SINGLE EXPONENTIAL PATH FINDING IN SEMIALGEBRAIC SETS
PART I: THE CASE OF A REGULAR BOUNDED HYPERSURFACE

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Abstract. *Let V be a bounded semialgebraic hypersurface defined by a regular polynomial equation and let x_1, x_2 be two points of V. Assume that x_1, x_2 are given by a boolean combination of polynomial inequalities. We describe an algorithm which decides in single exponential sequential time and polynomial parallel time whether x_1 and x_2 are contained in the same semialgebraically connected component of V. If they do, the algorithm constructs a continuous semialgebraic path of V connecting x_1 and x_2. By the way the algorithm constructs a roadmap of V. In particular we obtain that the number of semialgebraically connected components of V is computable within the mentioned time bounds.*

1. Introduction

Let R be a real closed field containing a subring A. Let $X_1, \ldots, X_n$ be indeterminates over A.

We consider the following path finding problem for semialgebraic sets: let V be a semialgebraic subset of R^n containing two points x_1 and x_2. Suppose that V and x_1, x_2 are described by boolean combinations of inequalities involving polynomials $F_1, \ldots, F_s \in A[X_1, \ldots, X_n]$. Let $D := \deg(F_1) + \cdots + \deg(F_s)$ be the sum of the total degrees of $F_1, \ldots, F_s$.

Our aim is to design (by means of an arithmetical network over A; see [Ga], [FGM]) an algorithm which decides whether x_1 and x_2 lie in the same semialgebraically connected component of V, and, if they do, constructs a continuous semialgebraic path connecting x_1 and x_2. By such an algorithm the exact number of semialgebraically connected components can easily be determined. Moreover if V is closed and bounded a roadmap construction of V is required.

By cylindrical algebraic decomposition it is possible to solve algorithmically this path finding problem for semialgebraic sets. Unfortunately this technique has its shortcomings since it produces quite elevated worst case complexity results: sequential time bounds of order $D^{n^{O(n)}}$ and parallel time bounds of order $n^{O(n)}(\log D)^{O(1)}$ (see [FGM] for details). This method has been used in [SS] for the solution of path finding problems in robotics.

However it is possible to obtain more realistic (we shall say "admissible") complexity bounds for the path finding problem: the sequential time complexity (i.e. the size of the corresponding arithmetical network) is of order $D^{n^{O(1)}}$ and the parallel time complexity (the depth of the network) is of order $n^{O(1)}(\log D)^{O(1)}$.

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In this paper we present the core of this complexity result considering the case where V is a bounded regular hypersurface of R^n given by a polynomial $F \in A[X_1, \ldots, X_n]$ with non vanishing gradient on V (Theorem 12). The methods developed in [GV 1], [HRS 2,3] allow to transfer this algorithmical result for bounded regular hypersurfaces to arbitrary semialgebraic sets (see Theorem 13 below and for proofs [HRS 4]).

The paper uses ideas of elementary differential geometry and combines them with techniques from computational algebraic geometry. This method has been introduced in [GV 1], [G] and was extended in [S 1], [HRS 1,2,3]. More specifically our tools are the ones used in the efficient quantifier elimination procedure for real closed fields ([HRS 2,3]).

Our algorithms are based on elementary techniques of computational linear algebra and hence efficiently parallelizable. In this point they differ substantially from the algorithmical methods of [GV 1] and [G], which use efficient polynomial factorization. At this moment this has no efficient parallel counterpart.

A first attempt to obtain efficiently parallelizable single exponential time bounds for the path finding problem and roadmap constructions in semialgebraic sets has been made in [C 2] and [C 3]. However in this work the semialgebraically input sets are subject to considerable geometric and computational restrictions: they are in general position or already equipped with a Whitney Stratification (see also the comments on the proofs of [C 2] and [C 3] in [T]).

Let us also point out that in the bit-model a single exponential sequential time algorithm for the path finding problem with a somewhat more precise complexity bound of $D^{O(n^{20})}$ has been independently obtained by D. Yu Grigor'ev and N.N. Vorobjov in [GV 2].

It is not too difficult to derive from our methods a "parametrized" version of our main Theorem 12. Taking into account the results of [HRS 4] (or [GV 2]) one obtains thus an admissible algorithm which solves the problem of defining the semialgebraically connected components of a given semialgebraic set. Details have been done in [CGV] for the sequential time complexity in the bit-model.

Surveys of these results are given in [HKRS] and [GHRSV].

2. Notions and Notations

Throughout this paper we fix a real closed field R and a subring A of R (for example $R := \mathbf{R}$ and $A := \mathbf{Z}$).

Let $X_1, \ldots, X_n$ be indeterminates (variables) over R. To a polynomial $F \in A[X_1, \ldots, X_n]$ we assign its total degree $\deg(F)$ and its number of variables n. For a finite set $\mathcal{F}$ of polynomials of $A[X_1, \ldots, X_n]$ we write $\deg \mathcal{F} := 2 + \sum_{F \in \mathcal{F}} \deg(F)$.

We call an element $z \in R$ *real algebraic* if it is algebraic over A.

We consider R^n, $n = 0, 1, \ldots$, as a topological space equipped with the euclidean topology. Let $x := (x_1, \ldots, x_n)$ and $y := (y_1, \ldots, y_n)$ be two points of R^n, we write

$$|x - y| := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

for their euclidean distance. If r is a positive element of R we write $B(x, r) := \{y \in R^n;\ |x - y| < r\}$ for the open ball of radius r centered at x.

If $x, y \in R$ (i.e. $n = 1$) we write $[x, y], (x, y), [x, y), (x, y]$ for the closed, open and half open intervals with boundaries x and y.

A *semialgebraic subset* of R^n (over A) is a set definable by a boolean combination of equalities and inequalities involving polynomials from $A[X_1, \ldots, X_n]$.

A semialgebraic set has only finitely many semialgebraically connected components which are all semialgebraic (see [BCR], Def. 2.4.2 and Th. 2.4.4).

Let $V \subset R^n$, $W \subset R^m$ be semialgebraic sets and $f : V \to W$ a map. We call f *semialgebraic* if its graph is a semialgebraic subset of R^{n+m}.

The image of a semialgebraic set by a semialgebraic function is semialgebraic too. This fact is called the *Tarski-Seidenberg Principle* (see [BCR], Théorème 2.2.1).

The Tarski-Seidenberg Principle can also be stated in terms of logics. It means that the elementary theory of real closed fields with constants from A admits quantifier elimination ([BCR], Prop. 5.2.2).

From this we obtain the following *Transfer Principle* ([BCR], Prop. 5.2.3): let R' be a real closed extension of R and let Φ be a formula without free variables (i.e. all the variables are quantified) in the elementary language $\mathcal{L}$ of real closed fields with constants from A. Then Φ is valid in R' iff Φ is valid in R.

Let $\Phi \in \mathcal{L}$ be a formula in the free variables $X_1, \ldots, X_n$. Φ defines a subset S of R^n. Since the elementary theory of R admits quantifier elimination we see that S is semialgebraic. On the other hand Φ can also be interpretated over R'. Thus Φ defines also a semialgebraic subset of R'^n which we denote by $S(R')$ (the independence of this notation from the particular defining formula Φ is justified by the Transfer Principle).

We call two formulas in the same free variables $X_1, \ldots, X_n$ *equivalent* if they define the same subset of R^n (this means they are equivalent with respect to the elementary theory of real closed fields).

A formula containing no quantifiers is called *quantifier free.* Thus the semialgebraic sets are those which are definable by quantifier free formulas. A formula is called *prenex* if all its quantifiers occur at the beginning.

Let $\Phi \in \mathcal{L}$ be a formula built up by atomic formulas involving a finite set $\mathcal{F}$ of polynomials of $A[X_1, \ldots, X_n]$. We write $\deg \Phi := \deg \mathcal{F}$. The *length* of Φ is denoted by $|\Phi|$.

An algorithm $\mathcal{N}$ (represented by a suitable family of arithmetical networks over A, see [Ga], [FGM]) which for given natural numbers D, n and any input set $\mathcal{F} \subset A[X_1, \ldots, X_n]$ subject to $\deg \mathcal{F} \leq D$ computes an output set $\mathcal{G} \subset A[x_1, \ldots, X_m]$, is called *admissible* if the following conditions are satisfied:

– $\deg \mathcal{G} = D^{n^{O(1)}}$.

– the sequential complexity of $\mathcal{N}$ is $D^{n^{O(1)}}$.

– the parallel complexity of $\mathcal{N}$ is $(n \cdot \log D)^{O(1)}$.

In the case that such an algorithm exists we say that $\mathcal{G}$ is computable from the input $\mathcal{F}$ in *admissible time.*

Under certain circumstances $\mathcal{F}$ and $\mathcal{G}$ may represent the polynomials involved in quantifier free formulas $\Phi, \Psi \in \mathcal{L}$ defining semialgebraic sets $V \subset R^n$ and $W \subset R^m$. If $\mathcal{N}$ is admissible and computes also from the input data Φ the boolean combination of atomic formulas representing Ψ, we say that W is *admissible computable* from V.

Note that the data structures corresponding to polynomials and to finite sets of polynomials are considered as coefficients vectors written in dense form.

We shall also make use of the notions of *critical point, Nash function* and *Nash variety.* For precise definitions, we refer to [BCR].

3. Algorithmical and mathematical tools

In this section we collect some algorithmical and mathematical tools we need for our later roadmap construction. We state first a theorem which expresses a local property for projections of semialgebraic sets. In terms of logics this theorem can also be interpretated as a statement about the local existence of continuous semialgebraic Skolem functions, computable in admissible time.

THEOREM 1 ([HRS 3]). *Let S be a semialgebraic subset of $R^k \times R^n$. It is possible to compute in admissible time the following items:*

- a partition of R^k into semialgebraic sets T_i, $1 \leq i \leq s$.

- for each $1 \leq i \leq s$ a finite family $(\xi_{ij})_{1 \leq j \leq \ell_i}$ of continuous semialgebraic functions from T_i to R^n such that for each $1 \leq j \leq \ell_i$ the graph of ξ_{ij} belongs to S and such that for each $x \in T_i$ each semialgebraically connected component of $S \cap (\{x\} \times R^n)$ contains at least one point of the graph of some ξ_{ij}. In particular, each connected component of $S \cap (T_i \times R^n)$ contains at least the graph of some ξ_{ij}. ♢

For a proof of Theorem 1, see [HRS 3], Théorème 7. A weaker version of this theorem appears in [G], [HRS 1] and [S 1].

Theorem 1 has a series of consequences (Corollaries 2, 3, 4):

COROLLARY 2. *Let S be a semialgebraic subset of $R \times R^n$. In admissible time it is possible to compute the following items:*

- a partition of R in semialgebraic intervals T_i, $1 \leq i \leq s$.

- for each $1 \leq i \leq s$ a finite family $(\xi_{ij})_{1 \leq j \leq \ell_i}$ of continuous semialgebraic functions (curves) from T_i to R^n such that the graph of each ξ_{ij} belongs to S and such that for each semialgebraically connected component of $S \cap (\{x\} \times R^n)$ contains at least one point of the graph of some ξ_{ij}. In particular, each connected component of $S \cap (T_i \times R^n)$ contains at least the graph of some ξ_{ij}.

Proof. Put $k := 1$ in Theorem 1 and observe that any semialgebraic subset of R can be decomposed in admissible time in finitely many intervals. ♢

Since the curves ξ_{ij} in Corollary 2 are defined on intervals, their images are semialgebraically connected. This observation will be relevant in later applications of this corollary.

COROLLARY 3 (efficient quantifier elimination, [HRS 2,3], [R]). *Let Φ be a prenex formula in the language $\mathcal{L}$ of ordered fields with constants in A. Suppose that Φ contains m blocks of quantifiers and n variables. Then it is possible to compute in sequential time $(\deg \Phi)^{n^{O(m)}} |\Phi|^{O(1)}$ and in parallel time $n^{O(m)} (\log \deg \Phi)^{O(1)} + (\log |\Phi|)^{O(1)}$ a quantifier free formula Ψ which is equivalent to Φ.* ♢

COROLLARY 4 (efficient curve selection lemma, [HRS 1], [S 1]). *Let S be a semialgebraic subset of R^n and let p be in the topological closure $\overline{S}$ of S. Assume that S and p are definable by quantifier free formulas of $\mathcal{L}$. Then there exists an admissible algorithm* *which computes positive elements $\delta, \varepsilon \in R$ and continuous semialgebraic curves $\gamma_i : [0,\delta] \to R^n$, $1 \le i \le s$, satisfying the following conditions:*

- $\gamma_i((0,\delta]) \subset S$ and $\gamma_i(0) = p$ for each $1 \le i \le s$.

- each semialgebraically connected component of $B(p,\varepsilon) \cap S$ contains $\gamma_i((0,\delta])$ for at least one i.

Proof. Apply Theorem 1 to the semialgebraic set S' defined by

$$S' := \{(r,x) \in R \times R^n;\ x \in S,\ |x-p|^2 \le r\}. \quad \diamondsuit$$

Let K be the fraction field of A. In the next lemma we consider an arithmetical network over K.

LEMMA 5. *Let $\mathcal{J}$ be a zero dimensional ideal of $K[X_1,\dots,X_n]$ given by a finite set $\mathcal{F}$ of generators. Denote by $Z(\mathcal{J})$ the set of zeros of $\mathcal{J}$ contained in R^n. Let U be a new indeterminate. In admissible time it is possible to compute polynomials $P, P_1,\dots,P_n \in K[U]$ such that for each element $x \in Z(\mathcal{J})$ there exists a root u of P contained in R satisfying $x = (P_1(u),\dots,P_n(u))$.* $\diamondsuit$

For a proof of Lemma 5 see [HRS 3], Proposition 7, or [C 1].

Let $F \in A[X_1,\dots,X_n]$ be a polynomial such that the closed semialgebraic set $V := \{F = 0\} := \{x \in R^n;\ F(x) = 0\}$ is bounded, and such that the gradient $\bigtriangledown F := \left(\frac{\partial F}{\partial X_1},\dots,\frac{\partial F}{\partial X_n}\right)$ vanishes nowhere on V. Thus V is a regular bounded hypersurface of R^n.

Let $G \in A[X_1,\dots,X_n]$ be a linear form and let $g : V \to R$ be the semialgebraic map induced by G on V.

We are going to apply to V and g concepts borrowed from elementary differential geometry. To be more precise, we consider V as a Nash subvariety of R^n and g as a Nash function defined on V (see [BCR], Chapitre 8); therefore the reader interested in more generality may replace in the next statements (Observation 6, Lemma 7, Propositions 8 and 9) the semialgebraic set V by any bounded and closed Nash subvariety of R^n and g by any Nash function mapping V into R.

The local inversibility theorem for Nash functions ([BCR], Proposition 2.9.5) implies the following basic fact:

OBSERVATION 6. Let x be a point of V and let $y_1,\dots,y_{n-1}$ be a system of local coordinates for x. Suppose that x is not a critical point of g. Then there exists $1 \le i \le n-1$ such that $g, y_1,\dots,y_{i-1},y_{i+1},\dots,y_{n-1}$ is a system of local coordinates for x.

LEMMA 7. *Let t be an element of R such that t is not a critical value of g and such that the fibre $g^{-1}(t)$ is non empty. Then there exists a positive element $\varepsilon \in R$ such that for the semialgebraically connected components $C_1,\dots,C_s$ of $g^{-1}((t-\varepsilon,t+\varepsilon))$ and for each $t' \in (t-\varepsilon,t+\varepsilon)$ the sets $C_1 \cap g^{-1}(t'),\dots,C_s \cap g^{-1}(t')$ are non empty and semialgebraically connected.*

Proof. Since the semialgebraically connected components of semialgebraic sets definable by polynomials of previously bounded degree are themselves uniformly semialgebraically definable we may argue by the Transfer Principle. Thus we may assume without loss of generality that $R := \mathbf{R}$. Thus V is compact.

By assumption no point of the fibre $g^{-1}(t)$ is critical. By Observation 6 we may therefore choose for each $x \in g^{-1}(t)$ an open neighborhood U_x in V, a positive element $\varepsilon_x \in \mathbf{R}$ and local coordinates of the form $g\big|_{U_x}, y_2, \ldots, y_{n-1}$ which map U_x homeomorphically onto the connected chart $(t-\varepsilon_x\,,\, t+\varepsilon_x)\times(-\varepsilon_x, \varepsilon_x)^{n-2}$ (here $g\big|_{U_x}$ denotes the restriction of g to U_x). From the particular form of this chart we infer that U_x and $U_x \cap g^{-1}(t')$ are connected for any $t' \in (t-\varepsilon_x, t+\varepsilon_x)$. Since $g^{-1}(t)$ is compact there exist finitely many points $x_1, \ldots, x_p \in g^{-1}(t)$ such that $U_{x_1}, \ldots, U_{x_p}$ cover $g^{-1}(t)$. Again by compactness we see that we may choose a positive $\varepsilon < \min\{\varepsilon_{x_j} : 1 \leq j \leq p\}$ such that for each $t' \in (t-\varepsilon, t+\varepsilon)$ the fibre $g^{-1}(t')$ is contained in $U_{x_1} \cup \ldots \cup U_{x_p}$ (this can be seen by the following indirect argument: suppose that there exists a sequence $(t_i)_{i\in\mathbf{N}}$ of real numbers converging to t such that for each $i \in \mathbf{N}$ there is a point $y_i \in (V \cap g^{-1}(t_i))\backslash(U_{x_1} \cup \ldots \cup U_{x_p})$; since V is compact we may suppose that the sequence $(y_i)_{i\in\mathbf{N}}$ converges to a point $y \in (V \cap g^{-1}(t))\backslash(U_{x_1} \cup \ldots \cup U_{x_p})$. Contradiction).

For $1 \leq j \leq p$ let $U_j := U_{x_j} \cap g^{-1}((t-\varepsilon, t+\varepsilon))$.

Choosing ε small enough and taking into account the "open ball" form of the charts $U_1, \ldots, U_p$ one can achieve the following: for each $1 \leq j, j' \leq p$ and for each $t' \in (t-\varepsilon, t+\varepsilon)$ the sets U_j are connected, $U_j \cap g^{-1}(t')$ is non empty and connected and its topological closure in $g^{-1}(t')$ coincides which $\overline{U_j} \cap g^{-1}(t')$.

For each $t' \in (t-\varepsilon, t+\varepsilon)$ we consider the following undirected graph $G_{t'}$: its vertex set is $\{1, \ldots, p\}$ and two vertices $j, j' \in \{1, \ldots, p\}$ form a "primitive edge" of $G_{t'}$ if $U_j \cap U_{j'} \cap g^{-1}(t') \neq \phi$. The edges of the graph $G_{t'}$ are defined by taking the transitive closure of the relation of $\{1, \ldots, p\}$ induced by the primitive edges. The connected components of the graph $G_{t'}$ describe the (topological) connected components of $g^{-1}(t')$ as unions of the connected sets $U_1 \cap g^{-1}(t'), \ldots, U_p \cap g^{-1}(t')$. We claim that $G_t = G_{t'}$ for all $t' \in (t-\varepsilon, t+\varepsilon)$ if $\varepsilon > 0$ is chosen small enough. Thus G_t describes the connected components of $g^{-1}((t-\varepsilon, t+\varepsilon))$ as unions of the connected sets U_j, $1 \leq j \leq p$. This implies Lemma 7.

Now we are going to prove our claim. First we show that G_t is a subgraph of $G_{t'}$ for all $t' \in (t-\varepsilon, t+\varepsilon)$ with $\varepsilon > 0$ sufficiently small. For this purpose let $t' \in (t-\varepsilon, t+\varepsilon)$ and let $j, j' \in \{1, \ldots, p\}$ be two vertices of G_t forming a primitive edge. Thus $U_j \cap U_{j'} \cap g^{-1}(t) \neq \phi$. Since $U_j \cap U_{j'}$ is a non empty open subset of the chart U_j its image by g contains an open interval of the form $(t-\varepsilon, t+\varepsilon)$. Therefore $U_j \cap U_{j'} \cap g^{-1}(t') \neq \phi$ for $t' \in (t-\varepsilon, t+\varepsilon)$. This means that j and j' form a primitive edge of $G_{t'}$ and our assertion follows now easily.

Let us now show the converse, namely that $G_{t'}$ is a subgraph of G_t for all $t' \in (t-\varepsilon, t+\varepsilon)$ where ε is chosen sufficiently small. Suppose that this is not the case. Then there exist vertices $j, j' \in \{1, \ldots, p\}$ and a sequence $(t_i)_{i\in\mathbf{N}}$ of real values converging to t such that $U_j \cap U_{j'} \cap g^{-1}(t_i) \neq \phi$ for all $i \in \mathbf{N}$ and such that $U_j \cap U_{j'} \cap g^{-1}(t) = \phi$. Thus we may choose without loss of generality a sequence $(y_i)_{i\in\mathbf{N}}$ of points $y_i \in U_j \cap U_{j'} \cap g^{-1}(t_i)$ converging to a point $y \in \overline{U_j} \cap \overline{U_{j'}} \cap g^{-1}(t)$. Let $k \in \{1, \ldots, p\}$ such that $y \in U_k \cap g^{-1}(t)$. Since $U_k \cap g^{-1}t$ is a neighborhood of y in $g^{-1}(t)$ and since the set $\overline{U_j} \cap g^{-1}(t)$, which contains y, is the closure of $U_j \cap g^{-1}(t)$ in $g^{-1}(t)$, we conclude that $U_j \cap U_k \cap g^{-1}(t) \neq \phi$.

By the same argument one infers that $U_{j'} \cap U_k \cap g^{-1}(t) \neq \phi$. Thus $\{j,k\}$ and $\{j',k\}$ are primitive edges of G_t which implies that $\{j,j'\}$ is an edge of G_t. Contradiction.

$\diamondsuit$

PROPOSITION 8. *Let $t_1 < t_2$ be two elements of R. Suppose that the interval (t_1,t_2) doesn't contain any critical value of g. Let $C_1,\dots,C_s$ be the semialgebraically connected components of $g^{-1}((t_1,t_2))$. Then for each $t \in (t_1,t_2)$, the sets $C_1 \cap g^{-1}(t),\dots,C_s \cap g^{-1}(t)$ are the semialgebraically connected components of $g^{-1}(t)$. In particular we obtain that the number of semialgebraically components remain constant when t ranges over (t_1,t_2).*

Proof. By the Transfer Principle we may assume $R := \mathbf{R}$.
In a first step of the proof we consider arbitrary real numbers t_1', t_2' with $t_1 < t_1' < t_2' < t_2$. Let $C_1',\dots,C_{s'}'$, be the connected components of the compact semialgebraic set $g^{-1}([t_1',t_2'])$. We claim that for any $t \in [t_1',t_2']$ the sets $C_1' \cap g^{-1}(t),\dots,C_{s'}' \cap g^{-1}(t)$ are the connected components of $g^{-1}(t)$. To see this we observe that $[t_1',t_2']$ can be covered by finitely many open intervals with the properties stated in Lemma 7. These intervals can be arranged in a chain of succesively overlaping members. Looking at the common values contained in any two of these overlaping intervals we infer our claim just by glueing connected sets together.
Proposition 8 follows now easily from our claim by considering any ascending chain of closed intervals $[t_1',t_2']$, $[t_1'',t_2'']$, ... converging to (t_1,t_2). $\diamondsuit$

PROPOSITION 9. *Let notations and assumptions be as in Proposition 8. Then for each $1 \leq j \neq j' \leq s$ and $k = 1,2$, the following is true:*
(i) *$\overline{C_j} \cap g^{-1}(t_k)$ is semialgebraically connected*
(ii) *each point of $\overline{C_j} \cap \overline{C_{j'}} \cap g^{-1}(t_k)$ is a critical point of g.*

Proof. By the Transfer Principle, one may again assume that $R = \mathbf{R}$.
We show first (i): Suppose on the contrary that there exists a connected component of $g^{-1}((t_1,t_2))$, say C_1, such that $\overline{C_1} \cap g^{-1}(t_1)$ is disconnected. Then there exist open subsets $\mathcal{O}$, $\mathcal{U}$ of $\mathbf{R}^n$ such that $\mathcal{O} \cap \mathcal{U} = \phi$, $\overline{C_1} \cap g^{-1}(t_1) \cap \mathcal{O} \neq \phi$, $\overline{C_1} \cap g^{-1}(t_1) \cap \mathcal{U} \neq \phi$ and $\overline{C_1} \cap g^{-1}(t_1) \subset \mathcal{O} \cup \mathcal{U}$. By compactness one sees then that there exists $t_1 < t < t_2$ such that $C_1 \cap g^{-1}(t) \cap \mathcal{O} \neq \phi$, $C_1 \cap g^{-1}(t) \cap \mathcal{U} \neq \phi$ and $C_1 \cap g^{-1}(t) \subset \mathcal{O} \cup \mathcal{U}$. This contradicts the conclusion of Proposition 8.
Now we show (ii): Suppose on the contrary that there exist two different connected components of $g^{-1}((t_1,t_2))$, say C_1 and C_2, such that $g^{-1}(t_1) \cap \overline{C_1} \cap \overline{C_2}$ contains a point x which is not critical for g. By Observation 6, we may chose an open neighborhood $\mathcal{U}$ of x in V, a real number $0 < \varepsilon < t_2 - t_1$ and local coordinates of the form $g\big|_{\mathcal{U}}, y_2, \dots, y_{n-1}$ which maps $\mathcal{U}$ homeomorphically onto $(t_1 - \varepsilon, t_1 + \varepsilon) \times (-\varepsilon, \varepsilon)^{n-2}$. Thus $\mathcal{U} \cap g^{-1}((t_1,t_2))$ is mapped onto $(t_1, t_1+\varepsilon) \times (-\varepsilon,\varepsilon)^{n-2}$ which is connected. Since x is in the closure of C_1 and C_2 we see that $C_1 \cap \mathcal{U} \neq \phi$ and $C_2 \cap \mathcal{U} \neq \phi$. Choose points $x_1 \in C_1 \cap \mathcal{U}$ and $x_2 \in C_2 \cap \mathcal{U}$. The images of x_1 and x_2 in $(t_1, t_1+\varepsilon) \times (-\varepsilon,\varepsilon)^{n-2}$ can be connected by a continuous path. Thus x_1 and x_2 can be connected by a continuous path in $\mathcal{U} \cap g^{-1}((t_1,t_2))$. This implies $C_1 = C_2$. Contradiction. $\diamondsuit$

We shall need the following concept of M-directions (M-functions) which share some basic properties with Morse functions.

Definition. Let notations be as before. We call $g : V \to R$ an *M-direction* (or *M-function*) of V if the set of critical points of g is nowhere dense.

Note that the set of critical points of an M-function is finite, since this set is semialgebraic and nowhere dense.

Observation 10. As a consequence of Sard's theorem ([BCR], Théorème 9.5.2), one obtains that the coefficient vectors of the linear forms of $R[X_1,\ldots,X_n]$ inducing M-directions on V form a dense subset of R^n (see [BCR], proof of Proposition 11.5.1 for details). Later we shall make use of the existence of an admissible algorithm which yields M-directions of V with *integer* coordinates.
Such an algorithm can be obtained combining the fact that being an M-direction is an elementary property and the mentioned density of M-directions with efficient quantifier elimination over R (Corollary 3).
A detailed account of this argument can be found in [S 2]. Compare also [HRS 3] Théorème 3, where a description of an admissible algorithm for finding M-directions which doesn't make use of Sard's theorem and efficient quantifier elimination is given (in fact, this algorithm constructs M-directions with coordinates contained in a real quadratic field extension of $\mathbf{Q}$).

4. The roadmap algorithm

During this section let $V := \{F = 0\}$ be a regular bounded hypersurface of R^n defined by a polynomial $F \in A[X_1,\ldots,X_n]$ of degree $d := \deg(F)$, whose gradient $\bigtriangledown F := \left(\frac{\partial F}{\partial X_1},\ldots,\frac{\partial F}{\partial X_n}\right)$ vanishes nowhere on V (we say in this case that F is a *regular* polynomial). The variables $X_1,\ldots,X_n$ induce on V continuous semialgebraic coordinate functions $\pi_1,\ldots,\pi_n$.

Definition. Let $G \in A[X_1,\ldots,X_n]$ be a linear form and let $g : V \to R$ be the semialgebraic map induced by G on V. A piecewise parametrized continuous semialgebraic curve σ of V is called a *roadmap of V with respect to g* if the following conditions are satisfied:

(i) for each $t \in R$, any semialgebraically connected component of $g^{-1}(t)$ cuts σ.
(ii) any two points of σ which lie on the same semialgebraically connected component of V lie on the same semialgebraically connected component of σ.

This definition of roadmap is closely related to the regular and bounded hypersurface V and the semialgebraic map g.

As a matter of fact any semialgebraic set of dimension one can be represented in admissible time as an union of continuously parametrized semialgebraic curves. This can easily be deduced from Corollary 2 (see [HRS 4] for details).

By Observation 10 we are able to construct (in admissible time) a $\mathbf{Z}$-linear combination of the variables $X_1,\ldots,X_n$ which induces on V an M-direction. Thus we suppose from now on that the coordinate function $\pi_n : V \to R$ is an M-direction of V. For the sake of notational simplicity we write $\pi := \pi_n$.

In this section we are going to describe an algorithm $\mathcal{N}(A,n)$ (depending on the parameters A and n) which constructs a roadmap of V with respect to the M-direction

π and which, for any two points x_1 and $x_2 \in V$, given by quantifier free formulas Φ_1 and Φ_2 of the elementary language $\mathcal{L}$, decides whether x_1 and x_2 lie in the same semialgebraically connected component of V. If x_1 and x_2 do so, the algorithm finds a continuous semialgebraic curve of V joining them. We shall represent the algorithm $\mathcal{N}(A,n)$ by a family of arithmetical networks over A depending on the parameter $D := \deg(\Phi_1) + \deg(\Phi_2) + d$ when n is fixed. We shall apply $\mathcal{N}(A,n)$ recursively in n changing at each call the base ring A. In order to obtain an admissible overall complexity in terms of arithmetical operations and sign evaluations in the original base ring, we shall use only constructions which are uniform on A.

In the next section we will show that the algorithm $\mathcal{N}(A,n)$ runs in admissible time in the inputs: n, d, for the construction of the roadmap, and n, $\deg(\Phi_1) + \deg(\Phi_2) + d$ for the decision and the construction of the curve.

During this section we suppose that for any subring A' of R and any bounded hypersurface of R^{n-1} defined by a regular polynomial $F' \in A'[X_1, \ldots, X_{n-1}]$ such an algorithm $\mathcal{N}(A', n-1)$ is already given.

We subdivide the description of our algorithm in several steps.

<u>Step 1</u>: The set $\mathcal{B} := \{x \in V;\, x \text{ is a critical point of } \pi\}$ is definable by the following quantifier free formula Φ:

$$F = 0 \wedge \frac{\partial F}{\partial X_1} = 0 \wedge \ldots \wedge \frac{\partial F}{\partial X_{n-1}} = 0$$

Since π is an M-direction, the set $\mathcal{B}$ is finite. Applying efficient quantifier elimination (Corollary 3) to the formulas $(\exists X_2) \ldots (\exists X_n)\Phi, \ldots, (\exists X_1) \ldots (\exists X_{n-1})\Phi$, we compute (in admissible time) univariate polynomials in $X_1, \ldots, X_n$ which represent the coordinates of the elements of $\mathcal{B}$ as their roots, suitably codified by Thom's Lemma (see [CR]). In this sense we are able to compute $\mathcal{B}$ "explicitely" and to handle its elements. In particular we are able to evaluate the sign of a given polynomial at any point of $\mathcal{B}$ (see [HRS 3] or [RS] for details). Observe also that the coordinates of the points of $\mathcal{B}$ are algebraic over A and zeros of polynomials of degree $d^{n^{O(1)}}$. In such a way we obtain the critical values of π explicitely.

<u>Step 2</u>: Using Corollary 2 we compute (in admissible time) the following:
– a partition of $\pi(V)$ in intervals T_i, $1 \le i \le s$, definable over A.
– for each $1 \le i \le s$ a finite family of continuous semialgebraic curves $\xi_{ij} : T_i \to V$, $1 \le j \le \ell_i$, such that for any $t \in T_i$ each semialgebraically connected component of $\pi^{-1}(t)$ contains at least one point of the form $\xi_{ij}(t)$, where $1 \le j \le \ell_i$, and such that $\pi \circ \xi_{ij}(t) = t$ for all $1 \le j \le \ell_i$ and all $t \in T_i$. Since V is a closed and bounded semialgebraic subset of R^n, each ξ_{ij} has a unique continuous semialgebraic extension to the boundary of T_i (by Corollary 3, this extension can be computed in admissible time). Thus we may assume that all intervals T_i are closed and have disjoint interiors. For $t \in \pi(V)$ let $V_t^+ := \{x \in V;\, \pi(x) > t\}$ and $V_t^- := \{x \in V,\, \pi(x) < t\}$. Given a point $x \in V$ corresponding to a critical value $t := \pi(x)$ such that $x \in \mathcal{B}$ or x is an endpoint of a curve ξ_{ij}, we construct (in admissible time) two families of continuous semialgebraic curves $(\xi_k^{(x)})_{1 \le k \le m_x}$ and $(\widetilde{\xi}_{\widetilde{k}}^{(x)})_{1 \le \widetilde{k} \le \widetilde{m}_x}$ as follows: assume first that x is contained in

the topological closure of V_t^+. Using the effective curve selection lemma (Corollary 4) we find (in admissible time) a closed interval T_x^+ of the form $T_x^+ = [t, t^+]$ with $t < t^+$, a positive element $\varepsilon_x^+ \in R$ and continuous semialgebraic curves $\xi_k^{(x)} : T_x^+ \to V$, $1 \le k \le m_x$, satisfying the following conditions:

- $\xi_k^{(x)}((t, t^+]) \subset V_t^+$, $\xi_k^{(x)}(t) = x$ and $\pi \circ \xi_k^{(x)}(t') = t'$ for each $t' \in T_x^+$ and each $1 \le k \le m_x$.
- each semialgebraically connected component of $B(x, \varepsilon_x^+) \cap V_t^+$ contains at least one $\xi_k^{(x)}((t, t^+])$.

If x is not contained in the closure of V_t^+ we put $m_x := 0$.
Assume now that x is contained in the closure of V_t^-. By the effective curve selection lemma we find (in admissible time) a closed interval T_x^- of the form $T_x^- = [t^-, t]$ with $t^- < t$, a positive element $\varepsilon_x^- \in R$ and continuous semialgebraic curves $\widetilde{\xi}_{\widetilde{k}}^{(x)} : T_x^- \to V$, $1 \le \widetilde{k} \le \widetilde{m}_x$, satisfying the analogous conditions as before.
If x is not contained in the closure of V_t^- we put $\widetilde{m}_x = 0$.
Let $\mathcal{K}$ be the set of all curves of the form ξ_{ij}, where $1 \le i \le s$, $1 \le j \le \ell_i$, or of the form $\xi_k^{(x)}, \widetilde{\xi}_{\widetilde{k}}^{(x)}$, where $x \in \mathcal{B}$, $1 \le k \le m_x$, $1 \le \widetilde{k} \le \widetilde{m}_x$.
Without loss of generality we may suppose that the curves contained in $\mathcal{K}$ have domains whose interior is not empty.
$\mathcal{K}$ is computable in admissible time and so is also the set of all "endpoints" of the curves contained in $\mathcal{K}$, namely

$$\mathcal{C} := \left\{ \begin{array}{c} x \in V;\ \text{there exists } \eta \in \mathcal{K} \text{ with domain } [t_1, t_2] \text{such that} \\ x = \eta(t_1) \text{ or } x = \eta(t_2) \end{array} \right\}$$

Summarizing all this we have constructed in admissible time the following items
– a finite set $\mathcal{K}$ of continuous semialgebraic curves of V having domains with non empty interiors.
– a finite set $\mathcal{C}$ consisting of all endpoints of the curves contained in $\mathcal{K}$.
Note in particular that the domains of the curves of $\mathcal{K}$ cover $\pi(V)$ and that $\mathcal{C}$ contains all the critical points of π. Observe also that the coordinates of the points of $\mathcal{C}$ are algebraic over A and that we have $\#\mathcal{C} \cap \mathcal{K} = d^{n^{O(1)}}$ (# denotes cardinality).
Subdividing the domains of the curves contained in $\mathcal{K}$ and increasing eventually $\mathcal{C}$ we may suppose that no curve of $\mathcal{K}$ ranges over an interval whose interior contains an element of $\pi(\mathcal{C})$. This implies that two curves of $\mathcal{K}$ defined on intervals whose interiors overlap, have the same domain. Moreover we may assume that at most one boundary point of the domain of any curve of $\mathcal{K}$ is a critical value.

Step 3: We are going to describe a roadmap σ of V with respect to π. For this purpose let us consider an arbitrary element t of $\pi(\mathcal{C})$ which is not a critical value of π. Note that under this circumstance $\pi^{-1}(t)$ is a bounded hypersurface of R^{n-1} defined by the regular polynomial $F(X_1, \ldots, X_{n-1}, t) \in A[t][X_1, \ldots, X_{n-1}]$. Thus we call $\pi^{-1}(t)$ a regular fibre. For each two points $x,\ x' \in \mathcal{C} \cap \pi^{-1}(t)$ we decide whether x and x' are in the same semialgebraically connected component of $\pi^{-1}(t)$ by means of the algorithm $\mathcal{N}(A[x, x'], n-1)$. If they do we find a continuous semialgebraic curve of $\pi^{-1}(t)$ connecting x and x'.

We define σ as the join of these connecting curves (for t ranging over all non critical values contained in $\pi(\mathcal{C})$) with the curves of $\mathcal{K}$ constructed in Step 2. Thus σ is a piecewise parametrized continuous semialgebraic curve of V.

LEMMA 11. *σ is a roadmap of V with respect to π.*

Proof. From Step 2 in the construction of σ it is clear that for each $t \in R$ any semialgebraically connected component of $\pi^{-1}(t)$ cuts σ. Let x_1, x_2 be two different points of σ lying in the same semialgebraically connected component of V. We are going to show that x_1 and x_2 lie in the same connected component of σ. From Steps 2 and 3 in the construction of σ we see that we may suppose without loss of generality that $x_1, x_2 \in \mathcal{C}$. Thus we can choose curves $\eta_1, \eta_2 \in \mathcal{K}$ such that x_1 is an endpoint of η_1 and x_2 is an endpoint of η_2.
Suppose first that η_1 and η_2 are defined on the same interval $[t, t']$ and that x_1 and x_2 are in the same semialgebraically connected component of $\pi^{-1}([t, t'])$. From Step 2 in the construction of σ we deduce that (t, t') contains no element of $\pi(\mathcal{C})$ and in particular no critical value of π.
Suppose first that t and t' are not critical values of π. By hypothesis $\eta_1([t, t'])$ and $\eta_2([t, t'])$ are contained in the same connected component C of $\pi^{-1}([t, t'])$. From Proposition 8 we infer that $C \cap \pi^{-1}(t)$ is a semialgebraically connected component of $\pi^{-1}(t)$ containing the points $\eta_1(t)$, $\eta_2(t) \in \mathcal{C}$. Therefore $\eta_1(t)$ and $\eta_2(t)$ are connected by a continuous semialgebraic path of σ (following Step 3 in the construction of our roadmap). This implies that $\eta_1([t, t'])$, $\eta_2([t, t'])$ and hence also x_1, x_2 are contained in the same semialgebraically connected component of σ.
Assume now that t is a critical value of π. Then, by Step 2 in the construction of the roadmap σ, t' is not a critical value. First suppose that $\eta_1((t, t'])$ and $\eta_2((t, t'])$ are contained in the same semialgebraically connected component C of $\pi^{-1}((t, t'])$. Since $(t, t']$ contains no critical value of π we deduce from Proposition 8 that $C \cap \pi^{-1}(t')$ is semialgebraically connected in the regular fibre $\pi^{-1}(t')$. Furthermore $C \cap \pi^{-1}(t')$ contains the points $\eta_1(t')$, $\eta_2(t')$ which are elements of $\mathcal{C}$. Thus, as before, we see that $\eta_1(t')$ and $\eta_2(t')$ are connected by a continuous semialgebraic arc of σ. This implies that x_1 and x_2 are contained in the same semialgebraically connected component of σ.
Now suppose that $\eta_1((t, t'])$ and $\eta_2((t, t'])$ are contained in distinct semialgebraically connected components C and C' of $\pi^{-1}((t, t'])$. Since $(t, t']$ contains no critical point and π is an M-function, we infer from Proposition 9 that the semialgebraically connected components of $\pi^{-1}([t, t'])$ are unions of the topological closures of the semialgebraically connected components of $\pi^{-1}((t, t'])$ and eventually some critical points contained in $\pi^{-1}(t)$. Since by hypothesis $\eta_1([t, t'])$ and $\eta_2([t, t'])$ are semialgebraically connected in $\pi^{-1}([t, t'])$ there exist distinct semialgebraically connected components $C_1, \ldots, C_s$ of $\pi^{-1}((t, t'])$ with $C_1 = C$, $C_s = C'$ such that $\overline{C}_1 \cap \overline{C}_2 \neq \phi, \ldots, \overline{C}_{s-1} \cap \overline{C}_s \neq \phi$. By Proposition 9, the sets $\overline{C}_1 \cap \overline{C}_2, \ldots, \overline{C}_{s-1} \cap \overline{C}_s$ consist only of critical points contained in $\pi^{-1}(t)$.
Thus by Step 2 in the construction of σ there exist continuous semialgebraic curves $\theta_1, \theta_2, \theta_2', \ldots, \theta_{s-1}, \theta_{s-1}', \theta_s \in \mathcal{K}$ defined on $[t, t']$ such that the following holds:

- $\theta_1([t, t'])$ is contained in $\overline{C}_1$,
  $\theta_2([t, t'])$ and $\theta_2'([t, t'])$ are contained in $\overline{C}_2$,

$\vdots$
$\theta_{s-1}([t,t'])$ and $\theta'_{s-1}([t,t'])$ are contained in $\overline{C}_{s-1}$,
$\theta_s([t,t'])$ is contained in $\overline{C}_s$, and
- $\theta_1(t) = \theta_2(t)$, $\theta'_2(t) = \theta_3(t), \ldots, \theta'_{s-2}(t) = \theta_{s-1}(t)$,
$\theta'_{s-1}(t) = \theta_s(t)$.

On the other hand $\eta_1(t')$ and $\theta_1(t')$ are elements of $\mathcal{C}$ lying in the same semialgebraically connected component $C_1 \cap \pi^{-1}(t')$ of the regular fibre $\pi^{-1}(t')$. Thus, by Step 3 of the construction of σ, the points $\eta_1(t')$ and $\theta_1(t')$ are connected by a continuous semialgebraic subarc of σ. This implies that $\eta_1(t')$ and $\theta_1(t')$ are in the same semialgebraically connected component of σ. Similarly one shows that $\theta_2(t')$ and $\theta'_2(t'), \ldots, \theta_{s-1}(t')$ and $\theta'_{s-1}(t')$, $\theta_s(t')$ and $\eta_2(t')$ lie pairwise in the same semialgebraically connected component of σ.

From this we conclude that one and the same connected component of σ contains all the endpoints of $\eta_1, \theta_1, \theta_2, \theta'_2, \theta_3, \ldots, \theta'_{s-2}, \theta_{s-1}$, $\theta'_{s-1}, \theta_s, \eta_2$. The same holds for the images of these curves. Thus x_1 and x_2 are in the same semialgebraically connected component of σ.

Now let $x_1 \neq x_2$ be arbitrary points of $\mathcal{C}$ lying in the same semialgebraically connected component of V. By [BCR], Définition et Proposition 2.5.11, there exists a continuous semialgebraic curve $\gamma : [0,1] \to V$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

Let r be the number of times that $\pi \circ \gamma$ "crosses" values of $\pi(\mathcal{C})$, i.e. let r be the number of semialgebraically connected components of the set $\{z \in [0,1];\ \pi \circ \gamma(z) \in \pi(\mathcal{C})\}$.

The case $r = 1$ is already treated.

Suppose that $r > 1$ and that the assertion is true for distinct points $x_1, x_2 \in \mathcal{C}$ which can be connected by a continuous semialgebraic curve which crosses less than r times values of $\pi(\mathcal{C})$. Choose $z \in [0,1]$ such that $t := \pi \circ \gamma(z)$ is an element of $\pi(\mathcal{C})$ different from $\pi(x_1) = \pi \circ \gamma(0)$ and $\pi(x_2) = \pi \circ \gamma(1)$. By Step 2 of the construction of σ there exists $x \in \mathcal{C}$ lying in the same semialgebraically connected component of $\pi^{-1}(t)$ as $\gamma(z)$. Thus x_1, x and x_2 lie in the same semialgebraically connected component of V and we can connect x_1 with x and x with x_2 by two continuous semialgebraic curves γ' and γ'' such that $\pi \circ \gamma'$ and $\pi \circ \gamma''$ cross less than r times values of $\pi(\mathcal{C})$. From our induction hypothesis we conclude now that x_1, x and x_2 lie on the same semialgebraically connected component of σ. This ends the proof of Lemma 11. $\diamondsuit$

To finish the description of our algorithm $\mathcal{N}(A,n)$ it suffices to explain how a point $x \in V$, given by a quantifier free formula of $\mathcal{L}$, can be connected with the roadmap σ. From this the reader infers easily an algorithm which for two points x_1, x_2 of V, given by quantifier free formulas of $\mathcal{L}$, decides whether x_1 and x_2 lie on the same semialgebraically connected component of V, and, if they do so, finds a continuous semialgebraic curve connecting x_1 and x_2. Let x be a point of V defined by a quantifier free formula Φ of $\mathcal{L}$. By Corollary 3 we compute x explicitely from Φ in admissible time. If $t := \pi(x)$ is a critical value of π we find by the effective curve selection lemma (Corollary 4) in admissible time a continuous semialgebraic curve $\theta : T \to V$, defined on a closed interval T, such that the following holds:
- the boundary points of T are t and some non critical value t'.
- $\theta(t) = x$.

Replacing the point x by $\theta(t')$ we may assume without loss of generality that t is not a critical value of π. Thus $\pi^{-1}(t)$ is a bounded hypersurface of R^{n-1} defined by the regular polynomial $F(X_1, \ldots, X_{n-1}, t) \in A[t][X_1, \ldots, X_{n-1}]$. For each curve $\eta \in \mathcal{K}$ whose domain contains t we compute $\eta(t)$. Thus we obtain a finite set $\mathcal{D}$, explicitely computable in admissible time, which for each semialgebraically connected component of $\pi^{-1}(t)$ contains at least one point (see Step 2 in the construction of the roadmap σ). For each point $x' \in \mathcal{D}$ we decide by means of the algorithm $\mathcal{N}(A[x, x'], n-1)$ whether x' and x lie on the same semialgebraically connected component of $\pi^{-1}(t)$. Finally we find a point $x' \in \mathcal{D}$ and a continuous semialgebraic curve τ in $\pi^{-1}(t)$ connecting x and x'. Since x' is a point of σ the curve τ connects x with the roadmap σ.
This ends the description of the roadmap algorithm $\mathcal{N}(A, n)$.

Finally we observe that the algorithm $\mathcal{N}(A, n)$ can also be used to compute the number of semialgebraically connected components of V exactly. For this purpose we note that the set $\mathcal{C}$ computed in Step 2 of our construction of the roadmap σ cuts each semialgebraically connected component of V in at least one point. By means of the algorithm $\mathcal{N}(A, n)$ we can decide whether two points of $\mathcal{C}$ lie in the same semialgebraically connected component of V. Thus we can find a system of points of $\mathcal{C}$ representing each semialgebraically connected component of V just once. The cardinality of this system is exactly the number of semialgebraically connected components of V.

5. *Complexity of the roadmap algorithm*

It is clear that iterating the procedure of the last section we obtain a recursive construction of an algorithm denoted by $\mathcal{N}(A, n)$ which accepts as input a regular polynomial $F \in [X_1, \ldots, X_n]$ defining a bounded hypersurface V of R^n and points $x_1, x_2 \in V$ given by quantifier free formulas $\Phi_1, \Phi_2 \in \mathcal{L}$. The algorithm $\mathcal{N}(A, n)$ computes for this input a roadmap σ and decides whether x_1 and x_2 are in the same semialgebraically connected component of V, and, if they do, constructs a continuous semialgebraic curve joining x_1 and x_2. The algorithm $\mathcal{N}(A, n)$ is realizable by an arithmetical network over A whose size and depth depends on n and $D := \deg F + \deg \Phi_1 + \deg \Phi_2$.

We fix the inputs V, x_1, x_2 (given by F, Φ_1, Φ_2) and the parameters n, D for the moment.

Reviewing the constructions of the last sections we observe that the sequential and parallel complexities of $\mathcal{N}(A, n)$ (the size and the depth of the arithmetical network realizing the algorithm for fixed n and D) doesn't really depend on the ground ring A but only on the parameters n and D. Thus, let $S(n, D)$ be the sequential and $P(n, D)$ the (simultaneous) parallel complexity of the algorithm $\mathcal{N}(A, n)$ applied to an input with parameters of size n and D.

In the last section, we constructed in admissible time the following items:

– a variable transformation corresponding to an M-direction π of V.
– certain continuous semialgebraic curves of V (e.g. the curves of $\mathcal{K}$).
– certain points of V, definable over $\mathcal{L}$, with algebraic coordinates.

All the constructions of this "preprocessing" are realizable by arithmetical networks over A of size $D^{n^{O(1)}}$ and depth $n^{O(1)}(\log D)^{O(1)}$. Then we used $D^{n^{O(1)}}$ calls of algorithms of the form $\mathcal{N}(A[x,x'],n-1)$, where x and x' were certain points of V definable over $\mathcal{L}$ (e.g. from $\mathcal{C}$) with $t := \pi(x) = \pi(x')$ and t being not a critical value of π.

Such an algorithm $\mathcal{N}(A[x,x'],n-1)$ is realizable by an arithmetical network over $A[x,x']$. Its inputs are the regular polynomial $F(X_1,\ldots,X_{n-1},t)$ and the points $x,x' \in A[x,x']^n$. This input has parameters of size $n-1$ and $\deg(F)+2n$, which are bounded by $n-1$ and D. Thus the arithmetical network *over* $A[x,x']$ which realizes this algorithm has sequential complexity $S(n-1,D)$ and parallel complexity $P(n-1,D)$.

We have to transform this algorithm into an arithmetical network *over* A.

Let K be the fraction field of A. Since x and x' are explicitely given they are zeros of a 0-dimensional ideal of $K[X_1,\ldots,X_n]$ with known generators whose number and degrees are of order $D^{n^{O(1)}}$. Using Lemma 5 we construct in sequential time $D^{n^{O(1)}}$ and parallel time $n^{O(1)}(\log D)^{O(1)}$ an algebraic element (over A) u of R, suitably codified by Thom's Lemma as a zero of an effectively given univariate polynomial $P \in A[U]$ with $\deg(P) = D^{n^{O(1)}}$, such that $K[x,x'] \subset K[u]$.

We interprete now our algorithm as an arithmetical network over $K[u]$. From [HRS 3], Proposition 3, we deduce that arithmetical operations and sign determinations in $K[u]$ can be performed by an arithmetical network over A of size $D^{n^{O(1)}}$ and depth $n^{O(1)}(\log D)^{O(1)}$. This implies that the algorithm $\mathcal{N}(A[x,x'],n-1)$ in question can be executed by an arithmetical network *over* A in sequential time $D^{n^{O(1)}} \cdot S(n-1,D)$.

Let us first estimate the sequential complexity $S(n,D)$ of the algorithm $\mathcal{N}(A,n)$. We have to consider the "preprocessing", which can be realized in sequential time $D^{n^{O(1)}}$, and $D^{n^{O(1)}}$ calls of algorithms of type $\mathcal{N}(A[x,x'],n-1)$, which can be executed by arithmetical networks over A of size $D^{n^{O(1)}}S(n-1,D)$.

Thus we obtain the following recursion formula:

$$S(n,D) = D^{n^{O(1)}}S(n-1,D)\ .$$

From this formula one infers easily that $S(n,D) = D^{n^{O(1)}}$.

Let us now estimate the parallel complexity $P(n,D)$ of the algorithm $\mathcal{N}(A,n)$.

The "preprocessing" can be executed in parallel time $n^{O(1)}(\log D)^{O(1)}$ and the $D^{n^{O(1)}}$ calls of the algorithms $\mathcal{N}(A[x,x'],n-1)$ can be made simultaneously. We are going to estimate the parallel complexity of such a call. Let $x,x' \in V$ and u be as before, and let $\mathcal{A}$ be the arithmetical network over A which realizes the algorithm $\mathcal{N}(A[x,x'],n-1)$. First observe that $\mathcal{A}$ simulates arithmetical operations and sign tests in $K[u]$ which come from the original version of the algorithm $\mathcal{N}(A[x,x'],n-1)$ as arithmetical network over $K[u]$. Here the elements of $K[u]$ are interpretated as polynomials of degree $D^{n^{O(1)}}$ in u and represented by their coefficient vectors which are suitably codified over A. Thus the element u, algebraic over A, may be replaced by an indeterminate U. We may now read our arithmetical network as follows:
$\mathcal{A}$ computes first certain polynomials $G_1,\ldots,G_s \in K[U]$ with $\deg(G_1)+\cdots+\deg(G_s) = D^{n^{O(1)}}$, evaluates then the signs of $G_1(u),\ldots,G_s(u)$ and finally simulates the original version of the algorithm $\mathcal{N}(A[x,x'],n-1)$, given as an arithmetical network over $K[u]$. Taking into account that the degrees of $G_1,\ldots,G_s$ are bounded by $D^{n^{O(1)}}$ and

that the sequential complexity of $\mathcal{A}$ is of order $D^{n^{O(1)}}$, we can realize the computation of the coefficients of $G_1, \ldots, G_s$ by an arithmetical network of size $D^{n^{O(1)}}$ and depth $n^{O(1)}(\log D)^{O(1)}$ using interpolation in $D^{n^{O(1)}}$ points of $\mathbf{Z}$ (see [VSBR]). The evaluation of the signs of $G_1(u), \ldots, G_s(u)$, which can be done simultaneously, costs $n^{O(1)}(\log D)^{O(1)}$ parallel steps. Using these data one simulates the original version of the algorithm $\mathcal{N}(A[x,x'], n-1)$ given as an arithmetical network over $K[u]$ by an arithmetical network over A of depth $P(n-1, D)$. If we add to this "preprocessing" and to the $D^{n^{O(1)}}$ calls of the algorithms the complexity of decide if two points of $\mathcal{C}$ can be joined in σ (which can be reduced to the computations of the connected components of a graph, represented here by the points of $\mathcal{C}$), we obtain the following recursion formula:

$$P(n, D) = n^{O(1)}(\log D)^{O(1)} + P(n-1, D) \, .$$

From this we infer the parallel complexity bound: $P(n, D) = n^{O(1)}(\log D)^{O(1)}$.

We summarize these complexity results in the following

Theorem 12. *Let V be a bounded hypersurface of R^n given by a regular polynomial $F \in A[X_1, \ldots, X_n]$ and let x_1, x_2 be points of V definable by two quantifier free formulas Φ_1, Φ_2. Let $g : V \to R$ be an M-direction of V given by a linear form $G \in A[X_1, \ldots, X_n]$. There exists an admissible algorithm which constructs, from the input F, Φ_1, Φ_2, G, a roadmap of V with respect to g and which decides whether x_1 and x_2 lie in the same semialgebraically connected component of V. If this is the case, the algorithm constructs a continuous semialgebraic curve contained in V connecting x_1 and x_2. In particular, the number of semialgebraically connected components of V is computable in admissible time.*

$\diamondsuit$

6. Conclusion

Let $V \subset R^n$ be a semialgebraic set defined by a quantifier free formula $\Phi \in \mathcal{L}$. Let $g : V \to R$ be a direction of V induced by a linear form $G \in A[X_1, \ldots, X_n]$. If V is closed and bounded it makes sense to define the notion of roadmap of V with respect to g in the same way as we did for bounded regular hypersurfaces of R^n. If V doesn't satisfy these conditions we are not going to speak about roadmaps. However, we can obtain the following result for the case of an arbitrary semialgebraic set:

Theorem 13. ([HRS 4]. See also [GV 2] for an analogous result in the sequential bit-complexity model). *Let $V \subset R^n$ be a semialgebraic set defined by a quantifier free formula Φ. Let x_1, x_2 be two points of V defined by quantifier free formulas $\Phi_1, \Phi_2 \in \mathcal{L}$. Then there exists an admissible algorithm which decides whether x_1 and x_2 are in the same semialgebraically connected component of V. If they do, the algorithm computes a continuous semialgebraic curve contained in V connecting x_1 and x_2. In particular the algorithm computes the exact number of semialgebraically connected components of V.*

If V is closed and bounded and if there is given a direction $g : V \to R$ induced by a linear form $G \in A[X_1, \ldots, X_n]$, the algorithm is able to compute a roadmap of V with respect to g. $\diamondsuit$

The proof of this theorem (see [HRS 4]) is based on Theorem 12 and combines efficient computations with infinitesimals over R developped in [GV 1] and [HRS 3].

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