p(1-\mu)}dx\le C\int_\Omega |f(y)|^p d(y)^{-n-p\mu}\left(\int_{B(y,d(y))}dx\right)dy\le C\int_\Omega |f(y)|^p d(y)^{-p\mu}dy$$

concluding the proof. $\square$

We can now prove the weighted Korn inequalities. We will use the following notations: for a vector function $u=(u_i)$, Δu is the vector with components Δu_i and, for a tensor $\sigma=(\sigma_{ij})$, $\operatorname{Div}\sigma$ is the vector with components $\sum_{j=1}^n\frac{\partial\sigma_{ij}}{x_j}$. In the proof we will make use of the following well known identity

$$\frac{\partial^2 v_i}{\partial x_j\partial x_k}=\frac{\partial\varepsilon_{ik}(v)}{\partial x_j}+\frac{\partial\varepsilon_{ij}(v)}{\partial x_k}-\frac{\partial\varepsilon_{jk}(v)}{\partial x_i} \tag{3.13}$$

Theorem 3.1. *Let Ω be a Hölder α domain and $1 < p < \infty$. Then, for $\alpha \le \beta \le 1$ the following inequality holds,*

$$\|d^{1-\beta}\nabla u\|_{L^p(\Omega)} \le C\left\{\|d^{\alpha-\beta}\varepsilon(u)\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right\} \tag{3.14}$$

where the constant C depends only on Ω and p.

Proof. Following [13] we can show that there exists $v \in W^{1,p}(\Omega)^n$ such that

$$\Delta v = \Delta u \qquad \text{in } \ \Omega \tag{3.15}$$

and

$$\|v\|_{W^{1,p}(\Omega)} \le C\|\varepsilon(u)\|_{L^p(\Omega)} \tag{3.16}$$

Indeed, define

$$F = \begin{cases} 2\varepsilon(u) - (\operatorname{tr}\varepsilon(u))I & \text{in } \Omega \\ 0 & \text{outside } \Omega, \end{cases}$$

Then, it is easy to check that $\operatorname{Div} F = \Delta u$ in Ω and so, one can obtain v by solving a Poisson equation in a smooth domain, for example a ball B_1, containing Ω. In fact, since $\operatorname{Div} F \in W^{-1,p}(B_1)^n$, there exists $v \in W_0^{1,p}(B_1)^n$ such that

$$\Delta v = \operatorname{Div} F$$

and (3.16) is satisfied in view of known a priori estimates for smooth domains.

Now, let B be a ball contained in Ω and $\phi \in C_0^\infty(B)$ be such that $\int_B \phi\, dx = 1$. For $i = 1, \cdots, n$ define the linear functions L_i as

$$L_i(x) = \left(\int_\Omega \nabla(u_i - v_i)\phi\, dx\right) \cdot x$$

and $L(x)$ as the vector with components $L_i(x)$.

Then we have

$$\nabla L = \int_B \nabla(u - v)\phi\, dx$$

and so, integrating by parts, we obtain

$$|\nabla L| \le \|u - v\|_{L^p(\Omega)}\|\nabla\phi\|_{L^q(\Omega)}$$

where q is the dual exponent of p. Therefore, it follows from (3.16) that there exists a constant C depending only on Ω, p and ϕ such that

$$\|\nabla L\|_{L^p(\Omega)} \le C\left\{\|\varepsilon(u)\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right\} \tag{3.17}$$

Let us now introduce

$$w = u - v - L$$

Then, in view of the bounds (3.16) and (3.17), to conclude the proof we have to estimate w. But, from (3.15) and the fact that L is linear we know that

$$\Delta w_i = 0$$

and consequently

$$\Delta\varepsilon_{ij}(w) = 0$$

therefore, we can apply Lemma 3.1 to obtain

$$\|d^{1-\gamma}\nabla\varepsilon_{ij}(w)\|_{L^p(\Omega)} \le C\|d^{-\gamma}\varepsilon_{ij}(w)\|_{L^p(\Omega)}$$

and using (3.13),

$$\|d^{1-\gamma}D^2 w\|_{L^p(\Omega)} \le C\|d^{-\gamma}\varepsilon(w)\|_{L^p(\Omega)} \tag{3.18}$$

where D^2 denote the tensor of second derivatives. Now, since $\int_\Omega \nabla w\phi\, dx = 0$ (indeed, we have defined L in order to have this property), it follows from Theorem 2.1 that

$$\|d^{1-\beta}\nabla w\|_{L^p(\Omega)} \le C\|d^{\alpha-\beta+1}D^2 w\|_{L^p(\Omega)}$$

and therefore, using (3.18) with $\gamma = \beta - \alpha$ we obtain

$$\|d^{1-\beta}\nabla w\|_{L^p(\Omega)} \le C\|d^{\alpha-\beta}\varepsilon(w)\|_{L^p(\Omega)}$$

which together with (3.16) and (3.17) concludes the proof. $\square$

The following example, based on that given by Weck in [19], shows that the result of the previous theorem is optimal.

Let $\chi \in C^\infty(\mathbb{R})$ non negative and such that $\mathrm{supp}(\chi) \subset [1,2]$. Consider the field

$$u(x) = \chi(\tau^{-1}x_3)(x_2, -x_1, 0).$$

If we set $\varepsilon_{ij} = \varepsilon_{ij}(u)$ we have

$$\varepsilon_{ij}(x) = \frac{1}{2\tau}\chi'(\tau^{-1}x_3)\begin{cases} x_2 & \text{for } i = 3, j = 1 \text{ or vice versa,} \\ -x_1 & \text{for } i = 3, j = 2 \text{ or vice versa,} \\ 0 & \text{otherwise.}\end{cases}$$

Let $\alpha \in (0,1)$ and $\gamma = \frac{1}{\alpha}$ and take Ω as the α-cusp defined by

$$\Omega = \{x \in \mathbb{R}^3 : 0 < x_3 < 1, x_3^{-\gamma}|(x_1,x_2)| < 1\}.$$

Then, one can check that

$$\sum_{i=1}^{3} |u_i|^p = \chi(\tau^{-1}x_3)^p(|x_1|^p + |x_2|^p)$$

$$\sum_{i,j=1}^{3} |\varepsilon_{ij}(x)|^p = \frac{1}{(2\tau)^p}\chi'(\tau^{-1}x_3)(|x_1|^p + |x_2|^p)$$

$$\frac{\partial u_1}{\partial x_2} = \chi(\tau^{-1}x_3)$$

$$d(x) \sim x_3^\gamma - \sqrt{x_1^2 + x_2^2}$$

and then it follows that (see [19] for details)

$$\|u\|_p^p \sim \tau^{\gamma(p+2)+1}, \qquad \|d^\delta\varepsilon(u)\|_p^p \sim \tau^{1-p+\gamma(2+p+\delta p)}, \qquad \left\|d^\beta\frac{\partial u_1}{\partial x_2}\right\|_p^p \sim \tau^{\gamma(\beta p+2)+1}.$$

Therefore, letting $\tau \to 0$, we conclude that an inequality like

$$\|d^\beta\nabla u\|_{L^p(\Omega)} \le C\left\{\|d^\delta\varepsilon(u)\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right\}$$

does not hold for Hölder α domains if $\beta < 1$ and $\delta - \beta > \alpha - 1$.

Remark 3.1. *Given* $1 < p < \infty$, *let* $V_p(\Omega) = \{u \in L^p(\Omega)^n : \varepsilon(u) \in L^p(\Omega)^{n\times n}\}$. *An important consequence of the classic Korn inequality for Lipschitz domains is the compactness of the inclusion of* $V_p \subset L^p(\Omega)^n$, *which follows from the well known Rellich-Kondrachov theorem for Sobolev spaces. For Hölder* α *domains, although the Korn inequality is not valid and consequently* $V_p(\Omega) \neq W^{1,p}(\Omega)^n$, *the compactness of the inclusion* $V_p(\Omega) \subset L^p(\Omega)^n$ *was proved in* [19] *for* $p = 2$ *under the restriction* $1/2 < \alpha$.

Our weighted Korn inequality provides a different proof of the result of Weck. Indeed, (1.3) shows that the space $V_p(\Omega)$ *is contained in a weighted Sobolev space. Therefore, combining our estimate (1.3) with known compactness results for weighted Sobolev spaces (see Theorem 19.11 in* [16, page 275]*), it follows immediately the compactness of the inclusion* $V_p(\Omega) \subset L^p(\Omega)^n$ *when* Ω *is a Hölder* α *domain with* $1/2 < \alpha$.

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