AAECC 2, 1–14 (1991)

AAECC
Applicable Algebra in Engineering, Communication and Computing

Effective Łojasiewicz Inequalities in Semialgebraic Geometry

Pablo Solernó

Working Group Noaï Fitchas. IAM. Viamonte 1636. ler. piso. (1055) Buenos Aires. Argentina. // Fac. Ciencias Exactas y Naturales. Universidad de Buenos Aires

Received May 9, 1990

Abstract. The main result of this paper can be stated as follows: let $V \subset \mathbb{R}^n$ be a compact semialgebraic set given by a boolean combination of inequalities involving only polynomials whose number and degrees are bounded by some $D > 1$. Let F, $G \in \mathbb{R}[X_1, \ldots, X_n]$ be polynomials with $\deg F, \deg G \leqq D$ inducing on V continuous semialgebraic functions $f, g: V \to R$. Assume that the zeros of f are contained in the zeros of g. Then the following effective Łojasiewicz inequality is true: there exists an universal constant $c_1 \in \mathbb{N}$ and a positive constant $c_2 \in \mathbb{R}$ (depending on V, f, g) such that $|g(x)|^{D^{c_1 \cdot n}} \leqq c_2 \cdot |f(x)|$ for all $x \in V$. This result is generalized to arbitrary given compact semialgebraic sets V and arbitrary continuous functions $f, g: V \to \mathbb{R}$. An effective global Łojasiewicz inequality on the minimal distance of solutions of polynomial inequalities systems and an effective Finiteness Theorem (with admissible complexity bounds) for open and closed semialgebraic sets are derived.

Keywords: Łojasiewicz inequalities, Real algebraic geometry, Computer algebra, Complexity

I. Introduction

In the present paper we consider different versions of the well-known Łojasiewicz-Inequality in semialgebraic geometry both from a quantitative and an algorithmic point of view.

Our main result is the following (see Theorem 3 (ii) below): let V be a compact semialgebraic subset of $\mathbb{R}^n$ defined by a boolean combination of inequalities involving only polynomials whose number and degrees are bounded by some $D > 1$. Let further $F, G \in \mathbb{R}[X_1, \ldots, X_n]$ be polynomials with $\deg F, \deg G \leqq D$ inducing on V continuous semialgebraic functions $f, g: V \to R$. Assume that the zeros of f are contained in the zeros of g. Then there exists an universal constant $c_1 \in \mathbb{N}$ and a positive constant $c_2 \in \mathbb{R}$, depending on V, f and g, such that for all $x \in V$ the inequality $|g(x)|^{D^{c_1 n}} \leqq c_2 |f(x)|$ holds. If all polynomials involved in the definition of V and F, G have integer coefficients of maximum modulus l, then c_2 may be chosen as $c_2 := l^{D^{\bar{c}_1 n^2}}$ where $\bar{c}_1$ is a suitable universal constant.

An inequality of type $|g(x)|^N \leqq c_2 \cdot |f(x)|$ for all $x \in V$ is called a Łojasiewicz-Inequality, N is called its exponent. If furthermore N is given explicitely as a function of D and n, we say that our Łojasiewicz-Inequality is *effective*, stressing its quantitative algebraic nature.

It is wickedly known (although it seems not to be published) that it is possible to show an effective Łojasiewicz-Inequality with an exponent which is polynomial in D but *doubly* exponential in n. One obtains this result as a consequence of "fast" cylindric algebraic decomposition (see [2], [12] or [4]) by a somewhat lengthy but rather straightforward proof.

In our effective Łojasiewicz-Inequality the exponent is polynomial in D but only *single exponential* in n.

This quantitative improvement is due to recent progress in computational commutative algebra, namely the effective (Hilberts-) Nullstellensatz with single exponential bounds (see e.g. [3] and the references given there). One of the algorithmical consequences of this result is a new and more efficient quantifier elimination procedure for real closed fields (see [7, 8]) on which our efficient Łojasiewicz-Inequality is based. Apart from this we follow the general lines of the proof of the "classical" Łojasiewicz-Inequality (without any bound) of [1] taking care of subtle details concerning the estimates.

We would like to stress here the fact that our bound on the exponent in the Łojasiewicz-Inequality is asymptotically optimal by the following example essentially due to Möller–Mora, Lazard, Masser, Philippon and many others (see e.g. [10]): consider the compact semialgebraic set $V := \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_1^2 + \cdots + x_n^2 \leqq 1\}$ and the polynomials $F := (X_2 - X_1^D)^2 + \cdots + (X_n - X_{n-1}^D)^2 + X_n^2$ and $G := X_1^2 + \cdots + X_n^2$, which induce on V continuous semialgebraic functions $f, g: V \to \mathbb{R}$. f and g have both only the origin as zero. For $u \in \mathbb{R}$ positive and sufficiently small the point $x(u) := (u, u^D, \ldots, u^{D^{n-1}})$ lies in the closed ball V and we have:

$$f(x(u)) = u^{2D^{n-1}} \quad \text{and} \quad g(x(u)) = |x(u)|^2 \approx u^2.$$

Thus any Łojasiewicz-Inequality $|g(x)|^N \leqq c_2 |f(x)|$ implies $u^{2N} \leqq c_2 u^{2D^{n-1}}$ for all positive and sufficiently small $u \in \mathbb{R}$. Therefore we obtain $N \geqq D^{n-1}$ for the exponent of any Łojasiewicz-Inequality for V, f and g.

We give also an adequate generalization of the mentioned effective Łojasiewicz-Inequality to arbitrary given compact semialgebraic sets V and arbitrary continuous semialgebraic functions $f, g: V \to \mathbb{R}$ (see Theorem 3(i) below). As an application of this result we obtain an effective (and algorithmic) Finiteness Theorem for any *open* semialgebraic set S of $\mathbb{R}^n$ given as before (Theorem 9 and Remark 10 below): we find in admissible time (i.e. in sequential time $D^{n^{O(1)}}$ and parallel time $(n \log_2 D)^{O(1)}$) a representation of S by basic open sets which involve only polynomials whose number and degrees are bounded by $D^{O(n^3)}$. (An analogous fact is true for closed semialgebraic sets.) This result simplifies the treatment of many computational problems dealing with open or closed semialgebraic sets within the complexity class of admissible algorithms since it allows the algorithmical reduction to basic (open or closed) semialgebraic sets.

In Theorem 7 below we prove for closed basic semialgebraic sets a "global" effective Łojasiewicz-Inequality with single exponential bounds. This result is of some use in approximatively solving of polynomial inequalities systems because it gives some a priori information where a solution of such a system can be found if there is any. An analogous result (with more precise bounds) in the context of algebraically closed fields equipped by an absolute value was independently obtained in [9].

II. Preliminaries

Throughout this paper we fix a real closed field R and a subring A of R (for example $R:=\mathbb{R}$ and $A:=\mathbb{Z}$).

We consider $R^n, n=0,1,\ldots$ as a topological space equipped with the euclidean topology. Let $x:=(x_1,\ldots,x_n)$ and $y:=(y_1,\ldots,y_n)$ be two points of R^n, we write

$$|x-y|:=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

for their euclidean distance. If $r>0$, is a positive element of R we write $B(x,r):=\{y\in R^n; |x-y|<r\}$ for the open ball of radius r centered at x.

If $x,y\in R$, we denote by $[x,y],(x,y),[x,y),(x,y]$ the closed, open and half open intervals with boundaries x and y.

A *semialgebraic subset* of R^n (over A) is a set definable by a boolean combination of equalities and inequalities involving polynomials from $A[X_1,\ldots,X_n]$ ($X_1,\ldots,X_n$ are indeterminates -variables- over R).

Let $V\subset R^n$, $W\subset R^n$ be semialgebraic sets and $f:V\to W$ a map. We call f *semialgebraic* if its graph is a semialgebraic subset of R^{n+m}.

The image of a semialgebraic set by a semialgebraic function is semialgebraic too. This fact is called the *Tarski–Seidenberg Principle* (see [1], Theorem 2.2.1).

The Tarski–Seidenberg Principle can also be stated in terms of logics: let $\mathscr{L}$ be the elementary language of real closed fields with constants from A, we call two formulas of $\mathscr{L}$, in the same free variables $X_1,\ldots,X_n$, *equivalent* (with respect to $\mathscr{L}$) if they define the same semialgebraic subset of R^n. The Tarski–Seidenberg Principle says that for each formula $\Phi\in\mathscr{L}$ there exists $\bar{\Phi}\in\mathscr{L}$, equivalent to Φ and without quantifiers (see [1] Sects. 2.2 and 5.2); in other words, if Φ is a formula in $\mathscr{L}$, with n free variables, the subset V of R^n defined by

$$V:=\{x\in R^n;\ x \text{ satisfies } \Phi\}$$

is a semialgebraic set.

We define now the parameters which are involved in our subsequent quantitative estimations:

– if $\mathscr{F}\subset A[X_1,\ldots,X_n]$ is a finite set of polynomials we write:

$$\deg(\mathscr{F}):=\sum_{f\in\mathscr{F}}\deg f$$

$$\sigma(\mathscr{F}):=\max\{\deg f; f\in\mathscr{F}\}$$

(deg f denotes the total degree of f)

$$l(\mathcal{F}):=\max\left\{\begin{array}{c}\text{absolute values of all the coefficients}\\ \text{of the polynomials in } \mathcal{F}\end{array}\right\}$$

– for each formula $\Phi\in\mathcal{L}$ built up by atomic formulas involving a finite set $\mathcal{F}_\Phi$ of polynomials of $A[X_1,\ldots,X_n]$, we write

$$|\Phi|:=\text{length of } \Phi$$

$$\deg(\Phi):=\deg(\mathcal{F}_\Phi)$$

$$\sigma(\Phi):=\sigma(\mathcal{F}_\Phi)$$

$$l(\Phi):=l(\mathcal{F}_\Phi)$$

– if the formula $\Phi\in\mathcal{L}$ is a *prenex* formula (i.e. all its quantifiers occur at the beginning of Φ) we define:

$$a(\Phi):=\text{the number of alternating blocks of quantifiers of } \Phi.$$

In order to define the notion of *algorithm*, we use the concept of arithmetical network over A (see [5]); the number of nodes and the depth of the associated graph will be the *sequential* and the *parallel complexity* respectively.

With this notions we can state our main tool, an effective version of Tarski–Seidenberg Principle:

Theorem 1. *Let Φ be a prenex formula in the elementary language $\mathcal{L}$ of real closed fields with constant in A. Let n be its number of variables,* $D:=\deg(\Phi)$ *and* $a:=a(\Phi)$. *It is possible to compute in sequential time* $D^{n^{O(a)}}\cdot|\Phi|^{O(1)}$ *and in parallel time* $n^{O(a)}\cdot(\log_2 D)^{O(1)}+(\log_2|\Phi|)^{O(1)}$ *a quantifier free formula $\bar{\Phi}$ equivalent to Φ. Moreover, there exists an universal constant $c\in\mathbb{N}$ (independent of Φ) such that the formula $\bar{\Phi}$ satisfies the following:*

$$-\deg(\bar{\Phi})\leqq D^{n^{c\cdot a}}$$

$$-\ l(\bar{\Phi})\leqq l(\Phi)^{D^{n^{c\cdot a}}}$$

In particular, when $a=1$, we have more precise bounds:

$$-\sigma(\bar{\Phi})\leqq D^{c\cdot n}$$

$$-\ l(\bar{\Phi})\leqq l(\Phi)^{D^{c\cdot n}}.$$

Proof. [7, 8]; see also [11]. ■

III. Łojasiewicz Inequalities

Notation. Let $V\subset R^n$ be a semialgebraic set and $f: V\to R$ a semialgebraic function, we write $Z(f):=\{x\in V; f(x)=0\}$.

Lemma 2. *Let $V\subset R^q$ be a non empty and closed semialgebraic set defined by a prenex formula $\Phi\in\mathcal{L}$. Let $f, g: V\to R$ be two continuous semialgebraic functions which satisfy $Z(f)\subset Z(g)$ and let $\Phi_1,\Phi_2\in\mathcal{L}$ be prenex formulas describing the graphs of f and g respectively. Let* $\mathcal{F}_\Phi\cup\mathcal{F}_{\Phi_1}\cup\mathcal{F}_{\Phi_2}\subset A[X_1,\ldots,X_n]$, $D:=\max\{\deg(\Phi),$

$\deg(\Phi_1), \deg(\Phi_2)\}$ *and* $a := \max\{a(\Phi), a(\Phi_1), a(\Phi_2)\}$. *Then there exists a constant* $c_1 \in \mathbb{N}$ *(independent of* V, f *and* g*) such that:*

(i) *if* $a > 0$, *the function* $\alpha: V \to R$ *defined by*

$$\alpha(x) := \begin{cases} \dfrac{g(x)^{D^{n^{c_1 \cdot a}}}}{f(x)} & \text{if} \quad x \notin Z(f) \\ 0 & \text{if} \quad x \in Z(f) \end{cases}$$

is a continuous semialgebraic function.

(ii) *if* $a = 0$ *(i.e. if* Φ, Φ_1 *and* Φ_2 *are quantifier free formulas), the function* $\alpha: V \to R$ *defined by*

$$\alpha(x) := \begin{cases} \dfrac{g(x)^{D^{c_1 \cdot n}}}{f(x)} & \text{if} \quad x \notin Z(f) \\ 0 & \text{if} \quad x \in Z(f) \end{cases}$$

is a continuous semialgebraic function.

Proof. Our proof follows the line of that of [1] Proposition 2.6.4 and we treat only (ii) since (i) can be shown in an analogous way. Without loss of generality we may suppose that $g \not\equiv 0$ (i.e. g is not identically zero).

1. For each $x \in V$ and $u \in R$ we denote by $V(x, u)$ the following bounded and closed semialgebraic subset of R^n:

$$V(x, u) := \overline{B(x, 1)} \cap \{y \in V; u.|g(y)| = 1\}.$$

(Observe that $g \not\equiv 0$ implies the existence of an element $(x, u) \in R^{n+1}$ such that $V(x, u) \neq \varnothing$.)

We shall consider the following formula $\psi \in \mathscr{L}$:

$$\Phi(X) \wedge (|Y - X|^2 \leqq 1) \wedge \Phi(Y) \wedge \Phi_2(Y, T) \wedge ((U.T)^2 = 1) \wedge (U > 0)$$

(with $X := (X_1, \ldots, X_n)$ and $Y := (Y_1, \ldots, Y_n)$).

ψ is a quantifier free formula in the $2n+2$ variables X, Y, T, U with $\deg(\psi) = O(D)$.

A point $(x, y, t, u) \in R^{2n+2}$ verifies ψ if and only if $y \in V(x, u)$ and $g(y) = t$.

2. We shall also consider the semialgebraic function $v: V \times R \to R$ defined by:

$$v(x, u) := \begin{cases} \max\left\{\dfrac{1}{|f(y)|}; y \in V(x, u)\right\} & \text{if} \quad V(x, u) \neq \varnothing \\ 0 & \text{if} \quad V(x, u) = \varnothing \end{cases}.$$

(Observe that $g(y) \neq 0$ implies $f(y) \neq 0$ for all $y \in V(x, u)$).

Let $\theta \in \mathscr{L}$ be the formula:

$$(\exists Y)(\exists T)(\exists W)(\psi(X, Y, T, U) \wedge \Phi_1(Y, W) \wedge |W| \cdot Z \leqq 1)$$

(where Z and W are two new variables).

Note that for $(x,u)\in R^{n+1}$ satisfying $V(x,u)\neq\varnothing$, the formula $\theta(x,u,Z)$ describes the half line $(-\infty, v(x,u)]\subset R$. Obviously $\deg(\theta)=O(D)$.

3. Applying Theorem 1 to the formula θ, one obtains a new quantifier free formula $\bar{\theta}$ in the $n+2$ variables X, U, Z, which is equivalent to θ and which satisfies $\sigma(\bar{\theta})\leqq D^{c.n}$. (Here the constant $c\in\mathbb{N}$ is independent of θ.)

We may assume that $\bar{\theta}$ is a disjunction of formulas which have the following form:

(*) $h_1(X,U,Z)>0\wedge\cdots\wedge h_k(X,U,Z)>0\wedge h_{k+1}(X,U,Z)=0\wedge\cdots\wedge h_s(X,U,Z)=0$

where $h_1,\ldots,h_s$ are polynomials of $A[X_1,\ldots,X_n,U,Z]$ and where k ranges between 0 and s. (Note that the formula (*) contains no strict inequality if $k=0$ and no equality if $k=s$.)

4. Let be given $x\in V$ and suppose first that $V(x,u)\neq\varnothing$ for some $u\in R$. According to (2) and (3), the formula $\bar{\theta}(x,u,Z)$ describes the half line $(-\infty, v(x,u)]$. Therefore $\bar{\theta}$ contains at least one conjunction of type (*) such that the following formula (in the only free variable Z):

$$h_1(x,u,Z)>0\wedge\cdots\wedge h_k(x,u,Z)>0\wedge h_{k+1}(x,u,Z)=0\wedge\cdots\wedge h_s(x,u,Z)=0$$

is consistent (i.e. satisfiable in R) and contains an equality. Thus there exists an index j ($k<j\leqq s$) for which:

$$-\ h_j(x,u,Z)\not\equiv 0 \quad \text{in } R[Z]$$
$$-\ h_j(x,u,v(x,u))=0.$$

Let $\tilde{\mathscr{F}}(x)\subset R[U,Z]$ be the set of polynomials which appear in $\bar{\theta}(x,U,Z)$. Applying cylindrical algebraic decomposition to $\tilde{\mathscr{F}}(x)$ in order to remove the variable Z (see for example [1] Theorem 2.3.1) one concludes that there exists an element $\beta(x)\in R$ such that one of the following conditions is satisfied:

a) $V(x,u)=\varnothing \quad \forall u\geqq\beta(x)$
b) $V(x,u)\neq\varnothing \quad \forall u\geqq\beta(x)$.

Moreover in the case b), for all $u\geqq\beta(x)$, the same conjunction of type (*) in $\bar{\theta}(x,u,Z)$:

$$h_1(x,u,Z)>0\wedge\cdots\wedge h_k(x,u,Z)>0\wedge h_{k+1}(x,u,Z)=0\wedge\cdots\wedge h_s(x,u,Z)=0$$

is consistent and the same index j ($k<j\leqq s$) verifies the conditions:

$$-\ h_j(x,u,Z)\not\equiv 0 \quad \text{for all} \quad u\geqq\beta(x)$$
$$-\ h_j(x,u,v(x,u))=0 \quad \text{for all} \quad u\geqq\beta(x).$$

In the case that $V(x,u)=\varnothing$ for all $u\in R$ we put $\beta(x):=1$.

5. For each $x\in V$ we define a polynomial $h\in A[X_1,\ldots,X_n,U,Z]$ (depending on the point x) as:

$h:=Z$ if for all $u\in R$, $V(x,u)=\varnothing$ or if a) in (4) is satisfied.

$h:=h_j(X_1,\ldots,X_n,U,Z)$ if b) in (4) is satisfied.

The polynomial h just defined verifies:

$$-h(x,u,Z)\not\equiv 0 \quad \text{for all} \quad u\geqq\beta(x)$$
$$-h(x,u,v(x,u))=0 \quad \text{for all} \quad u\geqq\beta(x)$$
$$-\deg h\leqq\sigma(\bar{\theta})\leqq D^{c\cdot n}.$$

From the construction of $\beta(x)$ and from the fact that $v(x,u)$ is a root of $h(x,u,Z)$, we conclude now that

$$|v(x,u)|\leqq\gamma(x)\cdot u^{p_0} \quad \text{for all} \quad u\geqq\beta(x)$$

where $\gamma(x)\in R$ is a positive constant depending on x and where p_0 (which also depends on x) is a natural number bounded by $\deg h\leqq\sigma(\bar{\theta})$.

Without loss of generality we may assume $\beta(x)\geqq 1$ for all $x\in V$. Thus we obtain

$$|v(x,u)|\leqq\gamma(x)\cdot u^{\sigma(\bar{\theta})} \quad \text{for all} \quad u\geqq\beta(x).$$

6. Let $p:=\sigma(\bar{\theta})+1$. We observe that $p\leqq D^{c_1\cdot n}$ where $c_1\in\mathbb{N}$ is a constant not depending on V,f,g.

We claim that the semialgebraic function $\alpha:V\to R$ defined by:

$$\alpha(x):=\begin{cases}\dfrac{g(x)^p}{f(x)} & \text{if} \quad x\notin Z(f)\\ 0 & \text{if} \quad x\in Z(f)\end{cases}$$

is a continuous function.

To prove this, it is enough to analyze the continuity of α in a point $x\in Z(f)$. Let $\varepsilon>0$ and $0<\delta<1$ be two elements of R such that $|g(y)|<\min\left\{\beta(x_0),\dfrac{\varepsilon}{\gamma(x_0)}\right\}$ for all $y\in B(x_0,\delta)\cap V$.

Consider $y\in B(x_0,\delta)\cap(V\setminus Z(g))$. We obtain that $y\in B\left(x_0,\dfrac{1}{|g(y)|}\right)$ and thus

$$\frac{1}{|f(y)|}\leqq v\left(x_0,\frac{1}{|g(y)|}\right).$$

From

$$v\left(x_0,\frac{1}{|g(y)|}\right)\leqq\gamma(x_0)\cdot\left|\frac{1}{g(y)}\right|^{\sigma(\bar{\theta})} \quad \text{(see (5))}$$

one deduces:

$$\frac{|g(y)|^{\sigma(\bar{\theta})}}{|f(y)|}\leqq\gamma(x_0) \quad \text{for all} \quad y\in B(x_0,\delta)\cap(V\setminus Z(g))$$

This implies $|\alpha(y)|<\varepsilon$ for all $y\in B(x_0,\delta)\cap(V\setminus Z(g))$.

In the case $y\in Z(g)$ we have $\alpha(y)=0$. ∎

From this lemma it is easy to deduce the following "Łojasiewicz Inequality":

Theorem 3. *Let $V\subset R^q$ be a non empty, closed and bounded semialgebraic set defined by a prenex formula $\Phi\in\mathscr{L}$. Let $f,g:V\to R$ be two continuous semialgebraic functions* *which satisfy $Z(f) \subset Z(g)$ and let $\Phi_1, \Phi_2 \in \mathscr{L}$ be formulas describing the graphs of f and g respectively. Let n, D, a be defined as in Lemma 2. Then there exists an universal constant $c_1 \in \mathbb{N}$ (not depending on V, f and g) and a positive element $c_2 \in \mathbb{R}$ (depending from them)* such that:

(i) *if* $a > 0$, $|g(x)|^{D^{n^{c_1 \cdot a}}} \leqq c_2 \cdot |f(x)|$ *for all* $x \in V$.

(ii) *if* $a = 0$ (*i.e. if* Φ, Φ_1 *and* Φ_2 *are quantifier free formulas*):

$$|g(x)|^{D^{c_1 \cdot n}} \leqq c_2 \cdot |f(x)| \quad \textit{for all } x \in V.$$

In the case $A := \mathbb{Z}$, $R := \mathbb{R}$, *and if* $l := \max\{l(\Phi), l(\Phi_1), l(\Phi_2)\}$, *we can choose* $c_2 = l^{D^{n^{\bar{c}_1 \cdot a}}}$ *in* (i) *and* $c_2 = l^{D^{\bar{c}_1 \cdot n^2}}$ *in* (ii). (*Here* $\bar{c}_1 \in \mathbb{N}$ *is an universal constant*).

Proof. We consider only the case $a = 0$ (the case $a > 0$ can be treated in an analogous way).

We observe that the hypothesis of Lemma 2 are verified. Let $\alpha: V \to R$ be the semialgebraic functions whose continuity is asserted in the lemma. Since V is a closed and bounded semialgebraic set, α takes a maximum there (see [1] Theorem 2.5.8). Thus we have

$$|g(x)|^{D^{c_1 \cdot n}} \leqq c_2 \cdot |f(x)| \quad \text{for all } x \in V \quad (c_2 := \max\{|\alpha(x)|; x \in V\}).$$

Finally in order to obtain the asserted estimation for c_2 in the case $A := \mathbb{Z}$ and $R := \mathbb{R}$ we consider the following formula $\Psi \in \mathscr{L}$:

$$(\exists X)(\exists W)(\exists T)(\Phi_1(X, W) \wedge \Phi_2(X, T) \wedge (T^{2D^{c_1 \cdot n}} \geqq (U \cdot W)^2) \wedge (W^2 > 0) \wedge (U \geqq 0))$$

where $X := (X_1, \ldots, X_n)$.

Ψ is a formula in $n+2$ quantified variables $(X_1, \ldots, X_n, W, T)$ and one free variable U, which describes the semialgebraic interval $[0, c_2] \subset R$. Note that $\deg(\Psi) = O(D^{c_1 \cdot n})$ and $l(\Psi) \leqq l$.

Applying Theorem 1 to the prenex formula Ψ, one concludes that there exists a polynomial $h \in \mathbb{Z}[U]$ such that:

$$- h \not\equiv 0 \quad \text{and} \quad \deg h = D^{O(n^2)}$$

$$- l(h) = l^{D^{O(n)}}$$

$$- h(c_2) = 0.$$

Since c_2 is a root of the polynomial h one infers now easily that there exists an universal constant $\bar{c}_1 \in \mathbb{N}$, not depending on V, f and g such that $c_2 \leqq l^{D^{\bar{c}_1 \cdot n^2}}$. ■

Remark 4. Theorem 1 and the proof of Theorem 3 show that in the case of an arbitrary real closed field it is possible to compute the bound c_2 by means of an algorithm which runs in sequential time $D^{n^{O(a)}} \cdot (|\Phi| + |\Phi_1| + |\Phi_2|)$ and parallel time $n^{O(a)} \cdot (\log_2 D)^{O(1)} + (\log_2(|\Phi| + |\Phi_1| + |\Phi_2|))^{O(1)}$ for $a > 0$, and $D^{n^{O(1)}}$ (sequential time), $(n \cdot \log_2 D)^{O(1)}$ (parallel time) for $a = 0$.

Lemma 5. *Let $V \subset R^q$ be a non empty and closed semialgebraic set defined by a prenex formula $\Phi \in \mathscr{L}$ and let $f: V \to R$ be a continuous semialgebraic function whose graph is given by a prenex formula $\Phi_1 \in \mathscr{L}$. Let $\mathscr{F}_\Phi \cup \mathscr{F}_{\Phi_1} \subset A[X_1, \ldots, X_n]$, $D := \max\{\deg(\Phi), \deg(\Phi_1)\}$, $a := \max\{a(\Phi), a(\Phi_1)\}$. Then there exists an universal constant $c_1 \in \mathbb{N}$ (not depending on V and f) and a positive element $c_2 \in R$ (depending* *from them) such that:*

(i) *if* $a>0$: $|f(x)| \leqq c_2 \cdot (1+|x|)^{D^{n^{c_1 \cdot a}}}$ *for all* $x \in V$.

(ii) *if* $a=0$ (*i.e.* Φ *and* Φ_1 *are quantifier free formulas*)

$$|f(x)| \leqq c_2 \cdot (1+|x|)^{D^{c_1 \cdot n}} \quad \textit{for all} \quad x \in V.$$

In the case $A := \mathbb{Z}$, $R := \mathbb{R}$, *if* $l := \max\{l(\Phi), l(\Phi_1)\}$, *we can choose* $c_2 = l^{D^{n^{\bar{c}_1 \cdot a}}}$ *in (i) and* $c_2 = l^{D^{\bar{c}_1 \cdot n}}$, where $\bar{c}_1 \in \mathbb{N}$ *is an universal constant.*

Proof. Our arguments are similar from the ones used in Lemma 2 and are based in [1] Proposition 2.6.2.

Again we consider only the case $a=0$.

1. For each $u \in R$, we define:

$$V(u) := \{x \in V; |x| = u\}$$

$V(u) \subset R^n$ is a bounded and closed semialgebraic set.

Let $\Psi \in \mathscr{L}$ be the formula in the $n+1$ free variables $X := (X_1, \dots, X_n)$, U, defined by:

$$\Psi: \Phi(X) \wedge (|X|^2 = U^2) \wedge (U \geqq 0)$$

It is clear that $\deg(\Psi) = O(D)$ and that $V(u) \neq \emptyset$ holds if and only if u satisfies the formula $(\exists X)(\Psi(X, U))$.

2. Let $v: R \to R$ be the following semialgebraic function:

$$v(u) := \begin{cases} \max\{|f(x)|; x \in V(u)\} & \text{if} \quad V(u) \neq \emptyset \\ 0 & \text{if} \quad V(u) = \emptyset \end{cases}$$

(Note that this function is well defined by means of [1] Theorem 2.5.8.) We consider the following formula θ:

$$(\exists X)(\exists W)(\Psi(X, U) \wedge \Phi_1(X, W) \wedge W^2 \geqq Z^2)$$

Obviously $\deg(\theta) = O(D)$.

Applying Theorem 1 to θ, one obtains a quantifier free formula $\bar{\theta}$, equivalent to θ in the variables U, Z with $\sigma(\bar{\theta}) = D^{c_1 \cdot n}$ ($c_1 \in \mathbb{N}$ is an universal constant).

We may assume that the formula $\bar{\theta}$ is a disjunction of formulas which have the following form:

$$(**) \quad h_1(U,Z) > 0 \wedge \cdots \wedge h_k(U,Z) > 0 \wedge h_{k+1}(U,Z) = 0 \wedge \cdots \wedge h_s(U,Z) = 0$$

where $h_1, \dots, h_s \in A[U, Z]$ and k ranges between 0 and s.

3. We fix $u \in R$. Thus the formula $\bar{\theta}(u, Z)$ defines a semialgebraic subset of R: either the empty set (in case $V(u) = \emptyset$), or the closed interval $[-v(u), v(u)]$ (in case $V(u) \neq \emptyset$). Therefore, if $u \in R$ verifies $V(u) \neq \emptyset$, there is in $\bar{\theta}$ at least one conjunction of type $(**)$ such that the formula (in the only free variable Z):

$$h_1(u,Z) > 0 \wedge \cdots \wedge h_k(u,Z) > 0 \wedge h_{k+1}(u,Z) = 0 \wedge \cdots \wedge h_s(u,Z) = 0$$

is consistent and contains an equality. Thus there exists an index j ($k < j \leqq s$) for which

$$-\, h_j(u,Z) \not\equiv 0$$
$$-\, \deg_Z h_j(u,Z) \geqq 1$$
$$-\, h_j(u,v(u)) = 0.$$

4. Applying cylindrical algebraic decomposition to the polynomial set $\mathscr{F}_{\bar{\theta}} \subset A[U,Z]$ in order to remove the variable Z (cf. [1] Theorem 2.3.1. or [4]) one deduces that there exists an element $\beta \in R$ such that one of the following conditions is satisfied:

a) $V(u) = \varnothing$ for all $u \geqq \beta$.

b) $V(u) \neq \varnothing$ for all $u \geqq \beta$.

Moreover, in case b) for all $u \geqq \beta$ the same conjunction of type (**) in $\bar{\theta}(U,Z)$ is consistent and the same index j $(k < j \leqq s)$ verifies the conditions:

$$-\, h_j(u,Z) \not\equiv 0 \quad \text{for all} \quad u \geqq \beta$$
$$-\, \deg_Z h_j(u,Z) \geqq 1 \quad \text{for all} \quad u \geqq \beta$$
$$-\, h_j(u,v(u)) = 0 \quad \text{for all} \quad u \geqq \beta.$$

5. We define a polynomial $h \in A[U,Z]$ as:

$h := Z$ if a) occurs in (4) or,
$h := h_j$ if b) occurs in (4).

In both cases the polynomial h verifies:

$$-\, h(u,Z) \not\equiv 0 \quad \text{for all} \quad u \geqq \beta$$
$$-\, \deg_Z h(u,Z) \geqq 1 \quad \text{for all} \quad u \geqq \beta$$
$$-\, h(u,v(u)) = 0 \quad \text{for all} \quad u \geqq \beta$$
$$-\, \deg h \leqq \deg h_j \leqq \sigma(\bar{\theta}) = D^{c_1 \cdot n}.$$

From the construction (by cylindrical algebraic decomposition) of β and the fact that $v(u)$ is a root of $h(u,Z)$, we infer that there exists a positive constant $k_1 \in R$ such that:

(***) $$|v(u)| \leqq k_1 |u|^p \quad \text{with} \quad p \leqq \deg h \leqq D^{c_1 \cdot n}, \quad \text{for all } u \geqq \beta.$$

Let $k_2 := \max\{|f(x)|; x \in V \text{and } |x| \leqq \beta\}$. (Observe that [1] Theorem 2.5.8 guarantees that k_2 is well defined).

We denote by $c_2 := \max\{k_1, k_2\}$. Thus we obtain:

$-$ if $x \in V$ and $|x| \leqq \beta$: $|f(x)| \leqq k_2 \leqq c_2 \leqq c_2(1+|x|)^p$

$-$ if $x \in V$ and $|x| \geqq \beta$: $|f(x)| \leqq \max\{|f(y)|; y \in V(|x|)\}$
$= v(|x|) \leqq k_1 \cdot |x|^p$ (by (***)), and finally $|f(x)| \leqq c_2(1+|x|)^p$.

We conclude that $|f(x)| \leqq c_2 \cdot (1+|x|)^{D^{c_1 \cdot n}}$ for all $x \in V$.

6. In the case $A := \mathbb{Z}$, $R := \mathbb{R}$ it is possible to found the values of k_1 and k_1 in order to compute c_2. For k_1 we observe that it suffices to bound the real roots of the polynomial $h(u,Z)$ for arbitrary high values $u \geqq \beta$. From Theorem 1 one infers

$\deg h \leqq D^{c_1 \cdot n}$ and $l(h) \leqq l(\bar{\theta}) = l^{D^{c_1 \cdot n}}$. The constant β comes from the elimination of one variable by cylindrical decomposition applied to $\mathscr{F}_{\bar{\theta}}$ and this procedure increases only polynomially the degrees. Finally for k_2 we apply again Theorem 1 to estimate an upper bound for $\max\{|f(x)|; x \leqq \beta\}$; for this purpose we consider the following formula φ:

$$(\exists X)(\exists W)((\Phi_1(X, W)) \wedge (W^2 \geqq T^2) \wedge (T \geqq 0) \wedge (|X|^2 \leqq \beta^2))$$

in the only free variable T. φ describes the closed interval $[0, k_2] \subset R$. Clearly $\deg(\varphi) = O(D)$ and $l(\varphi) = l^{D^{O(n)}}$. From Theorem 1 above we conclude that k_2 can be chosen as $l^{D^{O(n)}}$. ■

Remark 6. The proof of Lemma 5 gives an algorithm to compute effectively a bound for the constant c_2; the algorithm runs within the same time limits as in Remark 4.

Definition. Let $V \subset R^n$ be a closed semialgebraic set and let $x \in R^n$ be an arbitrary point. We denote by $d(x, V)$ the usual euclidean distance between x and V, i.e. $d(x, V) := \min\{|x - y|; y \in V\}$. (Note that this minimum exists by [1] Theorem 2.5.8.)

Now we are able to state another Łojasiewicz-type inequality:

Theorem 7. *Let* $f_1, \ldots, f_s \in A[X_1, \ldots, X_n]$, $D := \sum_{i=1}^{s} \deg f_i$, $V := \{x \in R^n; f_1(x) \geqq 0 \wedge \cdots \wedge f_k(x) \geqq 0 \wedge f_{k+1}(x) = 0 \wedge \cdots \wedge f_s(x) = 0\}$ *and assume that* V *is non empty. Then, there exist constants* $c_1 \in \mathbb{N}$ *(not depending on the* f_i*),* $m \in \mathbb{N}$ *and a positive* $c_2 \in R$ *(both depending on the* f_i*) such that*

$$d(x, V)^m \leqq c_2 \cdot \max_{1 \leqq i \leqq s} \{|f_i(x)|\} \cdot (1 + |x|)^{D^{n^{c_1}}} \quad \textit{for all} \quad x \in R^n,$$

where $m \leqq D^{n^{c_1}}$.

In the case $A := \mathbb{Z}$, $R := \mathbb{R}$, *if* $l := l(\{f_1, \ldots, f_s\})$ *we can choose* $c_2 := l^{D^{n^{\bar{c}_1}}}$ $(\bar{c}_1 \in \mathbb{N}$ *is an universal constant).*

Proof. Let $\Phi \in \mathscr{L}$ be the following formula which describes V:

$$f_1(X) \geqq 0 \wedge \cdots \wedge f_k(X) \geqq 0 \wedge f_{k+1}(X) = 0 \wedge \cdots \wedge f_s(X) = 0.$$

We have $\deg(\Phi) = D$ and $a(\Phi) = 0$ $(X := (X_1, \ldots, X_n))$.

We consider the following functions $f, g: R^n \to R$:

$$f(x) := \sqrt{\sum_{1 \leqq i \leqq s} f_i^2(x)}$$

$$g(x) := d(x, V).$$

f and g are continuous semialgebraic functions satisfying $Z(f) \subset Z(g)$.

The graphs of f and g can be described by the following formulas Φ_1 and Φ_2:

$$\Phi_1: \left(W^2 - \sum_{1 \leqq i \leqq s} f_i^2(X) = 0 \right) \wedge (W \geqq 0)$$

$$\Phi_2: (\exists Y)(\forall Y')(\neg \Phi(Y') \vee (\Phi(Y) \wedge (|X - Y|^2 = T^2) \wedge (T \geqq 0) \wedge (|X - Y'|^2 \geqq T^2))$$

$(Y := (Y_1, \ldots, Y_n),\ Y' := (Y'_1, \ldots, Y'_n))$.

We have $\deg(\Phi_i) = O(D)$ $(i = 1, 2)$, $a(\Phi_1) = 0$ and $a(\Phi_2) = 2$.

Applying Lemma 2 to the functions f and g we deduce that there exist $k_1 \in \mathbb{N}$ (not depending on the f_i) and a natural number $m \leqq D^{n^{k_1}}$ (depending from them) such that the function $\alpha: R^n \to R$:

$$\alpha(x) := \begin{cases} \dfrac{g(x)^m}{f(x)} & \text{if} \quad x \notin Z(f) \\ 0 & \text{if} \quad x \in Z(f) \end{cases}$$

is continuous.

The graph of α can be described by the following formula Φ_3:

$$(\exists W)(\exists T)(\Phi_1(X, W) \wedge \Phi_2(X, T) \wedge (((T^m - UW = 0) \wedge (W > 0)) \vee (W = 0 \wedge U = 0))),$$

which verifies $\deg(\Phi_3) = O(D)$ and $a(\Phi_3) = 3$.

By Lemma 5 applied to the function α we obtain that there exist a consistent $k_2 \in \mathbb{N}$ (not depending on the f_i) and a positive constant $k_3 \in R$ (depending from them) such that

$$|\alpha(x)| \leqq k_3 \cdot (1 + |x|)^{D^{n^{k_2}}} \quad \text{for all} \quad x \in R^n.$$

Thus,

$$d(x, V)^m \leqq k_3 \cdot f(x) \cdot (1 + |x|)^{D^{n^{k_2}}} \quad \text{for all} \quad x \in R^n.$$

Putting $c_2 := D \cdot k_3$ and $c_1 := \max\{k_1, k_2\}$ we conclude:

$$d(x, V)^m \leqq k_3 \cdot f(x)(1 + |x|)^{D^{n^{k_2}}} \leqq c_2 \cdot \max_{1 \leqq i \leqq s} \{|f_i(x)|\} \cdot (1 + |x|)^{D^{n^{c_1}}},$$

for all $x \in R^n$, with $m \leqq D^{n^{c_1}}$.

Finally, when $A := \mathbb{Z}$ and $R := \mathbb{R}$, the asserted bound for c_2 can be deduced from Lemma 5 taking into account $l(\Phi_3) \leqq l^2$. ∎

Remark 8. Remark 6 and the proof of Theorem 7 show that there exists an algorithm which computes the constant c_2 with sequential complexity $D^{n^{O(1)}}$ and parallel complexity $(n \cdot \log_2 D)^{O(1)}$.

IV. An Effective Finiteness Theorem

In this section we are going to apply the results of the last paragraph in order to obtain an effective representation theorem for open and for closed semialgebraic sets.

The qualitative part of this result (not involving any bounds) is also known as "Finiteness Theorem" (see [1] Theorem 2.7.1.).

Theorem 9. *Let $S \subset R^n$ be an open (resp. closed) semialgebraic set given by a quantifier free formula $\Phi \in \mathscr{L}$. Then there exists a decomposition of S:*

$$S = \bigcup_{j=1}^{N} S_j$$

where each S_j *is an open (resp. closed) semialgebraic set of the form*:

$$S_j := \{x \in R^n; f_{1j}(x) > 0 \wedge \cdots \wedge f_{s_j j}(x) > 0\}$$

$$(\textit{resp. } S_j := \{x \in R^n; f_{1j}(x) \geqq 0 \wedge \cdots \wedge f_{s_j j}(x) \geqq 0\})$$

where the f_{ij} *are polynomials of* $A[X_1, \ldots, X_n]$ $(1 \leqq i \leqq s_j, 1 \leqq j \leqq N)$. *Moreover for* $D := \deg(\Phi)$ *we have*:

$$-\deg f_{ij} = D^{O(n^3)} (1 \leqq i \leqq s_j, 1 \leqq j \leqq N)$$

$$-N \leqq D \quad (\textit{resp. } N = D^{O(n^4)})$$

$$-\sum_{i,j} \deg f_{ij} = D^{O(n^3)} (\textit{resp. } D^{O(n^4)}).$$

Proof. We treat here only the case of an open set following the arguments of the proof of [1] Theorem 2.7.1 (the case of closed sets is obtained taking complements). Let $\Phi = \bigvee_{j=1}^{N} \Phi^{(j)}$ be the disjunctive form of Φ, where each $\Phi^{(j)}$ is a formula of type:

$$h_1 > 0 \wedge \cdots \wedge h_k > 0 \wedge h_{k+1} = 0 \wedge \cdots \wedge h_s = 0,$$

with $h_r \in A[X_1, \ldots, X_n]$, $0 \leqq r \leqq s$, $0 \leqq k \leqq s$.

We may assume that $0 < k < s$. (The case $k = s$ is obvious and, if $k = 0$, we add one inequality $h_{k+1} + 1 > 0$).

Let us fix j $(1 \leqq j \leqq N)$, and let $S' \subset S$ be the semialgebraic set described by $\Phi^{(j)}$.

We define:

$$f := \sum_{r=k+1}^{s} h_r^2 \quad \text{and} \quad g := \prod_{r=1}^{k} (h_r + |h_r|).$$

Both are continuous semialgebraic functions defined on R^n.

Let $V \subset R^n$ be the closed semialgebraic set $R^n \backslash S$, described by the formula $\varphi := \neg \Phi$.

By [6] Lemma 1, one infers that φ can be chosen (eliminating superfluous conjunctions) in such a way that $\deg(\varphi) \leqq D^{4n}$.

Let Φ_1, $\Phi_2 \in \mathscr{L}$ be the following formulas, which define the graphs of $f|_V$ and of $g|_V$:

$$\Phi_1 : (W - f(X) = 0) \wedge \varphi(X)$$

$$\Phi_2 : (\varphi(X) \wedge \left(U - 2^k \prod_{r=1}^{k} h_r(X) = 0 \right) \wedge h_1(X) > 0 \wedge \cdots \wedge h_k(X) > 0)$$

$$\vee \bigvee_{r=1}^{k} (\varphi(X) \wedge (U = 0) \wedge (h_r(X) \leqq 0)).$$

$(X := (X_1, \ldots, X_n))$.

We have $D^{O(n)} = \max\{\deg(\Phi_1), \deg(\Phi_2), \deg(\varphi)\}$.

Since $Z(f) \cap V \subset Z(g) \cap V$ we can apply Lemma 2 (ii) to $f|_V, g|_V$. Thus the semialgebraic function $\alpha: V \to R$:

$$\alpha(x) := \begin{cases} \dfrac{g(x)^{D^{c_1 \cdot n^2}}}{f(x)} & \text{if } x \in V \backslash Z(f) \\ 0 & \text{if } x \in V \cap Z(f) \end{cases} \quad \text{is continuous} \quad (c_1 \in \mathbb{N}).$$

The graph of α can be described by the following quantifier free formula Φ_3:

$$\left(\left(2^k\cdot\prod_{r=1}^{k} h_r(X)\right)^{D^{c_1\cdot n^2}} - Z\cdot f(X)=0 \wedge h_1(X)>0 \wedge \cdots \wedge h_k(X)>0 \wedge \varphi(X)\right)$$

$$\vee \bigvee_{r=1}^{k} (\varphi(X) \wedge Z=0 \wedge h_r(X) \leqq 0).$$

We have $\deg(\Phi_3)=D^{O(n^2)}$.

From Lemma 5 we conclude that $|\alpha(x)| \leqq c_2\cdot(1+|x|^2)^{D^{c'_1\cdot n^3}}$ for all $x\in V$. (Here $c'_1\in\mathbb{N}$ is an universal constant and c_2 a positive element of R).

For $j=1,\ldots,N$ we consider the following open semialgebraic sets:

$$S_j:=\left\{x\in R^n; c_2 f(x)(1+|x|^2)^{D^{c'_1\cdot n^3}} < \left(2^k\prod_{r=1}^{k} h_r(x)\right)^{D^{c_1\cdot n^2}} \wedge h_1(x)>0 \wedge \cdots \wedge h_k(x)>0\right\}.$$

Clearly $S' \subset S_j$ and $S_j \cap V=\emptyset$, whence $S' \subset S_j \subset S$. Therefore $S=\bigcup_{j=1}^{N} S_j$ where each S_j is defined only by (strict) inequalities. ■

Remark 10. By means of Theorem 1 it is possible to describe effectively the consistent conjunctions appearing in the formula φ. Combining this with Remark 6 one can construct the set of polynomials $\{f_{ij}\}$ and conjunctions defining each S_j in sequential time $D^{n^{O(1)}}$ and in parallel time $(n\log_2 D)^{O(1)}$.

References

1. Bochnak, J., Coste, M., Roy, M-F.: Géométrie algébrique réelle. Berlin, Heidelberg, New York: Springer 1987
2. Collins, G.: Quantifier elimination for real closed fields by cylindric algebraic decomposition. Second GI Conference on Automata Theory and Formal Languages. Lecture Notes in Computer Sciences Vol. 33. pp. 134–183. Berlin, Heidelberg, New York: Springer 1975
3. Fitchas, N., Galligo, A.: Nullstellensatz effectif et Conjecture de Serre (Théorème de Quillen–Suslin) pour le Calcul Formel. Mathematische Nachrichten (to appear)
4. Fitchas, N., Galligo, A., Morgenstern, J.: Algorithmes rapides en séquentiel et en parallèle pour l'élimination des quantificateurs en géométrie élémentaire. Publ. Math. Univ. Paris VII. Sém. Struct. Alg. Ord. '84–'87 Vol. I, 103–145 (1990)
5. von zur Gathen, J.: Parallel arithmetic computations: a survey. Proc. 13th. Conf. MFCS. Lecture Notes in Computer Sciences, Vol. 233 pp. 93–112. Berlin, Heidelberg, New York: Springer 1986
6. Grigor'ev, D.: Complexity of deciding Tarski algebra. J. Sumb. Comput. **5**, 65–108 (1988)
7. Heintz, J., Roy, M.-F., Solernó, P.: Complexité du principe de Tarski–Seidenberg. Comptes–Rendus de l'Académie des Sciences Paris **309**, 825–830 (1989)
8. Heintz, J., Roy, M-F., Solernó, P.: Sur la complexité du principe de Tarski–Seidenberg. Bull. Soc. Math. France **118**, 101–126 (1990)
9. Ji, S., Kollár, J., Shiffman, B.: A global Łojasiewicz inequality for algebraic varieties. Trans. A.M.S. (to appear)
10. Möller, H., Mora, F.: Upper and lower bounds for the degree of Gröebner bases. Proc. EUROSAM 84., Cambridge, England. Lecture Notes in Comp. Sci. Vol. 174. pp. 172–183. Berlin, Heidelberg, New York: Springer 1984
11. Renegar, J.: On the computational complexity and geometry of the first order theory of the reals. Technical Report Vol. 856, Cornell University Ithaca (1989)
12. Wüthrich, H.: Ein Entscheidungsverfahren für die Theorie der reell-abgeschlossenen Körper. Komplexität von Entscheidungsproblemen, Lecture Notes in Comp. Sci. Vol. 43. pp. 138–162. Berlin, Heidelberg, New York: Springer 1976