Dimer models, Glauber dynamics and height fluctuations

F. Toninelli, CNRS and Université Lyon 1

SPA, Buenos Aires\textsuperscript{1}

\textsuperscript{1}Many thanks to Clay Math Institute & French Embassy in Buenos Aires for financial support
Plan

- Dimer models (perfect matchings) and height function
- Random perfect matchings
- Macroscopic shape and Gaussian fluctuations
- Glauber dynamics: approaching the macroscopic shape
- Beyond the solvable case: interacting dimers (and the GFF)
Perfect matchings of bipartite planar graphs
Perfect matchings of bipartite planar graphs
Height function

\[ h(f') - h(f) = \sum_{e \in C_{f \rightarrow f'}} \sigma_e (1_{e \in M} - 1/4) \]

where \( \sigma_e = +1/-1 \) if \( e \) crossed with white on the right/left.
Height function:

\[ h(f') - h(f) = \sum_{e \in C_{f \rightarrow f'}} \sigma_e (1_{e \in M} - 1/4) \]

where \( \sigma_e = +1/ -1 \) if \( e \) crossed with white on the right/left.
Definition is path-independent. Crucial: graph is bipartite.
A 2D statistical mechanics model

If \( \Lambda \) is a large domain, e.g. the \( 2L \times 2L \) square/torus, many (\( \approx \exp(cL^2) \)) perfect matchings exist. Call \( \langle \cdot \rangle_\Lambda \) the uniform measure.
A 2D statistical mechanics model

If $\Lambda$ is a large domain, e.g. the $2L \times 2L$ square/torus, many ($\approx \exp(cL^2)$) perfect matchings exist.

Call $\langle \cdot \rangle_\Lambda$ the uniform measure.

Observe:

- By symmetry, on the torus, $\langle 1_{e \in M} \rangle_\Lambda = 1/4$ for every $e$, so that $\langle h(f) - h(f') \rangle_\Lambda = 0$.
- Dimers do not interact (except for hard-core constraint).
A 2D statistical mechanics model

If $\Lambda$ is a large domain, e.g. the $2L \times 2L$ square/torus, many ($\approx \exp(cL^2)$) perfect matchings exist.

Call $\langle \cdot \rangle_\Lambda$ the uniform measure.

Observe:

- By symmetry, on the torus, $\langle 1_{e \in M} \rangle_\Lambda = 1/4$ for every $e$, so that $\langle h(f) - h(f') \rangle_\Lambda = 0$.

- Dimers do not interact (except for hard-core constraint).

Somewhat analogous to the critical Ising model: power-law decay of correlations, conformal invariance...
Kasteleyn theory (’61)

Partition functions and correlations given by determinants

Define a $|\Lambda|/2 \times |\Lambda|/2$ matrix $K$, indexed by white/black sites, as $K(x, x + (1, 0)) = 1$, $K(x, x + (0, 1)) = i$ and zero otherwise. Then,

$$Z_\Lambda = \#\{\text{perfect matchings of } \Lambda\} = \det(K)$$
Similarly, if \( e_1 = (b_1, w_1), e_2 = (b_2, w_2) \) are two bonds (\( b_i \) black site, neighboring white site \( w_i \)), then

\[
\langle 1_{e_1 \in M} 1_{e_2 \in M} \rangle_\Lambda = K(e_1)K(e_2) \det(R)
\]

with \( R \) the 2 \( \times \) 2 matrix with \( R_{ij} = K^{-1}(b_i, w_j) \).

Analogous expression for multi-dimer correlations
Macroscopic shape

[Cohn-Kenyon-Propp, JAMS 2001]

Scaling limit: lattice step $1/L \to 0$, domain $U \equiv \Lambda/L$ of size $O(1)$, boundary height $\varphi$ on $\partial U$.

**Theorem** The height function concentrates with high probability around a deterministic shape $\Phi : U \mapsto \mathbb{R}$. This minimizes a surface tension functional

$$\Gamma(\phi) = \int_U F(\nabla \phi) d^2 u$$

with $\phi_{\partial U} = \varphi$. $F$ is convex and explicitly known.
Macroscopic shape

[Cohn-Kenyon-Propp, JAMS 2001]

Scaling limit: lattice step $1/L \to 0$, domain $U \equiv \Lambda/L$ of size $O(1)$, boundary height $\varphi$ on $\partial U$.

**Theorem** The height function concentrates with high probability around a deterministic shape $\Phi : U \mapsto \mathbb{R}$. This minimizes a surface tension functional

$$\Gamma(\phi) = \int_U F(\nabla \phi) d^2 u$$

with $\phi_{\partial U} = \varphi$. $F$ is convex and explicitly known.

According to the boundary height, the minimizer $\Phi$ can be either $C^\infty$ or have “facets”.
An example with facets: arctic circle

[Cohn, Larsen, Propp ’98]
Fluctuations

Take periodic b.c.

- Dimer-dimer correlations decay slowly:

  \[
  \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda} \approx |e - e'|^{-2}
  \]
Fluctuations

Take periodic b.c.

- Dimer-dimer correlations decay slowly:

\[
\lim_{\Lambda \to \mathbb{Z}^2} \left\langle 1_{e \in M} ; 1_{e' \in M} \right\rangle_{\Lambda} \approx |e - e'|^{-2}
\]

- Height fluctuations grow logarithmically:

\[
\lim_{\Lambda \to \mathbb{Z}^2} \text{Var}_{\Lambda}(h(f) - h(f')) \sim \frac{1}{\pi^2} \log |f - f'| \quad \text{as} \quad |f - f'| \to \infty
\]

(see Kenyon-Okounkov-Sheffield for general bipartite graphs)
Fluctuations

Take periodic b.c.

- Dimer-dimer correlations decay slowly:

  \[
  \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda} \approx |e - e'|^{-2}
  \]

- Height fluctuations grow logarithmically:

  \[
  \lim_{\Lambda \to \mathbb{Z}^2} \text{Var}_{\Lambda}(h(f) - h(f')) \sim \frac{1}{\pi^2} \log |f - f'| \quad \text{as} \quad |f - f'| \to \infty
  \]

  (see Kenyon-Okounkov-Sheffield for general bipartite graphs)

- The height field is asymptotically Gaussian: for \( m \geq 3 \), the \( m^{th} \) cumulant of \( h(f) - h(f') \) is

  \[
  \langle h(f) - h(f'); m \rangle_{\Lambda} = o(\text{Var}_{\Lambda}(h(f) - h(f'))^{m/2}).
  \]
Glauber (stochastic) dynamics

Defines a continuous-time Markov chain

The unique stationary (reversible) measure is the uniform one, \( \langle \cdot \rangle_\Lambda \).

As \( t \to \infty \), convergence to \( \langle \cdot \rangle_\Lambda \): a way to sample random tilings.

Corresponds to zero-temperature dynamics of 3D Ising model.

Dynamics for Monotone surfaces (or lozenge tilings)

Theorem (Caputo, Martinelli, F. T. 2011)

\[ T_{mix} = \frac{1}{2} L^2 \left( \log L \right)^{\frac{1}{2}} \]

(Previously known bound was just \( T_{mix} \sim const L^4 \left( \log L \right)^2 \).)

mercoledì 11 maggio 2011
Glauber (stochastic) dynamics

Defines a continuous-time Markov chain

The unique stationary (reversible) measure is the uniform one, $\langle \cdot \rangle_\Lambda$. As $t \to \infty$, convergence to $\langle \cdot \rangle_\Lambda$: a way to sample random tilings.
Glauber (stochastic) dynamics

Defines a continuous-time Markov chain

The unique stationary (reversible) measure is the uniform one, $\langle \cdot \rangle_\Lambda$. As $t \to \infty$, convergence to $\langle \cdot \rangle_\Lambda$: a way to sample random tilings.

Corresponds to zero-temperature dynamics of 3D Ising model
Natural mathematical questions

Speed of convergence to equilibrium, mixing time, etc
[Theoretical computer science motivation: running time of algorithm, counting # of tilings]

Deterministic interface evolution on diffusive time-scales?
[MathPhys motivation: motion of interfaces. Similar questions e.g. for Ising interfaces at low temperature]

Influence of singularities of $\Phi$ on the dynamics?
Three types of particles (lozenges) exchanging randomly their positions.
Analog with Simple Exclusion Process suggests $T_{rel} \approx L^2$. 
Heuristics: diffusive scaling and hydrodynamic limit

Three types of particles (lozenges) exchanging randomly their positions.
Analogy with Simple Exclusion Process suggests $T_{rel} \approx L^2$.

After diffusive time rescaling (set $\tau = t/L^2$) expected convergence to deterministic evolution (hydrodynamic limit).

$$\partial_t \phi = \mu(\nabla \phi) \text{div}(\nabla F \circ \nabla \phi)$$

Idea: system decreases surface free energy $\Gamma(\phi) = \int F(\nabla \phi)$. 
“Rapid mixing”

Theorem: [Luby-Randall-Sinclair, Wilson, Randall-Tetali (theoretical computer science community)]

The mixing time grows at most as a polynomial of $L$, uniformly in the boundary height.

Based on “path coupling methods”; at best, these can give $T_{\text{mix}} \leq cL^{4+\epsilon}$. 

An almost optimal result

$h_t(\cdot)$: height function of the time-evolving discrete interface.

**Theorem:** [B. Laslier, F. T. ’13] Assume the macroscopic shape $\Phi$ is $C^\infty$. With probability close to 1,

\[
\|h_t(\cdot) - \Phi(\cdot)\|_\infty = o(1) \quad t \geq L^{2+\epsilon}
\]

\[
\|h_t(\cdot) - h_0(\cdot)\|_\infty = o(1) \quad t \ll L^2
\]

Uses refined results on equilibrium height fluctuations (L. Petrov).

If $\Phi$ is affine, see [Caputo, Martinelli, Toninelli ’12 and Laslier, Toninelli ’14]: mixing time of order $L^{2+o(1)}$. 
An almost optimal result

$h_t(\cdot)$: height function of the time-evolving discrete interface.

**Theorem:** [B. Laslier, F. T. ’13] Assume the macroscopic shape $\Phi$ is $C^\infty$. With probability close to 1,

\[
\|h_t(\cdot) - \Phi(\cdot)\|_\infty = o(1) \quad t \geq L^{2+\epsilon} \\
\|h_t(\cdot) - h_0(\cdot)\|_\infty = o(1) \quad t \ll L^2
\]

Uses refined results on equilibrium height fluctuations (L. Petrov)
An almost optimal result

$h_t(\cdot)$: height function of the time-evolving discrete interface.

**Theorem:** [B. Laslier, F. T. ’13] Assume the macroscopic shape $\Phi$ is $C^\infty$. With probability close to 1,

$$\|h_t(\cdot) - \Phi(\cdot)\|_\infty = o(1) \quad t \geq L^{2+\epsilon}$$

$$\|h_t(\cdot) - h_0(\cdot)\|_\infty = o(1) \quad t \ll L^2$$

Uses refined results on equilibrium height fluctuations (L. Petrov)

If $\Phi$ is affine, see [Caputo, Martinelli, Toninelli ’12 and Laslier, Toninelli ’14]: mixing time of order $L^{2+o(1)}$. 
Beyond the solvable case: interacting dimers

Associate an energy $\lambda \in \mathbb{R}$ to adjacent dimers:

I.e., with $N(M)$ the number of adjacent pairs of dimers in $M$,

$$\langle \cdot \rangle_{\Lambda,\lambda} = \frac{\sum_M e^{\lambda N(M)}}{Z_{\Lambda,\lambda}}$$

Beyond the solvable case: interacting dimers

Theorem [Giuliani, Mastropietro, T. 2014] If $|\lambda| \leq \lambda_0$ then:

- Fluctuations still grow logarithmically:
  $$\lim_{\Lambda \to \mathbb{Z}^2} \text{Var}_{\Lambda,\lambda}(h(f) - h(f')) \quad |f - f'| \to \infty \quad \frac{K(\lambda)}{\pi^2} \log |f - f'| + O(1)$$
  
  with $K(\cdot)$ analytic and $K(0) = 1$;

- for $m \geq 3$, the $m^{th}$ cumulant of $h(f) - h(f')$ is bounded:
  $$\sup_{f,f'} \lim_{\Lambda \to \mathbb{Z}^2} \langle h(f) - h(f'); m \rangle_{\Lambda,\lambda} \leq C(m).$$
Beyond the solvable case: interacting dimers

**Theorem** [Giuliani, Mastropietro, T. 2014] If $|\lambda| \leq \lambda_0$ then:

- **Convergence to Gaussian Free Field:** if $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \varphi(x)dx = 0$ then, as $\epsilon \to 0$,
  \[
  \epsilon^2 \sum_f \varphi(\epsilon f) h(f) \to \int_{\mathbb{R}^2} \varphi(x)X(x)dx
  \]

  with $X$ the Gaussian Free Field of covariance
  \[
  -\frac{K(\lambda)}{2\pi^2} \log |x - y|.
  \]
Universality or not? dimer correlations

Back to the non-interacting case. From Kasteleyn’s solution,

\[ \sigma_e \sigma_{e'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_e \in M; 1_{e'} \in M \rangle_{\Lambda, \lambda=0} \]

\[ = -\frac{1}{2\pi^2} \Re \left[ \frac{1}{(z_e - z_{e'})^2} \Delta z_e \Delta z_{e'} \frac{1}{(z_e - z_{e'})^2} \right] \]

\[ + \text{Osc}(z_e, z_{e'}) \frac{1}{|z_e - z_{e'}|^2} + O(|z_e - z_{e'}|^{-3}). \]
Universality or not? dimer correlations

Back to the non-interacting case. From Kasteleyn’s solution,

$$\sigma_e \sigma_{e'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda, \lambda=0}$$

$$= -\frac{1}{2\pi^2} \Re \left[ \Delta z_e \Delta z_{e'} \frac{1}{(z_e - z_{e'})^2} \right]$$

$$+ \text{Osc}(z_e, z_{e'}) \frac{1}{|z_e - z_{e'}|^2} + O(|z_e - z_{e'}|^{-3}).$$

$$\sum_{e \in C_{f \to f'}, e' \in C'_{f \to f'}} A_{e,e'} \sim -\frac{1}{2\pi^2} \Re \int_{f}^{f'} \frac{dzdz'}{(z - z')^2} = \frac{1}{\pi^2} \log |f - f'|$$
Universality or not? dimer correlations

If $\lambda$ is small, then [see also Falco, Phys Rev E 2013]

$$
\sigma_e \sigma_{e'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda, \lambda} = \frac{K(\lambda)}{2\pi^2} \Re \left[ \Delta z_e \Delta z_{e'} \frac{1}{(z_e - z_{e'})^2} \right] + O\left(\frac{1}{|z_e - z_{e'}|^2 + \eta(\lambda)}\right) + O(\lambda).
$$

with $K(\cdot), \eta(\cdot)$ analytic and $K(0) = 1, \eta(0) = 0.$
Universality or not? dimer correlations

If $\lambda$ is small, then [see also Falco, Phys Rev E 2013]

$$\sigma_e \sigma_{e'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda,\lambda}$$

$$= -\frac{K(\lambda)}{2\pi^2} \Re \left[ \Delta z_e \Delta z_{e'} \frac{1}{(z_e - z_{e'})^2} \right]$$

$$+ \text{Osc}(z_e, z_{e'}) \frac{1}{|z_e - z_{e'}|^{2+\eta(\lambda)}} + O(|z_e - z_{e'}|^{-3+O(\lambda)}).$$

with $K(\cdot)$, $\eta(\cdot)$ analytic and $K(0) = 1$, $\eta(0) = 0$.

- in the main term the critical exponent remains 2
- in the oscillating term it changes to $2 + \eta(\lambda)$ (non-universal).
A Renormalization Group approach

Algebraic identity: Determinants can be written as “Grassmann Gaussian integrals”, or “Lattice free fermions”.
A Renormalization Group approach

Algebraic identity: Determinants can be written as “Grassmann Gaussian integrals”, or “Lattice free fermions”.

To each lattice site, associate Grassmann variable $\psi_x$.

Anticommutation rule: $\psi_x \psi_y = -\psi_y \psi_x$

Then, with $(\psi, K\psi) = \sum_{b,w} \psi_w K(w, b) \psi_b$,

$$\det(K) = \int \prod_x d\psi_x e^{-\frac{1}{2} (\psi, K\psi)}$$

and

$$K^{-1}(b, w) = \frac{1}{\det(K)} \int \prod_x d\psi_x e^{-\frac{1}{2} (\psi, K\psi)} \psi_b \psi_w.$$
A Renormalization Group approach

Similarly, the partition function of the interacting model is written as

\[ Z_{\Lambda, \lambda} = \frac{1}{\det(K)} \int \prod d\psi_x \exp\left( -\frac{1}{2} (\psi, K\psi) + \lambda V(\psi) \right) \]

with \( V \) a non-quadratic polynomial of the \( \psi \).
A Renormalization Group approach

Similarly, the partition function of the interacting model is written as

$$Z_{\Lambda,\lambda} = \frac{1}{\det(K)} \int \prod d\psi_x \exp \left( -\frac{1}{2} (\psi, K\psi) + \lambda V(\psi) \right)$$

with $V$ a non-quadratic polynomial of the $\psi$.

Naive power series in $\lambda$ diverges. Constructive Renormalization Group methods (Benfatto, Brydges, Gallavotti, Gawedzki, Kupiainen, Mastropietro, Rivasseau, ... ≥ 1980’s) allow to obtain convergent expansion for correlation functions and to study large-distance behavior.
Open problems

- Effect of facets on Glauber dynamics?
Open problems

• Effect of facets on Glauber dynamics?
• Kenyon '00 proved conformal invariance of height moments e.g.

\[ g_D(x, y) = \lim_{L \to \infty} \langle (h_{xL} - \langle h_{xL} \rangle_\Lambda)(h_{yL} - \langle h_{yL} \rangle_\Lambda) \rangle_\Lambda \]

(lattice spacing \( 1/L \to 0 \), \( \Lambda \subset (\mathbb{Z}/L)^2 \) suitable discretization of domain \( \mathcal{D} \subset \mathbb{C} \) and \( x_L, y_L \) tend to distinct points \( x, y \))

Conformal invariance for the interacting dimer model?
Thank you!