

A linear algebra approach to the differentiation index of generic DAE systems

Lisi D'Alfonso · Gabriela Jeronimo ·
Pablo Solernó

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Abstract The notion of differentiation index for DAE systems of arbitrary order with generic second members is discussed by means of the study of the behavior of the ranks of certain Jacobian associated sub-matrices. As a by-product, we obtain upper bounds for the regularity of the Hilbert–Kolchin function and the order of the ideal associated to the DAE systems under consideration, not depending on characteristic sets. Some quantitative and algorithmic results concerning differential transcendence bases and induced equivalent explicit ODE systems are also established.

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L. D'Alfonso · G. Jeronimo · P. Solernó (✉)
Departamento de Matemática, Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires,
Ciudad Universitaria, 1428 Buenos Aires, Argentina
e-mail: psolerno@dm.uba.ar

L. D'Alfonso
Departamento de Ciencias Exactas, Ciclo Básico Común,
Universidad de Buenos Aires,
Ciudad Universitaria, 1428 Buenos Aires, Argentina
e-mail: lisi@dm.uba.ar

G. Jeronimo · P. Solernó
CONICET, Buenos Aires, Argentina
e-mail: jeronimo@dm.uba.ar

1 Introduction

This paper is devoted to the study, mainly from a quantitative point of view, of differential algebraic equation (DAE) systems of the form:

$$(\Sigma) := \begin{cases} f_1(X, U) &= \dot{X}_1 \\ &\vdots \\ f_n(X, U) &= \dot{X}_n \\ g_1(X, U, \dot{U}, \dots, U^{(e_1)}) &= Y_1 \\ &\vdots \\ g_r(X, U, \dot{U}, \dots, U^{(e_r)}) &= Y_r \end{cases},$$

where f_1, \dots, f_n are polynomials in the $n + m$ variables $X := X_1, \dots, X_n$, $U := U_1, \dots, U_m$, and, for every $1 \leq j \leq r$, g_j is a polynomial in the $n + (e_j + 1)m$ variables X, U and the derivatives $U^{(i)} := U_1^{(i)}, \dots, U_m^{(i)}$, $1 \leq i \leq e_j$, with coefficients in a differential field k (for instance $k := \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}(t)$, etc.). The constants $e_j \in \mathbb{N}_0$ denote the order of the respective equation g_j in the variables U . The variables $Y := Y_1, \dots, Y_r$ are a new set of differential indeterminates which can be viewed as parameters (while the variables X and U are the unknowns of the system). So, it is quite natural to extend the ground field k to the differential field $\mathbb{L} := k\langle Y \rangle$ (i.e., the smallest field containing k and all the successive derivatives of Y) and consider our input system also as a system over \mathbb{L} . Even if we do not assume, as customarily, a differential 0-dimensional situation (r may be strictly smaller than m), we will suppose that the last r equations are “independent” in a suitable natural way defined in Sect. 2.1.

DAE systems like (Σ) can be regarded from several points of view: for instance, this kind of systems arises in Control Theory (see for instance [8] and [3, Sect. 4]); they may also be interpreted as the equations defining the graph of a differential morphism (see [22]). The system (Σ) may be viewed as a family of usual polynomial DAE systems where the second member parametrizes the family and takes arbitrary values outside a suitable proper algebraic Zariski closed set (see [32, Sect. 5.2]). In this last sense we say that the system (Σ) is *generic*.

We will focus on several basic topics concerning the system (Σ) , including the number of differentiations that suffice to obtain explicit equations, a description of all the relations of a prescribed order that all the solutions must verify and the number of initial conditions that can be arbitrarily fixed. All these aspects have been studied extensively during the last two centuries beginning with two posthumous articles by Jacobi ([12, 13]). The present paper (as well as most of the previous ones on the subject) may be considered as a modern approach to part of the work done by Jacobi in these remarkable and not sufficiently known papers.

The main notion we will consider is the *differentiation index*, which is a well known and important invariant associated to a DAE system (customarily for first-order and 0-dimensional systems).

There are many different definitions of differentiation indices, not always equivalent and sometimes informal (see for instance [2, 3, 8, 16, 21, 25–29, 35]). Here we

are mainly interested in the so-called *global differentiation index* (see [2, Sect. 2.2]). Roughly speaking, the differentiation index represents the minimum number of times that all or part of a given DAE system must be differentiated in order to obtain an *equivalent explicit ordinary differential equation (ODE) system*; in other words, the number of differentiations required to determine the derivatives of a certain order of the unknowns as continuous functions of derivatives of lower order of the unknowns (see [2, Definition 2.2.2]). In some sense, the index can be regarded as a measure of the complexity of the DAE system. From the theoretical point of view it represents the distance between the given system and another one for which an existence and uniqueness theorem holds (see [28]). On the other hand, from the point of view of its numerical resolution, it is closely related to the condition number of the iteration matrix in the implicit Runge–Kutta method (see [2, Theorem 5.4.1]).

In this paper we give a precise, elementary algebraic definition of a differentiation index for DAE systems as (Σ) , *not necessarily of first order nor 0-dimensional* (see Definition 15 below), by means of certain stationary properties of the ranks of suitable Jacobian sub-matrices which are proved in Sect. 3. One of the main consequences of our approach is the formulation of an equivalent definition of this index in terms of a quite natural filtration given by the successive differentiation of the input equations, which is implicitly contained in Theorem 17 below. In particular, this last formulation shows how the differentiation index uncovers constraints that every solution must satisfy. This approach is closely related to the classical algorithmic definitions of the differentiation index by means of iterated *prolongations* (differentiations) and *projections* (eliminations) (see [5, 18, 28, 29]).

Our theoretical approach has the advantage of leading to a polynomial time algorithm which computes the index of the input system (Σ) by simple comparison of ranks of Jacobian matrices (Sect. 5.3.2; see also [36]). For previous work related to the computation of differentiation indices of DAE systems, we refer the reader to [5, 18, 19, 25–29, 37].

As we said before, the notion of differentiation index is closely related to the possibility of writing certain derivatives of the unknowns in terms of derivatives of lower order of these unknowns. Unfortunately, in general, this cannot be done using *only* the algebraic relations induced by the original equations. For example, if all the equations have order one, this nice situation, which is usually ensured by the Theorem of Implicit Functions, corresponds exactly to those systems whose associated index is 0. However, in the general case, by successive differentiations (as many as the index) we can always obtain such a situation. Evidently, the new explicit system comes out of the frame of the polynomial (or even rational) systems. Under certain additional conditions on the system (Σ) , we are able to compute an implicit ODE system which is equivalent to (Σ) (see Sect. 5); in particular, in the case of a first-order system, we give an implicit but simple polynomial way to describe it, distinguishing the variables by their interrelations (namely, “free variables”, “implicit variables”, etc.). Moreover, we can estimate degree and order upper bounds for the implicit equations involved in the equivalent ODE systems (see Sect. 5.3.1) and give an algorithm to compute them with polynomial complexity in terms of an intrinsic parameter related to the geometric degree of suitable associated algebraic varieties (Proposition 33).

Our method also allows us to give a new upper bound for the *regularity* of the *Hilbert–Kolchin function* (or *differential Hilbert function*) associated to the DAE system (Σ) . The Hilbert–Kolchin function is introduced in [14, Chapter II] in order to estimate, for each non-negative integer i , the degree of freeness of the first i -derivatives of the unknowns modulo the relations induced by the input equation system (see also Sect. 4.2 below for a precise definition in our case).

As it happens for the classical Hilbert function associated to homogeneous polynomial ideals (see for instance [1, Chapter 11]), the Hilbert–Kolchin function becomes a well defined polynomial for sufficiently big arguments i (in the ordinary differential setting, this polynomial is extremely simple since its degree is at most 1). The *regularity* of the Hilbert–Kolchin function is defined to be the first non-negative integer from where the function and the polynomial coincide. It is well known that this regularity can be exactly described in terms of the orders of the elements in a characteristic set associated to any orderly ranking (see the proof of [14, Chapter II, Sect. 12, Theorem 6] or [4, Theorem 3.3]). In this article (see Theorem 20 below) we show that $\max\{1, e_1, \dots, e_r\} - 1$ is an upper bound for the regularity of the Hilbert function for the system (Σ) over the field \mathbb{L} (notice that no characteristic sets are involved in the bound). In particular, if (Σ) is a first-order system, the regularity of the Hilbert–Kolchin function is 0; in other words, the Hilbert–Kolchin function and the associated polynomial coincide for all i .

As further consequences of our techniques we deduce Bézout-type bounds for the differentiation index and the order of the input system (Σ) in terms of the orders of its defining equations (see Remark 16; Sect. 4.2.2). A similar bound for the differentiation index may also be obtained by rewriting methods as in [32, Sect. 5.2] (see also [8, Sect. 3] for an analogous bound for 0-dimensional first-order systems). Concerning the order, we recover Ritt's differential analogue to Bézout's Theorem ([30, Ch. VII, p.135]). A more precise bound for the order of a 0-dimensional system was conjectured by Jacobi (see [12]) and proved in [15] under additional hypotheses which are met in our situation (see also [20, 24, 31]). We point out that a convenient refinement of our approach may be applied to obtain Jacobi-type bounds for both the order and the differentiation index, which will be the subject of a forthcoming paper.

The paper is organized as follows: basic definitions and notations are introduced in Sect. 2. Section 3 is devoted to the study of the behavior of an integer sequence which is strongly related to the ranks of suitable Jacobian sub-matrices of the input equations and their successive derivatives. In Sect. 4 we give a precise algebraic definition of the differentiation index based on the behavior of this sequence, and an equivalent description of it in terms of the variety of constraints (Sect. 4.1). Our results on the Hilbert–Kolchin function of the differential ideals associated with the considered DAE systems are presented in Sect. 4.2: we estimate the regularity of the Hilbert–Kolchin function in Sect. 4.2.1 and the order of the induced differential ideal in terms of the orders of the equations in the given system in Sect. 4.2.2; further, a first result related to equivalent explicit ODE systems is given for the differential 0-dimensional case in Sect. 4.2.3. This result is refined in Sect. 5 by introducing a more accurate definition of the index, which is done in Sect. 5.1, and generalized for first-order systems with positive differential dimension in Sect. 5.2. Finally, we present quantitative and algorithmic considerations concerning these results in Sect. 5.3.

2 Preliminaries

2.1 Basic definitions and notations

Let k be a characteristic zero field equipped with a derivation δ (for instance $k = \mathbb{Q}$, \mathbb{R} or \mathbb{C} with $\delta := 0$, or $k = \mathbb{Q}(t)$ with the usual derivation $\delta(t) = 1$, etc.).

For an arbitrary set of (differential) indeterminates $Z := Z_1, \dots, Z_\alpha$ over k we denote the l th successive derivative of a variable Z_j as $Z_j^{(l)}$ (as customarily, the first derivatives are also denoted by \dot{Z}_j); we write $Z^{(l)} := \{Z_1^{(l)}, \dots, Z_\alpha^{(l)}\}$ and $Z^{[l]} := \{Z^{(i)}, 0 \leq i \leq l\}$. The (non-noetherian) polynomial ring $k[Z^{(l)}, l \in \mathbb{N}_0]$, called the ring of *differential polynomials*, is denoted by $k\{Z_1, \dots, Z_\alpha\}$ and its fraction field by $k\langle Z_1, \dots, Z_\alpha \rangle$. Given a finite set of (differential) polynomials $H_1, \dots, H_\beta \in k\{Z_1, \dots, Z_\alpha\}$, we write $[H_1, \dots, H_\beta]$ to denote the smallest ideal of $k\{Z_1, \dots, Z_\alpha\}$ stable under differentiation, i.e., the smallest ideal containing H_1, \dots, H_β and all their derivatives of arbitrary order. The ideal $[H_1, \dots, H_\beta]$ is called the *differential ideal* generated by H_1, \dots, H_β .

We deal with a particular class of differential algebraic equation (DAE) systems:

$$\left\{ \begin{array}{lcl} f_1(X, U) & = & \dot{X}_1 \\ & \vdots & \\ f_n(X, U) & = & \dot{X}_n \\ g_1(X, U, \dot{U}, \dots, U^{(e_1)}) & = & Y_1 \\ & \vdots & \\ g_r(X, U, \dot{U}, \dots, U^{(e_r)}) & = & Y_r \end{array} \right. \quad (1)$$

where f_1, \dots, f_n are polynomials in the $n + m$ variables $X := X_1, \dots, X_n, U := U_1, \dots, U_m$, and, for every $1 \leq j \leq r$, g_j is a polynomial in the $n + (e_j + 1)m$ variables X, U and the derivatives $U^{(i)} := U_1^{(i)}, \dots, U_m^{(i)}, 1 \leq i \leq e_j$, with coefficients in the field k . The constants $e_j \in \mathbb{N}_0$ denote the order of the respective equation g_j in the variables U , i.e., the order of the highest derivative of a variable in U appearing with non-zero coefficient in this polynomial. The variables $Y := Y_1, \dots, Y_r$ form a new set of indeterminates which we regard as parameters (while the variables X and U are the unknowns of the system). We allow n to be equal to zero (i.e., no variables X appear in the system).

In addition, we will assume that the polynomials g_i are *differentially algebraically independent over k* as elements of the fraction field $\text{Frac}(k\{Y, X, U\}/[f_1 - \dot{X}_1, \dots, f_n - \dot{X}_n])$, i.e., there is no non-trivial differential relation involving the classes of the differential polynomials g_1, \dots, g_r over k . This assumption guarantees that the differential ideal of $k\{Y, X, U\}$ associated to system (1) does not contain a non-zero differential polynomial involving only variables Y .

For any differential polynomial g lying in a differential polynomial ring $k\{Z_1, \dots, Z_\alpha\}$ the following recursive relations hold for the successive derivatives of g :

$$g^{(0)} := g,$$

$$g^{(l)} := \delta(g^{(l-1)}) + \sum_{i,j} \frac{\partial g^{(l-1)}}{\partial Z_j^{(i)}} Z_j^{(i+1)}, \quad \text{for } l \geq 1,$$

where $\delta(g^{(l-1)})$ denotes the polynomial obtained from $g^{(l-1)}$ by applying the derivative δ to all its coefficients (for instance, if k is a field of constants, this term is always zero).

Concerning the system (1) we also introduce the following definitions and notations:

- \mathbb{L} denotes the fraction field $k\langle Y \rangle$.
- $e := \max\{1, e_1, e_2, \dots, e_r\}$.
- For every $l \in \mathbb{N}_0$ we set:

$$F_i^{(l)} := f_i^{(l)} - X_i^{(l+1)} \in k[X^{[l+1]}, U^{[l]}] \quad i = 1, \dots, n,$$

$$G_j^{(l)} := g_j^{(l)} - Y_j^{(l)} \in k[Y^{[l]}, X^{[l]}, U^{[l+e_j]}] \quad j = 1, \dots, r.$$

- For every $l \in \mathbb{N}$, A_l denotes the polynomial ring $A_l := \mathbb{L}[X^{[l]}, U^{[l]}]$ and $\Delta_l \subset A_{l-1+e}$ the ideal generated by $F^{[l-1]}, G^{[l-1]}$ (observe that the ideal Δ_l is contained in A_{l-1+e} because the orders of $F^{(l-1)}$ and $G^{(l-1)}$ are at most $l-1+e$). We set $\Delta_0 := (0)$.
- $\Delta := [F, G] \subset \mathbb{L}\{X, U\}$ is the differential ideal generated by the polynomials $F := F_1, \dots, F_n$ and $G := G_1, \dots, G_r$.
- \mathbb{K} is the differential field $k(X)\langle U \rangle$ with the derivation induced by $\dot{X}_i = f_i(X, U)$, $i = 1, \dots, n$.

The hypothesis on the differential algebraic independence of the polynomials g_j ($1 \leq j \leq r$) easily implies that *the ideal Δ has differential dimension $m - r$ over \mathbb{L}* (see for instance [7, Proposition 12]). Furthermore, the map

$$\dot{X}_j \mapsto f_j, \quad j=1, \dots, n, \quad Y_j \mapsto g_j, \quad j=1, \dots, r, \quad \text{and} \quad U_j \mapsto U_j, \quad j=1, \dots, m,$$

induces an isomorphism between the fraction field $\text{Frac}(\mathbb{L}\{X, U\}/\Delta)$ and the differential field $\mathbb{K} = k(X)\langle U \rangle$ (see [7, Remark 7]).

In the sequel, in order to simplify notations, for any $g \in \mathbb{L}\{X, U\}$, we will also write g for its class in $\mathbb{L}\{X, U\}/\Delta$ or in its fraction field \mathbb{K} (the ring where we consider the object will be clear by the context). In the same way, \dot{g} will denote either the derivative of g in $\mathbb{L}\{X, U\}$ or its derivative as an element of \mathbb{K} , and so on.

2.2 Associated Jacobian sub-matrices

Here, we introduce the Jacobian matrices and sub-matrices we will deal with throughout the paper. From the considered differential system (1), we define a family of sub-matrices constructed from the (infinite) Jacobian matrix associated to the (infinitely many) polynomials $F^{(l)}$ and $G^{(l)}$ with respect to the (infinitely many) variables $X^{(j)}$ and $U^{(j)}$.

Definition 1 For each $k \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq e-1}$ (i.e., $i \in \mathbb{Z}$ and $i \geq e-1$), we define the matrix $\mathfrak{J}_{k,i} \in \mathbb{K}^{k(n+r) \times k(n+m)}$ as follows:

$$\mathfrak{J}_{k,i} := \begin{pmatrix} \frac{\partial F^{(i-e+1)}}{\partial X^{(i+1)}} & \frac{\partial F^{(i-e+1)}}{\partial U^{(i+1)}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial G^{(i-e+1)}}{\partial X^{(i+1)}} & \frac{\partial G^{(i-e+1)}}{\partial U^{(i+1)}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial F^{(i-e+k)}}{\partial X^{(i+1)}} & \frac{\partial F^{(i-e+k)}}{\partial U^{(i+1)}} & \frac{\partial F^{(i-e+k)}}{\partial X^{(i+2)}} & \frac{\partial F^{(i-e+k)}}{\partial U^{(i+2)}} & \cdots & \frac{\partial F^{(i-e+k)}}{\partial X^{(i+k)}} & \frac{\partial F^{(i-e+k)}}{\partial U^{(i+k)}} \\ \frac{\partial G^{(i-e+k)}}{\partial X^{(i+1)}} & \frac{\partial G^{(i-e+k)}}{\partial U^{(i+1)}} & \frac{\partial G^{(i-e+k)}}{\partial X^{(i+2)}} & \frac{\partial G^{(i-e+k)}}{\partial U^{(i+2)}} & \cdots & \frac{\partial G^{(i-e+k)}}{\partial X^{(i+k)}} & \frac{\partial G^{(i-e+k)}}{\partial U^{(i+k)}} \end{pmatrix}.$$

In other words, $\mathfrak{J}_{k,i}$ is the Jacobian matrix of the polynomials

$$F^{(i-e+1)}, G^{(i-e+1)}, \dots, F^{(i-e+k)}, G^{(i-e+k)} \in \mathbb{L}[X^{[i-e+k+1]}, U^{[i+k]}]$$

with respect to the variables $X^{(i+1)}, U^{(i+1)}, \dots, X^{(i+k)}, U^{(i+k)}$, where the entries are regarded as elements in \mathbb{K} .

Observe that the block triangular form of $\mathfrak{J}_{k,i}$ follows from the fact that the differential polynomials $F^{(i-e+l)}$ and $G^{(i-e+l)}$ have order bounded by $i+l$. Hence, their derivatives with respect to the variables $X^{(i+j)}$ and $U^{(i+j)}$ are identically zero for $j \geq l+1$.

The matrices $\mathfrak{J}_{k,i}$ are strongly related with some algebraic facts concerning the (algebraic) ideals Δ_l introduced in the previous subsection:

Proposition 2 Let $k \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq e-1}$. Then:

(i) The transcendence degree of the field extension

$$\text{Frac}(A_i / (\Delta_{i-e+1+k} \cap A_i)) \hookrightarrow \text{Frac}(A_{i+k} / \Delta_{i-e+1+k})$$

equals the dimension of the kernel of $\mathfrak{J}_{k,i}$.

(ii) The following identity holds:

$$\text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i / \Delta_{i-e+1+k} \cap A_i)) = (m-r)(i+1) + e(n+r) - \dim_{\mathbb{K}}(\ker(\mathfrak{J}_{k,i}^t)),$$

where $\mathfrak{J}_{k,i}^t$ denotes the usual transpose of the matrix $\mathfrak{J}_{k,i}$.

Proof In order to prove (i) we follow closely the proof of [7, Proposition 16] (see also [22, Theorem 10]): since the polynomials $F_i^{(p)}$ and $G_j^{(p)}$ have order $p+1$ and $p+e_j$, respectively, we conclude that for $i > e-1$, the polynomials $F^{[i-e]}$ and $G^{[i-e]}$ belong to the ring A_i . In any case (even if $i = e-1$), we may consider $\text{Frac}(A_{i+k} / \Delta_{i-e+1+k})$ as the fraction field of the domain

$$R := K[X^{(i+1)}, U^{(i+1)}, \dots, X^{(i+k)}, U^{(i+k)}] / (F^{(i-e+1)}, G^{(i-e+1)}, \dots, F^{(i-e+k)}, G^{(i-e+k)}),$$

where K denotes the field $\text{Frac}(A_i / (\Delta_{i-e+1+k} \cap A_i))$.

Then, the transcendence degree we want to compute is the dimension over the field $\text{Frac}(R)$ (or equivalently, over $\text{Frac}(A_{i+k}/\Delta_{i-e+1+k})$) of the kernel of the Jacobian matrix associated to the K -algebra R (see for instance [17, Chapter VI, §1, Theorem 1.15]). This Jacobian matrix is exactly the matrix $\mathfrak{J}_{k,i}$. To finish the proof of (i), it suffices to show that the dimension of both kernels (namely, over $\text{Frac}(R)$ and over \mathbb{K}) of $\mathfrak{J}_{k,i}$ is the same.

Let us observe that the entries of $\mathfrak{J}_{k,i}$ can be regarded as polynomials in the ring $k[X, U^{[i+k]}] \subset \mathbb{K}$, which is isomorphic to $k[Y^{[i-e+k]}, X^{[i+k]}, U^{[i+k]}/(F^{[i-e+k]}, G^{[i-e+k]})$. After tensoring by \mathbb{L} , this last ring becomes $A_{i+k}/\Delta_{i-e+1+k}$. Now, since $\mathbb{L} \cap \Delta = 0$, the rank of $\mathfrak{J}_{k,i}$ is preserved after that tensoring and so, its rank over the fraction field of $k[X, U^{[i+k]}]$ (and therefore, over \mathbb{K}) is equal to its rank over the fraction field of $A_{i+k}/\Delta_{i-e+1+k}$ (namely, $\text{Frac}(R)$). This finishes the proof of (i).

For part (ii), it is easy to see that the polynomials $F^{(p)}, G^{(p)}$ form a regular sequence (see for instance [7, Corollary 9]) and hence, the field extension $\mathbb{L} \hookrightarrow \text{Frac}(A_{i+k}/\Delta_{i-e+1+k})$ has transcendence degree equal to the number of variables in A_{i+k} minus the number of generators of the ideal $\Delta_{i-e+1+k}$, that is $(i+k+1)(n+m) - (i-e+k+1)(n+r) = (m-r)(i+k+1) + e(n+r)$. The result follows by considering the tower of fields

$$\mathbb{L} \hookrightarrow \text{Frac}(A_i/\Delta_{i-e+1+k} \cap A_i) \hookrightarrow \text{Frac}(A_{i+k}/\Delta_{i-e+1+k})$$

and part (i), noticing that $\dim_{\mathbb{K}} \ker(\mathfrak{J}_{k,i}) = k(m-r) + \dim_{\mathbb{K}} \ker(\mathfrak{J}_{k,i}^t)$. \square

Remark 3 The equality in part (ii) of Proposition 2 also holds for $k = 0$ defining $\dim_{\mathbb{K}}(\ker(\mathfrak{J}_{0,i}^t)) := 0$ for $i \geq e-1$. This follows from the fact that $\Delta_{i-e+1} \subset A_i$ is generated by a regular sequence, as in the proof of that Proposition.

Remark 4 Fix an index $i \geq e-1$. Note that the prime ideals $(\Delta_{i-e+1+k} \cap A_i)_{k \in \mathbb{N}_0}$ form an increasing chain of ideals. Since A_i is a Noetherian ring, this chain must be stationary for k big enough, and so, from condition (ii) of Proposition 2, the sequence of integers $(\dim_{\mathbb{K}} \ker(\mathfrak{J}_{k,i}^t))_{k \in \mathbb{N}_0}$ is non-decreasing and becomes stationary for k big enough.

In the next section we proceed to study more closely this kind of stationary properties related to the rank of the matrices $\mathfrak{J}_{k,i}$.

3 The rank of Jacobian sub-matrices

The aim of this section is to study the behavior of the ranks of the matrices $\mathfrak{J}_{k,i}$ when k and/or i run over \mathbb{N} and $\mathbb{N}_{\geq e-1}$ respectively, which will provide us with some information about certain invariants of the system (1), namely, the differentiation index and the regularity of the Hilbert–Kolchin function.

We introduce a double sequence $\mu_{k,i}$ of non-negative integers associated with the matrices $\mathfrak{J}_{k,i}$:

Definition 5 For $k \in \mathbb{N}_0$ and $i \in \mathbb{N}_{\geq e-1}$, we define $\mu_{k,i} \in \mathbb{N}_0$ as follows:

- $\mu_{0,i} := 0$, for every $i \in \mathbb{N}_{\geq e-1}$;
- $\mu_{k,i} := \dim_{\mathbb{K}} \ker(\mathfrak{J}_{k,i}^t) = k(n+r) - \text{rank}_{\mathbb{K}}(\mathfrak{J}_{k,i})$, for $k \geq 1$ and $i \in \mathbb{N}_{e-1}$.

Now, we will focus on the study of certain stationarity properties of the sequence $(\mu_{k,i})_{k,i}$. We begin by analyzing the behavior of the sequence $(\mu_{k,i})_k$ when the index i is fixed, which will be done by comparing the matrices $\mathfrak{J}_{k,i}$ for $k \in \mathbb{N}$.

First, let us observe the following obvious recursive relation which holds for every $k \geq 1$:

$$\mathfrak{J}_{k+1,i} = \begin{pmatrix} \boxed{\mathfrak{J}_{k,i}} & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \frac{\partial F^{(i-e+k+1)}}{\partial X^{(i+1)}} & \frac{\partial F^{(i-e+k+1)}}{\partial U^{(i+1)}} & \dots & \frac{\partial F^{(i-e+k+1)}}{\partial X^{(i+k+1)}} & \frac{\partial F^{(i-e+k+1)}}{\partial U^{(i+k+1)}} \\ \frac{\partial G^{(i-e+k+1)}}{\partial X^{(i+1)}} & \frac{\partial G^{(i-e+k+1)}}{\partial U^{(i+1)}} & \dots & \frac{\partial G^{(i-e+k+1)}}{\partial X^{(i+k+1)}} & \frac{\partial G^{(i-e+k+1)}}{\partial U^{(i+k+1)}} \end{pmatrix}. \quad (2)$$

When the differential system (1) is linear (for instance, if $k := \mathbb{Q}$ and the system (1) is of type $AU + B\dot{U} = Y$, where A and B belong to $\mathbb{Q}^{r \times m}$), the matrices $\mathfrak{J}_{k,i}$ have a nice Hankel-block type form, but this is not exactly our situation. However, there is a main relation arising from their underlying differential structure which enables us to study them also in our non linear setting:

Proposition 6 Let $Z := Z_1, \dots, Z_\alpha$ be differential independent variables and let H be a differential polynomial in $k\{Z\}$. For all $l, j \in \mathbb{N}_0$ the following relation holds in \mathbb{K} :

$$\left(\frac{\partial H^{(l)}}{\partial Z^{(j+1)}} \right)^* = \frac{\partial H^{(l+1)}}{\partial Z^{(j+1)}} - \frac{\partial H^{(l)}}{\partial Z^{(j)}}. \quad (3)$$

In particular, if $H \in \{F, G\}$ and $Z \in \{X, U\}$, we have that $\frac{\partial H^{(l)}}{\partial Z^{(j+1)}} = 0$ for every $j \geq l + e$, since the order of $H^{(l)}$ is at most $l + e$, and therefore identity (3) implies that

$$\frac{\partial H^{(l+1)}}{\partial Z^{(j+1)}} = \frac{\partial H^{(l)}}{\partial Z^{(j)}} \quad \forall j \geq l + e. \quad (4)$$

Note that these identities are also valid over the differential field \mathbb{K} , where the derivation is now the one induced by $\dot{X}_j = f_j(X, U)$ in \mathbb{K} and all partial derivatives are regarded as elements in \mathbb{K} .

Due to the triangular form of $\mathfrak{J}_{k,i}$ and condition (4), all the matrices $\mathfrak{J}_{k,i}$ have the same $(n+r) \times (n+m)$ -block $\mathfrak{J}_{1,i}$ in their main diagonal.

Proof Straightforward from the definitions and the Chain Rule. \square

We are now ready to prove the first stationarity property of the sequence $(\mu_{k,i})_{k,i}$ (see also Remark 4 above):

Proposition 7 *For each fixed $i \in \mathbb{N}_{\geq e-1}$, the sequence $(\mu_{k,i})_{k \in \mathbb{N}_0}$ is non-decreasing and verifies the inequality*

$$n \min\{k, e-1\} + \sum_{j=1}^r \min\{k, e-e_j\} \leq \mu_{k,i} \leq \min\{k, e\} (n+r). \quad (5)$$

In particular, there exists $k \in \mathbb{N}_0$ (depending on i), $0 \leq k \leq e+n + \sum_{j=1}^r e_j$, such that $\mu_{k,i} = \mu_{k+1,i}$.

Proof The fact that $(\mu_{k,i})_k$ is a non-decreasing sequence follows immediately by observing that $\ker(\mathfrak{J}_{k,i}^t) \times \{0\} \subset \ker(\mathfrak{J}_{k+1,i}^t)$ for every $k \in \mathbb{N}$ (see (2), also Remark 4).

For every non-negative integer $k \in \mathbb{N}_0$, the matrix $\mathfrak{J}_{k,i}$ has $k(n+r)$ rows. Therefore, $\dim \ker(\mathfrak{J}_{k,i}^t) \leq k(n+r)$. On the other hand, due to Proposition 2, Remark 3 and the definition of $\mu_{k,i}$, we have that $\text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/\Delta_{i-e+1+k} \cap A_i)) = (m-r)(i+1) + e(n+r) - \mu_{k,i}$. Now, $\text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/\Delta_{i-e+1+k} \cap A_i)) \geq \text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/\Delta \cap A_i))$, since $\Delta_{i-e+1+k} \cap A_i \subset \Delta \cap A_i$, and so, the fact that the differential dimension of Δ is $m-r$ implies that $\text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/\Delta \cap A_i)) \geq (m-r)(i+1)$. Hence, $\mu_{k,i} \leq e(n+r)$ holds.

In order to show the other inequality, we observe that, since the order of the polynomial $G_j^{(p)}$ is $p+e_j$ ($1 \leq j \leq r$), the partial derivatives $\partial G_j^{(p)}/\partial X^{(q)}$ and $\partial G_j^{(p)}/\partial U^{(q)}$ are all zero for $q > p+e_j$. In particular, we conclude that for $t, s \in \mathbb{N}$, with $t \geq s$, the derivatives $\partial G_j^{(i-e+t)}/\partial X^{(i+s)}$ and $\partial G_j^{(i-e+t)}/\partial U^{(i+s)}$ are zero if $i+s > i-e+t+e_j$, or equivalently, if $e-e_j > t-s$. So, each polynomial G_j induces $\min\{k, e-e_j\}$ many null rows in the matrix $\mathfrak{J}_{k,i}$. Analogously, each polynomial F_j induces $\min\{k, e-1\}$ many null rows in this matrix.

We conclude that the matrix $\mathfrak{J}_{k,i}$ has at least $n \min\{k, e-1\} + \sum_{j=1}^r \min\{k, e-e_j\}$ null rows. Thus, the dimension of the kernel of the transpose matrix $\mathfrak{J}_{k,i}^t$ (i.e., $\mu_{k,i}$) is at least $n \min\{k, e-1\} + \sum_{j=1}^r \min\{k, e-e_j\}$.

The second assertion follows directly from the fact that for every $k \geq e$, the inequality (5) reads $n(e-1) + \sum_{j=1}^r (e-e_j) \leq \mu_{k,i} \leq e(n+r)$. \square

In fact, in Theorem 9 we are able to prove a more precise result than that of Proposition 7: the sequence $(\mu_{k,i})_{k \in \mathbb{N}_0}$ is strictly increasing up to a certain index $k_i \leq e+n + \sum_{j=1}^r e_j$ where it becomes stationary.

For the sake of simplicity, we will use the following notations:

Notation 8 *The variables X, U involved in system (1) are renamed in the following way: $Z_j := X_j$ for $j = 1, \dots, n$, and $Z_{n+j} := U_j$ for $j = 1, \dots, m$ (and the same is done for their corresponding formal derivatives). Analogously, the polynomials are renamed as: $H_j := F_j$ for $j = 1, \dots, n$, and $H_{n+j} := G_j$ for $j = 1, \dots, r$.*

With these notations, the matrix $\mathfrak{J}_{k,i}$ involves exactly the derivatives of the polynomials $H^{(i-e+1+p)}$ with respect to the variables $Z^{(i+q)}$, with $p = 0, \dots, k-1$ and $q = 1, \dots, k$.

Theorem 9 Fix an index $i \in \mathbb{N}_{\geq e-1}$ and let $k_i \in \mathbb{N}_0$ be the minimum of all the k 's in \mathbb{N}_0 such that $\mu_{k+1,i} = \mu_{k,i}$ (this minimum is well defined due to Proposition 7). Then $\mu_{k,i} = \mu_{k_i,i}$ for every $k \geq k_i$.

Proof According to Notation 8 we will rename variables and equations as $Z := (X, U)$ and $H := (F, G)$.

The result is clear for $k_i = 0$: in this case, $\mu_{1,i} = 0$, which is equivalent to the fact that the matrix $\mathfrak{J}_{1,i} = \frac{\partial H^{(i-e+1)}}{\partial Z^{(i+1)}}$ has full row rank. From identity (4) in Proposition 6 we conclude that $\mathfrak{J}_{k,i}$ has full row rank too or, equivalently, that $\mu_{k,i} = 0$ for all k .

Now, let us assume that $k_i \geq 1$. It suffices to show that the equality $\mu_{k,i} = \mu_{k-1,i}$ for an arbitrary index $k \geq 2$, implies $\mu_{k+1,i} = \mu_{k,i}$.

In the sequel, for a vector $v \in \mathbb{K}^{l(n+r)}$ we will write its description as a block vector $v = (v_1, \dots, v_l)$ with $v_j \in \mathbb{K}^{n+r}$.

Due to the recursive relation (2), the identity $\ker(\mathfrak{J}_{k,i}^t) \times \{0\} = \ker(\mathfrak{J}_{k+1,i}^t) \cap \{v_{k+1} = 0\}$ holds in $\mathbb{K}^{(k+1)(n+r)}$ for every $k \in \mathbb{N}$ and so, the equality $\mu_{k,i} = \mu_{k+1,i}$ is equivalent to the inclusion $\ker(\mathfrak{J}_{k+1,i}^t) \subset \{v_{k+1} = 0\}$. Then, the theorem is a consequence of the following recursive principle:

Claim: For all $k \in \mathbb{N}$, $\ker(\mathfrak{J}_{k,i}^t) \subset \{v_k = 0\}$ implies $\ker(\mathfrak{J}_{k+1,i}^t) \subset \{v_{k+1} = 0\}$.

Proof of the Claim. Let us show that if $(v_1, \dots, v_{k+1}) \in \ker(\mathfrak{J}_{k+1,i}^t)$ then, the vector $w = (w_1, \dots, w_k) \in \mathbb{K}^{k(n+r)}$ defined as

$$w_k := v_{k+1}, \quad w_j := v_{j+1} - \dot{w}_{j+1}, \quad j = k-1, \dots, 1,$$

lies in $\ker(\mathfrak{J}_{k,i}^t)$, which implies the Claim.

Our assumptions on the order of the equations imply that $\frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j)}} = 0$ for $\ell < j$. Hence, we have that $w \in \ker(\mathfrak{J}_{k,i}^t)$ if and only if the following identities hold over \mathbb{K} for every $1 \leq j \leq k$:

$$\sum_{\ell=j}^k w_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j)}} = 0.$$

We will proceed recursively for $j = k, k-1, \dots, 1$. For $j = k$, the definition of w and identity (4) imply that

$$w_k \frac{\partial H^{(i-e+k)}}{\partial Z^{(i+k)}} = v_{k+1} \frac{\partial H^{(i-e+k+1)}}{\partial Z^{(i+k+1)}} = 0.$$

Now, assume that $\sum_{\ell=j+1}^k w_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j+1)}} = 0$. Differentiating this identity in \mathbb{K} and using identity (3) we get:

$$\sum_{\ell=j+1}^k \dot{w}_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j+1)}} + \sum_{\ell=j+1}^k w_\ell \left(\frac{\partial H^{(i-e+\ell+1)}}{\partial Z^{(i+j+1)}} - \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j)}} \right) = 0.$$

This implies that

$$\begin{aligned} \sum_{\ell=j}^k w_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j)}} &= w_j \frac{\partial H^{(i-e+j)}}{\partial Z^{(i+j)}} + \sum_{\ell=j+1}^k \dot{w}_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j+1)}} + \sum_{\ell=j+1}^k w_\ell \frac{\partial H^{(i-e+\ell+1)}}{\partial Z^{(i+j+1)}} \\ &= \sum_{\ell=j+1}^k (\dot{w}_\ell + w_{\ell-1}) \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j+1)}} + w_k \frac{\partial H^{(i-e+k+1)}}{\partial Z^{(i+j+1)}} \\ &= \sum_{\ell=j+1}^{k+1} v_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j+1)}} = \sum_{\ell=1}^{k+1} v_\ell \frac{\partial H^{(i-e+\ell)}}{\partial Z^{(i+j+1)}} = 0, \end{aligned}$$

where the second equality follows from identity (4) and the third one is simply the definition of w . This concludes the proof of the theorem. \square

So far, we have studied the behavior of the sequence $(\mu_{k,i})_k$ for an arbitrary (but fixed) index $i \in \mathbb{N}_{\geq e-1}$. In the remaining part of the section we will analyze the sequence $(\mu_{k,i})_i$ fixing the index $k \in \mathbb{N}$.

We start by exhibiting a (non- \mathbb{K} -linear) bijection between the kernels of the matrices $\mathfrak{J}_{k,i}$ and $\mathfrak{J}_{k,i+1}$ for any index $k \in \mathbb{N}$.

Lemma 10 *Let (v_1, \dots, v_k) and (w_1, \dots, w_k) be arbitrary elements in $\mathbb{K}^{k(n+m)}$ (here v_j and w_j denote vectors in \mathbb{K}^{n+m} for every index j). Then, for each $i \in \mathbb{N}_{\geq e-1}$, the function $\theta : \mathbb{K}^{k(n+m)} \rightarrow \mathbb{K}^{k(n+m)}$ defined as $\theta(v_1, \dots, v_k) = (v_1, v_2 - \dot{v}_1, v_3 - \dot{v}_2, \dots, v_k - \dot{v}_{k-1})$ maps $\ker(\mathfrak{J}_{k,i+1})$ into $\ker(\mathfrak{J}_{k,i})$. Moreover, θ is a bijection between $\ker(\mathfrak{J}_{k,i+1})$ and $\ker(\mathfrak{J}_{k,i})$, with inverse $\theta^{-1}(w_1, \dots, w_k) = (w_1, w_2 + \dot{w}_1, w_3 + \dot{w}_2 + w_1^{(2)}, \dots, w_k + \dot{w}_{k-1} + \dots + w_1^{(k-1)})$.*

Proof It is easy to see that θ is a bijection in $\mathbb{K}^{k(n+m)}$ with the inverse given in the statement of the Lemma. Let us show that it maps $\ker(\mathfrak{J}_{k,i+1})$ to $\ker(\mathfrak{J}_{k,i})$.

We keep the notations introduced in Notation 8.

For arbitrary vectors (v_1, \dots, v_k) and (w_1, \dots, w_k) in $\mathbb{K}^{k(n+m)}$, consider the following two families of sums ($p = 0, \dots, k-1$):

$$E_{i+1,p}(v) := \sum_{j=1}^k \frac{\partial H^{(i-e+2+p)}}{\partial Z^{(i+1+j)}} v_j \quad \text{and} \quad E_{i,p}(w) := \sum_{j=1}^k \frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+j)}} w_j.$$

Note that $v \in \ker(\mathfrak{J}_{k,i+1})$ if and only if $E_{i+1,p}(v) = 0$ for $p = 0, \dots, k-1$ and $w \in \ker(\mathfrak{J}_{k,i})$ if and only if $E_{i,p}(w) = 0$ for $p = 0, \dots, k-1$.

First, we compare the vectors $\mathfrak{J}_{k,i+1} v$ and $\mathfrak{J}_{k,i} \theta(v)$ for a given vector $v = (v_1, \dots, v_k) \in \mathbb{K}^{k(n+m)}$: let us observe that $E_{i+1,0}(v) = E_{i,0}(v) = E_{i,0}(\theta(v))$, since $\frac{\partial H^{(i-e+2)}}{\partial Z^{(i+1+j)}} = \frac{\partial H^{(i-e+1)}}{\partial Z^{(i+j)}} = 0$ for every $j \geq 2$ (recall that the order of H is smaller

than e), $\frac{\partial H^{(i-e+2)}}{\partial Z^{(i+2)}} = \frac{\partial H^{(i-e+1)}}{\partial Z^{(i+1)}}$ [from (4)] and $\theta(v)_1 = v_1$. Now, for $p > 0$, we have:

$$\begin{aligned} & E_{i+1,p}(v) - (E_{i+1,p-1}(v))^* \\ &= \sum_{j=1}^k \left(\frac{\partial H^{(i-e+2+p)}}{\partial Z^{(i+1+j)}} - \left(\frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+1+j)}} \right)^* \right) v_j - \sum_{j=1}^k \frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+1+j)}} \dot{v}_j \\ &= \sum_{j=1}^k \frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+j)}} v_j - \sum_{j=2}^{k+1} \frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+j)}} \dot{v}_{j-1} \\ &= \frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+1)}} v_1 + \sum_{j=2}^k \frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+j)}} (v_j - \dot{v}_{j-1}) = E_{i,p}(\theta(v)), \end{aligned}$$

where the second equality follows from identity (3), and the third one from the fact that $\frac{\partial H^{(i-e+1+p)}}{\partial Z^{(i+k+1)}} = 0$ for $p \leq k-1$.

These equalities imply straightforwardly that θ maps $\ker(\mathfrak{J}_{k,i+1})$ into $\ker(\mathfrak{J}_{k,i})$.

In order to prove that it is onto, we may argue recursively: if $w \in \ker(\mathfrak{J}_{k,i})$, then $E_{i,p}(w) = 0$ for $p = 0, \dots, k-1$. Now, $E_{i+1,0}(\theta^{-1}(w)) = E_{i+1,0}(w) = E_{i,0}(w) = 0$. Assuming that $E_{i+1,p-1}(\theta^{-1}(w)) = 0$ has already been proved, we deduce that $E_{i+1,p}(\theta^{-1}(w)) = E_{i,p}(w) + (E_{i+1,p-1}(\theta^{-1}(w)))^*$ also equals 0. We conclude that $\theta^{-1}(w) \in \ker(\mathfrak{J}_{k,i+1})$. \square

Even though the bijection θ between $\ker(\mathfrak{J}_{k,i+1})$ and $\ker(\mathfrak{J}_{k,i})$ shown in the previous Lemma is not a \mathbb{K} -linear map, it enables us to prove the following:

Proposition 11 *Let $k \in \mathbb{N}_0$ and $i \in \mathbb{N}_{\geq e-1}$ be arbitrary integers. Then $\mu_{k,i} = \mu_{k,i+1}$.*

Proof The result is immediate if $k = 0$. Now, let $k \in \mathbb{N}$ be a positive integer. In order to prove that $\dim_{\mathbb{K}}(\ker(\mathfrak{J}_{k,i}^t)) = \dim_{\mathbb{K}}(\ker(\mathfrak{J}_{k,i+1}^t))$, it suffices to show that $\mathfrak{J}_{k,i}$ and $\mathfrak{J}_{k,i+1}$ have the same rank, since they are two matrices of the same size.

For each pair of indices j, t , $1 \leq j \leq k$ and $1 \leq t \leq n+m$, set

$$C_{j,t} := \frac{\partial H^{[i-e+1, i-e+k]}}{\partial Z_t^{(i+j)}} \quad \text{and} \quad D_{j,t} := \frac{\partial H^{[i-e+2, i-e+k+1]}}{\partial Z_t^{(i+1+j)}}$$

for the corresponding columns of the matrices $\mathfrak{J}_{k,i}$ and $\mathfrak{J}_{k,i+1}$, respectively.

Assume that a column D_{j_0,t_0} of the matrix $\mathfrak{J}_{k,i+1}$ is a \mathbb{K} -linear combination of the columns $D_{j,t}$ to its right. Then, there exist elements $\alpha_{j,t} \in \mathbb{K}$ such that

$$D_{j_0,t_0} = \sum_{t=j_0+1}^{n+m} \alpha_{j_0,t} D_{j_0,t} + \sum_{j=j_0+1}^k \sum_{t=1}^{n+m} \alpha_{j,t} D_{j,t},$$

and so, the vector $v := (\vec{0}, \dots, \vec{0}, -\alpha_{j_0}, -\alpha_{j_0+1}, \dots, -\alpha_k) \in \mathbb{K}^{k(n+m)}$ belongs to the kernel of $\mathfrak{J}_{k,i+1}$, where $\vec{0}$ denotes the null vector in \mathbb{K}^{n+m} , $\alpha_{j_0} := (0, \dots, 0, 1, \alpha_{j_0, t_0+1}, \dots, \alpha_{j_0, k})$ and $\alpha_j := (\alpha_{j,1}, \dots, \alpha_{j,k})$ for $j \geq j_0 + 1$. By Lemma 10, the vector $\theta(v)$ belongs to the kernel of $\mathfrak{J}_{k,i}$ and, due to the particular form of the application θ , it turns out that the column C_{j_0, t_0} is a \mathbb{K} -linear combination of the columns to its right. Hence, the rank of $\mathfrak{J}_{k,i}$ is lower than or equal to the rank of $\mathfrak{J}_{k,i+1}$.

By means of the inverse map θ^{-1} , one proves in an analogous way that the rank of $\mathfrak{J}_{k,i+1}$ is lower than or equal to the rank of $\mathfrak{J}_{k,i}$. \square

Remark 12 The following alternative proof of this proposition was kindly suggested to us by Prof. F. Ollivier. From [14, Prop. 10, Ch. IV] the dimensions of the kernels of $\mathfrak{J}_{k,i}$ and $\mathfrak{J}_{k,i+1}$ are the differential dimensions of the ideals generated in a differential polynomial ring in $k(n+m)$ new variables by the linear equations defined by these matrices. Now, Proposition 11 follows since the bijection θ induces a differential isomorphism between these fields.

The previous proposition states that the sequence $\mu_{k,i}$ does not depend on the index i ; therefore:

Notation 13 In the sequel, we will write μ_k instead of $\mu_{k,i}$, for any $i \in \mathbb{N}_{\geq e-1}$.

So, Theorem 9 can be restated as follows:

Corollary 14 There exists a non-negative integer σ such that $\mu_k < \mu_{k+1}$ for every $k < \sigma$ and $\mu_k = \mu_{k+1}$ for every $k \geq \sigma$. \square

4 The differentiation index

Here we apply the properties of the matrices $\mathfrak{J}_{k,i}$ established in the previous section to the study of well-known invariants of our DAE system (1): the differentiation index, the regularity of the Hilbert–Kolchin function and the order of its associated differential ideal.

We start by introducing the notion of differentiation index of the DAE system (1) (see also [2, 3, 8, 21]):

Definition 15 The non-negative integer σ introduced in Corollary 14 is called the *differentiation index* of the system (1).

Corollary 14 states that the differentiation index σ is the smallest non-negative integer k where the sequence μ_k becomes stationary. Since $\mu_\sigma = \mu_k$ for all $k \geq \sigma$, we deduce from Proposition 7:

Remark 16 The following inequalities hold:

$$(e-1)n + \sum_{j=1}^r (e - e_j) \leq \mu_\sigma \leq e(n+r); \quad (6)$$

$$0 \leq \sigma \leq \min \left\{ e(n+r), e+n + \sum_{j=1}^r e_j \right\}. \quad (7)$$

For the last inequality see also [32, Sect. 5.2].

4.1 The manifold of constraints

For every $i \in \mathbb{N}_{\geq e-1}$, the differentiation index σ is strongly related to the minimum number of derivatives of the system (1) required to obtain the intersection of the whole differential ideal Δ with the polynomial ring A_i , namely those polynomials in Δ which involve only derivatives up to order i (a similar but less precise result can also be obtained using characteristic sets by means of [32, Theorem 27]; see also [7, Lemma 14] and [22, Lemma 9]):

Theorem 17 *Let $\sigma \in \mathbb{N}_0$ be the differentiation index of the system (1). Then, for every $i \in \mathbb{N}_{\geq e-1}$, the equality of ideals*

$$\Delta_{i-e+1+\sigma} \cap A_i = \Delta \cap A_i$$

holds in the polynomial ring A_i . In particular, from inequality (7) it follows that the identity $\Delta_{i+1+n+\sum_{j=1}^r e_j} \cap A_i = \Delta \cap A_i$ holds for every $i \in \mathbb{N}_{\geq e-1}$.

Furthermore, for every $i \in \mathbb{N}_{\geq e-1}$, the differentiation index σ verifies:

$$\sigma = \min\{h \in \mathbb{N}_0 : \Delta_{i-e+1+h} \cap A_i = \Delta \cap A_i\}. \quad (8)$$

Taking $i = e - 1$, we have

$$\sigma = \min\{h \in \mathbb{N}_0 : \Delta_h \cap A_{e-1} = \Delta \cap A_{e-1}\}. \quad (9)$$

Proof Fix an index $i \in \mathbb{N}_{\geq e-1}$. Let us consider the increasing chain $(\Delta_{i-e+1+k} \cap A_i)_{k \in \mathbb{N}_0}$ of prime ideals in the polynomial ring A_i . From Proposition 2, Remark 3 and the definition of the sequence μ_k , for every $k \in \mathbb{N}_0$, we have

$$\text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/\Delta_{i-e+1+k} \cap A_i)) = (m-r)(i+1) + e(n+r) - \mu_k. \quad (10)$$

Since μ_k is stationary for $k \geq \sigma$ (Corollary 14), the previous equality implies that all the prime ideals $\Delta_{i-e+1+k} \cap A_i$ have the same dimension for $k \geq \sigma$, and so, they coincide because they form an increasing chain of ideals. On the other hand, any finite system of generators of the prime ideal $\Delta \cap A_i \subset A_i$ belongs to $\Delta_{i-e+1+k} \cap A_i$ for all k big enough, which finishes the proof of the first assertion of the Theorem.

In order to prove the second part of the statement, for each $i \in \mathbb{N}_{\geq e-1}$, let h_i be the smallest non-negative integer such that $\Delta_{i-e+1+h_i} \cap A_i = \Delta \cap A_i$. By the definition of h_i , the transcendence degrees $\text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/\Delta_{i-e+1+k} \cap A_i))$ coincide for $k \geq h_i$, and so, μ_k is constant for $k \geq h_i$ [see identity (10) above]. This implies that $\sigma \leq h_i$. The equality follows from the first part of the statement and the minimality of h_i . \square

Remark 18 Identities (8) or (9) in Theorem 17 can be regarded as *alternative definitions of the differentiation index* introduced in Definition 15.

In particular, identity (9) gives the following interpretation of the differentiation index (see [8] for the case $e = 1$):

Remark 19 If the differentiation index σ of system (1) equals 0, there are no constraints on initial conditions for the system (recall that $\Delta_0 := (0)$). In the case when $\sigma \geq 1$, the quantity $\sigma - 1$ is the minimal number of derivatives of the equations in the system needed to obtain all the relations that the initial conditions must satisfy (the so called “manifold of constraints on initial conditions”).

Another fundamental property of the differentiation index, concerning the number of derivatives of the equations required to obtain an explicit equivalent ODE system, will be considered in Sects. 4.2.3 and 5 below.

First we will show how our previous results can be applied in order to estimate the regularity of the Hilbert–Kolchin function.

4.2 The regularity of the Hilbert–Kolchin function and applications

We recall the main basic facts of the Hilbert–Kolchin function for the particular case of our input system (1) and our ideal Δ (see [14, Chapter II] for the general theory).

The *Hilbert–Kolchin function* $H_\Delta : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ of the differential ideal $\Delta \subset \mathbb{L}\{X, U\}$ is defined as

$$H_\Delta(i) := \text{trdeg}_{\mathbb{L}}(\text{Frac}(A_i/(\Delta \cap A_i)))$$

for every $i \in \mathbb{N}_0$. Since the ideal Δ we are considering has differential dimension $m - r$, the identity $H_\Delta(i) = (m - r)(i + 1) + \text{ord}_{\mathbb{L}}(\Delta)$ holds for i sufficiently big (see for instance [14, Ch. II, Sec. 12, Th. 6]), where $\text{ord}_{\mathbb{L}}(\Delta)$ is a non-negative integer depending only on the differential ideal Δ which is called the *order* of the ideal. The polynomial

$$\mathcal{H}_\Delta(T) := (m - r)(T + 1) + \text{ord}_{\mathbb{L}}(\Delta)$$

is called the *Hilbert–Kolchin polynomial* associated to the ideal Δ . The *regularity* of the Hilbert–Kolchin function H_Δ is defined to be the minimum integer $i_0 \in \mathbb{N}_0$ such that $H_\Delta(i) = \mathcal{H}_\Delta(i)$ holds for all $i \geq i_0$.

4.2.1 The regularity of the Hilbert–Kolchin function

The theory of *characteristic sets* (see for instance [30]) can be used to give a precise estimation of the regularity of H_Δ : let \mathcal{C} be a characteristic set of the differential ideal $\Delta \subset \mathbb{L}\{X, U\}$ for an *orderly ranking* (see [30]) in the variables X, U . Then, the regularity of H_Δ is equal to $\max\{\text{ord}(C) : C \in \mathcal{C}\} - 1$ (this fact follows as an immediate consequence of the proof of [14, Ch. II, Sect. 12, Th. 6 (d)], see also [4, Theorem 3.3]).

The results developed so far enable us to exhibit the following simple upper bound for this regularity, which depends only on the order of the polynomials involved in the system (1):

Theorem 20 *The regularity of the Hilbert–Kolchin function of the ideal Δ over the ground field \mathbb{L} is bounded by $e - 1$.*

In particular, for first-order systems of type (I) (in other words, for the case $e = 1$), the Hilbert–Kolchin function of Δ coincides with the associated Hilbert–Kolchin polynomial.

Proof It suffices to show that the equality $H_{\Delta}(i + 1) = H_{\Delta}(i) + (m - r)$ holds for every $i \in \mathbb{N}_{\geq e-1}$.

Fix an index $i \in \mathbb{N}_{\geq e-1}$. Due to Theorem 17, we have that $\Delta \cap A_i = \Delta_{i+1-e+\sigma} \cap A_i$ and $\Delta \cap A_{i+1} = \Delta_{i+2-e+\sigma} \cap A_{i+1}$. So, $H_{\Delta}(i+1) = \text{trdeg}_{\mathbb{L}}(A_{i+1}/(\Delta_{i+2-e+\sigma} \cap A_{i+1}))$ and $H_{\Delta}(i) = \text{trdeg}_{\mathbb{L}}(A_i/(\Delta_{i+1-e+\sigma} \cap A_i))$. Thus, using Proposition 2 (recall also Notation 13) we obtain:

$$\begin{aligned} H_{\Delta}(i + 1) &= (m - r)(i + 2) + e(n + r) - \mu_{\sigma}, \\ H_{\Delta}(i) &= (m - r)(i + 1) + e(n + r) - \mu_{\sigma}. \end{aligned}$$

Hence, the equality $H_{\Delta}(i + 1) = H_{\Delta}(i) + (m - r)$ holds. \square

As we have already pointed out, the regularity of the Hilbert–Kolchin function can be obtained from the maximal order appearing in any characteristic set associated to the differential system (1) for an orderly ranking. Therefore, our upper bound on the regularity implies an upper bound for this maximum:

Corollary 21 *Let \mathcal{C} be a characteristic set of $\Delta \subset \mathbb{L}\{X, U\}$ with respect to an orderly ranking on the variables X, U . Then, the order of each of the differential polynomials in \mathcal{C} is bounded by the maximal order $e := \max\{1, e_1, e_2, \dots, e_r\}$.* \square

4.2.2 Upper bounds for the order of the ideal Δ

The proof of Theorem 20 also shows an indirect relation between the Hilbert–Kolchin polynomial \mathcal{H}_{Δ} and the differentiation index σ . More precisely:

Remark 22 The Hilbert–Kolchin polynomial $\mathcal{H}_{\Delta}(T)$ of the ideal Δ can be written as $\mathcal{H}_{\Delta}(T) = (m - r)(T + 1) + e(n + r) - \mu_{\sigma}$. Equivalently, $\text{ord}_{\mathbb{L}}(\Delta) = e(n + r) - \mu_{\sigma}$.

From inequality (6) and Remark 22 we deduce that the order of the ideal Δ may be bounded in terms of the orders of the defining equations as follows:

$$\text{ord}_{\mathbb{L}}(\Delta) \leq n + \sum_{j=1}^r e_j. \quad (11)$$

For general 0-dimensional DAE systems J. Ritt in [30, Ch. VII, p.135] (see also [14, Chapter IV, Proposition 9] and [33]) states a more precise bound, showing that the order of the ideal is always bounded by the sum of the maxima of the orders of each variable in the given differential polynomials. This bound is called by Ritt a *differential analogue to Bézout’s Theorem*. For the systems considered in the present

article, we are able to give a simple proof of the following extension of this fact with no assumption on the differential dimension:

Corollary 23 *For each $i = 1, \dots, m$ set $\epsilon_i := \max\{\text{ord}_{U_i} G_j, j = 1, \dots, r\}$. We have the following inequality:*

$$\text{ord}_{\mathbb{L}}(\Delta) \leq n + \sum_{i=1}^m \epsilon_i.$$

Proof We transform our input system (1) in an equivalent *first-order* system by introducing new variables and equations in the usual way: for each $i = 1, \dots, m$ and $\ell = 0, \dots, \epsilon_i$ we define $U_{i,\ell} := U_i^{(\ell)}$. Making this substitution in the original system and adding the equations $U_{i,\ell} = \dot{U}_{i,\ell-1}$ ($1 \leq i \leq m, 1 \leq \ell \leq \epsilon_i$) we obtain a first-order DAE system (Σ') with a structure similar to the original system (1), but such that the order of each parametric equation $g_j = Y_j$ is 0 (in other words, the new e_j 's are zero for $j = 1, \dots, r$).

Set \mathcal{U} for the set of variables $U_{i,\ell}$ with $1 \leq i \leq m, 0 \leq \ell \leq \epsilon_i$. For each integer $p \geq 0$, denote by $\Delta' \subset k[X, \mathcal{U}]$ the differential ideal generated by the equations defining (Σ') and by A'_p the polynomial ring $\mathbb{L}[X^{[p]}, \mathcal{U}^{[p]}]$.

It is easy to see that for each $p > e$ the map

$$U_i^{(j)} \mapsto \begin{cases} U_{i,j} & \text{if } 0 \leq j \leq \epsilon_i \\ U_{i,\epsilon_i}^{(j-\epsilon_i)} & \text{if } \epsilon_i \leq j \leq p \end{cases}$$

induces a monomorphism $A_p/(\Delta \cap A_p) \hookrightarrow A'_p/(\Delta' \cap A'_p)$. Hence, for p sufficiently big, the inequality $\mathcal{H}_{\Delta}(p) \leq \mathcal{H}_{\Delta'}(p)$ holds. Since the differential dimension of both systems (1) and (Σ') is $m - r$, we conclude that $\text{ord}_{\mathbb{L}}(\Delta) \leq \text{ord}_{\mathbb{L}}(\Delta')$.

The proof finishes by applying (11) to the first-order system (Σ') , and observing that in this case the bound is simply the number of “non parametric” equations (since the new parametric ones have order zero), which is exactly $n + \sum_{i=1}^m \epsilon_i$. \square

Ritt's order bound was refined in 1959 by Greenspan [9] in the frame of difference algebra and proved in 1980 by Cohn [6] for the differential case (always for 0-differential dimensional components). In the case of system (1) for $r = m$, this bound is:

$$\text{ord}_{\mathbb{L}}(\Delta) \leq n + \sum_{i=1}^m \epsilon_i - \max\{\eta_j, 1 \leq j \leq r\} \quad (12)$$

where $\eta_j := \max\{\eta \in \mathbb{N}_0 \text{ such that } \text{ord}_{U_i} G_j^{(\eta)} \leq \epsilon_i, \forall i = 1, \dots, m\}$.

The most interesting and precise Bézout-type bound for 0-dimensional systems, which includes both (11) and Corollary 23 is discussed by Jacobi [12] (see also [30, Ch.VII, Sect. 6] and [23] for an historical survey). Even though Jacobi's bound remains conjectural for the general case, in [15] (based on [31]) and recently in [24], it has been proved to hold for a class of 0-dimensional differential ideals including those

considered in this paper. More precisely, suppose $m = r$ and set e_{ij} for the maximum of the orders of a derivative of the variable U_i occurring in g_j , putting $e_{ij} := -\infty$ if the variable U_i does not appear in g_j . Then, Jacobi's inequality states

$$\text{ord}_{\mathbb{L}}(\Delta) \leq n + \max_{\tau \in S_m} \sum_{j=1}^m e_{\tau(j)j},$$

where S_m is the permutation group of order m .

The following example shows that Jacobi's bound is reached and improves the previous ones.

Example 24 Consider the following first-order DAE system with coefficients in $k := \mathbb{Q}$:

$$\begin{cases} Y_1 = U_1 \\ Y_2 = U_2 + \dot{U}_1 \\ \vdots \\ Y_m = U_m + \dot{U}_{m-1} \end{cases}$$

with $m > 2$. Here, both upper bounds for $\text{ord}_{\mathbb{L}}(\Delta)$ given by (11) and Corollary 23 equal $m - 1$, Greenspan's bound (12) is $m - 2$, while Jacobi's upper bound is $0 = \text{ord}_{\mathbb{L}}(\Delta)$.

4.2.3 The index and an equivalent explicit ODE system

The estimation for the regularity of the Hilbert–Kolchin function allows us to give a first result concerning the number of derivatives of the input equations required to obtain an equivalent explicit ODE system in the 0-dimensional case. We will show that this number is at most the differentiation index of the system:

Theorem 25 *Let (Σ) be a DAE system as in (I) of differential dimension 0 (or equivalently, $r = m$), maximal order bounded by e and differentiation index σ . Let $\Xi = \{\xi_1^{(\ell_1)}, \dots, \xi_s^{(\ell_s)}\} \subseteq \{X^{[e-1]}, U^{[e-1]}\}$ be an algebraic transcendence basis of the fraction field of $A_{e-1}/\Delta \cap A_{e-1}$ over the field \mathbb{L} . Then:*

1. *for each $i = 1, \dots, s$ there exists a non-zero separable polynomial P_i with coefficients in the base field k , such that $P_i(Y^{[\sigma]}, \Xi, \xi_i^{(e)}) \in (F^{[\sigma]}, G^{[\sigma]}) \subset k[Y^{[\sigma]}, X^{[\sigma+1]}, U^{[\sigma+e]}]$;*
2. *set $\{\eta_{s+1}, \dots, \eta_{n+m}\} := \{X, U\} \setminus \{\xi_1, \dots, \xi_s\}$. Then, for all $i = s+1, \dots, n+m$, there exists a non-zero separable polynomial P_i with coefficients in the base field k , such that $P_i(Y^{[\sigma-1]}, \Xi, \eta_i^{(e-1)}) \in (F^{[\sigma-1]}, G^{[\sigma-1]}) \subset k[Y^{[\sigma-1]}, X^{[\sigma]}, U^{[\sigma+e-1]}]$.*

In particular, for every $\eta \in \{X, U\}$ there exists a separable non-trivial polynomial relation between $\eta^{(e)}$ and Ξ modulo Δ which can be obtained using at most σ derivations of the input equations.

Proof In order to prove the first statement, we fix a variable ξ_{i_0} . Since we are in a differential 0-dimensional situation, from the upper bound on the regularity of the

Hilbert–Kolchin function (Theorem 20), we have that the set Ξ is also an algebraic transcendence basis of the fraction field of the ring $A_e/\Delta \cap A_e$. Then, there exists a polynomial Q in $s + 1$ variables with coefficients in \mathbb{L} , such that $Q(\Xi, \xi_{i_0}^{(e)})$ belongs to the ideal $\Delta \cap A_e = \Delta_{\sigma+1} \cap A_e$ (Theorem 17). Clearly, this polynomial can be chosen so that $\frac{\partial Q}{\partial \xi_{i_0}^{(e)}} \notin \Delta$ (i.e., separable). Now, the polynomial P_{i_0} of the statement can be easily obtained from the previous Q by multiplying it by an adequate factor in $k\{Y\}$ and evaluating the superfluous variables $Y^{(l)}$ for all $l \geq \sigma + 1$ at suitably chosen elements of the base field k .

The second assertion follows similarly, but in this case we use the fact that the family $\{\eta_i^{(e-1)}, \Xi\}$ is algebraically dependent when regarded in the fraction field of $A_{e-1}/\Delta \cap A_{e-1}$ over \mathbb{L} , for all $i = s + 1, \dots, n + m$. \square

The next section is devoted to showing slightly more precise results in the same spirit of the previous Theorem.

5 Toward an equivalent explicit ODE system

In this section we make use of Theorem 25 in order to give more precise results on the number of derivations required to obtain from a system of type (1) an equivalent explicit ODE system. In the first subsection, we show that for 0-dimensional systems there exists a more accurate upper bound for the number of derivatives and the order of the involved variables by means of an adequate modification of the notion of differentiation index. In the second subsection, we consider the case of positive dimension for first-order systems. Finally, we discuss some quantitative and symbolic algorithmic aspects.

5.1 Equivalent explicit ODE systems and an alternative notion of differentiation index for zero-dimensional systems

Throughout this subsection, our input system defined in (1) is considered to be 0-dimensional and further, it is assumed that no variables X appear (i.e., $r = m$ and $n = 0$). More explicitly, we consider the following DAE system with generic second members:

$$(\Sigma) := \begin{cases} g_1(U, \dot{U}, \dots, U^{(e_1)}) = Y_1 \\ \vdots \\ g_r(U, \dot{U}, \dots, U^{(e_r)}) = Y_r \end{cases}, \quad (13)$$

where $U := U_1, \dots, U_r$ and $Y := Y_1, \dots, Y_r$ are sets of differential indeterminates, $e_j \in \mathbb{N}_0$ for every $1 \leq j \leq r$, and $e := \max\{e_1, \dots, e_r\}$. We suppose $e \geq 1$.

For each $1 \leq i \leq r$ we set ϵ_i for the maximum of the orders of the variable U_i in the polynomials g_1, \dots, g_r . Since we assume a differential 0-dimensional situation, all variables must appear in the system; that is, we have that $0 \leq \epsilon_i \leq e$ for all i .

We introduce a new set of differential variables $Z := Z_1, \dots, Z_r$ verifying:

$$U_i = Z_i^{(e-\epsilon_i)} \quad (14)$$

for $i = 1, \dots, r$. Writing the input system (Σ) in terms of the indeterminates Z we obtain a new DAE system:

$$(\tilde{\Sigma}) := \begin{cases} g_1(Z, \dot{Z}, \dots, Z^{(e)}) = Y_1 \\ \vdots \\ g_r(Z, \dot{Z}, \dots, Z^{(e)}) = Y_r \end{cases} \quad (15)$$

Note that each variable Z_i appears at order exactly e in at least one of the equation in the system $(\tilde{\Sigma})$, and e is the maximum of the orders of all variables appearing in $(\tilde{\Sigma})$.

We denote $\tilde{\sigma}$ the differentiation index of the system $(\tilde{\Sigma})$. The number $\tilde{\sigma}$ can be regarded as an invariant of the DAE system (Σ) and it represents, in some sense, a more accurate version of the global differentiation index introduced above, allowing us to distinguish the variables more precisely according to their maximal orders (c.f. the notion of *structural index* introduced in [25, 26]). We first show that the differentiation index σ of (Σ) is always an upper bound for $\tilde{\sigma}$:

Lemma 26 *The inequality $\tilde{\sigma} \leq \sigma$ holds.*

Proof Let \tilde{G} be the polynomials defining the system $(\tilde{\Sigma})$, and let $\tilde{\Delta}$ be the differential ideal generated by the differential polynomials \tilde{G} in the differential polynomial ring $\mathbb{L}\{Z\}$. For each $p \in \mathbb{N}_0$, we set \tilde{A}_p for the polynomial ring $\mathbb{L}[Z^{[p]}]$ and $\tilde{\Delta}_p$ for the ideal generated by the polynomials $\tilde{G}^{[p-1]}$ in the ring \tilde{A}_{p-1+e} (note that the maximum of the orders of the polynomials \tilde{G} equals e).

Denote by \mathcal{Z} the set of indeterminates $Z_i^{(\ell_i)}$ where $i = 1, \dots, r$ and $0 \leq \ell_i < e - \epsilon_i$ (if $e = \epsilon_i$ no derivatives of Z_i appear in the set \mathcal{Z}). The variables \mathcal{Z} will be considered as algebraic independent variables with respect to the original variables U and their successive derivatives. We also use the following standard notation: if A is a ring, $\mathfrak{A} \subset A$ an ideal and X a set of indeterminates over A , $\mathfrak{A}[X]$ denotes the ideal of $A[X]$ consisting of all the polynomials whose coefficients belong to \mathfrak{A} .

In order to prove the inequality $\tilde{\sigma} \leq \sigma$, applying Theorem 17 to $(\tilde{\Sigma})$, it suffices to show that the inclusion of ideals

$$\tilde{\Delta} \cap \tilde{A}_{e-1} \subseteq \tilde{\Delta}_\sigma \cap \tilde{A}_{e-1}$$

holds.

Let $H \in \tilde{\Delta} \cap \tilde{A}_{e-1}$. Since $H \in \tilde{\Delta}$, there exists a sufficiently big index $q \in \mathbb{N}$ such that $H \in \tilde{\Delta}_q$. So, we can suppose that H is a polynomial combination involving only variables Z of order at most q of polynomials of type $\tilde{G}_j^{(p)}$ with $p \leq q$.

Replacing $Z_i^{(e-\epsilon_i+\ell)}$ with $U_i^{(\ell)}$, for $i = 1, \dots, r$ and each non negative integer ℓ , we see that the polynomial H lies in the ideal $\Delta_q[\mathcal{Z}] \cap \mathbb{L}[U_1^{[\epsilon_1-1]}, \dots, U_r^{[\epsilon_r-1]}][\mathcal{Z}]$ (note that $\mathbb{L}[U_1^{[\epsilon_1-1]}, \dots, U_r^{[\epsilon_r-1]}][\mathcal{Z}] = \tilde{A}_{e-1}$ and that if $\epsilon_i = 0$ neither U_i nor its

successive derivatives appear in this polynomial ring). Since $\epsilon_i \leq e$ for all i , we conclude that $H \in \Delta_q[\mathcal{Z}] \cap A_{e-1}[\mathcal{Z}] = (\Delta_q \cap A_{e-1})[\mathcal{Z}]$. Now, due to identity (9) in the statement of Theorem 17, we have that $\Delta_q \cap A_{e-1} \subseteq \Delta_\sigma \cap A_{e-1}$. We infer that $H \in (\Delta_\sigma \cap A_{e-1})[\mathcal{Z}] = \Delta_\sigma[\mathcal{Z}] \cap A_{e-1}[\mathcal{Z}]$. The result follows observing that $\Delta_\sigma[\mathcal{Z}] = \widetilde{\Delta}_\sigma$ and $A_{e-1}[\mathcal{Z}] = \widetilde{A}_{e-1}$. \square

The following pendulum-type system shows that the inequality in the previous Lemma may be strict:

Example Consider the DAE system:

$$(\Sigma) := \begin{cases} U_1^{(2)} + U_1 U_3 = Y_1 \\ U_2^{(2)} + U_2 U_3 = Y_2 \\ U_1^2 + U_2^2 - 1 = Y_3 \end{cases}.$$

The corresponding matrix $\mathfrak{J}_{5,1}$ (see Definition 1) is

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline U_3 & 0 & U_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_3 & U_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2U_1 & 2U_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 3U_3 & 0 & 3U_1 & U_3 & 0 & U_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3\dot{U}_3 & 3\dot{U}_2 & 0 & U_3 & U_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6\dot{U}_1 & 6\dot{U}_2 & 0 & 2U_1 & 2U_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 6U_3^{(2)} & 0 & 6U_1^{(2)} & 4\dot{U}_3 & 0 & 4\dot{U}_1 & U_3 & 0 & U_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6U_3^{(2)} & 6U_2^{(2)} & 0 & 4\dot{U}_3 & 4\dot{U}_2 & 0 & U_3 & U_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 12U_1^{(2)} & 12U_2^{(2)} & 0 & 8\dot{U}_1 & 8\dot{U}_2 & 0 & 2U_1 & 2U_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The dimension (over the field $k\langle U \rangle$) of the corresponding kernels of $\mathfrak{J}_{i,1}^t$ are $\mu_i = i$, for $i = 0, \dots, 4$, and $\mu_5 = 4$. Therefore, the differentiation index of the system (Σ) is $\sigma = 4$.

Changing the variables as in (14): $Z_1 := U_1$, $Z_2 := U_2$ and $Z_3^{(2)} = U_3$, we transform the previous system into:

$$(\widetilde{\Sigma}) := \begin{cases} Z_1^{(2)} + Z_1 Z_3^{(2)} = Y_1 \\ Z_2^{(2)} + Z_2 Z_3^{(2)} = Y_2 \\ Z_1^2 + Z_2^2 - 1 = Y_3 \end{cases}.$$

Here, the corresponding Jacobian sub-matrix $\tilde{\mathfrak{J}}_{3,1}$ is:

$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & Z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \dot{Z}_1 & 1 & 0 & Z_1 & 0 & 0 & 0 \\ 0 & 0 & \dot{Z}_2 & 0 & 1 & Z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline Z_3^{(2)} & 0 & Z_1^{(2)} & 0 & 0 & 2\dot{Z}_1 & 1 & 0 & Z_1 \\ 0 & Z_3^{(2)} & Z_2^{(2)} & 0 & 0 & 2\dot{Z}_2 & 0 & 1 & Z_2 \\ 2Z_1 & 2Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

and the dimensions of the kernels of the matrices $\tilde{\mathfrak{J}}_{k,1}^t$ (for $k = 0, 1, 2, 3$) are $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = 2 = \mu_3$. Therefore, the differentiation index of the system $(\tilde{\Sigma})$ is $\tilde{\sigma} = 2$, which is smaller than the index $\sigma = 4$ of the original system (Σ) .

Remark 27 Applying a standard change of variables to the system (Σ) defined in (13) we obtain an equivalent first-order system $(\tilde{\Sigma})$ with differentiation index $\hat{\sigma}$. With a similar argument to that of Lemma 26 it is easy to see that $\tilde{\sigma} \leq \hat{\sigma} \leq \sigma$. In the previous example, we have that $\hat{\sigma} = 3$, which shows that the inequalities may be strict.

By means of the modified differentiation index $\tilde{\sigma}$, it is possible to give a more accurate version of Theorem 25:

Theorem 28 Let (Σ) be a DAE system as in (13). For $i = 1, \dots, r$ we denote ϵ_i the maximum of the orders of the variable U_i in the equations defining (Σ) . Then there exists a set $\mathcal{U} \subset \{U_1^{[\epsilon_1-1]}, \dots, U_r^{[\epsilon_r-1]}\}$ (here, if $\epsilon_i = 0$, no variable $U_i^{(p)}$ appears) and separable polynomials P_1, \dots, P_r such that

$$P_i(Y^{[\tilde{\sigma}]}, \mathcal{U}, U_i^{(\epsilon_i)}) \in (G_1^{[\tilde{\sigma}]}, \dots, G_r^{[\tilde{\sigma}]}).$$

In other words, there exist non-trivial (separable) polynomial relations between each $U_i^{(\epsilon_i)}$ and a fixed family of derivatives $\mathcal{U} \subset \{U_1^{[\epsilon_1-1]}, \dots, U_r^{[\epsilon_r-1]}\}$ (which is a family of order strictly lower than $\max_i \{\epsilon_i\} - 1$) that can be obtained from the first $\tilde{\sigma}$ many derivatives of the input equations.

Proof Let $(\tilde{\Sigma})$ be the DAE system obtained from (Σ) after the change of variables (14). For $j = 1, \dots, r$, denote by \tilde{G}_j the polynomial obtained from G_j after this change.

Applying Theorem 25 to $(\tilde{\Sigma})$, there exists a subset $\Xi \subset \{Z_1^{[e-1]}, \dots, Z_r^{[e-1]}\}$ such that for $i = 1, \dots, r$ there exists a separable polynomial Q_i satisfying

$$Q_i(Y^{[\tilde{\sigma}]}, \Xi, Z_i^{(e)}) \in (\tilde{G}^{[\tilde{\sigma}]}) \subset k[Y^{[\tilde{\sigma}]}, Z^{[\tilde{\sigma}+e]}]. \quad (16)$$

Fix an index i and its polynomial Q_i . Writing $U_j^{(\ell)}$ for each variable $Z_j^{(e-\epsilon_j+\ell)}$, we have that the set Ξ may be decomposed as $\Xi = \Xi_U \cup \Xi_Z$ where $\Xi_U \subset$

$\{U_1^{[\epsilon_1-1]}, \dots, U_r^{[\epsilon_r-1]}\}$ and $\Xi_Z \subset \{Z_1^{[e-\epsilon_1-1]}, \dots, Z_r^{[e-\epsilon_r-1]}\}$. With this notation, condition (16) can be written as $Q_i(Y^{[\tilde{\sigma}]}, \Xi_U, \Xi_Z, U_i^{(\epsilon_i)}) \in (G^{[\tilde{\sigma}]})$, where the ideal $(G^{[\tilde{\sigma}]})$ is considered in the polynomial ring $k[Y^{[\tilde{\sigma}]}, Z_1^{[e-\epsilon_1-1]}, \dots, Z_r^{[e-\epsilon_r-1]}, U_1^{[\tilde{\sigma}+\epsilon_1]}, \dots, U_r^{[\tilde{\sigma}+\epsilon_r]}]$.

Then, the Theorem follows from (16) taking $\mathcal{U} := \Xi_U$ and evaluating the variables $Z_j^{[e-\epsilon_j-1]}$ for $j = 1, \dots, r$ in generic elements of k . \square

We remark that Theorem 28 improves upon Theorem 25 in at least two points: first, the number of derivatives required to obtain the polynomial relations is the modified differentiation index $\tilde{\sigma}$ instead of σ . On the other hand, the order of each variable U_i in the corresponding relation P_i is the maximum order of the *single* variable U_i in the system (namely, ϵ_i) instead of the maximum of the orders of *all* variables (namely, e).

5.2 Differential transcendence bases and equivalent ODE form for generic first-order DAE systems of positive dimension

In the previous sections (more precisely, in Theorems 25 and 28) we saw how to obtain implicit equations for derivatives of low order of each variable in a DAE system after at most σ (or $\tilde{\sigma}$) many differentiations of the input equations. In both results a “square” 0-dimensional situation was assumed.

Here we show a result of the same kind in the case of *positive* differential dimension for generic *first-order* systems of type (1). Even though the straightforward idea could be applied (namely, the localization into a differential transcendence basis to reduce the problem to a 0-dimensional situation), different transcendence bases may lead to systems with different quantitative properties. For instance, the number of derivatives required to obtain an equivalent explicit ODE system may change. In this sense, there exist transcendence bases which are more adequate for our global analysis. Before stating the precise results we consider a simple example in order to illustrate this kind of phenomena.

5.2.1 Example

Consider the following 1-dimensional first-order DAE system with coefficients in $k := \mathbb{Q}$ borrowed essentially from [8, Sect. 3.4]:

$$\begin{cases} Y_1 = U_1 + \dot{U}_m \\ Y_2 = U_2 + \dot{U}_1 \\ \vdots \\ Y_{m-1} = U_{m-1} + \dot{U}_{m-2} \end{cases} \quad (17)$$

with $m > 2$.

Here, we have $n = 0$ and $r = m - 1$. The associated field \mathbb{K} is $\mathbb{Q}\langle U \rangle$ and the matrix $\mathfrak{J}_{1,0} \in \mathbb{K}^{(m-1) \times m}$ is

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \end{pmatrix},$$

which has full row rank $m - 1$. Hence, we have $\mu_1 = \mu_0 = 0$ and so, Definition 15 states that the differentiation index σ of the system (17) is equal to 0.

Following the slightly informal definition of the differentiation index given in the Introduction, the fact that $\sigma = 0$ should imply that all the derivatives \dot{U}_i can be written in terms of the variables U_1, \dots, U_m using the equations (i.e., the variables Y_1, \dots, Y_{m-1}). While this is trivially true for all the variables different from U_{m-1} ,

$$\begin{aligned} \dot{U}_m &= Y_1 - U_1 \\ \dot{U}_1 &= Y_2 - U_2 \\ &\vdots \\ \dot{U}_{m-2} &= Y_{m-1} - U_{m-1}, \end{aligned} \tag{18}$$

it is not so clear how to find a similar relation for \dot{U}_{m-1} , because it does not appear in the equations.

It is quite natural to consider U_{m-1} as a free variable and then to interpret the previous system as a 0-dimensional system over the field $k_1 := \mathbb{Q}\langle U_{m-1} \rangle$. Now, the matrix $\mathfrak{J}_{1,0}$ is the $(m - 1)$ -square matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},$$

which is clearly non singular. Hence $\mu_1 = \mu_0 = 0$ and so, the new associated differentiation index σ_1 is equal to 0. The relations (18) can be seen now as a full rewriting of the derivatives in terms of the variables (cf. Theorem 25). Furthermore, we observe that the order of the ideal associated to the system does not change by the extension of the ground field from \mathbb{Q} to $\mathbb{Q}\langle U_{m-1} \rangle$: following Remark 22, we have $\text{ord}_{\mathbb{L}\langle U_{m-1} \rangle}(\Delta) = \text{ord}_{\mathbb{L}}(\Delta) = m - 1$, since the sequence $(\mu_k)_k$ is the same in both cases.

On the other hand, if in the system (17) we take U_m as the free variable and consider the original system as a $(m - 1)$ -square 0-dimensional DAE system over the ground

differential field $k_2 := \mathbb{Q}\langle U_m \rangle$, the matrix $\mathfrak{J}_{1,0}$ is the $(m-1)$ -square matrix

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

So, $\mu_1 \neq 0 = \mu_0$ and therefore, the differentiation index σ_2 of the system (17) over k_2 is strictly positive. In fact, it is easy to see that $\sigma_2 = m-1$. Now, Theorem 25 ensures that the order of derivatives of the equations required to express the derivatives $\dot{U}_1, \dots, \dot{U}_{m-1}$ in terms of U_1, \dots, U_{m-1} is at most $m-1$. Indeed, the variables U_1, U_2, \dots, U_{m-1} can be written in terms of derivatives of the equations (i.e., derivatives of the variables Y_1, \dots, Y_{m-1}) and elements of the base field k_2 as follows:

$$\begin{aligned} U_1 &= Y_1 - \dot{U}_m \\ U_2 &= Y_2 - \dot{Y}_1 + U_m^{(2)} \\ U_3 &= Y_3 - \dot{Y}_2 + Y_1^{(2)} - U_m^{(3)} \\ &\vdots \\ U_{m-1} &= Y_{m-1} - \dot{Y}_{m-2} + \cdots + (-1)^{m-2} Y_1^{(m-2)} + (-1)^{m-1} U_m^{(m-1)}. \end{aligned}$$

Observe also that the order of the ideal is not preserved after localization in U_m , since $\text{ord}_{\mathbb{L}\langle U_m \rangle}(\Delta) = 0 \neq m-1 = \text{ord}_{\mathbb{L}}(\Delta)$.

The previous example shows two different situations arising when considering different transcendence bases. The first localization of the system (17) seems to follow the behavior of the original system more closely than the second one, since in that case the derivatives of the unknowns which are not in the transcendence bases could be written in terms of the unknowns themselves using as many derivatives of the equations as the differentiation index of the original system.

In the next two subsections we will show that for any first-order system of type (1) there are suitable differential transcendence bases which enable us to obtain relations between the remaining differential dependent variables using “few” (as many as the differentiation index) derivatives of the equations.

5.2.2 Differential transcendence basis preserving the order

In the sequel, we suppose that the input system (Σ) is of first order (or equivalently, $e = 1$). The notations correspond to those introduced in the previous sections.

We will denote $\mathcal{F}_i := \text{Frac}(A_i/(\Delta \cap A_i))$ for every $i \in \mathbb{N}_0$. As before, we will use the same notation for an element of A_i or its class in \mathcal{F}_i whenever the ring in which it is considered is clear from the context. The fact that $\mathcal{F}_i \hookrightarrow \mathcal{F}_{i+1}$ for every $i \in \mathbb{N}_0$ allows us to consider any subset of \mathcal{F}_i as a subset of \mathcal{F}_j for every $j \geq i$, which will also be done without changing notations.

Lemma 29 *Let $\mathcal{B} \subset A_i$ and let $\zeta \in A_i$ be a polynomial such that its class $\zeta \in \mathcal{F}_i$ is algebraic over $\mathbb{L}(\mathcal{B})$. Then, $\dot{\zeta} \in \mathcal{F}_{i+1}$ is algebraic over $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}})$, where $\dot{\mathcal{B}}$ denotes the set of classes of all derivatives of elements in \mathcal{B} . In particular, if $\mathbb{L}(\mathcal{B}) \hookrightarrow \mathcal{F}_i$ is an algebraic field extension, then $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}}) \hookrightarrow \mathcal{F}_{i+1}$ is also algebraic.*

Proof The result is immediate if $\zeta \in \mathcal{B}$. So, let us consider the case when $\zeta \notin \mathcal{B}$. Let $P \in \mathbb{L}(\mathcal{B})[T]$ be the minimal polynomial of ζ with respect to the field extension $\mathbb{L}(\mathcal{B}) \hookrightarrow \mathcal{F}_i$. Multiplying it by a non-zero element in $\mathbb{L}[\mathcal{B}]$, we may assume that $P \in \mathbb{L}[\mathcal{B}, T]$ and has non-zero leading coefficient.

We have $P(\mathcal{B}, \zeta) \in \Delta \cap A_i$, and so $\dot{P}(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta, \dot{\zeta}) \in \Delta \cap A_{i+1}$. Now, $\dot{P}(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta, \dot{\zeta}) = Q(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta) + \frac{\partial P}{\partial T}(\mathcal{B}, \zeta) \dot{\zeta}$ for some polynomial Q . As $\deg_T(\frac{\partial P}{\partial T}) < \deg_T(P)$, the minimality of P implies that $\frac{\partial P}{\partial T}(\mathcal{B}, \zeta) \notin \Delta$ and so, $\dot{P}(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta, T)$ is a non-zero polynomial in $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta)[T]$ annihilating $\dot{\zeta}$ in \mathcal{F}_{i+1} . This implies that $\dot{\zeta}$ is algebraic over $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta)$.

Since the field sub-extension $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}}) \hookrightarrow \mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}}, \zeta)$ of $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}}) \hookrightarrow \mathcal{F}_{i+1}$ is algebraic, we conclude that $\dot{\zeta}$ is algebraic over $\mathbb{L}(\mathcal{B} \cup \dot{\mathcal{B}})$. \square

Proposition 30 *Let $s := \text{ord}_{\mathbb{L}}(\Delta)$. There exists disjoint subsets $W := \{W_1, \dots, W_{m-r}\}$ and $\xi := \{\xi_1, \dots, \xi_s\}$ of the set $\{X_1, \dots, X_n, U_1, \dots, U_m\}$ such that $\mathcal{B}_i := \{W_1^{[i]}, \dots, W_{m-r}^{[i]}, \xi_1, \dots, \xi_s\}$ is a transcendence basis of the algebraic field extension $\mathbb{L} \hookrightarrow \mathcal{F}_i$ for every $i \in \mathbb{N}_0$. In particular, the set $\{W_1, \dots, W_{m-r}\}$ is a differential transcendence basis of the differential field extension $\mathbb{L} \hookrightarrow \text{Frac}(\mathbb{L}\{X, U\}/\Delta)$.*

Proof Let $\mathcal{B}_0 \subset \{X_1, \dots, X_n, U_1, \dots, U_m\}$ be a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_0$. Then, $\mathbb{L}(\mathcal{B}_0) \hookrightarrow \mathcal{F}_0$ is an algebraic field extension and so, due to Lemma 29, $\mathbb{L}(\mathcal{B}_0 \cup \dot{\mathcal{B}}_0) \hookrightarrow \mathcal{F}_1$ is also algebraic. Hence, $\mathcal{B}_0 \cup \dot{\mathcal{B}}_0$ contains a transcendence basis of \mathcal{F}_1 over \mathbb{L} . Since \mathcal{B}_0 is algebraically independent over \mathbb{L} , and $\text{trdeg}_{\mathbb{L}}(\mathcal{F}_1) = m - r + \text{trdeg}_{\mathbb{L}}(\mathcal{F}_0)$ (see the proof of Theorem 20), there exists a subset $\tilde{\mathcal{B}}_0 \subset \dot{\mathcal{B}}_0$ with $m - r$ elements such that $\mathcal{B}_1 := \mathcal{B}_0 \cup \tilde{\mathcal{B}}_0$ is a transcendence basis of the extension $\mathbb{L} \hookrightarrow \mathcal{F}_1$.

Let us denote W_1, \dots, W_{m-r} the variables whose first derivatives are all the elements in $\tilde{\mathcal{B}}_0$ (note that $\{W_1, \dots, W_{m-r}\} \subset \mathcal{B}_0$) and let $\{\xi_1, \dots, \xi_s\} := \mathcal{B}_0 \setminus \{W_1, \dots, W_{m-r}\}$ (observe that $\#\mathcal{B}_0 = (m - r) + s$ since the Hilbert–Kolchin function coincide with its associated polynomial as shown in Theorem 20). We will show that, for every $i \in \mathbb{N}$, the set $\mathcal{B}_i := \{W_1^{[i]}, \dots, W_{m-r}^{[i]}, \xi_1, \dots, \xi_s\}$ is a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_i$.

The case when $i = 1$ follows from our previous construction. Let us assume now that \mathcal{B}_i is a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_i$ for a fixed positive integer $i \in \mathbb{N}$. Then, by Lemma 29, $\mathbb{L}(\mathcal{B}_i \cup \dot{\mathcal{B}}_i) = \mathbb{L}(\mathcal{B}_{i+1} \cup \{\dot{\xi}_1, \dots, \dot{\xi}_s\}) \hookrightarrow \mathcal{F}_{i+1}$ is an algebraic field extension. Now, $\mathbb{L}(\mathcal{B}_{i+1}) \hookrightarrow \mathbb{L}(\mathcal{B}_{i+1} \cup \{\dot{\xi}_1, \dots, \dot{\xi}_s\})$ is an algebraic sub-extension of $\mathbb{L}(\mathcal{B}_{i+1}) \hookrightarrow \mathcal{F}_{i+1}$, since each of the elements $\dot{\xi}_j$ is obviously algebraic as an element of $\mathbb{L}(\mathcal{B}_{i+1})$ over $\mathbb{L}(\mathcal{B}_i)$. Therefore, $\mathbb{L}(\mathcal{B}_{i+1}) \hookrightarrow \mathcal{F}_{i+1}$ is an algebraic extension and, taking into account that $\text{trdeg}_{\mathbb{L}}(\mathcal{F}_{i+1})$ equals the cardinality of \mathcal{B}_{i+1} (see Theorem 20), we conclude that \mathcal{B}_{i+1} is a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_{i+1}$. \square

It is clear from the proof that the differential transcendence basis $\{W_1, \dots, W_{m-r}\}$ preserves the order of the ideal (the constant term of the Hilbert–Kolchin polynomial) after extending the ground field \mathbb{L} to $\mathbb{L}\langle W_1, \dots, W_{m-r} \rangle$.

5.2.3 An equivalent ODE system

Theorem 20 and Proposition 30 enable us to deduce the following “implicit function type” result in terms of the differentiation index σ introduced in Definition 15 (see also [8, Sect. 3]), an analogue of Theorem 25 for first-order non-square generic DAE systems:

Corollary 31 *The variables X, U can be split into three subsets $W := \{W_1, \dots, W_{m-r}\}$, $\xi := \{\xi_1, \dots, \xi_s\}$ and $\eta := \{\eta_{s+1}, \dots, \eta_{n+r}\}$ so that:*

1. W is a differential transcendence basis of $\mathbb{L} \hookrightarrow \text{Frac}(\mathbb{L}\{X, U\}/\Delta)$;
2. $W^{[j]} \cup \xi$ is an algebraic transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_j$ for all $j \in \mathbb{N}_0$;
3. for each $i = 1, \dots, s$ there exists a non-zero separable polynomial P_i with coefficients in the base field k , such that $P_i(Y^{[\sigma]}, W, \dot{W}, \xi, \xi_i) \in (F^{[\sigma]}, G^{[\sigma]}) \subset k[Y^{[\sigma]}, X^{[\sigma+1]}, U^{[\sigma+1]}]$;
4. for each $i = s+1, \dots, n+r$, there exists a non-zero separable polynomial P_i with coefficients in the base field k , such that $P_i(Y^{[\sigma-1]}, W, \xi, \eta_i) \in (F^{[\sigma-1]}, G^{[\sigma-1]}) \subset k[Y^{[\sigma-1]}, X^{[\sigma]}, U^{[\sigma]}]$.

Proof Let W and ξ be subsets of variables as in Proposition 30. Then, the first and second conditions in the statement hold. Let $\eta := \{\eta_{s+1}, \dots, \eta_{n+r}\}$ be the set of the remaining unknowns X, U (i.e., those different from the W 's and the ξ 's).

Now the proof runs *mutatis mutandis* as in Theorem 25. For instance, for the fourth item we observe that for every i , $s+1 \leq i \leq n+r$, the element $\eta_i \in \mathcal{F}_0$ is algebraic over $\mathbb{L}(W \cup \xi)$, since $W \cup \xi$ is a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_0$. Hence, there is a non-zero polynomial \hat{P}_i with coefficients in \mathbb{L} such that $\hat{P}_i(W, \xi, \eta_i) \in \Delta \cap A_0 = \Delta_\sigma \cap A_0$ (see Theorem 17). In the same way, for the third item we have that $W \cup \dot{W} \cup \xi$ is a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_1$ which ensures the existence of non-zero polynomials \hat{P}_i ($i = 1, \dots, s$) with coefficients in \mathbb{L} verifying $\hat{P}_i(W, \dot{W}, \xi, \xi_i) \in \Delta \cap A_1 = \Delta_{\sigma+1} \cap A_1$.

The polynomials P_i 's that we are looking for can be easily obtained from the previous \hat{P}_i 's, by multiplying them by adequate factors in $k\{Y\}$ and evaluating the superfluous variables $Y^{(l)}$ at suitably chosen elements of the base field k (for all $l \geq \sigma$ in the case of $i \geq s+1$ and for all $l \geq \sigma+1$ for $i \leq s$). \square

Formally, the fourth statement of the previous Corollary makes no sense if the differentiation index σ is zero. However, it admits a natural interpretation also in this case: this situation corresponds exactly to the case where the Theorem of Implicit Functions can be applied in order to write each derivative of the variables $\{X, U\} \setminus \{W\}$ in terms of the same variables and derivatives of the W 's (obviously not in a polynomial nor a rational way). So, the fourth item must be empty. More precisely:

Remark 32 Under the conditions of Corollary 31, suppose also that the differentiation index σ is zero. Following Remark 22, the Hilbert–Kolchin polynomial of Δ over \mathbb{L} is $\mathcal{H}_\Delta(T) = (m-r)(T+1) + (n+r)$ and the order of this ideal is $n+r$ (recall that we assume $e = 1$ and μ_0 is defined to be 0). So, the set η is empty, or equivalently $\xi = \{X, U\} \setminus \{W\}$. Then, there exist non-zero polynomials P_i , $i = 1, \dots, n+r$, such that $P_i(Y, W, \dot{W}, \xi, \xi_i) \in (F, G) \subset k[Y, X, \dot{X}, U, \dot{U}]$.

5.3 Quantitative and algorithmic aspects

5.3.1 Degree bounds of the implicit equations

The non-zero polynomials P_i of Theorems 25, and 28 and Corollary 31, are not uniquely determined without additional requirements (as minimality of order and degree, irreducibility, etc.). However, the conditions stated in those results allow us to choose a family of such polynomials that can be regarded as eliminating polynomials of suitable algebraic-geometric situations, which enables us to estimate their degrees.

In order to illustrate these facts consider for instance the situation of Corollary 31: let \bar{k} be a fixed algebraic closure of the ground field k . For each $N \in \mathbb{N}$ we denote by \mathbb{A}^N the affine space \bar{k}^N equipped with the Zariski topology. Set $N_0 := (r+n+m)\sigma + n+m$ and $N_1 := (r+n+m)(\sigma+1) + n+m$ and let $\mathbb{V}_0 \subset \mathbb{A}^{N_0}$ and $\mathbb{V}_1 \subset \mathbb{A}^{N_1}$ be the algebraic varieties defined by the ideals $(F^{[\sigma-1]}, G^{[\sigma-1]}) \subset k[Y^{[\sigma-1]}, X^{[\sigma]}, U^{[\sigma]}]$ and $(F^{[\sigma]}, G^{[\sigma]}) \subset k[Y^{[\sigma]}, X^{[\sigma+1]}, U^{[\sigma+1]}]$, respectively, that is:

$$\mathbb{V}_0 := \{F^{[\sigma-1]} = 0, G^{[\sigma-1]} = 0\} \quad \text{and} \quad \mathbb{V}_1 := \{F^{[\sigma]} = 0, G^{[\sigma]} = 0\}. \quad (19)$$

Note that both varieties are irreducible complete intersection and their dimensions are $m(\sigma+1) + n$ and $m(\sigma+2) + n$ respectively.

Let $W := \{W_1, \dots, W_{m-r}\}$, $\xi := \{\xi_1, \dots, \xi_s\}$ and $\eta := \{\eta_{s+1}, \dots, \eta_{n+r}\}$ be a partition of the set of variables $\{X, U\}$ as in Corollary 31. We define linear projections θ_i ($i = 1, \dots, s$) and π_i ($i = s+1, \dots, n+r$) as follows:

$$\begin{aligned} \theta_i : \mathbb{V}_1 &\rightarrow \mathbb{A}^{r(\sigma-1)+2m+s+1}, \quad \theta_i(y^{[\sigma]}, x^{[\sigma+1]}, u^{[\sigma+1]}) := (y^{[\sigma]}, w, \dot{w}, \xi, \dot{\xi}_i); \\ \pi_i : \mathbb{V}_0 &\rightarrow \mathbb{A}^{r\sigma+m-r+s+1}, \quad \pi_i(y^{[\sigma-1]}, x^{[\sigma]}, u^{[\sigma]}) := (y^{[\sigma-1]}, w, \xi, \eta_i). \end{aligned}$$

From Proposition 30, we deduce that the set $\{Y^{[\sigma]}, W, \dot{W}, \xi\}$ (resp. $\{Y^{[\sigma-1]}, W, \xi\}$) is algebraically independent in the fraction field $k(\mathbb{V}_1)$ (resp. $k(\mathbb{V}_0)$) over k . On the other hand, due to Corollary 31, the set $\{Y^{[\sigma]}, W, \dot{W}, \xi, \dot{\xi}_i\}$ (resp. $\{Y^{[\sigma-1]}, W, \xi, \eta_i\}$) is algebraically dependent. Thus, the closure of the image of the map θ_i (resp. π_i) is a k -definable irreducible hypersurface in the corresponding ambient space and so, it can be defined by a single polynomial lying in the ideal $(F^{[\sigma]}, G^{[\sigma]})$ (resp. $(F^{[\sigma-1]}, G^{[\sigma-1]})$) whose total degree (see [10, Lemma 2]) is bounded by $\deg \mathbb{V}_1$ (resp. $\deg \mathbb{V}_0$).

Applying the Bezout Inequality (see [10, Theorem 1]) to obtain an upper bound for $\deg \mathbb{V}_1$ (resp. $\deg \mathbb{V}_0$), we conclude that there exist polynomials P_i , for $i = 1, \dots, s$ (resp. $i = s+1, \dots, n+r$) meeting the conditions of Corollary 31 whose total degrees can be bounded by $\deg P_i \leq d^{(\sigma+1)(n+r)}$ (resp. $\deg P_i \leq d^{\sigma(n+r)}$), where d denotes an upper bound for the total degree of the polynomials in the input system.

A similar result can be obtained in the case of Theorem 25 by considering the linear projections

$$\begin{aligned} (y^{[\sigma]}, x^{[\sigma+1]}, u^{[\sigma+e]}) &\mapsto (y^{[\sigma]}, \Xi, \xi_i^{(e)}), \\ (y^{[\sigma-1]}, x^{[\sigma]}, u^{[\sigma+e-1]}) &\mapsto (y^{[\sigma-1]}, \Xi, \eta_i^{(e-1)}), \end{aligned}$$

whose domains are the irreducible varieties \mathbb{V}_1 and \mathbb{V}_0 defined in (19), respectively.

Analogously, the projections which allow us to estimate the degrees of the polynomials in Theorem 28 are

$$(y^{[\tilde{\sigma}]}, u^{[\tilde{\sigma}+e]}) \mapsto (y^{[\tilde{\sigma}]}, \mathcal{U}, u_i^{(\epsilon_i)}),$$

for $i = 1, \dots, r$, all of them defined over the irreducible variety

$$\tilde{\mathbb{V}}_1 := \{G^{[\tilde{\sigma}]} = 0\}. \quad (20)$$

5.3.2 Algorithmic issues

This section presents algorithmic procedures for the computation of the following objects:

- (1) The differentiation index of the system (1).
- (2) A differential transcendence basis of a first-order system of type (1) preserving the order.
- (3) The implicit relations in separated variables given by Theorems 25 and 28, and Corollary 31.

For algorithmic reasons, we will assume that our base field k is either \mathbb{Q} or $\mathbb{Q}(t)$ with the usual derivation (in the case when $k = \mathbb{Q}(t)$, the input polynomials will be assumed to have coefficients in $\mathbb{Q}[t]$).

We start with a brief description of the computational model.

Basic algorithmic notions

The objects our algorithms deal with are multivariate polynomials which will be encoded by means of *straight-line programs* (i.e., arithmetic circuits which enable us to evaluate them at any given point). The number of instructions in the program is called the *length* of the straight-line program. For a brief description of the algorithmic model and the data structure we will use, we refer the reader to [7, Sect. 2.2] and the references therein.

The basic subroutine we use is a polynomial-time probabilistic procedure for the computation of the rank of a matrix with polynomial entries. Roughly speaking, the problem is reduced to the computation of the rank of a matrix with entries in \mathbb{Q} by randomly choosing integer values for the variables and evaluating all matrix entries at them (see [7, Lemma 24] for the error probability analysis).

Our algorithms take as input a straight-line program of length L encoding the polynomials $f_1, \dots, f_n, g_1, \dots, g_r$ appearing in the system (1). However, the intermediate computations involve not only these polynomials but also their successive derivatives and so, straight-line programs for these derivatives are needed as well. The existence of “short” straight-line programs encoding them is ensured by [7, Lemma 21] (see also [22, Sect. 5.2]).

We recall that all our computations should be performed over the differential field \mathbb{K} with the derivation induced by $\dot{X}_i = f_i$ ($i = 1, \dots, r$) but, since the number of derivatives involved in each computation is controlled, they can be achieved over

the polynomial rings $k[X, U^{[l]}]$ for adequate choices of l . We assume that an upper bound $d \in \mathbb{N}$ for the degrees of the polynomials $f_1, \dots, f_n, g_1, \dots, g_r$ is known. In order to estimate complexities and error probabilities, we also need upper bounds for the degrees of the polynomials obtained by successive differentiation of the input polynomials (i.e., those giving the isomorphism between $\text{Frac}(k\{Y, X, U\}/\Delta)$ and \mathbb{K}), which can be found in [7, Notation 6 and Remark 25].

Computation of the differentiation index

According to Definition 15 and Notation 13, we have $\sigma := \min\{k \in \mathbb{N}_0 / \mu_k = \mu_{k+1}\}$. This minimum is obtained by computing and comparing the ranks of the matrices $\mathbb{J}_{k,e-1}$ (which are computed over polynomial rings) for successive values of $k \in \mathbb{N}$. The algorithm finishes, since we have an a priori upper bound for σ [see (7)].

The previous computation of σ can be achieved with error probability bounded by ε within a complexity which is polynomial in n, m, r , and linear in $\log d, \log \varepsilon$ and L (recall that d and L are upper bounds for the degrees and the size of the straight-line program representation of the input polynomials). Note that this algorithm can also be applied for the computation of the modified index $\tilde{\sigma}$ introduced in Sect. 5.1 with the same complexity bounds.

Computation of a differential transcendence basis for first-order systems

The algorithmic computation of a differential transcendence basis of the differential field extension $\mathbb{L} \hookrightarrow \text{Frac}(\mathbb{L}\{X, U\}/\Delta)$ preserving the order after localization follows the procedure underlying the proof of Proposition 30:

- compute a transcendence basis $\mathcal{B}_0 \subset \{X_1, \dots, X_n, U_1, \dots, U_m\}$ of $\mathbb{L} \hookrightarrow \mathcal{F}_0$;
- choose a subset $\tilde{\mathcal{B}}_0 \subset \mathcal{B}_0$ with $m - r$ elements such that $\mathcal{B}_1 := \mathcal{B}_0 \cup \tilde{\mathcal{B}}_0$ is a transcendence basis of $\mathbb{L} \hookrightarrow \mathcal{F}_1$.

Then, the variables W_1, \dots, W_{m-r} whose derivatives lie in $\tilde{\mathcal{B}}_0$ form a differential transcendence basis of $\mathbb{L} \hookrightarrow \text{Frac}(\mathbb{L}\{X, U\}/\Delta)$ with the required property.

The set \mathcal{B}_0 is constructed recursively by adding one variable at a time. In order to determine whether a subset of variables in \mathcal{F}_0 (resp. \mathcal{F}_1) is transcendental over the field \mathbb{L} , we use the fact that $\mathcal{F}_0 \hookrightarrow A_\sigma/\Delta_\sigma$ (resp. $\mathcal{F}_1 \hookrightarrow A_{\sigma+1}/\Delta_{\sigma+1}$). Thus, the problem amounts to determine whether a subset of variables in a quotient of a polynomial ring by a prime ideal is transcendental over the base field, which is done by applying the Jacobian criterion from commutative algebra (see [7, Lemma 19]).

Computation of the implicit equations

As we have shown in Sect. 5.3.1, the polynomials P_i 's of Theorems 25 and 28, and Corollary 31 can be interpreted as eliminating polynomials of the image of the algebraic varieties defined in (19) and (20) under suitable linear projections. Therefore, they can be computed by means of an algorithm based on standard algebraic elimination procedures (see [11, 34]). For simplicity, we assume $k := \mathbb{Q}$. The following complexity result can be obtained:

Proposition 33 *There is a probabilistic algorithm which computes the polynomials P_i of Theorems 25 and 28, and Corollary 31 with error probability bounded by ε , with $0 < \varepsilon < 1$, and within complexity $O(\log(1/\varepsilon)d^2L) \Pi(n + m, \max_i \deg \mathbb{V}_i)$, where Π is a suitable two-variate universal polynomial.* \square

We omit the proof of this result in the present article, since it is rather long and technical, and follows closely the proof of [7, Proposition 46].

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