

ADDENDUM

EQUIVALENCE OF TANGENT BUNDLES

The fact that all reasonable candidates for the tangent bundle of M turn out to be essentially the same is stated precisely as follows.

4. THEOREM*. If we have a bundle $T'M$ over M for each M , and a bundle map (f_{\sharp}, f) for each C^{∞} map $f: M \rightarrow N$ satisfying

- (1) of Theorem 1,
- (2) of Theorem 1, for certain equivalences t'^n ,
- (3) of Theorem 1, for certain equivalences $T'U \simeq (T'M)|U$,

then there are equivalences

$$e_M: TM \rightarrow T'M$$

such that the following diagram commutes for every C^{∞} map $f: M \rightarrow N$.

$$\begin{array}{ccc} TM & \xrightarrow{f_*} & TN \\ e_M \downarrow & & \downarrow e_N \\ T'M & \xrightarrow{f_{\sharp}} & T'N \end{array}$$

PROOF. The details of this proof are so horrible that you should probably skip it (and you should definitely quit when you get bogged down); the welcome symbol \diamond occurs quite a ways on. Nevertheless, the idea behind the proof is simple enough. If (x, U) is a chart on M , then both $(TM)|U$ and $(T'M)|U$ “look like” $x(U) \times \mathbb{R}^n$, so there ought to be a map taking the fibres of one to the fibres of the other. What we have to hope is that our conditions on TM and $T'M$ make them “look alike” in a sufficiently strong way for this idea to really work out. Those who have been through this sort of rigamarole before know (i.e., have faith) that it’s going to work out; those for whom this sort of proof is a new experience should find it painful and instructive.

Functorites will notice that Theorems 1 and 4 say that there is, up to natural equivalence, a unique functor from the category of C^{∞} manifolds and C^{∞} maps to the category of bundles and bundle maps which is naturally equivalent to $(\varepsilon^n, \text{old } f_)$ on Euclidean spaces, and to the restriction of the functor on open submanifolds.

Let (x, U) be a coordinate system on M . Then we have the following string of equivalences. Two of them, which are denoted by the same symbol \simeq , are the equivalences mentioned in condition (3). Let α_x denote the composition $\alpha_x = (t^n|_{x(U)}) \circ \simeq \circ x_* \circ (\simeq)^{-1}$.

$$(TM)|_U \xleftarrow{\simeq} TU \xrightarrow{x_*} T(x(U)) \xrightarrow{\simeq} (T\mathbb{R}^n)|_{x(U)} \xrightarrow{t^n|_{x(U)}} \varepsilon^n(\mathbb{R}^n)|_{x(U)}$$

α_x

Similarly, using equivalence \simeq' for T' , we can define β_x .

$$(T'M)|_U \xleftarrow{\simeq'} T'U \xrightarrow{x_{\sharp}} T'(x(U)) \xrightarrow{\simeq'} (T'\mathbb{R}^n)|_{x(U)} \xrightarrow{t'^n|_{x(U)}} \varepsilon^n(\mathbb{R}^n)|_{x(U)}$$

β_x

Then

$$\beta_x^{-1} \circ \alpha_x: (TM)|_U \rightarrow (T'M)|_U$$

is an equivalence, so it takes the fibre of TM over p isomorphically to the fibre of $T'M$ over p for each $p \in U$. Our main task is to show that this isomorphism between the fibres over p is independent of the coordinate system (x, U) . This will be done in three stages.

(I) Suppose $V \subset U$ is open and $y = x|_V$. We will need to name all the inclusion maps

$$\begin{aligned} i: U &\rightarrow M \\ \bar{i}: V &\rightarrow M \\ j: V &\rightarrow U \\ k: y(V) &\rightarrow x(U). \end{aligned}$$

To compare α_x and α_y , consider the following diagram.

$$\begin{array}{ccccccc} (TM)|_U & \xleftarrow{\simeq} & TU & \xrightarrow{x_*} & T(x(U)) & \xrightarrow{\simeq} & (T\mathbb{R}^n)|_{x(U)} \xrightarrow{t^n|_{x(U)}} \varepsilon^n(\mathbb{R}^n)|_{x(U)} \\ \uparrow \subset & \textcircled{1} & \uparrow j_* & \textcircled{2} & \uparrow k_* & \textcircled{3} & \uparrow \subset \textcircled{4} \uparrow \subset \\ (TM)|_V & \xleftarrow{\simeq} & TV & \xrightarrow{y_*} & T(y(V)) & \xrightarrow{\simeq} & (T\mathbb{R}^n)|_{y(V)} \xrightarrow{t^n|_{y(V)}} \varepsilon^n(\mathbb{R}^n)|_{y(V)} \end{array}$$

(1)

Each of the four squares in this diagram commutes. To see this for square ①, we enlarge it, as shown below. The two triangles on the left commute by condition (3) for TM , and the one on the right commutes because $i \circ j = \bar{i}$.

$$\begin{array}{ccc}
 (TM)|U & & \\
 \downarrow \subset & \swarrow \cong & \\
 & TU & \\
 \uparrow \subset & \nwarrow i_* & \\
 TM & & \\
 \uparrow \subset & \nwarrow \bar{i}_* & \\
 (TM)|V & & \\
 & \swarrow \cong & \\
 & TV & \\
 & \uparrow j_* &
 \end{array}$$

Square ② commutes because $k \circ y = x \circ j$. Square ③ commutes for the same reason as square ①; the inclusions $x(U) \rightarrow \mathbb{R}^n$ and $y(V) \rightarrow \mathbb{R}^n$ come into play. Square ④ obviously commutes. Chasing through diagram (1) now shows that the following commutes.

$$\begin{array}{ccc}
 (TM)|U & \xrightarrow{\alpha_x} & \varepsilon^n(\mathbb{R}^n)|x(U) \\
 \uparrow \subset & & \uparrow \subset \\
 (TM)|V & \xrightarrow{\alpha_y} & \varepsilon^n(\mathbb{R}^n)|y(V)
 \end{array}$$

This means that for $p \in V$, the isomorphism α_y between the fibres over p is the same as α_x . Clearly the same is true for β_x and β_y , since our proof used only properties (1), (2), and (3), not the explicit construction of TM . Thus $\beta_y^{-1} \circ \alpha_y = \beta_x^{-1} \circ \alpha_x$ on the fibres over p , for every $p \in V$.

(II) We now need a Lemma which applies to both TM and $T'M$. Again, it will be proved for TM (where it is actually obvious), using only properties (1), (2), and (3), so that it is also true for $T'M$.

LEMMA. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are open, and $f: A \rightarrow B$ is C^∞ , then the following diagram commutes.

$$\begin{array}{ccccc} TA & \xrightarrow{\cong} & (T\mathbb{R}^n)|A & \xrightarrow{t^n|A} & \varepsilon^n(\mathbb{R}^n)|A \\ f_* \downarrow & & & & \downarrow \text{old } f_* \\ TB & \xrightarrow{\cong} & (T\mathbb{R}^m)|B & \xrightarrow{t^m|B} & \varepsilon^m(\mathbb{R}^m)|B \end{array}$$

PROOF. Case 1. There is a map $\bar{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\bar{f} = f$ on A . Consider the following diagram, where $i: A \rightarrow \mathbb{R}^n$ and $j: B \rightarrow \mathbb{R}^m$ are the inclusion maps.

$$\begin{array}{ccccccc} & & & (T\mathbb{R}^n)|A & \xrightarrow{t^n|A} & \varepsilon^n(\mathbb{R}^n)|A & \\ & \nearrow \cong & & \downarrow \subset & & \downarrow \subset & \\ TA & \xrightarrow{i_*} & T\mathbb{R}^n & \xrightarrow{t^n} & \varepsilon^n(\mathbb{R}^n) & & \\ f_* \downarrow & & \downarrow \bar{f}_* & & \downarrow \text{old } \bar{f}_* & & \\ TB & \xrightarrow{j_*} & T\mathbb{R}^m & \xrightarrow{t^m} & \varepsilon^m(\mathbb{R}^m) & & \\ & \searrow \cong & & \uparrow \subset & & \uparrow \subset & \\ & & & (T\mathbb{R}^m)|B & \xrightarrow{t^m|B} & \varepsilon^m(\mathbb{R}^m)|B & \end{array}$$

Everything in this diagram obviously commutes. This implies that the two compositions

$$TA \xrightarrow{\cong} (T\mathbb{R}^n)|A \xrightarrow{t^n|A} \varepsilon^n(\mathbb{R}^n)|A \xrightarrow{\subset} \varepsilon^n(\mathbb{R}^n) \xrightarrow{\text{old } \bar{f}_*} \varepsilon^m(\mathbb{R}^m)$$

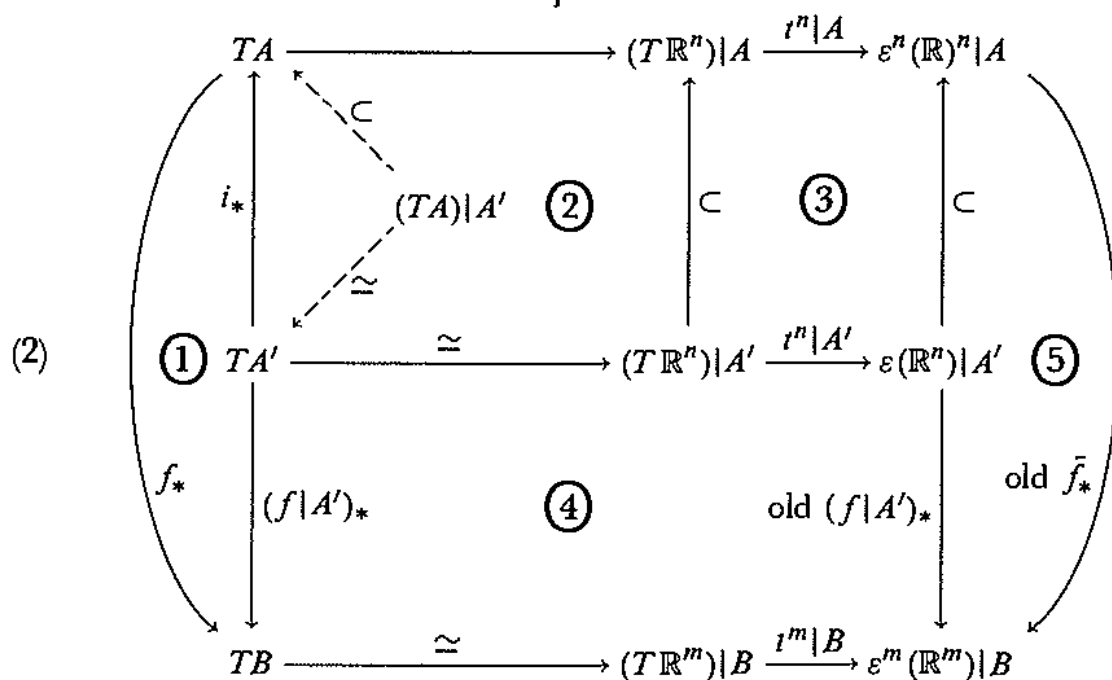
and

$$TA \xrightarrow{f_*} TB \xrightarrow{\cong} (T\mathbb{R}^m)|B \xrightarrow{t^m|B} \varepsilon^m(\mathbb{R}^m)|B \xrightarrow{\subset} \varepsilon^m(\mathbb{R}^m)$$

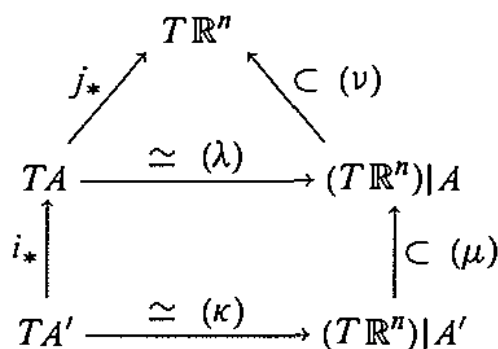
are equal and this proves the Lemma in Case 1, since the maps “old \bar{f}_* ” and “old f_* ” are equal on A .

Case 2. General case. For each $p \in A$, we want to show that two maps are the same on the fibre over p . Now there is a map $\bar{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\bar{f} = f$ on an open set A' , where $p \in A' \subset A$. We then have the following diagram, where every \cong comes from the fact that some set is an open submanifold of another,

and $i: A' \rightarrow A$ is the inclusion map.



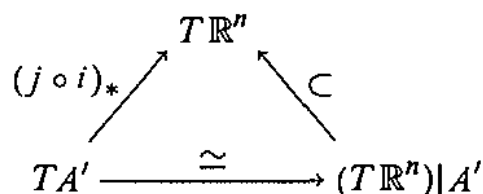
Boxes (1), (3), and (5) obviously commute, and (4) commutes by *Case 1*. To see that square (2) (which has a triangle within it) commutes, we imbed it in a larger diagram, in which $j: A \rightarrow \mathbb{R}^n$ is the inclusion map, and other maps have also been named, for ease of reference.



To prove that $\lambda \circ i_* = \mu \circ \kappa$, it suffices to prove that

$$\nu \circ \lambda \circ i_* = \nu \circ \mu \circ \kappa,$$

since ν is one-one. Thus it suffices to prove $j_* \circ i_* = \nu \circ \mu \circ \kappa$, which amounts to proving commutativity of the following diagram.



Since $j \circ i$ is just the inclusion of A' in \mathbb{R}^n , this does commute.

Commutativity of diagram (2) shows that the composition

$$TA \xrightarrow{f_*} TB \xrightarrow{\simeq} (T\mathbb{R}^m)|_B \xrightarrow{t^m|_B} \varepsilon^m(\mathbb{R}^m)|_B$$

coincides, on the subset $(TA)|_{A'}$, with the composition

$$TA \xrightarrow{\simeq} (T\mathbb{R}^n)|_A \xrightarrow{t^n|_A} \varepsilon^n(\mathbb{R}^n)|_A \xrightarrow{\text{old } \tilde{f}_*} \varepsilon^m(\mathbb{R}^m)|_B,$$

and on A' we can replace “old \tilde{f}_* ” by “old f_* ”. In other words, the two compositions are equal in a neighborhood of any $p \in A$, and are thus equal, which proves the Lemma.

(III) Now suppose (x, U) and (y, V) are any two coordinate systems with $p \in U \cap V$. To prove that $\beta_y^{-1} \circ \alpha_y$ and $\beta_x^{-1} \circ \alpha_x$ induce the same isomorphism on the fibre of TM at p , we can assume without loss of generality that $U = V$, because part (I) applies to x and $x|_{U \cap V}$, as well as to y and $y|_{U \cap V}$.

Assuming $U = V$, we have the following diagram.

$$(3) \quad \begin{array}{ccccc} & & T(x(U)) & \xrightarrow{\simeq} & (T\mathbb{R}^n)|_{x(U)} & \xrightarrow{t^n|_{x(U)}} & \varepsilon^n(\mathbb{R}^n)|_{x(U)} \\ & \nearrow x_* & \downarrow (y \circ x^{-1})_* & & \downarrow \text{old } (y \circ x^{-1})_* & & \downarrow \\ (TM)|_U & \xleftarrow{\simeq} & TU & & & & \\ & \searrow y_* & T(y(U)) & \xrightarrow{\simeq} & (T\mathbb{R}^n)|_{y(U)} & \xrightarrow{t^n|_{y(U)}} & \varepsilon^n(\mathbb{R}^n)|_{y(U)} \end{array}$$

The triangle obviously commutes, and the rectangle commutes by part (II). Diagram (3) thus shows that

$$\alpha_y = \text{old } (y \circ x^{-1})_* \circ \alpha_x.$$

Exactly the same result holds for T' :

$$\beta_y = \text{old } (y \circ x^{-1})_* \circ \beta_x.$$

The desired result $\beta_y^{-1} \circ \alpha_y = \beta_x^{-1} \circ \alpha_x$ follows immediately.

Now that we have a well-defined bundle map $TM \rightarrow T'M$ (the union of all $\beta_x^{-1} \circ \alpha_x$), it is clearly an equivalence e_M . The proof that $e_N \circ f_* = f_{\#} \circ e_M$ is left as a masochistic exercise for the reader. ♦