

Error estimates for anisotropic finite elements and applications

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Abstract. The finite element method is one of the most frequently used techniques to approximate the solution of partial differential equations. It consists in approximating the unknown solution by functions which are polynomials on each element of a given partition of the domain, made of triangles or quadrilaterals (or their generalizations to higher dimensions).

A fundamental problem is to estimate the error between the exact solution u and its computable finite element approximation. In many situations this error can be bounded in terms of the best approximation of u by functions in the finite element space of piecewise polynomial functions. A natural way to estimate this best approximation is by means of the Lagrange interpolation or other similar procedures.

Many works have considered the problem of interpolation error estimates. The classical error analysis for interpolations is based on the so-called *regularity assumption*, which excludes elements with different sizes in each direction (called *anisotropic*). The goal of this paper is to present a different approach which has been developed by many authors and can be applied to obtain error estimates for several interpolations under more general hypotheses.

An important case in which anisotropic elements arise naturally is in the approximation of convection-diffusion problems which present boundary layers. We present some applications to these problems.

Finally, we consider the finite element approximation of the Stokes equations and present some results for non-conforming methods.

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1. Introduction

The finite element method in its different variants is one of the most frequently used techniques to approximate the solution of partial differential equations. The general idea is to use weak or variational formulations in an infinite dimensional space and to replace that space by a finite dimensional one made of piecewise polynomial functions. In this way, the original differential equation is transformed into an algebraic problem which can be solved by computational methods. Although the main idea goes back to the works of Galerkin and Ritz in the early twentieth-century (or even to previous works, see for example [10] for a discussion of the history of these ideas), the finite element method became more popular since the middle of the twentieth century mainly because of its application by engineers to structural mechanics. On the other hand, the general mathematical analysis started only around forty years ago.

The theory of finite elements can be divided into *a priori* and *a posteriori* error analysis. The main goals of the *a priori* analysis are to prove convergence of the methods, to know the order of convergence (in terms of parameters associated with the finite dimensional problem, such as degree of approximation, mesh-size, size of the discrete problem, geometry of the elements, etc.) and the dependence of the error on properties of the unknown exact solution (such as its smoothness, which in many cases is already known from the theory of partial differential equations). Instead, the goals of the *a posteriori* error analysis are to obtain more quantitative information on the error and to develop self-adaptive methods to improve the approximation iteratively.

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In this paper we consider several problems related with *a priori* error estimates. We will deal mainly with the error analysis for flat or anisotropic elements, which arise naturally in several applications.

Let us begin by recalling the basic ideas of weak formulations of differential equations and finite element approximations. A general abstract formulation for linear problems is given by

$$B(u, v) = F(v) \quad \forall v \in V \quad (1.1)$$

where u is the solution to be found, V is a Hilbert space, F is a continuous linear form and B is a continuous bilinear form, i.e., there exists a constant $M > 0$ such that

$$|B(u, v)| \leq M \|u\| \|v\|$$

where $\|\cdot\|$ is the norm in the Hilbert space V .

To approximate the solution, we want to introduce a finite dimensional space $V_h \subset V$. The usual way to do this, is to introduce a partition \mathcal{T}_h of the domain Ω where we want to solve the differential equation usually made of triangular or quadrilateral elements (or their generalizations in 3D). The parameter h is usually related to the mesh size. Then, the space V_h consist of functions which restricted to each element of the partition are polynomials.

The approximate solution of our problem is $u_h \in V_h$ that satisfies

$$B(u_h, v) = F(v) \quad \forall v \in V_h.$$

Assume that the form B is coercive, namely, that there exists a constant $\alpha > 0$ such that

$$B(v, v) \geq \alpha \|v\|^2 \quad \forall v \in V. \quad (1.2)$$

Then, the classic error analysis is based on Cea's lemma (see [15]), which states that

$$\|u - u_h\| \leq \frac{M}{\alpha} \|u - v\| \quad \forall v \in V_h. \quad (1.3)$$

Notice that this condition also guarantees existence and uniqueness of solution in V as well as in V_h , thanks to the well-known Lax-Milgram's theorem.

If this condition does not hold, but the form B satisfies the so-called *inf-sup conditions*, that is, there exists $\beta > 0$ such that

$$\inf_{u \in V_h} \sup_{v \in V_h} \frac{B(u, v)}{\|u\| \|v\|} \geq \beta \quad (1.4)$$

$$\inf_{v \in V_h} \sup_{u \in V_h} \frac{B(u, v)}{\|u\| \|v\|} \geq \beta, \quad (1.5)$$

then we also have

$$\|u - u_h\| \leq \frac{M}{\beta} \|u - v\| \quad \forall v \in V_h. \quad (1.6)$$

If the above inf-sup conditions hold in V , we also have uniqueness and existence of solution. However, this is not sufficient to obtain (1.6), as the inf-sup conditions are not inherited by subspaces. This is the main difference between error analysis of coercive and non-coercive forms which satisfy (1.4) and (1.5).

The classical example of a form B which satisfies the inf-sup conditions but is not coercive, is the form associated to the Stokes equations of fluid dynamics (see for example [14, 21]).

In view of (1.3) and (1.6), in order to obtain an estimate for $\|u - u_h\|$, it is enough to bound $\|u - v\|$ for a function $v \in V_h$. Therefore, this is one of the most important problems in the theory of finite elements. Usually, the function v is taken to be a Lagrange interpolation of u . However, in some cases it is more convenient to use different approximations.

In many problems it is convenient to use spaces V_h which are not contained in V . These methods are called non-conforming and in this case the right hand sides of (1.3) and (1.6) are modified by adding the

so called “consistency terms”. One of the best-known method of this type is that of Crouzeix-Raviart, which is closely related with the mixed finite elements of Raviart-Thomas (see [9, 24]).

The goal of this paper is to present general ideas to obtain error estimates for different interpolations valid under very general hypotheses on the elements, in particular, allowing meshes with flat or anisotropic elements. We consider Lagrange and other kind of interpolations arising in mixed methods and give some applications to the approximation of convection-diffusion equations for which anisotropic elements are needed due to the presence of boundary layers.

Finally, we consider the finite element approximation of the Stokes equations and recall some results for non-conforming methods.

2. Notation and some basic inequalities

The classic finite element analysis for triangular elements requires the so-called regularity assumption, i.e.,

$$\frac{h_T}{\rho_T} \leq C \quad (2.1)$$

where h_T and ρ_T are the outer and inner diameter, respectively (see Figure 1). In other words, the constants in the error estimates depend on C (see for example [12, 15]).

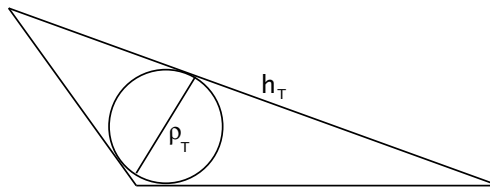


Figure 1.

The same hypothesis is also needed for the analysis of mixed and non-conforming methods (see [16] and [25]).

For standard Lagrange interpolation on conforming elements, since the works of Babuska-Azis [11] and Jamet [22], it is well known that the regularity assumption can be relaxed. For example, in the case of triangles, it can be replaced by the weaker maximum angle condition (i.e. angles bounded away from π). For rectangular elements, optimal error estimates can be obtained for arbitrary rectangles (while the regularity assumption requires that the edge sizes be comparable). In the case of general quadrilaterals, the situation is more complicated and several conditions, weaker than regularity, have been introduced to prove the error estimates (see, for example, [3]).

The standard method to prove error estimates is to obtain them first in a reference element and then to make a change of variables (see [15]). A different approach is to work directly in a given element and to use Poincaré type inequalities. The main idea is that the interpolation error usually has some vanishing averages (on the element, or edges, or faces, depending of the kind of interpolation considered). In this approach, the reference element is sometimes used to obtain the Poincaré type inequalities but, since one is bounding an L^2 -norm, the constants appearing in the estimates are independent of the aspect ratio of the element.

We will use the following notation. By $H^1(\Omega)$ we mean the usual Sobolev space of L^2 functions with distributional first derivatives in L^2 and by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary.

Similarly, $W^{k,p}(\Omega)$, for $1 \leq p \leq \infty$, indicates the Sobolev space of $L^p(\Omega)$ functions with distributional derivatives of order k in $L^p(\Omega)$. When $p = 2$, we set $H^k(\Omega) = W^{k,2}(\Omega)$.

Here $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain. For a general triangle T , h_T is its diameter, \mathbf{p}_0 is a vertex (arbitrary unless otherwise specified), $\mathbf{v}_1, \mathbf{v}_2$ (with $v_i := \|\mathbf{v}_i\| = 1$) are the directions of the

edges ℓ_1, ℓ_2 sharing \mathbf{p}_0 (see Figure 2) and ν_i is the exterior unit normal to the side ℓ_i (with obvious generalizations to 3D). We also use the standard notation \mathcal{P}_k for polynomials of total degree less than or equal to k , and \mathcal{Q}_k for polynomials of degree less than or equal to k in each variable. We call \hat{T} the reference triangle with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$, and $F : \hat{T} \rightarrow T$ the affine transformation $F(\hat{x}) = B\hat{x} + \mathbf{p}_0$ with $B\mathbf{e}_i = l_i\nu_i$, where \mathbf{e}_i are the canonical vectors.

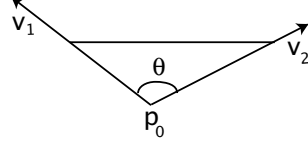


Figure 2.

The following two results are the classic Poincaré inequality and a generalization of it (first given in [11]) written in a convenient way for our purposes.

Lemma 2.1. *Let T be a triangle (resp.: tetrahedron) and let $f \in H^1(T)$ be a function with vanishing average on T . Then, there exists a constant C independent of T and of f such that*

$$\|f\|_{L^2(T)} \leq C \sum_{j=1}^n |\ell_j| \left\| \frac{\partial f}{\partial \nu_j} \right\|_{L^2(T)} \quad (2.2)$$

Proof. It follows from the Poincaré inequality on \hat{T} and making the change of variables F . □

Lemma 2.2. *Let T be a triangle (resp.: tetrahedron) and ℓ be any of its edges (resp.: faces). Let $f \in H^1(T)$ be a function with vanishing average on ℓ . Then, there exists a constant C independent of T such that*

$$\|f\|_{L^2(T)} \leq C \sum_{j=1}^n |\ell_j| \left\| \frac{\partial f}{\partial \nu_j} \right\|_{L^2(T)} \quad (2.3)$$

Proof. It is enough to prove that, on the reference element \hat{T} ,

$$\|f\|_{L^2(\hat{T})} \leq C \|\nabla f\|_{L^2(\hat{T})}. \quad (2.4)$$

Then, for a general triangle, the result follows by making the change of variables F .

The estimate (2.4) can be proved by a standard compactness argument (as was done in [11]). A different proof can be given by using (2.2) and a trace theorem. Indeed, if $f_{\hat{\ell}}$ and $f_{\hat{T}}$ denote the averages on $\hat{\ell}$ and \hat{T} , respectively, and we assume that $f_{\hat{\ell}} = 0$, we have

$$\|f\|_{L^2(\hat{T})} = \|f - f_{\hat{\ell}}\|_{L^2(\hat{T})} \leq \|f - f_{\hat{T}}\|_{L^2(\hat{T})} + \|f_{\hat{T}} - f_{\hat{\ell}}\|_{L^2(\hat{T})}.$$

But,

$$f_{\hat{T}} - f_{\hat{\ell}} = \frac{1}{|\hat{\ell}|} \int_{\hat{\ell}} (f_{\hat{T}} - f)$$

and therefore, an application of a standard trace theorem gives

$$\|f_{\hat{T}} - f_{\hat{\ell}}\|_{L^2(\hat{T})} \leq C \{ \|f - f_{\hat{T}}\|_{L^2(\hat{T})} + \|\nabla f\|_{L^2(\hat{T})} \}$$

with a constant which depends only on the reference element. So, (2.4) follows from (2.2). □

3. Error estimates for Lagrange interpolation

3.1. The two-dimensional case. To introduce the general idea we present first two simple classic cases: the Lagrange interpolation for lowest degree finite elements in triangles or rectangles. The argument is essentially that given in [11] for triangles. In the case of rectangles, an extra step is required due to the presence of a non-vanishing second derivative of the interpolating function.

Given a triangle T we denote with $I_1 u \in \mathcal{P}_1$ the Lagrange interpolation of u , i.e., the affine function which equals u on the vertices of T . $D^2 u$ denotes the sum of the absolute values of second derivatives of u .

Theorem 3.1. *There exists a constant C such that, if θ is the maximum angle of T ,*

$$\|\nabla(u - I_1 u)\|_{L^2(T)} \leq \frac{C}{\sin \theta} h_T \|D^2 u\|_{L^2(T)}.$$

Proof. Observe that, for $i = 1, 2$, $\nabla(u - I_1 u) \cdot v_i$, has vanishing average on one side of T , therefore, applying Lemma 2.2 and using that the second derivatives of $I_1 u$ vanish, we obtain

$$\|\nabla(u - I_1 u) \cdot v_i\|_{L^2(T)} \leq C \left\{ |\ell_1| \left\| \frac{\partial \nabla u \cdot v_i}{\partial v_1} \right\|_{L^2(T)} + |\ell_2| \left\| \frac{\partial \nabla u \cdot v_i}{\partial v_2} \right\|_{L^2(T)} \right\}.$$

Then, if we choose \mathbf{p}_0 as the vertex corresponding to the maximum angle of T , we have

$$|\nabla(u - I_1 u)| \leq \frac{C}{\sin \theta} \{ |\nabla(u - I_1 u) \cdot v_1| + |\nabla(u - I_1 u) \cdot v_2| \}$$

and therefore, the theorem is proved. \square

We consider now the case of rectangles. We use the same notation, $I_1 u$, for the interpolation which now belongs to \mathcal{Q}_1 . The proof for this case is analogous to the previous one, with the only difference that $\frac{\partial^2 I_1 u}{\partial x \partial y}$ does not vanish.

Let R be a rectangle and ℓ_1, ℓ_2 two adjacent sides. Clearly, the result of Lemma 2.2 holds for this case also.

Theorem 3.2. *There exists a constant C , independent of the relation between $|\ell_1|$ and $|\ell_2|$, such that*

$$\left\| \frac{\partial}{\partial x} (u - I_1 u) \right\|_{L^2(R)} \leq C \left\{ |\ell_1| \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(R)} + |\ell_2| \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)} \right\} \quad (3.1)$$

and

$$\left\| \frac{\partial}{\partial y} (u - I_1 u) \right\|_{L^2(R)} \leq C \left\{ |\ell_1| \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)} + |\ell_2| \left\| \frac{\partial^2 u}{\partial y^2} \right\|_{L^2(R)} \right\}. \quad (3.2)$$

Proof. Proceeding as in the case of triangles, we have

$$\left\| \frac{\partial}{\partial x} (u - I_1 u) \right\|_{L^2(R)} \leq C \left\{ |\ell_1| \left\| \frac{\partial^2 (u - I_1 u)}{\partial x^2} \right\|_{L^2(R)} + |\ell_2| \left\| \frac{\partial^2 (u - I_1 u)}{\partial x \partial y} \right\|_{L^2(R)} \right\}. \quad (3.3)$$

But, $\frac{\partial^2 I_1 u}{\partial x^2} = 0$ and an elementary computation shows that

$$\int_R \frac{\partial^2 I_1 u}{\partial x \partial y} = \int_R \frac{\partial^2 u}{\partial x \partial y}$$

i.e., $\frac{\partial^2 I_1 u}{\partial x \partial y}$ is the average of $\frac{\partial^2 u}{\partial x \partial y}$ on R . Then,

$$\left\| \frac{\partial^2 I_1 u}{\partial x \partial y} \right\|_{L^2(R)} \leq \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)}$$

and therefore (3.1) holds. Obviously, the proof of (3.2) is analogous. \square

Remark 3.3. If the function $u \in H^3(R)$, then the last term on the right hand side of (3.3) is of higher order. Indeed, that term is the difference between $\frac{\partial^2 u}{\partial x \partial y}$ and its average. Therefore, we have the estimate

$$\left\| \frac{\partial}{\partial x}(u - I_1 u) \right\|_{L^2(R)} \leq C|\ell_1| \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(R)} + \text{higher order terms.}$$

3.2. The three-dimensional case. Many results on finite elements can be extended almost straightforward from 2D to 3D. However, this is not the case of error estimates for anisotropic elements. Indeed, counterexamples for an estimate analogous to (3.1) in the 3D case have been given in [6] and [27]. They show that the constant in the estimate blows-up when a rectangular reference tetrahedron (or cube) is compressed in one direction.

Many papers have been published considering the 3D case. For example, in the case of tetrahedra, Krížek [23] introduced a natural generalization of the maximum angle condition: if the angles between faces and the angles in the faces are bounded away from π , he obtained error estimates for smooth functions, namely, $u \in W^{2,\infty}$. In [17] the results of Krížek were extended to functions in $W^{2,p}$ with $2 < p < \infty$ (and, moreover, for functions in an intermediate Orlicz space between H^2 and $W^{2,p}$, $p > 2$). Therefore, although the estimate fails for functions in H^2 , it is valid for functions only slightly more regular. Let us mention that the reason why the arguments applied in 2D cannot be generalized, is that the estimate given in Lemma 2.2 is not true in 3D if ℓ is an edge instead of a face (note that the interpolation error for the Lagrange interpolation has vanishing integral on edges).

On the other hand, many papers have considered error estimates for different interpolations (see for example [1, 5, 17, 18]), namely, different variants of average interpolators. This kind of interpolations have been introduced to approximate non-smooth functions (for which the Lagrange interpolation is not even defined). However, they have as well better approximation properties on anisotropic elements for functions in H^2 . Indeed, using average interpolations, the 2D results can be generalized to 3D. Observe that, in view of (1.3) and (1.6), error estimates for an average interpolation will give bounds for finite element approximations.

4. Applications to convection-diffusion equations

A very important application in which anisotropic elements are needed is the approximation of convection-diffusion problems in which boundary layers arise.

Consider for example the model problem

$$\begin{aligned} -\varepsilon \Delta u + b \cdot \nabla u + cu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

where $\Omega = (0, 1)^2$ and $\varepsilon > 0$ is a small parameter.

It is well known that the numerical approximation of this equation requires some special method in order to obtain good results when the problem is convection dominated, due to the presence of boundary or interior layers. In the case of boundary layers, one possibility is to use appropriate refined meshes near the boundary; this methodology gives rise to anisotropic elements. Using estimates (3.1) and (3.2) it is possible to obtain quasi-optimal order convergence (with respect to the number of nodes) in the ε -norm defined by

$$\|v\|_\varepsilon^2 = \|v\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v\|_{L^2(\Omega)}^2$$

for the standard \mathcal{Q}_1 approximation on appropriate graded meshes.

This problem can be written in the general form (1.1) with $V = H_0^1(\Omega)$,

$$B(u, v) = \int_{\Omega} (\varepsilon \nabla u \nabla v + b \cdot \nabla u v + cuv) \, dx$$

and

$$F(v) = \int_{\Omega} f v \, dx.$$

Assuming that there exists a constant μ independent of ε such that

$$c - \frac{\operatorname{div} b}{2} \geq \mu > 0, \quad (4.2)$$

the bilinear form B is coercive in the ε -norm uniformly in ε (see [26]), i.e., the constant α in (1.2) is independent of ε . However, the continuity of B is not uniform in ε and this is one of the reasons why it is not possible to apply directly the general result (1.3) to obtain error estimates valid uniformly in ε . Therefore, a special analysis is required and this was the object of [19]. It was proved in that paper that

$$\|u - u_h\|_\varepsilon \leq C \frac{\log^2(1/\varepsilon)}{\sqrt{N}}$$

where N is the number of nodes and $h > 0$ is a parameter associated with the meshes. Observe that this order of convergence is quasi-optimal in the sense that, up to the logarithm factor, is the same order that one obtains for a smooth solution of a problem with $\varepsilon = O(1)$ using uniform meshes.

Assuming that the coefficient b is such that the boundary layers are close to $x = 0$ and $y = 0$, the meshes \mathcal{T}_h are such that the grading in each direction is given by

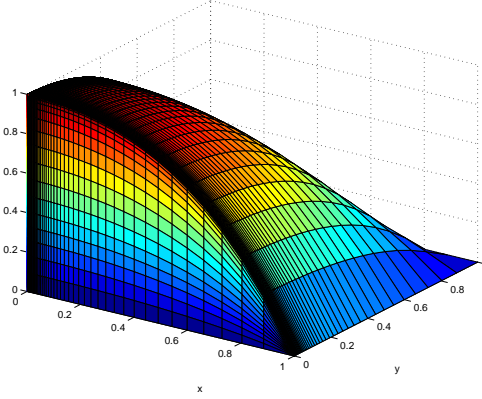


Figure 3.

$$\begin{cases} \xi_0 = 0 \\ \xi_i = ih\varepsilon & \text{for } 1 \leq i < \frac{1}{h} + 1 \\ \xi_{i+1} = \xi_i + h\xi_i & \text{for } \frac{1}{h} + 1 \leq i \leq M - 2 \\ \xi_M = 1 \end{cases} \quad (4.3)$$

where M is such that $\xi_{M-1} < 1$ and $\xi_{M-1} + h\xi_{M-1} \geq 1$. We assume that the last interval $(\xi_{M-1}, 1)$ is not too small in comparison with the previous one (ξ_{M-2}, ξ_{M-1}) (if this is not the case, we just eliminate the node ξ_{M-1}).

Figure 3 shows the approximate solution of (4.1) for

$$\varepsilon = 10^{-6}, \quad b = (1 - 2\varepsilon)(-1, -1), \quad c = 2(1 - \varepsilon)$$

and

$$f(x, y) = - \left[x - \left(\frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) + y - \left(\frac{1 - e^{-\frac{y}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) \right] e^{x+y}.$$

Observe that no oscillations arise although we are using the standard \mathcal{Q}_1 finite element method.

The graded meshes are an alternative to the well-known Shishkin meshes which have been widely analyzed for convection-diffusion problems (see for example [26]).

From the error analysis given in [19] one can see that a graded mesh designed for a value of ε works well also for larger values of ε . This is not the case for Shishkin meshes. Table 1 shows the values of the ε -norm of the error for different values of ε , solving the problem with the mesh corresponding to $\varepsilon = 10^{-6}$, using graded meshes and Shishkin meshes.

ε	Error
10^{-6}	0.040687
10^{-5}	0.033103
10^{-4}	0.028635
10^{-3}	0.024859
10^{-2}	0.02247
10^{-1}	0.027278

Graded meshes, $N = 10404$

ε	Error
10^{-6}	0.0404236
10^{-5}	0.249139
10^{-4}	0.623650
10^{-3}	0.718135
10^{-2}	0.384051
10^{-1}	0.0331733

Shishkin meshes, $N = 10609$

Table 1.

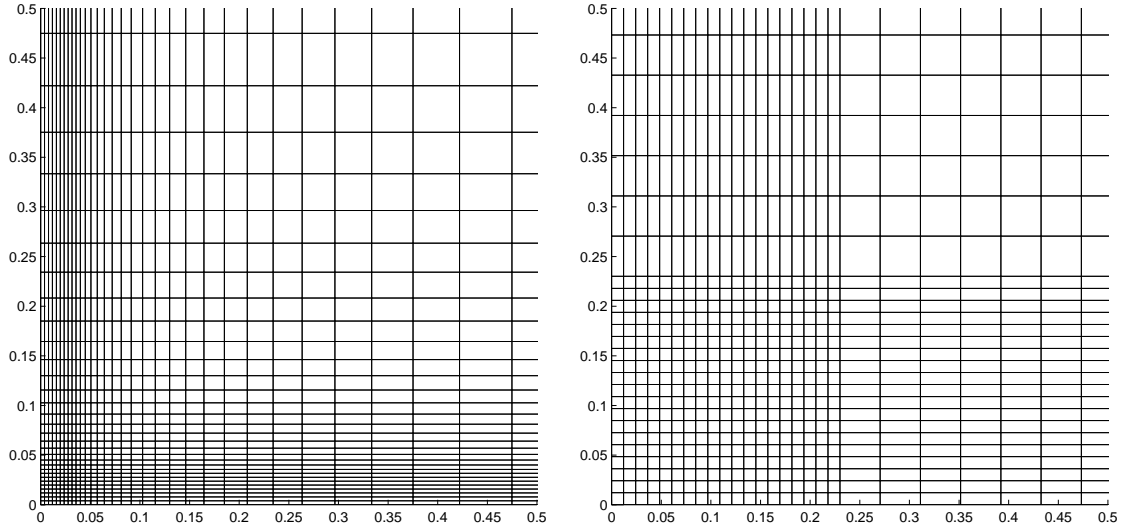


Figure 4.

To see the different structures, we show in Figure 4 a Shishkin mesh (on the right) and one of our graded meshes (on the left) having the same number of nodes. For the sake of clarity, we show only the part of the meshes corresponding to $(0, 1/2) \times (0, 1/2)$ and $\varepsilon = 10^{-\frac{3}{2}}$.

5. Error estimates for Raviart-Thomas interpolation

5.1. The two dimensional case. The Raviart-Thomas spaces were introduced in [25] to approximate vector fields $\mathbf{u} \in H(\text{div}, \Omega)$ where

$$H(\text{div}, \Omega) = \{\mathbf{u} \in L^2(\Omega) : \text{div } \mathbf{u} \in L^2\}.$$

For any integer $k \geq 0$, the space \mathcal{RT}_k on a triangle T is defined by

$$\mathcal{RT}_k(T) = \mathcal{P}_k^2(T) \oplus (x, y)\mathcal{P}_k(T).$$

Calling P_k the L^2 orthogonal projection on $\mathcal{P}_k(T)$, it is known (see [25]) that there exists an operator $RT_k : H^1(T)^2 \rightarrow \mathcal{RT}_k(T)$ satisfying the following commutative diagram property:

$$\begin{array}{ccc} H^1(T)^2 & \xrightarrow{\text{div}} & L^2(T) \\ RT_k(T) \downarrow & & \downarrow P_k \\ \mathcal{RT}_k & \xrightarrow{\text{div}} & \mathcal{P}_k(T) \longrightarrow 0 \end{array} \quad (5.1)$$

For the case of anisotropic elements, only the lowest degree case \mathcal{RT}_0 has been considered. Error estimates for this case have been obtained in [2].

Below we will show how the arguments can be generalized to obtain error estimates for the case of \mathcal{RT}_1 . Higher order approximations can be treated similarly although this extension is not straightforward.

Let us first recall the results for \mathcal{RT}_0 . Again, the results follow by the generalized Poincaré inequality given in Lemma 2.2 as we show in the next theorem.

Theorem 5.1. *There exists a constant C such that, if θ is the maximum angle of T ,*

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq \frac{C}{\sin \theta} \sum_{k=1}^2 |\ell_k| \left(\left\| \frac{\partial \mathbf{u}}{\partial v_k} \right\|_{L^2(T)} + \|\text{div } \mathbf{u}\|_{L^2(T)} \right).$$

Proof. Since $(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i$ has zero mean value on ℓ_i , it follows from Lemma 2.2 that

$$\|(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i\|_{L^2(T)} \leq C \sum_{k=1}^2 |\ell_k| \left\| \frac{\partial(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i}{\partial v_k} \right\|_{L^2(T)}. \quad (5.2)$$

But, it is easy to check that

$$\frac{\partial(RT_0\mathbf{u} \cdot \nu_i)}{\partial v_k} = \frac{1}{2}(\text{div } RT_0\mathbf{u}) v_k \cdot \nu_i.$$

On the other hand, using the commutative diagram property (5.1), we have

$$\|\text{div } RT_0\mathbf{u}\|_{L^2(T)} \leq \|\text{div } \mathbf{u}\|_{L^2(T)}$$

and so, it follows from (5.2) that

$$\|(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i\|_{L^2(T)} \leq C \sum_{k=1}^2 |\ell_k| \left(\left\| \frac{\partial \mathbf{u}}{\partial v_k} \right\|_{L^2(T)} + \|\text{div } \mathbf{u}\|_{L^2(T)} |\nu_i \cdot v_k| \right). \quad (5.3)$$

Up to now the constant C is independent of T . If we want to bound $\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)}$, it is natural to expect that the constant will depend on the geometry of the element.

In view of (5.3), it would be enough to control $\mathbf{u} - RT_0\mathbf{u}$ in terms of its components in the directions of the normals to the edges. For a fixed triangle the estimate

$$|\mathbf{u} - RT_0\mathbf{u}| \leq C\{ |(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_1| + |(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_2| \}$$

holds. Moreover, for a family of triangles, the constant C will not degenerate if the angle between ν_1 and ν_2 does not go to 0 or π or, equivalently, if the angle between the corresponding edges does not go to 0 or π . Therefore, the constant will be uniformly bounded for a family of elements with maximum angle bounded away from π . More precisely, we have

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq \frac{C}{\sin \theta} \sum_{i=1}^2 \|(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i\|_{L^2(T)} \quad (5.4)$$

where θ is the maximum angle of T . Indeed, if N is the matrix which has ν_1 and ν_2 as its rows, then,

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq \|N^{-1}\| \sum_{i=1}^2 \|(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i\|_{L^2(T)}$$

where $\|\cdot\|$ denotes the matrix norm associated with the euclidean norm. But, since ν_i are unit vectors, it follows that $\|N^{-1}\| \leq \frac{C}{|\det N|}$ and $|\det N| = \sin \theta_1$ where θ_1 is the angle between ν_1 and ν_2 . If the vertex \mathbf{p}_0 is the one corresponding to the maximum angle, $\theta_1 = \pi - \theta$ and so (5.4) holds and the theorem is proved. \square

Similar arguments can be applied for the analysis of higher order elements. However, the extension is not straightforward. In what follows we consider the case of \mathcal{RT}_1 . This case requires the following generalization of the Poincaré inequality.

Lemma 5.2. *Let T be a triangle and ℓ one of its sides. If $f \in H^2(T)$ satisfies*

$$\int_{\ell} fp = 0 \quad \forall p \in \mathcal{P}_1(\ell) \quad \text{and} \quad \int_T f = 0$$

then,

$$\|f\|_{L^2(T)} \leq Ch_T^2 \|D^2 f\|_{L^2(T)}$$

with a constant independent of the shape of the triangle.

Proof. Observing that, if $f \in \mathcal{P}_1$ satisfies the three hypotheses of the lemma, then $f = 0$, it follows by standard compactness arguments that

$$\|f\|_{L^2(\hat{T})} \leq C \|D^2 f\|_{L^2(\hat{T})}.$$

Then, an affine change of variables concludes the proof. \square

To obtain the error estimate for the RT_1 interpolation we will need to have a bound for the gradient of the P_1 projection. This is the goal of the next lemma.

Lemma 5.3. *If $f \in H^1(T)$ we have*

$$\|\nabla P_1 f\|_{L^2(T)} \leq C \|\nabla f\|_{L^2(T)}$$

with a constant C depending only on the maximum angle of T .

Proof. We will prove that for the triangle with vertices at $(0, 0)$, $(h, 0)$ and $(0, 1)$ we have

$$\|\nabla P_1 f\|_{L^2(T)} \leq 6 \|\nabla f\|_{L^2(T)}.$$

Then, the general result follows by an affine change of variables.

Let M_i , $i = 1, 2, 3$ be the mid-side points of T . Since the quadrature rule obtained by interpolating at these points is exact for quadratic polynomials, it is easy to see that the functions

$$\phi_1 = \left(\frac{6}{h}\right)^{1/2} (1 - 2y) \quad , \quad \phi_2 = \left(\frac{6}{h}\right)^{1/2} \left(2y + \frac{2x}{h} - 1\right) \quad \text{and} \quad \phi_3 = \left(\frac{6}{h}\right)^{1/2} \left(1 - \frac{2x}{h}\right)$$

form an orthonormal basis of $\mathcal{P}_1(T)$. Then,

$$P_1 f = \sum_{i=1}^3 c_i \phi_i$$

with $c_i = \int_T f \phi_i$. Therefore,

$$\frac{\partial P_1 f}{\partial x} = \frac{2\sqrt{6}}{h^{3/2}} \int_T f(\phi_2 - \phi_3) = \frac{24}{h^2} \int_T f(x, y) \left(y + \frac{2x}{h} - 1\right) dx dy.$$

Now, observe that for any $y \in (0, 1)$

$$\int_0^{h(1-y)} \left(y + \frac{2x}{h} - 1 \right) dx = 0$$

and so, if we call $\bar{f}(y) = \frac{1}{h(1-y)} \int_0^{h(1-y)} f(x, y) dx$ we have

$$\frac{\partial P_1 f}{\partial x} = \frac{24}{h^2} \int_0^1 \int_0^{h(1-y)} (f(x, y) - \bar{f}(y)) \left(y + \frac{2x}{h} - 1 \right) dx dy.$$

But using the one dimensional Poincaré inequality we have

$$\int_0^{h(1-y)} |f(x, y) - \bar{f}(y)| dx \leq \frac{h}{2} \int_0^{h(1-y)} \left| \frac{\partial f}{\partial x}(x, y) \right| dx$$

and, since $\left| y + \frac{2x}{h} - 1 \right| \leq 1$, it follows that

$$\left| \frac{\partial P_1 f}{\partial x} \right| \leq \frac{12}{h} \int_0^1 \int_0^{h(1-y)} \left| \frac{\partial f}{\partial x}(x, y) \right| dx dy.$$

Therefore,

$$\left| \frac{\partial P_1 f}{\partial x} \right| \leq \frac{12}{h} \left\| \frac{\partial f}{\partial x} \right\|_{L^1(T)} \leq \frac{12}{h} |T|^{\frac{1}{2}} \left\| \frac{\partial f}{\partial x} \right\|_{L^2(T)}$$

and consequently

$$\left\| \frac{\partial P_1 f}{\partial x} \right\|_{L^2(T)} \leq 6 \left\| \frac{\partial f}{\partial x} \right\|_{L^2(T)}.$$

Clearly, the same arguments can be applied to bound the derivative with respect to y . \square

Theorem 5.4. *There exists a constant C depending only on the maximum angle of T such that*

$$\|\mathbf{u} - RT_1 \mathbf{u}\|_{L^2(T)} \leq Ch_T^2 \|D^2 \mathbf{u}\|_{L^2(T)}.$$

Proof. From the definition of $RT_1 \mathbf{u}$ we know that, for $i = 1, 2, 3$, $(\mathbf{u} - RT_1 \mathbf{u}) \cdot \nu_i$ satisfies the hypotheses of Lemma 5.2 and then,

$$\|(\mathbf{u} - RT_1 \mathbf{u}) \cdot \nu_i\|_{L^2(T)} \leq Ch_T^2 \|D^2(\mathbf{u} - RT_1 \mathbf{u})\|_{L^2(T)}.$$

So, in order to estimate the component of $\mathbf{u} - RT_1 \mathbf{u}$ in the direction ν_i , we need to bound the second derivatives of $RT_1 \mathbf{u}$ in terms of $D^2 \mathbf{u}$.

But, an easy computation shows that, for any $\mathbf{v} \in \mathcal{R}_T(T)$,

$$\frac{\partial^2 \mathbf{v}}{\partial x^2} = \frac{2}{3} \left(\frac{\partial(\operatorname{div} \mathbf{v})}{\partial x}, 0 \right), \quad \frac{\partial^2 \mathbf{v}}{\partial y^2} = \frac{2}{3} \left(0, \frac{\partial(\operatorname{div} \mathbf{v})}{\partial y} \right)$$

and

$$\frac{\partial^2 \mathbf{v}}{\partial x \partial y} = \frac{1}{3} \left(\frac{\partial(\operatorname{div} \mathbf{v})}{\partial y}, \frac{\partial(\operatorname{div} \mathbf{v})}{\partial x} \right).$$

Therefore, we have

$$\|(\mathbf{u} - RT_1 \mathbf{u}) \cdot \nu_i\|_{L^2(T)} \leq Ch_T^2 \left\{ \|D^2 \mathbf{u}\|_{L^2(T)} + \|\nabla(\operatorname{div} RT_1 \mathbf{u})\|_{L^2(T)} \right\}. \quad (5.5)$$

Now, from (5.1) we know that

$$\nabla(\operatorname{div} RT_1 \mathbf{u}) = \nabla(P_1 \operatorname{div} \mathbf{u})$$

so, applying Lemma 5.3 we obtain

$$\|\nabla(\operatorname{div} RT_1 \mathbf{u})\|_{L^2(T)} \leq C \|\nabla(\operatorname{div} \mathbf{u})\|_{L^2(T)}$$

and replacing in (5.5) we obtain the estimates for the normal components of $(\mathbf{u} - RT_1 \mathbf{u})$. Then, to conclude the proof of the theorem we proceed as in the case of RT_0 . \square

5.2. The three-dimensional case. As in the case of the Lagrange interpolation, the 3D case presents some important differences with the 2D one. We recall that the definition of \mathcal{RT}_k can be extended straightforwardly to the 3D case, indeed, for T a tetrahedron we have

$$\mathcal{RT}_k(T) = \mathcal{P}_k^3(T) \oplus (x, y, z)\mathcal{P}_k(T).$$

The *maximum angle condition* can be generalized in different ways. The first one, introduced in [2], is the *Regular Vertex Property*. We say that a tetrahedron satisfies this property with a constant $\bar{c} > 0$ if it has a vertex \mathbf{p}_0 such that $|\det M| \geq \bar{c} > 0$, where M is the matrix which has v_i , $i = 1, 2, 3$ as rows (where we are using the obvious generalization of the notation of the 2D case).

Under this hypothesis, Theorem 5.1 can be generalized almost straightforwardly. Indeed, the basic result given in Lemma 2.2 is valid now for functions with vanishing average on a face of T , and using this result we can prove, using the same arguments as in the 2D case, that

$$\|(\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i\|_{L^2(T)} \leq C \sum_{k=1}^3 |\ell_k| \left(\left\| \frac{\partial \mathbf{u}}{\partial v_k} \right\|_{L^2(T)} + \|\operatorname{div} \mathbf{u}\|_{L^2(T)} |\nu_i \cdot v_k| \right)$$

and as a consequence we obtain the following estimate.

Theorem 5.5. *Let T be a tetrahedron satisfying the regular vertex property with a constant $\bar{c} > 0$. Then, there exists a constant C depending only on \bar{c} such that*

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq C \sum_{k=1}^3 |\ell_k| \left(\left\| \frac{\partial \mathbf{u}}{\partial v_k} \right\|_{L^2(T)} + \|\operatorname{div} \mathbf{u}\|_{L^2(T)} \right). \quad (5.6)$$

The other “natural” generalization of the 2D maximum angle condition is the condition introduced by Krížek [23]. We say that a family of tetrahedra satisfies the *maximum angle condition* with a constant $\bar{\psi} < \pi$ if the angles inside the faces and the angles between faces are bounded above by $\bar{\psi}$.

It is easy to see that in the 2D case the *regular vertex property* is equivalent to the *maximum angle condition*. However, the situation is different in the 3D case. In fact, the family in Figure 5, with arbitrary lengths h_1, h_2, h_3 , satisfies uniformly the *maximum angle condition* but not the *regular vertex property* (take for example $h_1 = h_3 = h^2$, and $h_2 = h$). On the other hand, the *regular vertex property* implies the *maximum angle condition* (see [2]).

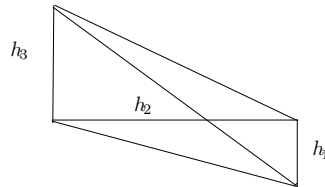


Figure 5.

A natural question is whether or not error estimates for the RT_0 interpolation hold under the *maximum angle condition*. The answer is positive: in [2] it was proved that

Theorem 5.6. *If T is a tetrahedron satisfying the maximum angle condition with a constant $\bar{\psi}$. Then, there exists a constant C depending only on $\bar{\psi}$ such that*

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq Ch_T \|D\mathbf{u}\|_{L^2(T)}. \quad (5.7)$$

Again the basic tool to obtain this estimate is the generalization to 3D of Lemma 2.2. Indeed, consider the face mean average interpolator introduced in [16], namely, $\Pi : H^1(T) \rightarrow \mathcal{P}_1(T)$ given by

$$\int_S \Pi w = \int_S w$$

for any face S of T .

Lemma 5.7. *The following error estimates hold with a constant C independent of T :*

$$\|w - \Pi w\|_{L^2(T)} \leq C \sum_{j=1}^3 |\ell_j| \left\| \frac{\partial w}{\partial v_j} \right\|_{L^2(T)} \quad (5.8)$$

$$\left\| \frac{\partial \Pi w}{\partial \xi} \right\|_{L^2(T)} \leq \left\| \frac{\partial w}{\partial \xi} \right\|_{L^2(T)} \quad (5.9)$$

$$\left\| \frac{\partial(w - \Pi w)}{\partial \xi} \right\|_{L^2(T)} \leq C \sum_{j=1}^3 |\ell_j| \left\| \frac{\partial^2 w}{\partial v_j \partial \xi} \right\|_{L^2(T)} \quad (5.10)$$

where $\frac{\partial}{\partial \xi}$ is a derivative in any direction.

Proof. Since $w - \Pi w$ has vanishing mean value on the faces of T , it follows from Lemma 2.2 that

$$\|w - \Pi w\|_{L^2(T)} \leq C \sum_{j=1}^3 |\ell_j| \left\| \frac{\partial(w - \Pi w)}{\partial v_j} \right\|_{L^2(T)}. \quad (5.11)$$

Now, it follows from the definition of Π that

$$\int_T \frac{\partial \Pi w}{\partial \xi} = \int_T \frac{\partial w}{\partial \xi}$$

or, in other words, the constant $\frac{\partial \Pi w}{\partial \xi}$ is the average on T of $\frac{\partial w}{\partial \xi}$ and so (5.9) holds and (5.10) follows from Lemma 2.1. Finally, (5.8) is a consequence of (5.11) and (5.9). \square

Now, it is not difficult to check that, for any $\mathbf{u} \in H^1(T)^3$,

$$RT_0 \mathbf{\Pi} \mathbf{u} = RT_0 \mathbf{u}$$

where $\mathbf{\Pi}$ is the vector version of Π . Consequently,

$$\|\mathbf{u} - RT_0 \mathbf{u}\|_{L^2(T)} \leq \|\mathbf{u} - \mathbf{\Pi} \mathbf{u}\|_{L^2(T)} + \|\mathbf{\Pi} \mathbf{u} - RT_0 \mathbf{\Pi} \mathbf{u}\|_{L^2(T)}$$

and therefore, in view of (5.8), to prove (5.7) it is enough to prove the error estimate for $\mathbf{u} \in \mathcal{P}_1(K)^3$. In this way, the problem is reduced to a finite dimensional one and the error estimate (5.7) can be proved under the *maximum angle condition* (see [2] for details).

6. The Stokes equations

The Stokes equations are given by

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= f & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= 0 & \text{on } \partial\Omega \end{aligned}$$

where \mathbf{u} is the velocity and p the pressure of a fluid contained in Ω .

This problem can be written in the form (1.1) with $V = H_0^1(\Omega)^n \times L_0^2(\Omega)$ where

$$L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_{\Omega} f = 0\},$$

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = \sum_{i,j=1}^n \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} - \int_{\Omega} p \operatorname{div} \mathbf{v} - \int_{\Omega} q \operatorname{div} \mathbf{u}$$

and

$$F(\mathbf{v}, q) = \int_{\Omega} f \mathbf{v}.$$

Then, to obtain a finite element approximation we need to use a space W_h for the velocity and a space Q_h for the pressure. Note that, since in this case the form B is symmetric, the two conditions (1.4) and (1.5) are exactly the same. From the classic theory for mixed finite elements of Brezzi [13] we know that to obtain (1.4) for the space $V_h = W_h \times Q_h$ it is enough to prove that there exists $\gamma > 0$, independent of h , such that

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in W_h} \frac{\int_{\Omega} q \operatorname{div} \mathbf{v}}{\|p\|_{L^2} \|\mathbf{v}\|_{H_0^1}} \geq \gamma. \quad (6.1)$$

A lot of work has been done to prove this inf-sup condition for different choices of spaces W_h and Q_h . We refer for example to the books [14, 21]. However, most proofs require the regularity assumption (2.1) on the elements although it is not known whether it is essential or not.

One of the main tools to prove (6.1) is the so called *Fortin's operator* introduced in [20] which in the case of the Stokes equations is an operator $\mathbf{\Pi} : H_0^1(\Omega)^n \rightarrow W_h$ such that

$$\int_{\Omega} q \operatorname{div} (\mathbf{v} - \mathbf{\Pi} \mathbf{v}) = 0 \quad \forall q \in Q_h$$

and

$$\|\mathbf{\Pi} \mathbf{v}\|_{H_0^1} \leq C \|\mathbf{v}\|_{H_0^1} \quad (6.2)$$

with a constant C independent of h .

Consider for example the non-conforming method of Crouzeix-Raviart, namely, W_h are the $(\mathcal{P}_1)^n$ functions in each element which are also continuous at the midpoints of the edges or faces of the partition, and Q_h are piecewise constant functions. Error estimates for anisotropic elements for this method have been proved in [2, 7].

The Fortin operator for this case is the edge (or face) mean average interpolator $\mathbf{\Pi}$ defined in the previous section. In view of (5.9), estimate (6.4) holds with a constant independent of the geometry of the elements which can be taken to be one. However, this is a non-conforming method (because $W_h \not\subset H_0^1(\Omega)^2$) and therefore, to obtain error estimates, some consistency terms have to be bounded. This can be done by using the RT_0 interpolation analyzed in the previous section. In this way it is possible to obtain optimal error estimates for this method under the maximum angle condition (see [2]).

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