ADDENDUM
EQUIVALENCE OF TANGENT BUNDLES

The fact that all reasonable candidates for the tangent bundle of M turn out to be essentially the same is stated precisely as follows.

4. THEOREM*. If we have a bundle $T'M$ over M for each M, and a bundle map $(f_\sharp, f)$ for each C^∞ map $f: M \to N$ satisfying

(1) of Theorem 1,

(2) of Theorem 1, for certain equivalences t'^n,

(3) of Theorem 1, for certain equivalences $T'U \simeq (T'M)|U$,

then there are equivalences

$$e_M : TM \to T'M$$

such that the following diagram commutes for every C^∞ map $f: M \to N$.

$$\begin{tikzcd} TM \arrow[r, "f_*"] \arrow[d, "e_M"'] & TN \arrow[d, "e_N"] \\ T'M \arrow[r, "f_\sharp"] & T'N \end{tikzcd}$$

PROOF. The details of this proof are so horrible that you should probably skip it (and you should definitely quit when you get bogged down); the welcome symbol ❖ occurs quite a ways on. Nevertheless, the idea behind the proof is simple enough. If (x, U) is a chart on M, then both $(TM)|U$ and $(T'M)|U$ "look like" $x(U) \times \mathbb{R}^n$, so there ought to be a map taking the fibres of one to the fibres of the other. What we have to hope is that our conditions on TM and $T'M$ make them "look alike" in a sufficiently strong way for this idea to really work out. Those who have been through this sort of rigamarole before know (i.e., have faith) that it's going to work out; those for whom this sort of proof is a new experience should find it painful and instructive.

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*Functorites will notice that Theorems 1 and 4 say that there is, up to natural equivalence, a unique functor from the category of C^∞ manifolds and C^∞ maps to the category of bundles and bundle maps which is naturally equivalent to $(\varepsilon^n, \text{old } f_*)$ on Euclidean spaces, and to the restriction of the functor on open submanifolds.

Let (x, U) be a coordinate system on M. Then we have the following string of equivalences. Two of them, which are denoted by the same symbol $\simeq$, are the equivalences mentioned in condition (3). Let α_x denote the composition $\alpha_x = (\iota^n|x(U)) \circ \simeq \circ\, x_* \circ (\simeq)^{-1}$.

$$\begin{tikzcd}
(TM)|U \arrow[rrrr, bend right, "\alpha_x"'] & TU \arrow[l, "\simeq"'] \arrow[r, "x_*"] & T(x(U)) \arrow[r, "\simeq"] & (T\mathbb{R}^n)|x(U) \arrow[r, "\iota^n|x(U)"] & \varepsilon^n(\mathbb{R}^n)|x(U)
\end{tikzcd}$$

Similarly, using equivalence $\simeq'$ for T', we can define β_x.

$$\begin{tikzcd}
(T'M)|U \arrow[rrrr, bend right, "\beta_x"'] & T'U \arrow[l, "\simeq'"'] \arrow[r, "x_\sharp"] & T'(x(U)) \arrow[r, "\simeq'"] & (T\mathbb{R}^n)|x(U) \arrow[r, "\iota'^n|x(U)"] & \varepsilon^n(\mathbb{R}^n)|x(U)
\end{tikzcd}$$

Then

$$\beta_x{}^{-1} \circ \alpha_x \colon (TM)|U \to (T'M)|U$$

is an equivalence, so it takes the fibre of TM over p isomorphically to the fibre of $T'M$ over p for each $p \in U$. Our main task is to show that this isomorphism between the fibres over p is independent of the coordinate system (x, U). This will be done in three stages.

(I) Suppose $V \subset U$ is open and $y = x|V$. We will need to name all the inclusion maps

$$\begin{aligned}
i &: U \to M \\
\bar{\imath} &: V \to M \\
j &: V \to U \\
k &: y(V) \to x(U).
\end{aligned}$$

To compare α_x and α_y, consider the following diagram.

$$\text{(1)}\qquad \begin{tikzcd}
(TM)|U & TU \arrow[l, "\simeq"'] \arrow[r, "x_*"] & T(x(U)) \arrow[r, "\simeq"] & (T\mathbb{R}^n)|x(U) \arrow[r, "\iota^n|x(U)"] & \varepsilon^n(\mathbb{R}^n)|x(U) \\
(TM)|V \arrow[u, "\subset"'] \arrow[ur, phantom, "1"] & TV \arrow[l, "\simeq"'] \arrow[r, "y_*"] \arrow[u, "j_*"'] \arrow[ur, phantom, "2"] & T(y(V)) \arrow[r, "\simeq"] \arrow[u, "k_*"'] \arrow[ur, phantom, "3"] & (T\mathbb{R}^n)|y(V) \arrow[r, "\iota^n|y(V)"] \arrow[u, "\subset"'] \arrow[ur, phantom, "4"] & \varepsilon^n(\mathbb{R}^n)|y(V) \arrow[u, "\subset"']
\end{tikzcd}$$

Each of the four squares in this diagram commutes. To see this for square (1), we enlarge it, as shown below. The two triangles on the left commute by condition (3) for TM, and the one on the right commutes because $i \circ j = \bar{\imath}$.

$$\begin{tikzcd}
(TM)|U \arrow[dd, "\subset"] & \\
 & TU \arrow[ul, "\simeq"'] \arrow[dl, "i_*"'] \\
TM & \\
 & TV \arrow[uu, "j_*"'] \arrow[ul, "\bar{\imath}_*"] \arrow[dl, "\simeq"] \\
(TM)|V \arrow[uu, "\subset"'] &
\end{tikzcd}$$

Square (2) commutes because $k \circ y = x \circ j$. Square (3) commutes for the same reason as square (1); the inclusions $x(U) \to \mathbb{R}^n$ and $y(V) \to \mathbb{R}^n$ come into play. Square (4) obviously commutes. Chasing through diagram (1) now shows that the following commutes.

$$\begin{tikzcd}
(TM)|U \arrow[r, "\alpha_x"] & \varepsilon^n(\mathbb{R}^n)|x(U) \\
(TM)|V \arrow[u, "\subset"] \arrow[r, "\alpha_y"] & \varepsilon^n(\mathbb{R}^n)|y(V) \arrow[u, "\subset"']
\end{tikzcd}$$

This means that for $p \in V$, the isomorphism α_y between the fibres over p is the same as α_x. Clearly the same is true for β_x and β_y, since our proof used only properties (1), (2), and (3), not the explicit construction of TM. Thus ${\beta_y}^{-1} \circ \alpha_y = {\beta_x}^{-1} \circ \alpha_x$ on the fibres over p, for every $p \in V$.

(II) We now need a Lemma which applies to both TM and $T'M$. Again, it will be proved for TM (where it is actually obvious), using only properties (1), (2), and (3), so that it is also true for $T'M$.

LEMMA. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are open, and $f\colon A \to B$ is C^∞, then the following diagram commutes.

$$
\begin{tikzcd}
TA \arrow[r, "\simeq"] \arrow[d, "f_*"'] & (T\mathbb{R}^n)|A \arrow[r, "t^n|A"] & \varepsilon^n(\mathbb{R})^n|A \arrow[d, "\text{old } f_*"] \\
TB \arrow[r, "\simeq"] & (T\mathbb{R}^m)|B \arrow[r, "t^m|B"] & \varepsilon^m(\mathbb{R})^m|B
\end{tikzcd}
$$

PROOF. Case 1. There is a map $\bar{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ *with* $\bar{f} = f$ *on* A. Consider the following diagram, where $i\colon A \to \mathbb{R}^n$ and $j\colon B \to \mathbb{R}^m$ are the inclusion maps.

$$
\begin{tikzcd}
 & (T\mathbb{R})^n|A \arrow[r, "t^n|A"] \arrow[d, "\subset"] & \varepsilon^n(\mathbb{R}^n)|A \arrow[d, "\subset"] \\
TA \arrow[ur, "\simeq"] \arrow[r, "i_*"] \arrow[d, "f_*"'] & T\mathbb{R}^n \arrow[r, "t^n"] \arrow[d, "\bar{f}_*"] & \varepsilon^n(\mathbb{R}^n) \arrow[d, "\text{old } \bar{f}_*"] \\
TB \arrow[r, "j_*"] \arrow[dr, "\simeq"'] & T\mathbb{R}^m \arrow[r, "t^m"] & \varepsilon^m(\mathbb{R}^m) \\
 & (T\mathbb{R}^m)|B \arrow[r, "t^m|B"] \arrow[u, "\subset"'] & \varepsilon^m(\mathbb{R}^m)|B \arrow[u, "\subset"']
\end{tikzcd}
$$

Everything in this diagram obviously commutes. This implies that the two compositions

$$
TA \xrightarrow{\simeq} (T\mathbb{R}^n)|A \xrightarrow{t^n|A} \varepsilon^n(\mathbb{R}^n)|A \xrightarrow{\subset} \varepsilon^n(\mathbb{R}^n) \xrightarrow{\text{old } \bar{f}_*} \varepsilon^m(\mathbb{R}^m)
$$

and

$$
TA \xrightarrow{f_*} TB \xrightarrow{\simeq} (T\mathbb{R}^m)|B \xrightarrow{t^m|B} \varepsilon^m(\mathbb{R}^m)|B \xrightarrow{\subset} \varepsilon^m(\mathbb{R}^m)
$$

are equal and this proves the Lemma in Case 1, since the maps "old $\bar{f}_*$" and "old f_*" are equal on A.

Case 2. General case. For each $p \in A$, we want to show that two maps are the same on the fibre over p. Now there is a map $\bar{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ with $\bar{f} = f$ on an open set A', where $p \in A' \subset A$. We then have the following diagram, where every $\simeq$ comes from the fact that some set is an open submanifold of another, and $i\colon A' \to A$ is the inclusion map.

(2)
$$
\begin{tikzcd}
TA \arrow[rr] \arrow[ddd, bend right=50, "f_*"'] & & (T\mathbb{R}^n)|A \arrow[r, "\iota^n|A"] & \varepsilon^n(\mathbb{R})^n|A \arrow[ddd, bend left=50, "\text{old } \bar{f}_*"] \\
 & (TA)|A' \arrow[ul, dashed, "\subset"'] \arrow[dl, dashed, "\simeq"] \arrow[r, phantom, "2"] & {} \arrow[r, phantom, "3"] & {} \\
TA' \arrow[uu, "i_*"] \arrow[d, "(f|A')_*"'] \arrow[rr, "\simeq"] \arrow[uu, phantom, "1"', shift right=6] & & (T\mathbb{R}^n)|A' \arrow[uu, "\subset"'] \arrow[r, "\iota^n|A'"] \arrow[d, phantom, "4"] & \varepsilon(\mathbb{R}^n)|A' \arrow[uu, "\subset"'] \arrow[d, "\text{old } (f|A')_*"'] \arrow[uu, phantom, "5", shift left=8] \\
TB \arrow[rr, "\simeq"] & & (T\mathbb{R}^m)|B \arrow[r, "\iota^m|B"] & \varepsilon^m(\mathbb{R}^m)|B
\end{tikzcd}
$$

Boxes 1, 3, and 5 obviously commute, and 4 commutes by *Case 1*. To see that square 2 (which has a triangle within it) commutes, we imbed it in a larger diagram, in which $j\colon A \to \mathbb{R}^n$ is the inclusion map, and other maps have also been named, for ease of reference.

$$
\begin{tikzcd}
 & T\mathbb{R}^n & \\
TA \arrow[ur, "j_*"] \arrow[rr, "\simeq\ (\lambda)"] & & (T\mathbb{R}^n)|A \arrow[ul, "\subset\ (\nu)"'] \\
TA' \arrow[u, "i_*"] \arrow[rr, "\simeq\ (\kappa)"] & & (T\mathbb{R}^n)|A' \arrow[u, "\subset\ (\mu)"']
\end{tikzcd}
$$

To prove that $\lambda \circ i_* = \mu \circ \kappa$, it suffices to prove that

$$\nu \circ \lambda \circ i_* = \nu \circ \mu \circ \kappa,$$

since ν is one-one. Thus it suffices to prove $j_* \circ i_* = \nu \circ \mu \circ \kappa$, which amounts to proving commutativity of the following diagram.

$$
\begin{tikzcd}
 & T\mathbb{R}^n & \\
TA' \arrow[ur, "(j \circ i)_*"] \arrow[rr, "\simeq"] & & (T\mathbb{R}^n)|A' \arrow[ul, "\subset"']
\end{tikzcd}
$$

Since $j \circ i$ is just the inclusion of A' in $\mathbb{R}^n$, this does commute.

Commutativity of diagram (2) shows that the composition

$$TA \xrightarrow{f_*} TB \xrightarrow{\simeq} (T\mathbb{R}^m)|B \xrightarrow{t^m|B} \varepsilon^m(\mathbb{R}^m)|B$$

coincides, *on the subset* $(TA)|A'$, with the composition

$$TA \xrightarrow{\simeq} (T\mathbb{R}^n)|A \xrightarrow{t^n|A} \varepsilon^n(\mathbb{R}^n)|A \xrightarrow{\text{old } \bar{f}_*} \varepsilon^m(\mathbb{R}^m)|B,$$

and on A' we can replace "old $\bar{f}_*$" by "old f_*". In other words, the two compositions are equal in a neighborhood of any $p \in A$, and are thus equal, which proves the Lemma.

(III) Now suppose (x, U) and (y, V) are any two coordinate systems with $p \in U \cap V$. To prove that ${\beta_y}^{-1} \circ \alpha_y$ and ${\beta_x}^{-1} \circ \alpha_y$ induce the same isomorphism on the fibre of TM at p, we can assume without loss of generality that $U = V$, because part (I) applies to x and $x|U \cap V$, as well as to y and $y|U \cap V$.

Assuming $U = V$, we have the following diagram.

(3)
$$\begin{tikzcd}
& & T(x(U)) \arrow[r, "\simeq"] \arrow[dd, "(y\circ x^{-1})_*"] & (T\mathbb{R}^n)|x(U) \arrow[r, "t^n|x(U)"] & \varepsilon^n(\mathbb{R}^n)|x(U) \arrow[dd, "\text{old }(y\circ x^{-1})_*"'] \\
(TM)|U & TU \arrow[l, "\simeq"'] \arrow[ur, "x_*"] \arrow[dr, "y_*"'] & & & \\
& & T(y(U)) \arrow[r, "\simeq"] & (T\mathbb{R}^n)|y(U) \arrow[r, "t^n|y(U)"] & \varepsilon^n(\mathbb{R}^n)|y(U)
\end{tikzcd}$$

The triangle obviously commutes, and the rectangle commutes by part (II). Diagram (3) thus shows that

$$\alpha_y = \text{old } (y \circ x^{-1})_* \circ \alpha_x.$$

Exactly the same result holds for T':

$$\beta_y = \text{old } (y \circ x^{-1})_* \circ \beta_x.$$

The desired result ${\beta_y}^{-1} \circ \alpha_y = {\beta_x}^{-1} \circ \alpha_x$ follows immediately.

Now that we have a well-defined bundle map $TM \to T'M$ (the union of all ${\beta_x}^{-1} \circ \alpha_x$), it is clearly an equivalence e_M. The proof that $e_N \circ f_* = f_\sharp \circ e_M$ is left as a masochistic exercise for the reader. ❖