Dimer models, Glauber dynamics and height fluctuations

F. Toninelli, CNRS and Université Lyon 1

SPA, Buenos Aires[1]

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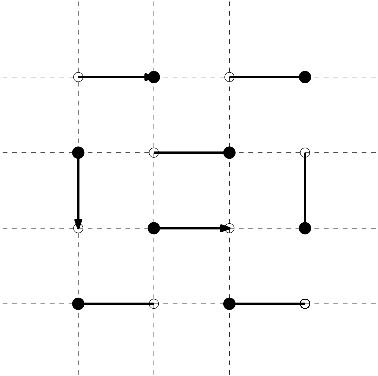
Plan

- Dimer models (parfect matchings) and height function
- Random perfect matchings
- Macroscopic shape and Gaussian fluctuations
- Glauber dynamics: approaching the macroscopic shape
- Beyond the solvable case: interacting dimers (and the GFF)

Perfect matchings of bipartite planar graphs



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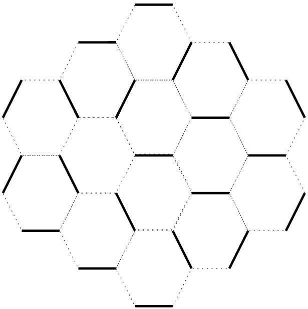
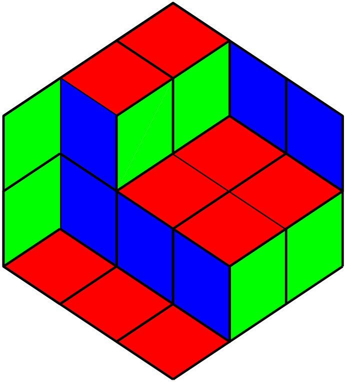
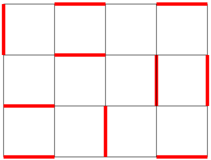
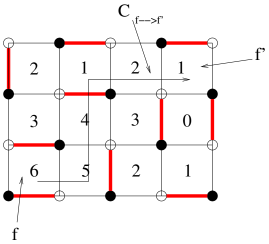




Height function





Height function:

$$h(f') - h(f) = \sum_{e \in C_{f \to f'}} \sigma_e (1_{e \in M} - 1/4)$$

where $\sigma_e = +1/-1$ if e crossed with white on the right/left.

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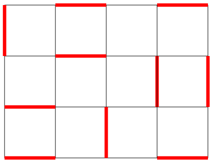

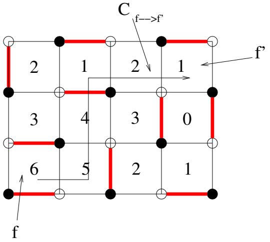

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Definition is path-independent. Crucial: graph is bipartite.

A 2D statistical mechanics model

If Λ is a large domain, e.g. the $2L \times 2L$ square/torus, many ($\approx \exp(cL^2)$) perfect matchings exist.

Call $\langle \cdot \rangle_\Lambda$ the uniform measure.

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Somewhat analogous to the critical Ising model: power-law decay of correlations, conformal invariance...

Kasteleyn theory ('61)

Partition functions and correlations given by determinants

Define a $|\Lambda|/2 \times |\Lambda|/2$ matrix K, indexed by white/black sites, as $K(x, x+(1,0)) = 1, K(x, x+(0,1)) = i$ and zero otherwise. Then,

$$Z_\Lambda = \#\{\text{perfect matchings of } \Lambda\} = \det(K)$$

Kasteleyn theory and determinantal representation

Similarly, if $e_1 = (b_1, w_1), e_2 = (b_2, w_2)$ are two bonds (b_i black site, neighboring white site w_i), then

$$\langle 1_{e_1 \in M} 1_{e_2 \in M} \rangle_\Lambda = K(e_1) K(e_2) \det(R)$$

with R the 2×2 matrix with $R_{ij} = K^{-1}(b_i, w_j)$.

Analogous expression for multi-dimer correlations

Macroscopic shape

[Cohn-Kenyon-Propp, JAMS 2001]

Scaling limit: lattice step $1/L \to 0$, domain $U \equiv \Lambda/L$ of size $O(1)$, boundary height φ on ∂U.

Theorem The height function concentrates with high probability around a deterministic shape $\Phi : U \mapsto \mathbb{R}$. This minimizes a surface tension functional

$$\Gamma(\phi) = \int_U F(\nabla \phi) d^2 u$$

with $\phi_{\partial U} = \varphi$. F is convex and explicitly known.

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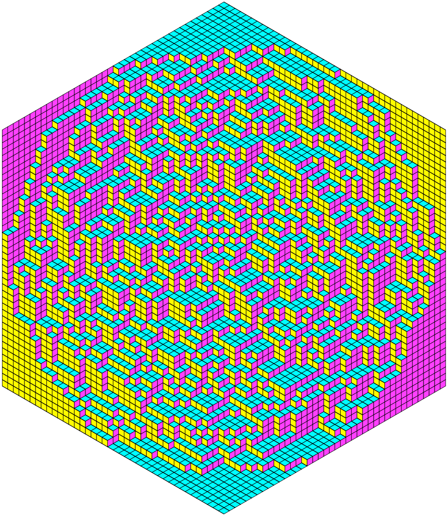
According to the boundary height, the minimizer Φ can be either C^∞ or have "facets".

An example with facets: arctic circle

[Cohn, Larsen, Propp '98]



Fluctuations

Take periodic b.c.

- Dimer-dimer correlations decay slowly:

$$\lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_\Lambda \approx |e - e'|^{-2}$$

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$$\lim_{\Lambda \to \mathbb{Z}^2} Var_\Lambda(h(f) - h(f')) \sim \frac{1}{\pi^2} \log |f - f'| \quad \text{as} \quad |f - f'| \to \infty$$

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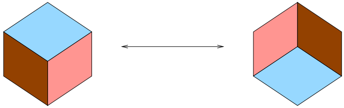
- the height field is asymptotically Gaussian: for $m\geq 3$, the m^{th} cumulant of $h(f)-h(f')$ is

$$\langle h(f)-h(f'); m\rangle_\Lambda = o(Var_\Lambda(h(f)-h(f'))^{m/2}).$$

Glauber (stochastic) dynamics



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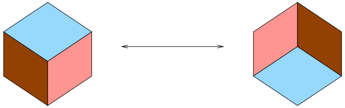

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The unique stationary (reversible) measure is the uniform one, $\langle \cdot \rangle_\Lambda$.
As $t \to \infty$, convergence to $\langle \cdot \rangle_\Lambda$: a way to sample random tilings.

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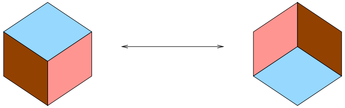

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Corresponds to zero-temperature dynamics of 3D Ising model

Natural mathematical questions

Speed of convergence to equilibrium, mixing time, etc
[Theoretical computer science motivation: running time of algorithm, counting # of tilings]

Deterministic interface evolution on diffusive time-scales?
[MathPhys motivation: motion of interfaces. Similar questions e.g. for Ising interfaces at low temperature]

Influence of singularities of Φ on the dynamics?

Heuristics: diffusive scaling and hydrodynamic limit

Three types of particles (lozenges) exchanging randomly their positions.

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Three types of particles (lozenges) exchanging randomly their positions.
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After diffusive time rescaling (set $\tau = t/L^2$) *expected* convergence to deterministic evolution (hydrodynamic limit).

$$\partial_t \phi = \mu(\nabla \phi)\mathrm{div}(\nabla F \circ \nabla \phi)$$

Idea: system decreases surface free energy $\Gamma(\phi) = \int F(\nabla \phi)$.

"Rapid mixing"

Theorem: [Luby-Randall-Sinclair, Wilson, Randall-Tetali (theoretical computer science community)]

The mixing time grows at most as a polynomial of L, uniformly in the boundary height.

Based on "path coupling methods"; at best, these can give $T_{mix} \leq cL^{4+\epsilon}$.

An almost optimal result

$h_t(\cdot)$: height function of the time-evolving discrete interface.

Theorem: [B. Laslier, F. T. '13] Assume the macroscopic shape Φ is C^∞. With probability close to 1,

$$\|h_t(\cdot) - \Phi(\cdot)\|_\infty = o(1) \qquad t \geq L^{2+\epsilon}$$

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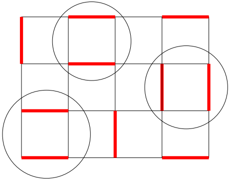
If Φ is affine, see [Caputo, Martinelli, Toninelli '12 and Laslier, Toninelli '14]: mixing time of order $L^{2+o(1)}$.

Beyond the solvable case: interacting dimers

Associate an energy $\lambda \in \mathbb{R}$ to adjacent dimers:



I.e., with $N(M)$ the number of adjacent pairs of dimers in M,

$$\langle \cdot \rangle_{\Lambda,\lambda} = \frac{\sum_M e^{\lambda N(M)} \cdot}{Z_{\Lambda,\lambda}}$$

[Alet et al., Phys. Rev. Lett 2005]

Beyond the solvable case: interacting dimers

Theorem [Giuliani, Mastropietro, T. 2014] If $|\lambda| \leq \lambda_0$ then:

- Fluctuations still grow logarithmically:

$$\lim_{\Lambda\to\mathbb{Z}^2} Var_{\Lambda,\lambda}(h(f)-h(f')) \overset{|f-f'|\to\infty}{=} \frac{K(\lambda)}{\pi^2}\log|f-f'|+O(1)$$

  with $K(\cdot)$ analytic and $K(0)=1$;

- for $m \geq 3$, the m^{th} cumulant of $h(f)-h(f')$ is *bounded*:

$$\sup_{f,f'}\lim_{\Lambda\to\mathbb{Z}^2}\langle h(f)-h(f');m\rangle_{\Lambda,\lambda} \leq C(m).$$

Beyond the solvable case: interacting dimers

Theorem [Giuliani, Mastropietro, T. 2014] If $|\lambda| \le \lambda_0$ then:

- Convergence to Gaussian Free Field: if $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \varphi(x)dx = 0$ then, as $\epsilon \to 0$,

$$\epsilon^2 \sum_f \varphi(\epsilon f) h(f) \to \int_{\mathbb{R}^2} \varphi(x) X(x) dx$$

with X the Gaussian Free Field of covariance

$$-\frac{K(\lambda)}{2\pi^2} \log |x-y|.$$

Universality or not? dimer correlations

Back to the non-interacting case. From Kasteleyn's solution,

$$\sigma_e \sigma_{e'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda, \lambda=0}$$

$$= -\frac{1}{2\pi^2} \Re\left[\Delta z_e \Delta z_{e'} \frac{1}{(z_e - z_{e'})^2} \right]$$

$$+ Osc(z_e, z_{e'}) \frac{1}{|z_e - z_{e'}|^2} + O(|z_e - z_{e'}|^{-3}).$$

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$$\sum_{e\in C_{f\to f'}, e'\in C'_{f\to f'}} A_{e,e'} \sim -\frac{1}{2\pi^2}\Re\int_f^{f'} \frac{dzdz'}{(z-z')^2} = \frac{1}{\pi^2}\log|f-f'|$$

Universality or not? dimer correlations

If λ is small, then [see also Falco, Phys Rev E 2013]

$$
\begin{aligned}
&\sigma_e \sigma_{e'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{e \in M}; 1_{e' \in M} \rangle_{\Lambda,\lambda} \\
&= -\frac{K(\lambda)}{2\pi^2} \Re\Big[\Delta z_e \Delta z_{e'} \frac{1}{(z_e - z_{e'})^2} \Big] \\
&+ Osc(z_e, z_{e'}) \frac{1}{|z_e - z_{e'}|^{2+\eta(\lambda)}} + O(|z_e - z_{e'}|^{-3+O(\lambda)}).
\end{aligned}
$$

with $K(\cdot)$, $\eta(\cdot)$ analytic and $K(0) = 1, \eta(0) = 0$.

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with $K(\cdot)$, $\eta(\cdot)$ analytic and $K(0) = 1, \eta(0) = 0$.

- in the main term the critical exponent remains 2
- in the oscillating term it changes to $2 + \eta(\lambda)$ (non-universal).

A Renormalization Group approach

Algebraic identity: Determinants can be written as "Grassmann Gaussian integrals", or "Lattice free fermions".

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To each lattice site, associate Grassmann variable ψ_x.
Anticommutation rule: $\psi_x \psi_y = -\psi_y \psi_x$
Then, with $(\psi, K\psi) = \sum_{b,w} \psi_w K(w,b) \psi_b$,

$$\det(K) = \int \prod_x d\psi_x e^{-\frac{1}{2}(\psi, K\psi)}$$

and

$$K^{-1}(b,w) = \frac{1}{\det(K)} \int \prod_x d\psi_x e^{-\frac{1}{2}(\psi, K\psi)} \psi_b \psi_w.$$

A Renormalization Group approach

Similarly, the partition function of the interacting model is written as

$$Z_{\Lambda,\lambda} = \frac{1}{\det(K)} \int \prod d\psi_x \exp\Big(-\frac{1}{2}(\psi, K\psi) + \lambda V(\psi)\Big)$$

with V a non-quadratic polynomial of the ψ.

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Naive power series in λ diverges

Constructive Renormalization Group methods (Benfatto, Brydges, Gallavotti, Gawedzki, Kupiainen, Mastropietro, Rivasseau, ... $\geq 1980's$) allow to obtain convergent expansion for correlation functions and to study large-distance behavior.

Open problems

- Effect of facets on Glauber dynamics?

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- Kenyon '00 proved conformal invariance of height moments e.g.

$$g_{\mathcal{D}}(x,y) = \lim_{L\to\infty} \langle (h_{x_L} - \langle h_{x_L}\rangle_\Lambda)(h_{y_L} - \langle h_{y_L}\rangle_\Lambda)\rangle_\Lambda$$

  (lattice spacing $1/L \to 0$, $\Lambda \subset (\mathbb{Z}/L)^2$ suitable discretization of domain $\mathcal{D} \subset \mathbb{C}$ and x_L, y_L tend to distinct points x, y)

  Conformal invariance for the interacting dimer model?

Thank you!