Random Surfaces and Quantum Loewner Evolution

Jason Miller and Scott Sheffield

Massachusetts Institute of Technology

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Overview

Part I: Picking surfaces at random

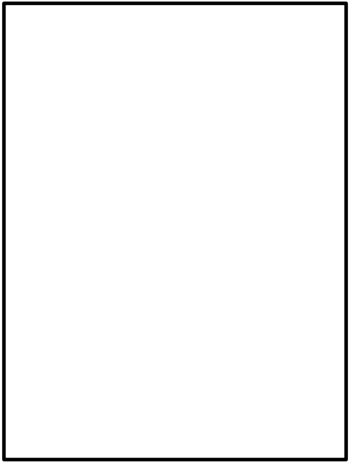
1. Discrete: random planar maps
2. Continuum: Liouville quantum gravity
3. Conjectured relationship

Part II: Quantum Loewner evolution

1. New universal family of growth processes
2. Tool to relate random planar maps to Liouville quantum gravity
3. Connected to many different topics in probability:
   RPM, LQG, TBM, GFF, SLE, DLA, FPP, DBM, KPZ, KPZ

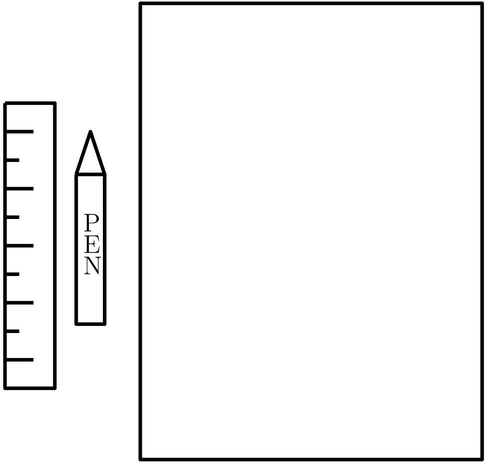
Part I: Picking surfaces at random

Quadrangulations



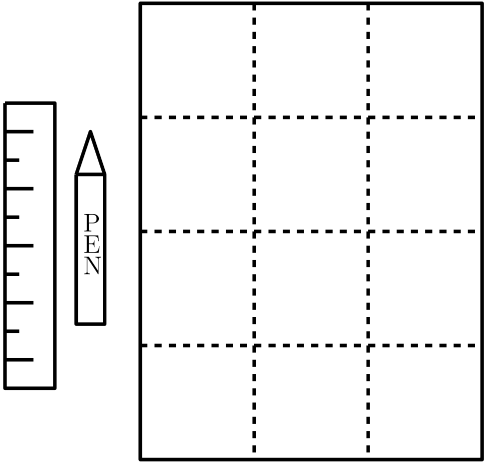
Start out with a sheet of paper

Quadrangulations



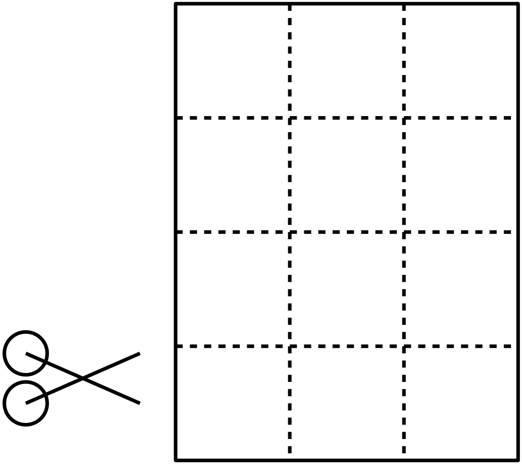
Get out pen and ruler

Quadrangulations



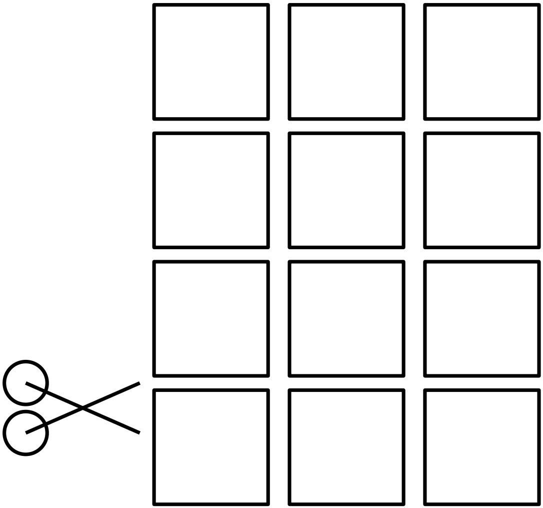
Measure and mark squares squares of equal size

Quadrangulations



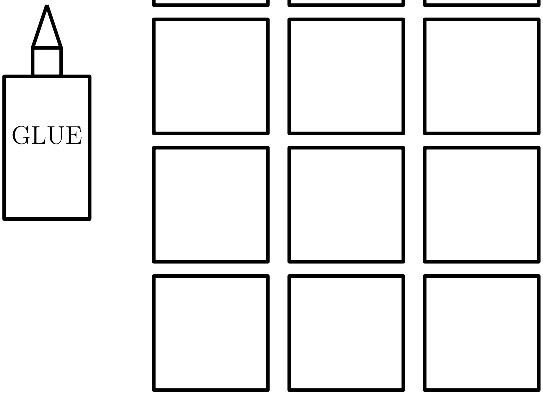
Get out scissors

Quadrangulations



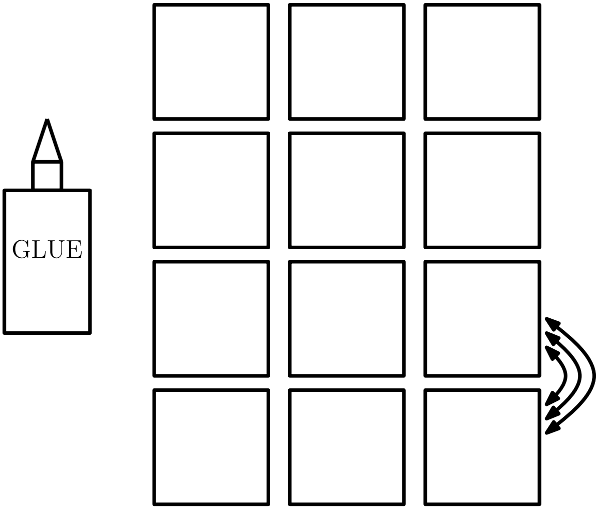
Cut into squares

Quadrangulations



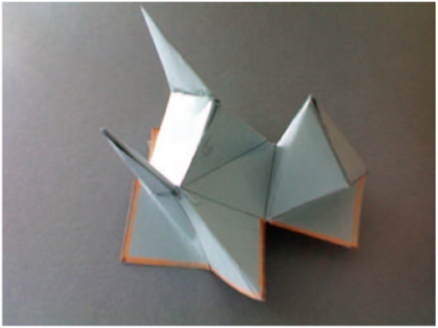
Get out bottle of glue

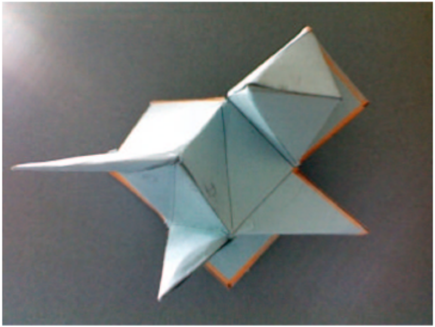
Quadrangulations



Attach squares along boundaries with glue to form a surface "without holes."

Quadrangulations





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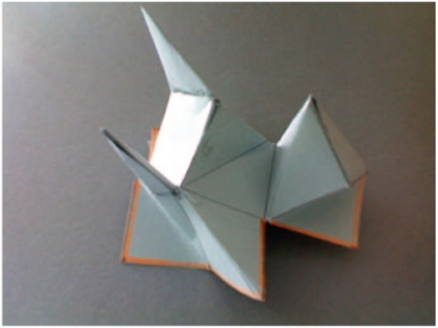
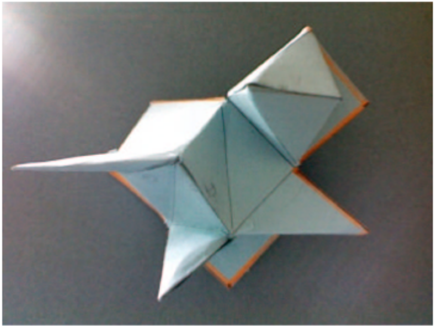
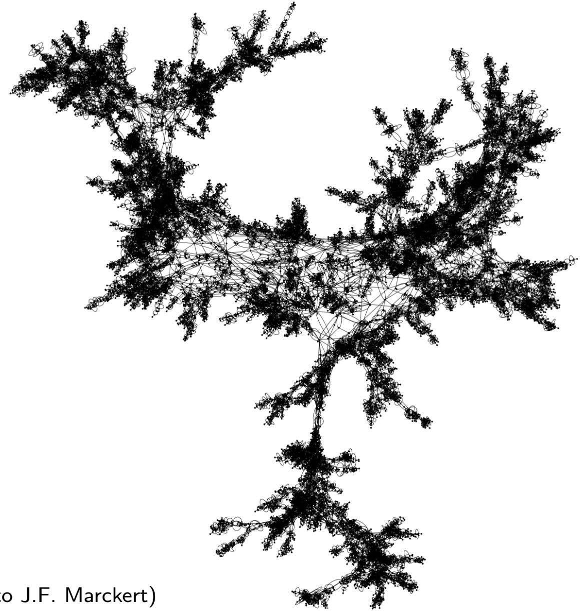




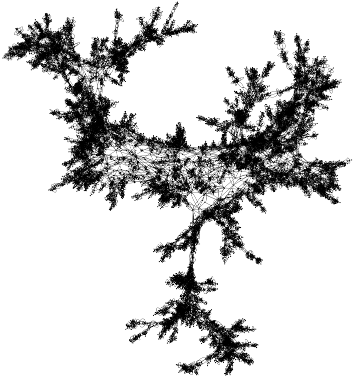
What is the structure of a typical quadrangulation when the number of faces is large?

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Background



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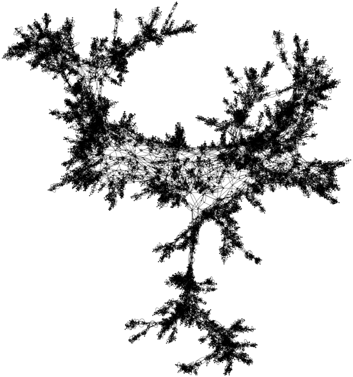
First studied by Tutte in 1960s while working on the four color theorem

- **Combinatorics**: enumeration formulas
- **Probability**: "uniformly random surface," Brownian surface
- **Physics**: statistical physics models: random walks, percolation, Ising model, uniform spanning tree, etc ...

Structure of large random planar maps

- RPM as a **metric space**. Is there a limit?



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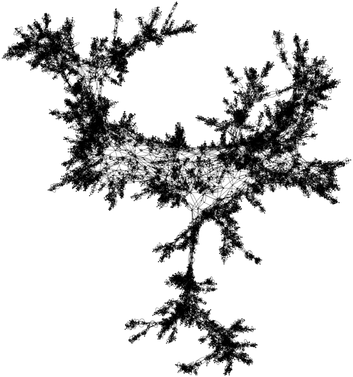

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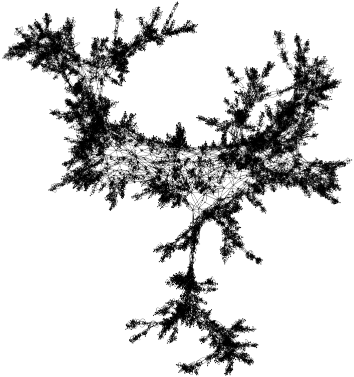

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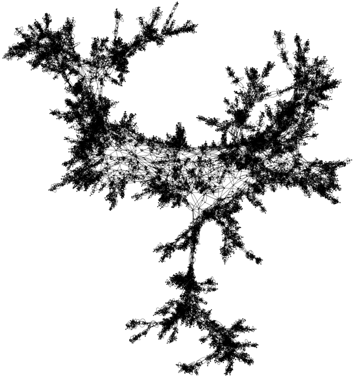

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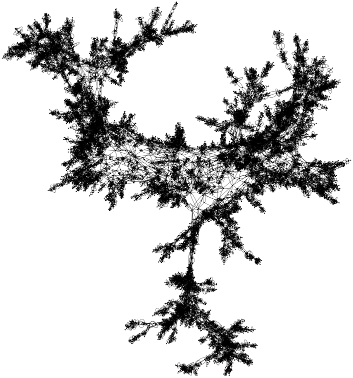

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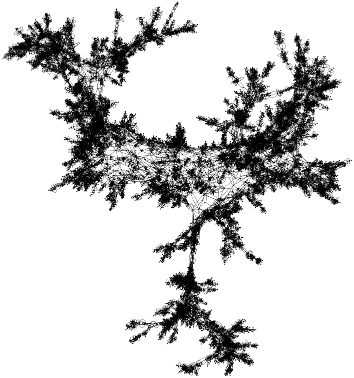

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Important tool: bijections which encode the surface using a gluing of a pair of trees

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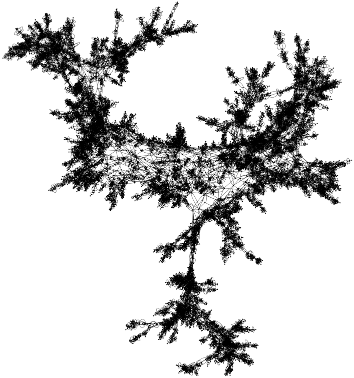

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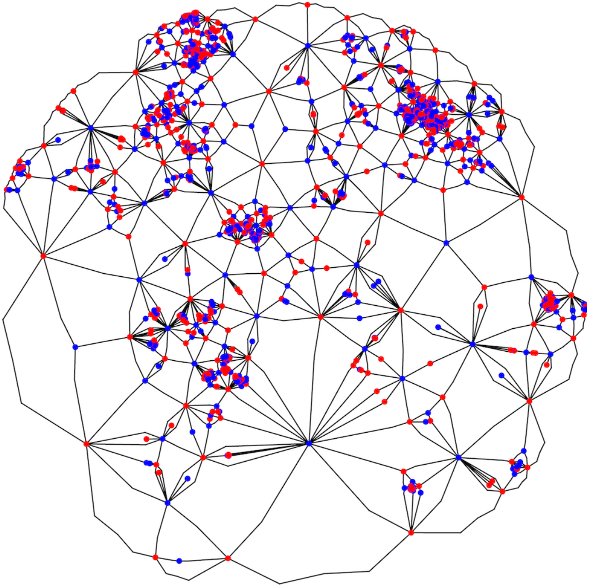
Brownian map also described in terms of trees (CRT)

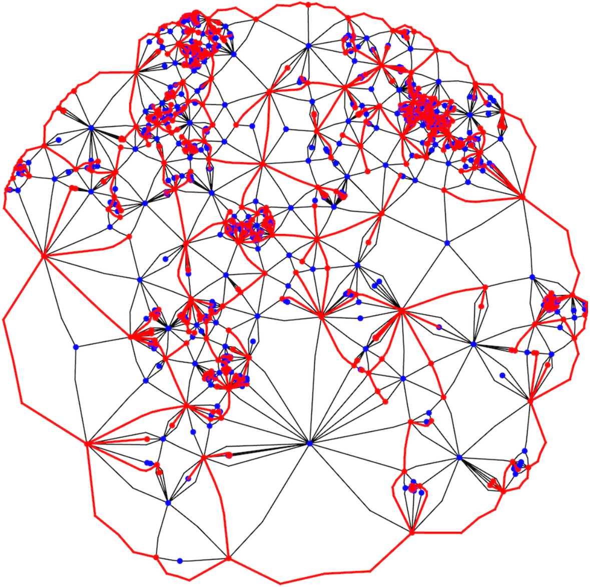
(Markert-Mokkadem)

Random quadrangulation



Sampled using H-C bijection.

Red tree



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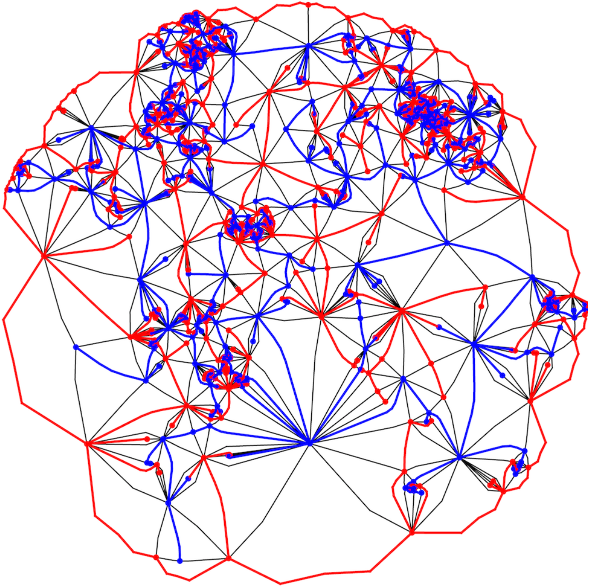
Red and blue trees

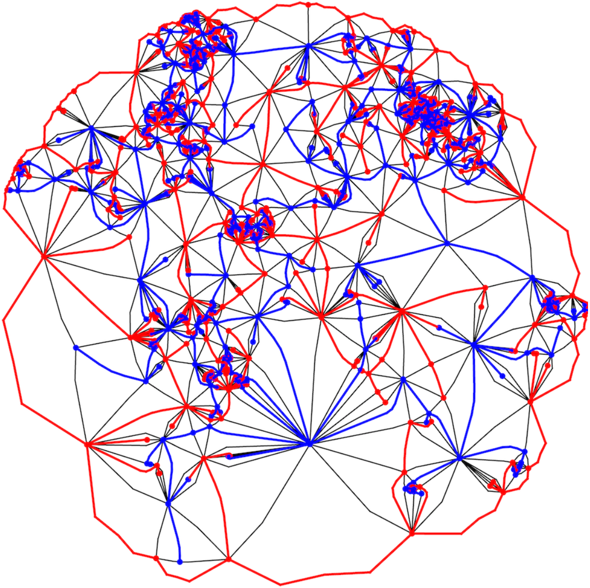


Sampled using H-C bijection.

Red and blue trees alone do not determine the map structure

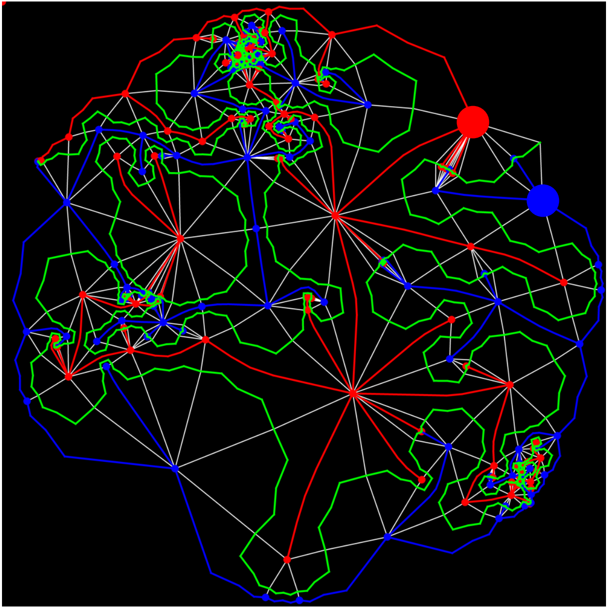
Sampled using H-C bijection.

Random quadrangulation with red and blue trees



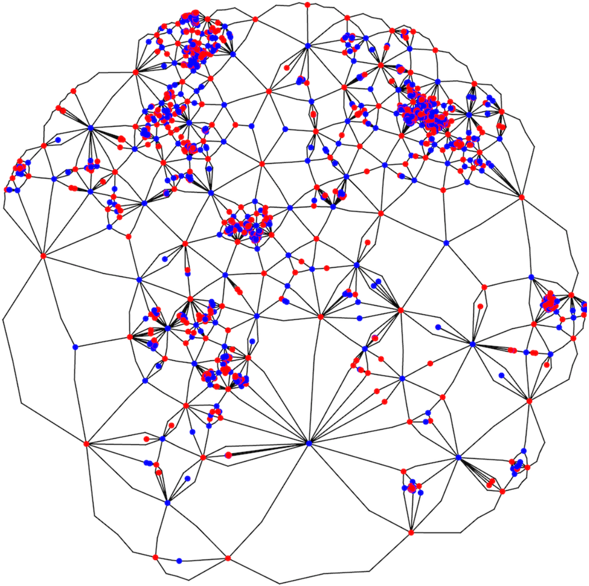
Sampled using H-C bijection.

Path snaking between the trees. Encodes the trees and how they are glued together.



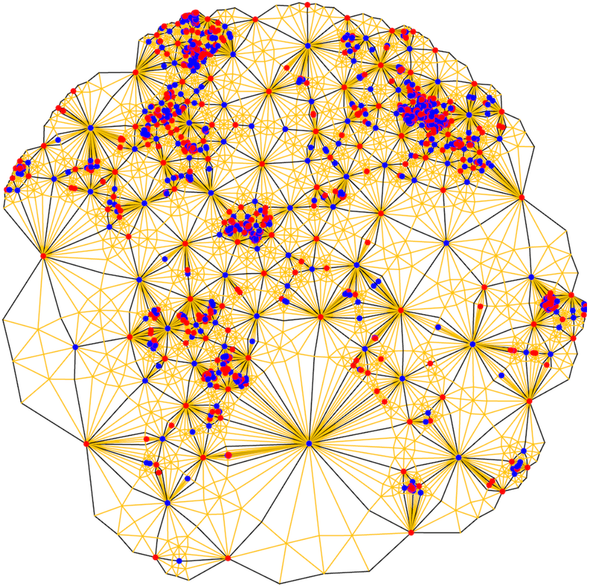
Sampled using H-C bijection.

How was the graph embedded into $\mathbf{R}^2$?



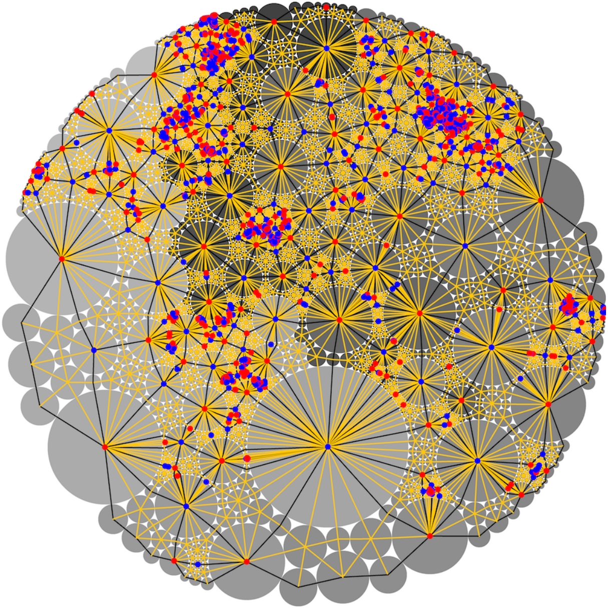
Sampled using H-C bijection.

Can subivide each quadrilateral to obtain a triangulation without multiple edges.



Sampled using H-C bijection.

Circle pack the resulting triangulation.



Sampled using H-C bijection. Packed with Stephenson's CirclePack.

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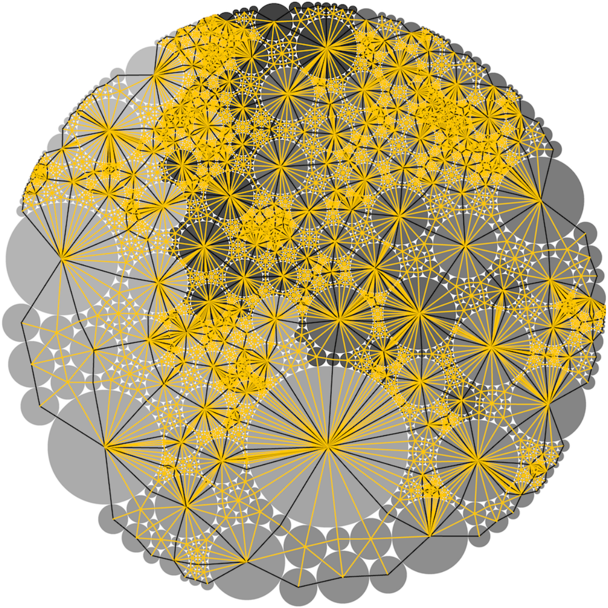

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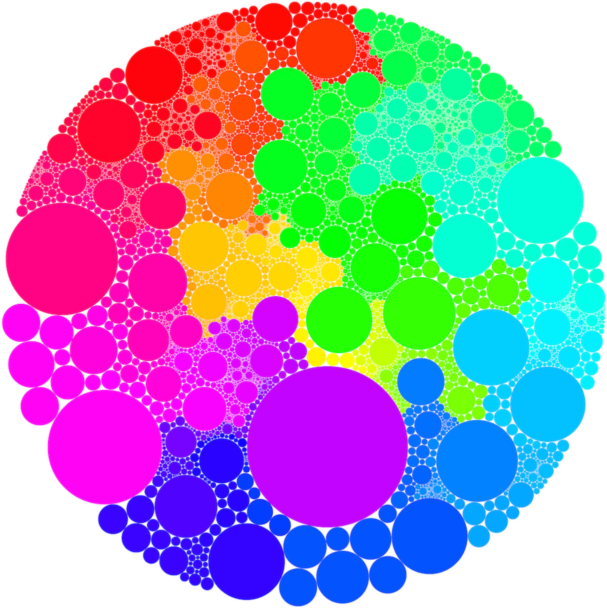
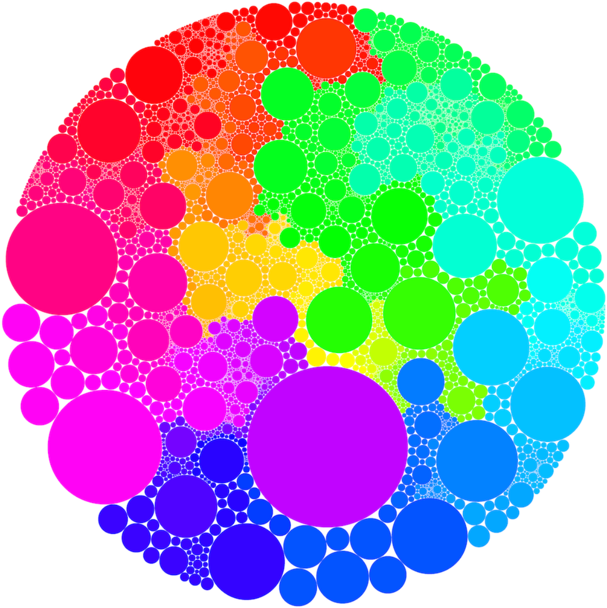


Sampled using H-C bijection. Packed with Stephenson's CirclePack.

What is the "limit" of this embedding? Circle packings are related to conformal maps.



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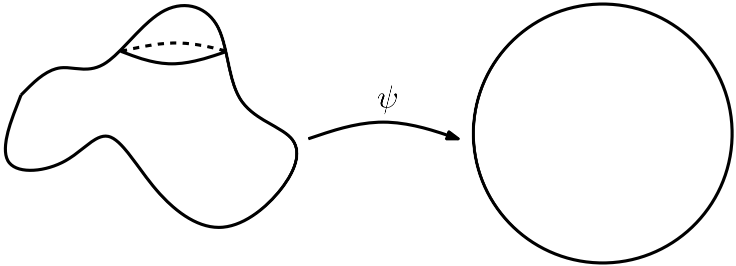
Picking a surface at random in the continuum

Uniformization theorem: every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$



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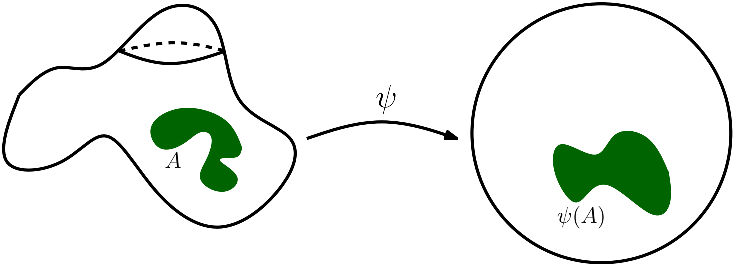

Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}dz$ for some smooth function ρ where dz is the Euclidean metric.

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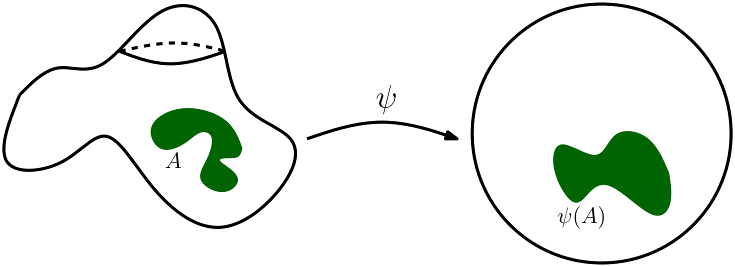

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$\Rightarrow$ Can parameterize the space of surfaces with smooth functions.

- If $\rho = 0$, get the same surface
- If $\Delta\rho = 0$, i.e. if ρ is harmonic, the surface described is flat

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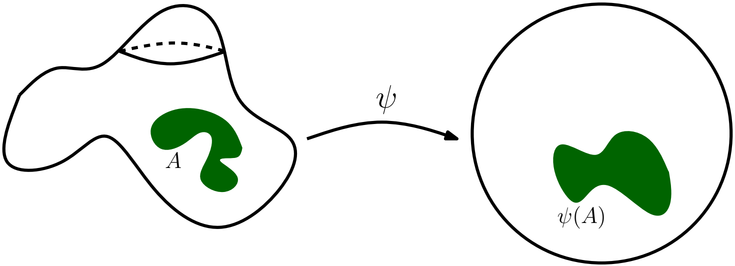

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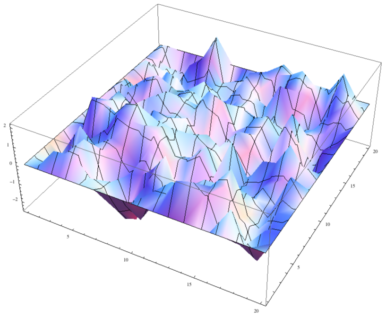
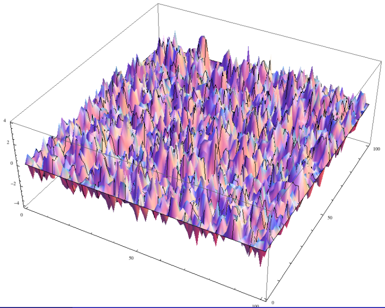
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.





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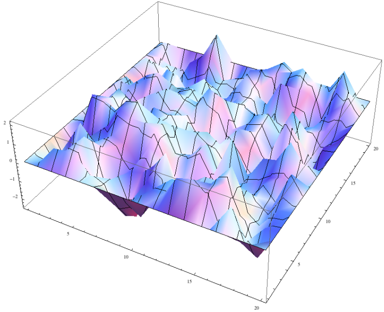
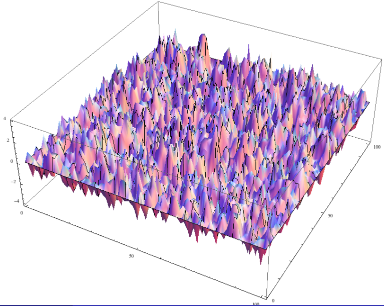
- Measure on functions $h \colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$





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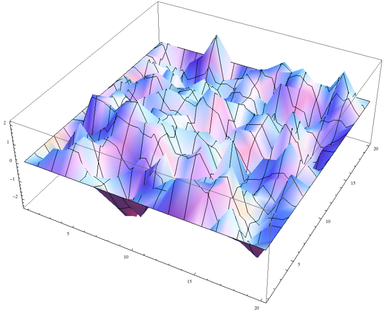

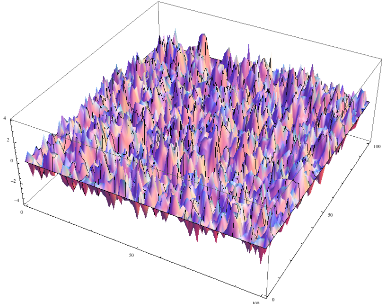

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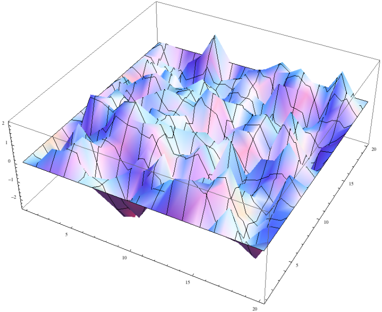
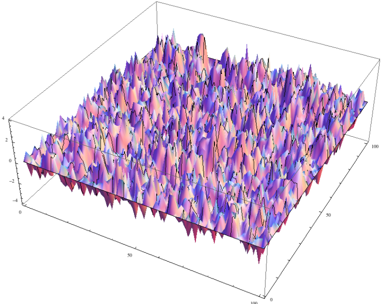
- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

$$(f, g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx.$$





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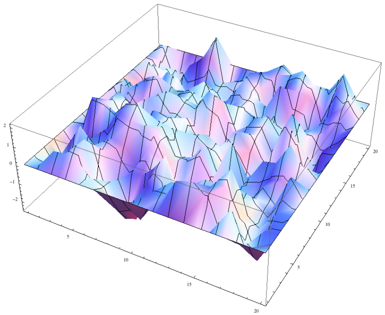
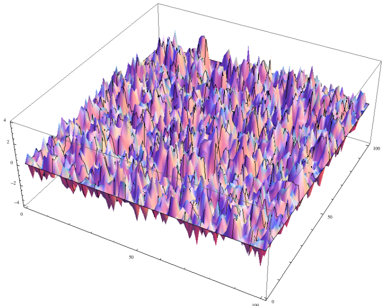
- Continuum GFF not a function — only a generalized function

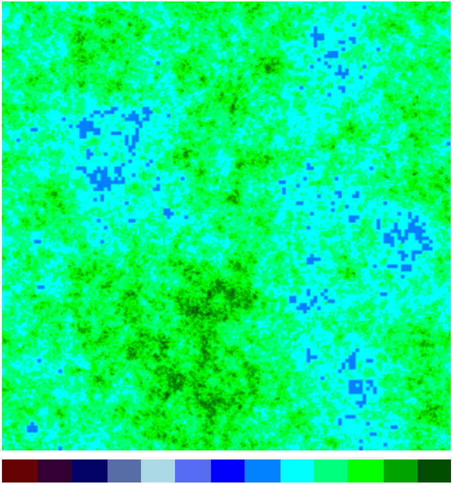




Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$

$\gamma = 0.5$



(Number of subdivisions)

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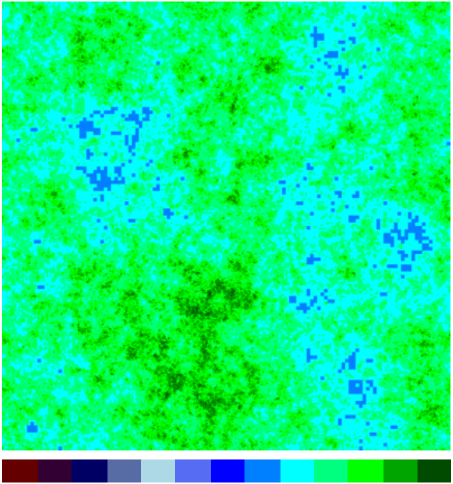

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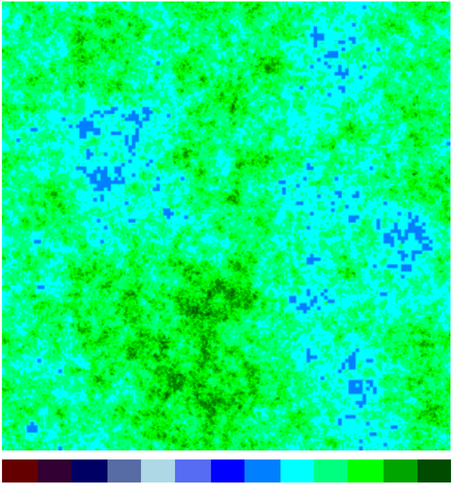

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- Does not make literal sense since h takes values in the space of distributions
- Can be made sense of as a random area measure using a regularization procedure
    - Can compute areas of regions and lengths of curves

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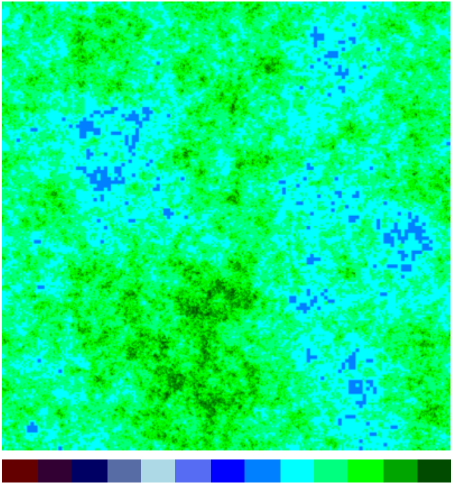

(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$
- Introduced by Polyakov in the 1980s
- Does not make literal sense since h takes values in the space of distributions
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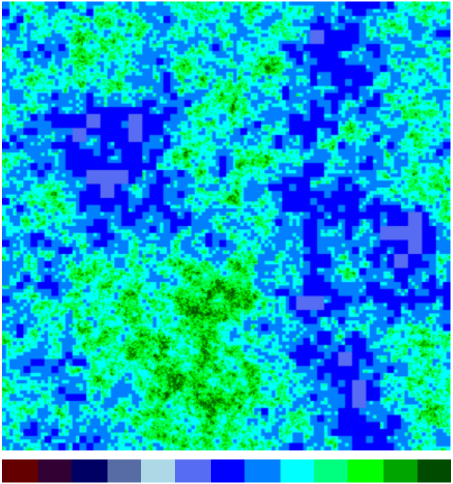

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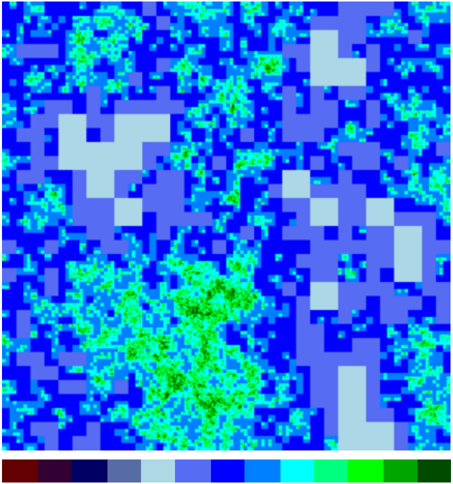

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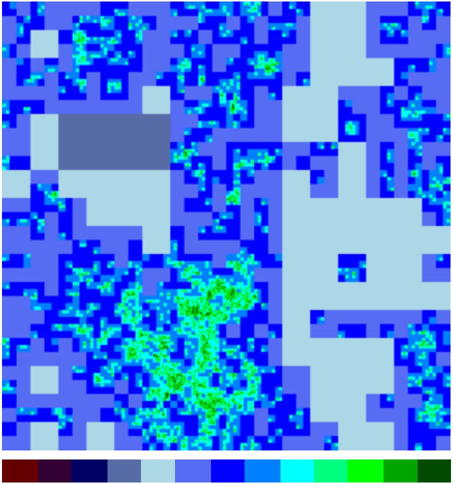
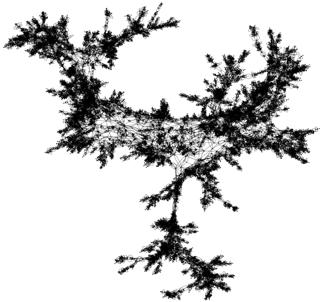
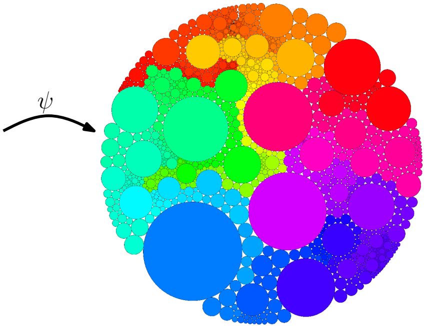


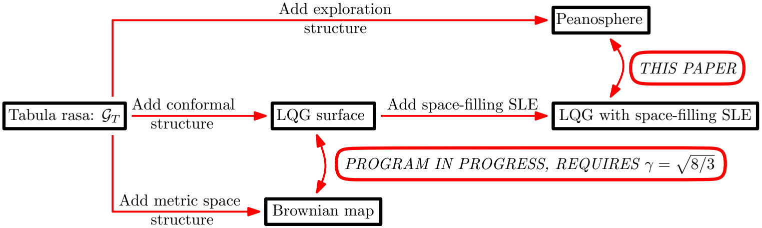
(Number of subdivisions)

Paper with Duplantier/Miller gives form of convergence

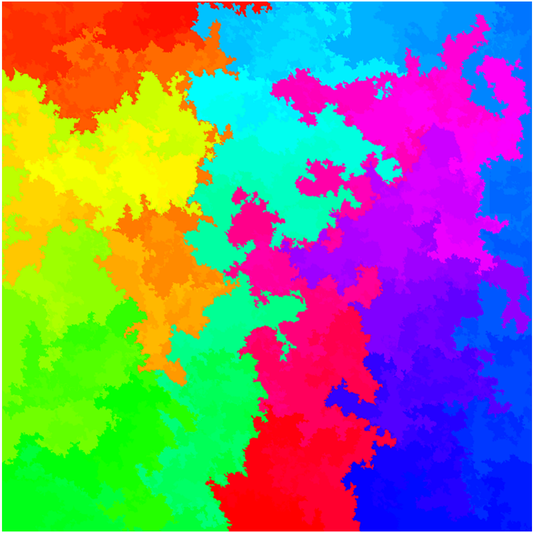




(Simulation due to J.-F. Marckert)



Continuum space-filling path



Space-filling SLE_6 on a LQG surface. Random path which encodes the limit of a RPM.

Part II:
Quantum Loewner Evolution

Recap

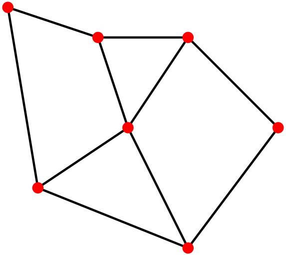
Two natural ways to pick surfaces at random

- **Discrete:** random planar maps
- **Continuum:** Liouville quantum gravity $e^{\gamma h(z)}dz$, h a GFF
- Conjectured to be the same for $\gamma = \sqrt{8/3}$
- LQG only made sense of so far as a **measure space**

Next part: describe new growth process which can be used to endow $\sqrt{8/3}$-LQG with a metric space structure

Detour: first passage percolation (FPP)

- Associate with a graph (V, E) i.i.d. $\exp(1)$ edge weights



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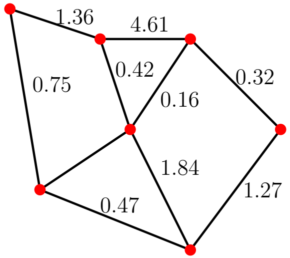

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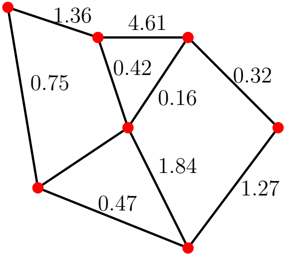

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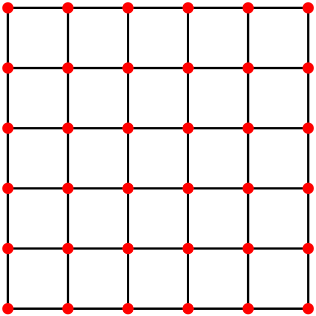

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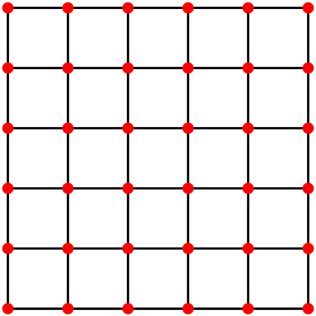

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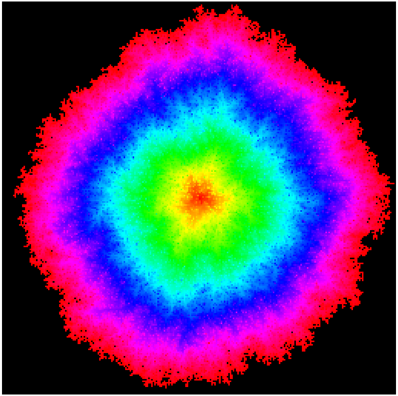

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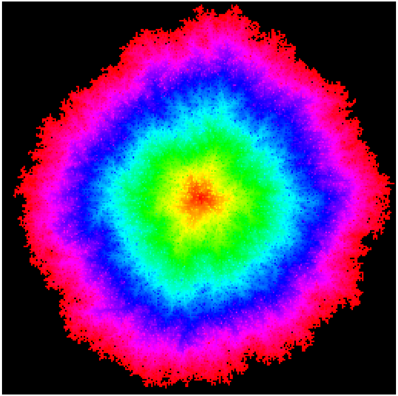

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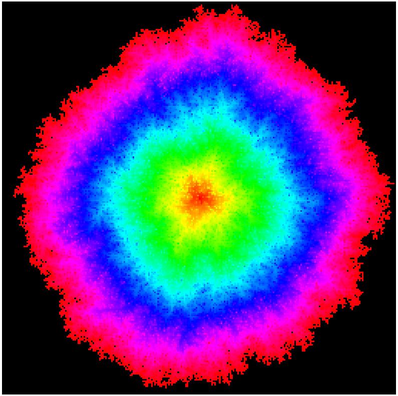

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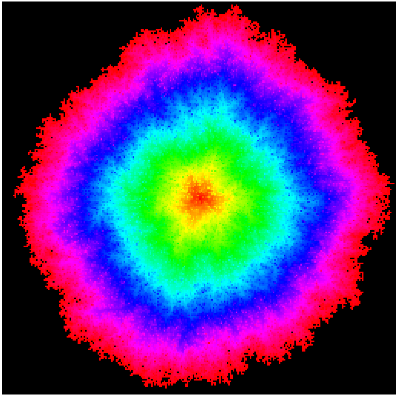

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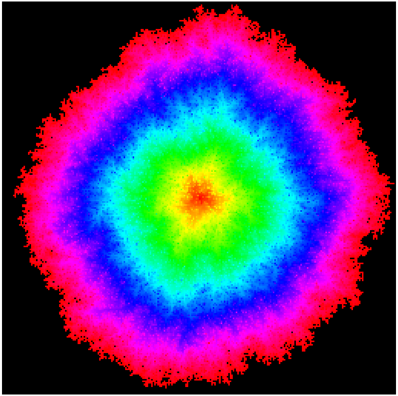
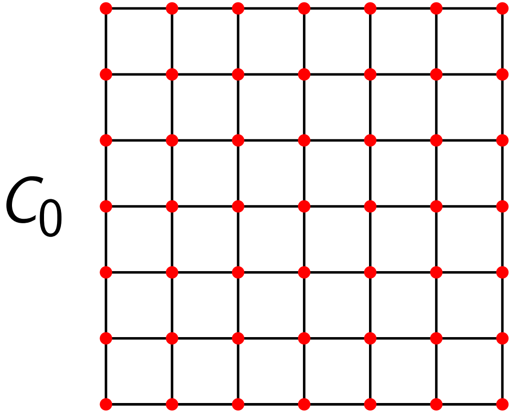
- Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if $\mathbf{Z}^2$ is replaced by the Voronoi tesselation associated with a Poisson process



Markovian formulation

Rather than sampling all of the edge weights at once, can explore the FPP metric ball starting from a point in a Markovian way.



Due to the memoryless property of the exponential distribution, can sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it.

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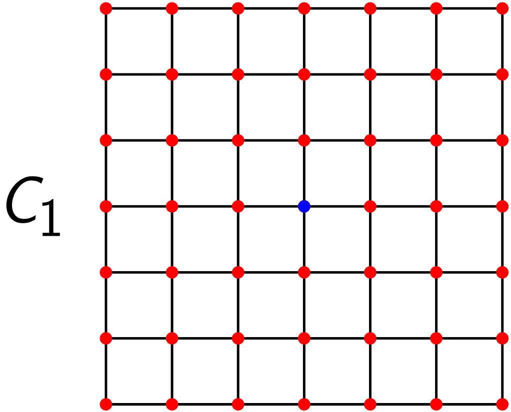

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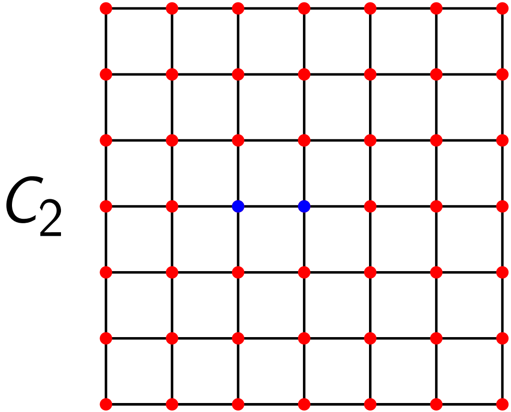

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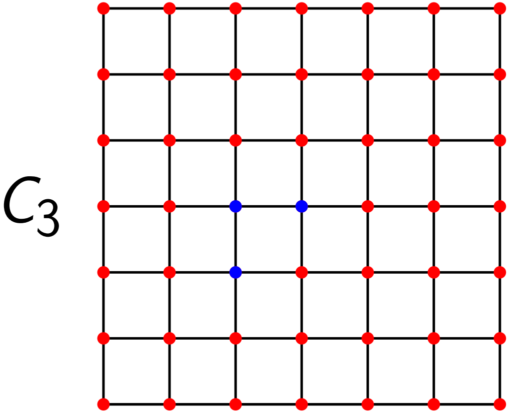

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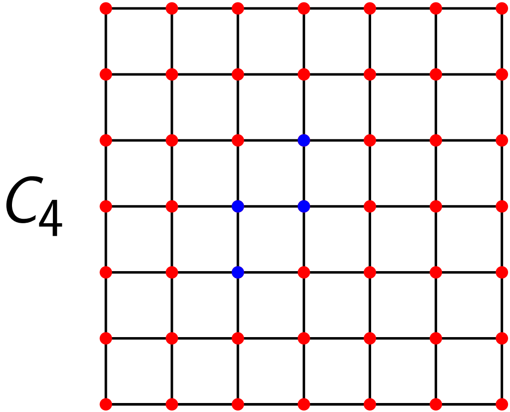

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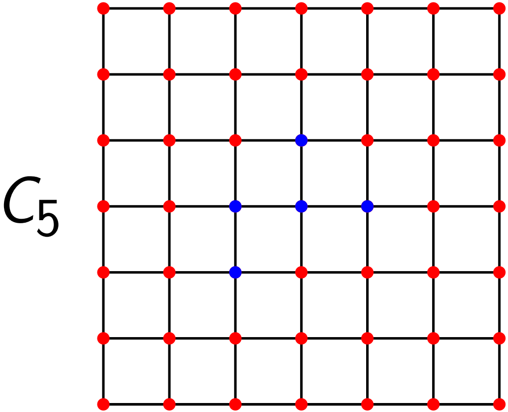

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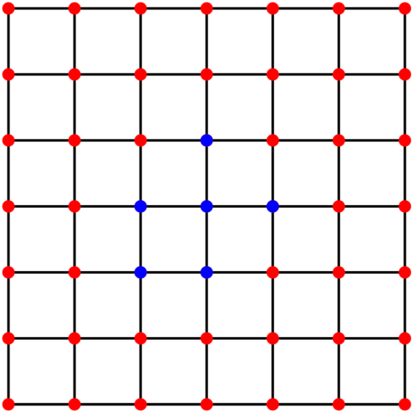

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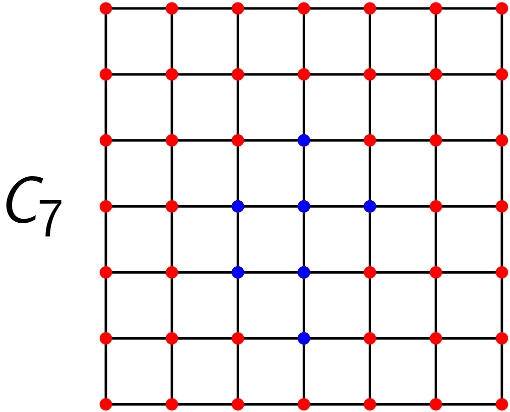

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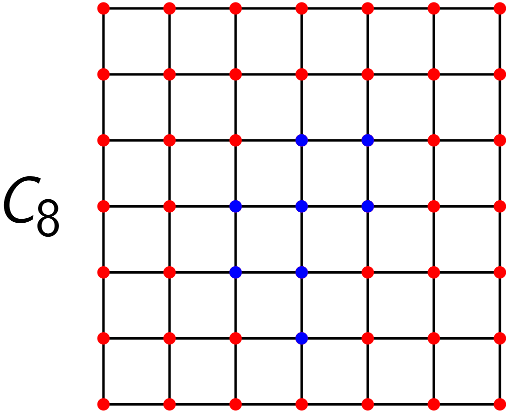

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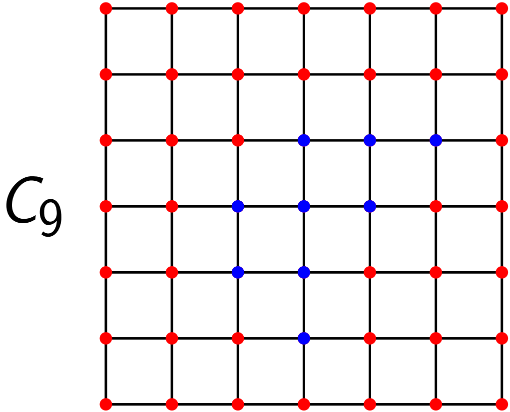

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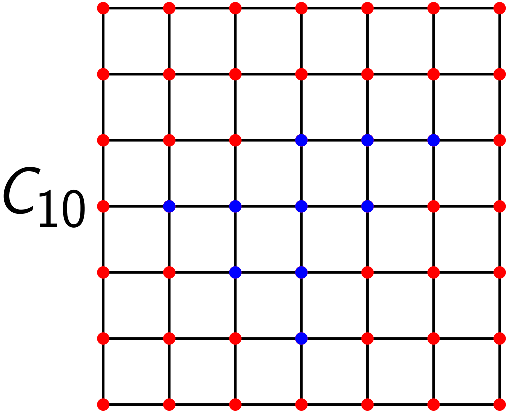

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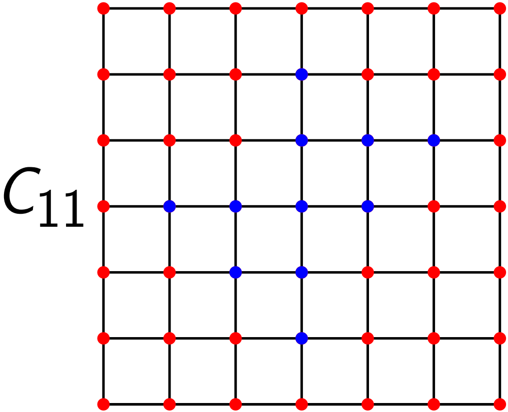

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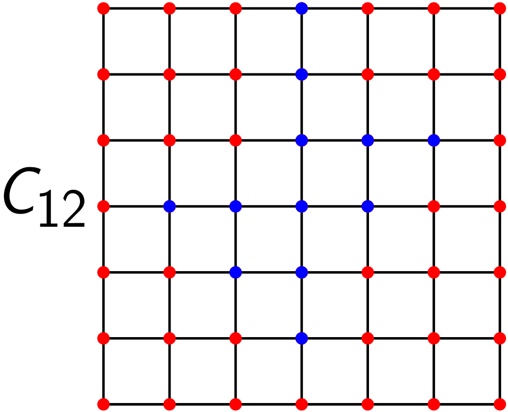

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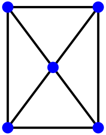
- Random planar map, random vertex x. Perform FPP from x.



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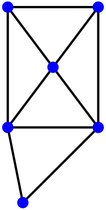

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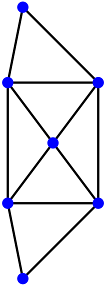

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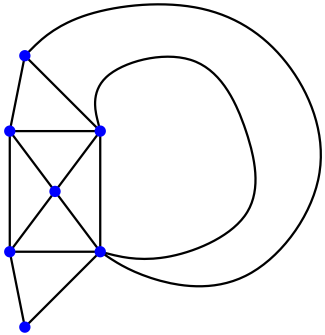

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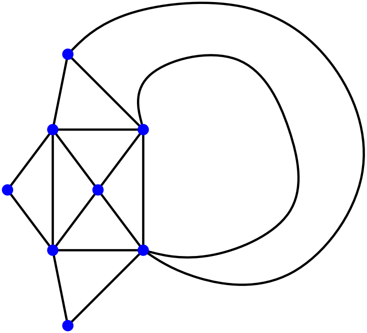

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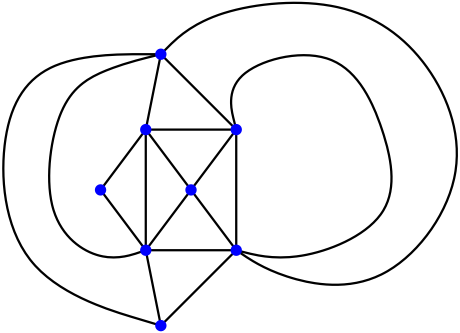

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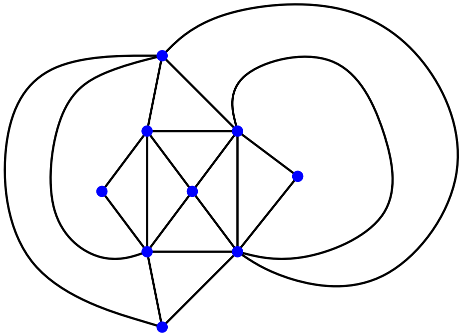

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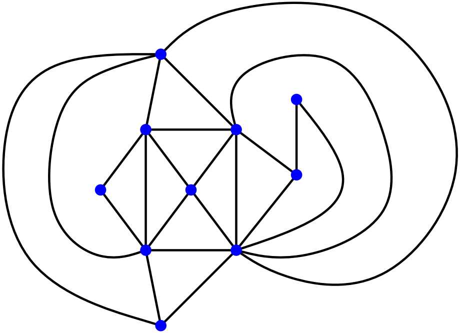

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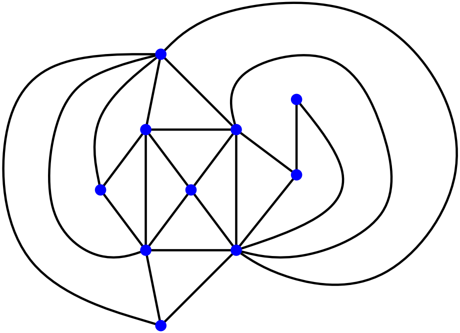

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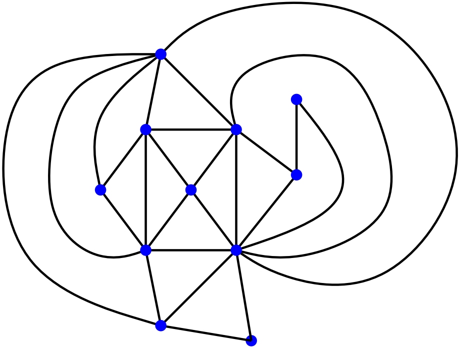

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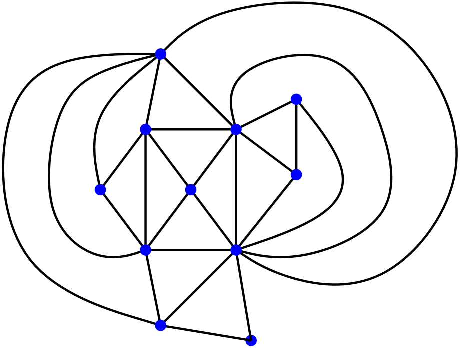

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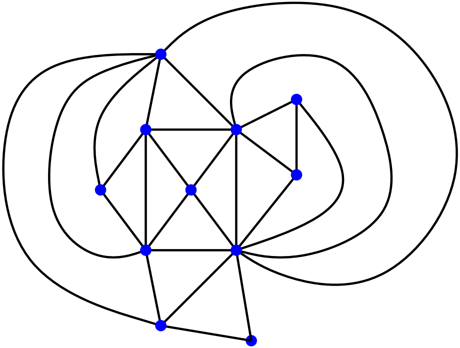

Important observations:

- Conditional law of map given ball at time n only depends on the boundary lengths of the outside components.

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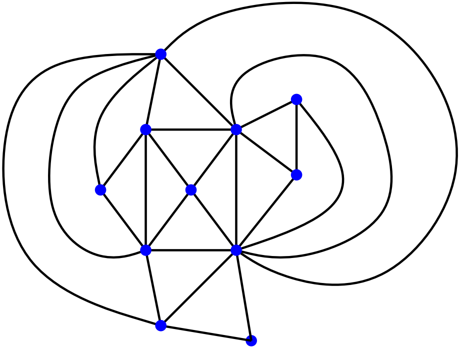

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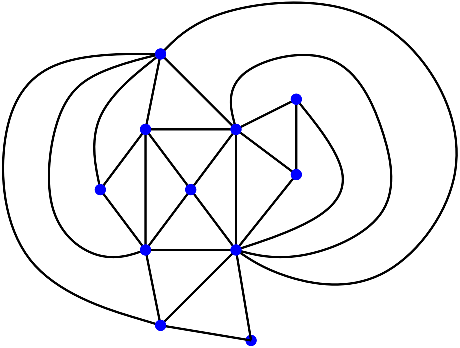

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- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length

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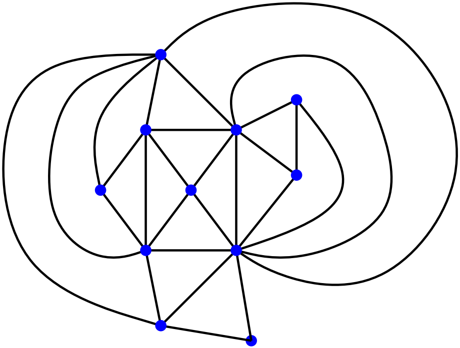

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Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric

First passage percolation on random planar maps II

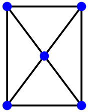
Goal: Make sense of FPP in the continuum on top of a LQG surface

- We do not know how to take a continuum limit of FPP on a random planar map and couple it directly with LQG
- Explain a discrete variant of FPP that involves two operations that we do know how to perform in the continuum:
  - Sample random points according to boundary length
  - Draw (scaling limits of) critical percolation interfaces (SLE_6)

First passage percolation on random planar maps III

Variant:

- Pick two edges on outer boundary of cluster



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Variant:

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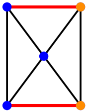

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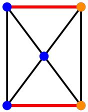

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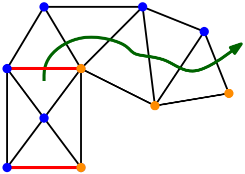

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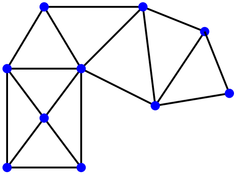

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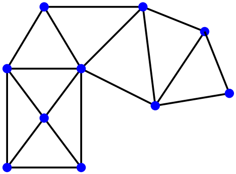

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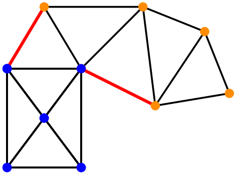

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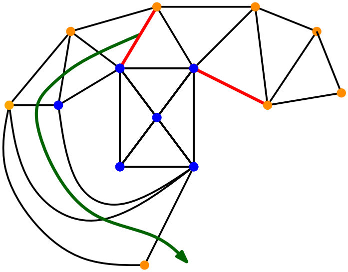

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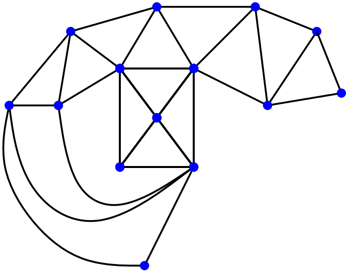

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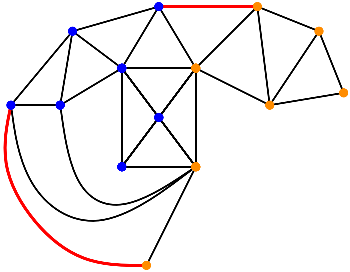

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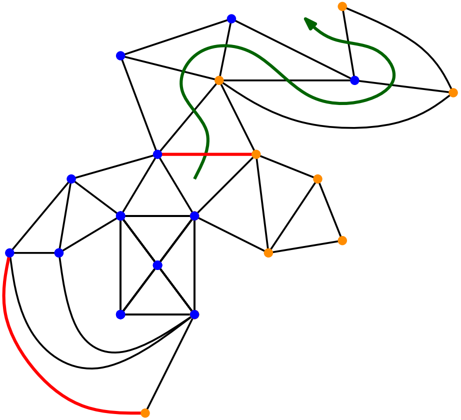

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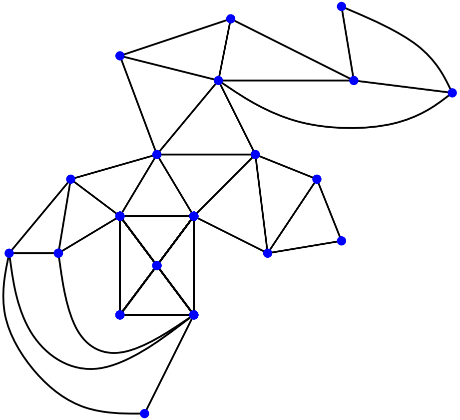

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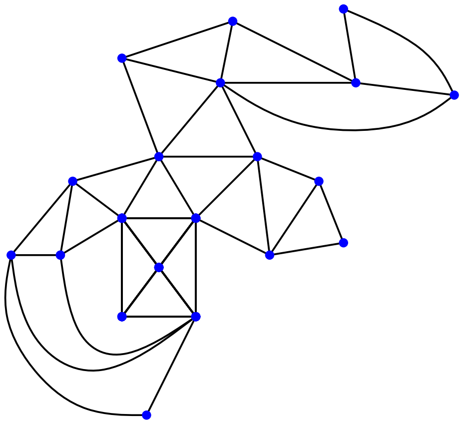

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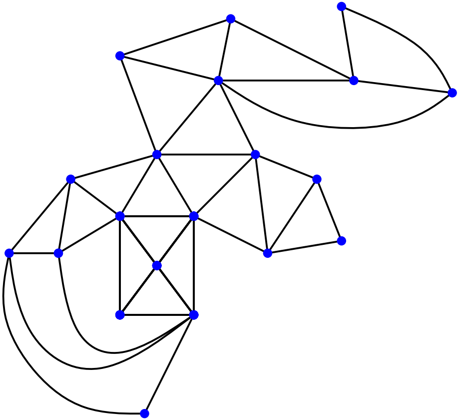

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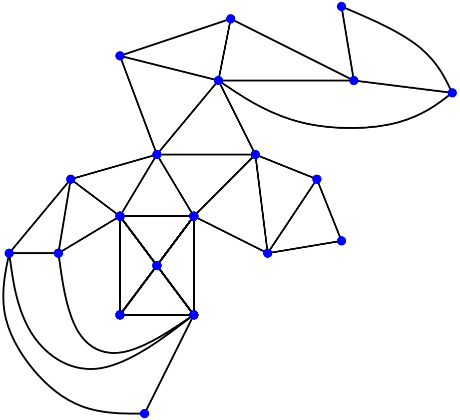
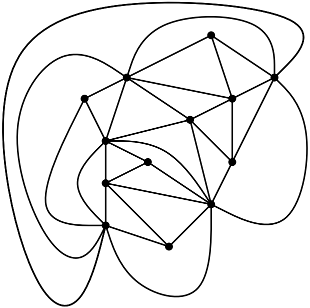


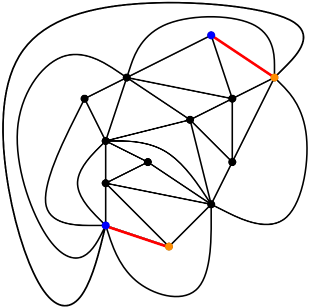
- *This exploration also respects the Markovian structure of the map.*
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
- Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball

Continuum limit ansatz



- Sample a random planar map

Continuum limit ansatz



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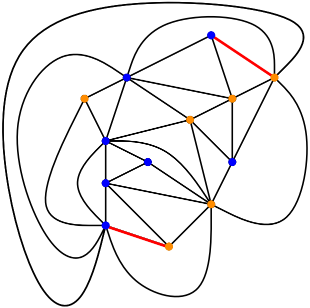

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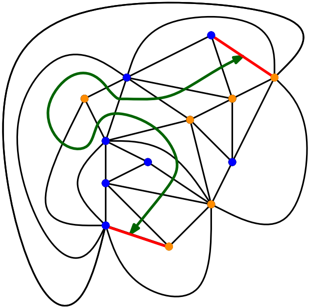

- Sample a random planar map and two edges uniformly at random
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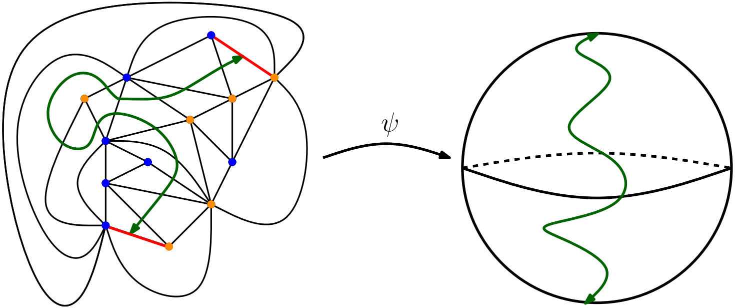

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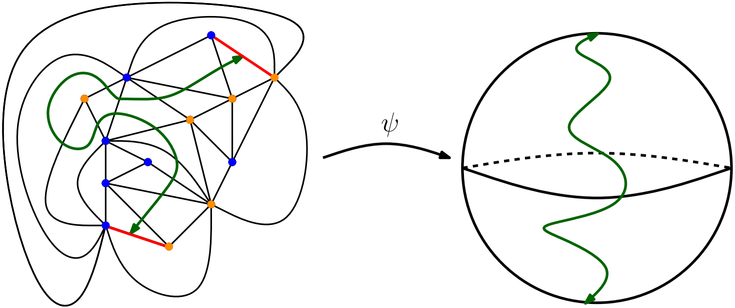

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Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x

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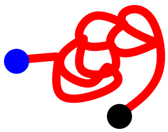

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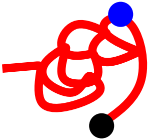

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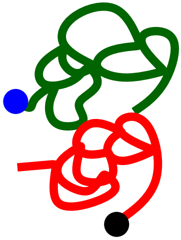

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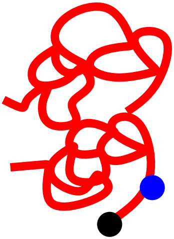

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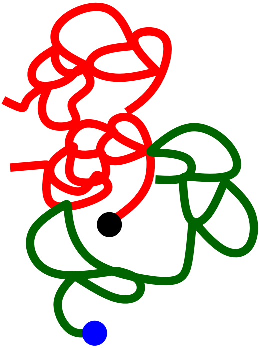

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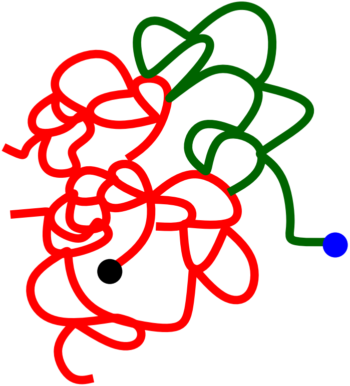

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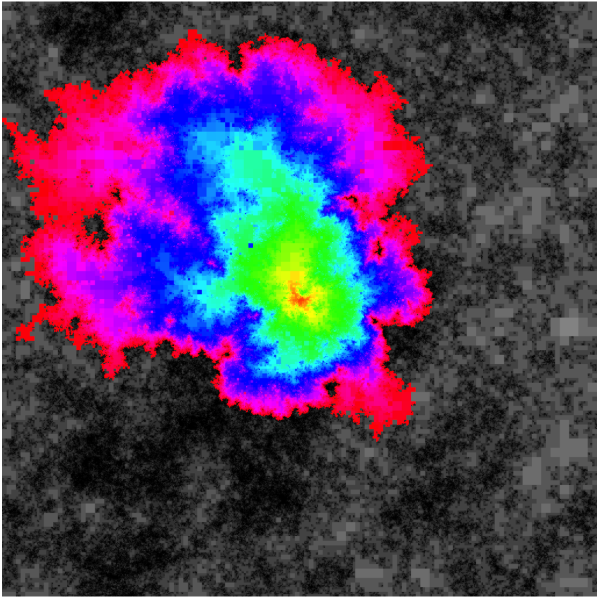
$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**.



Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG

What is $\mathrm{QLE}(\gamma^2, \eta)$?

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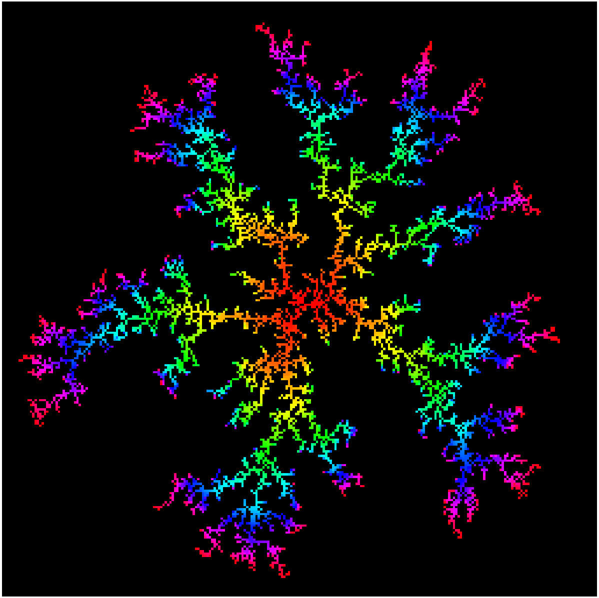
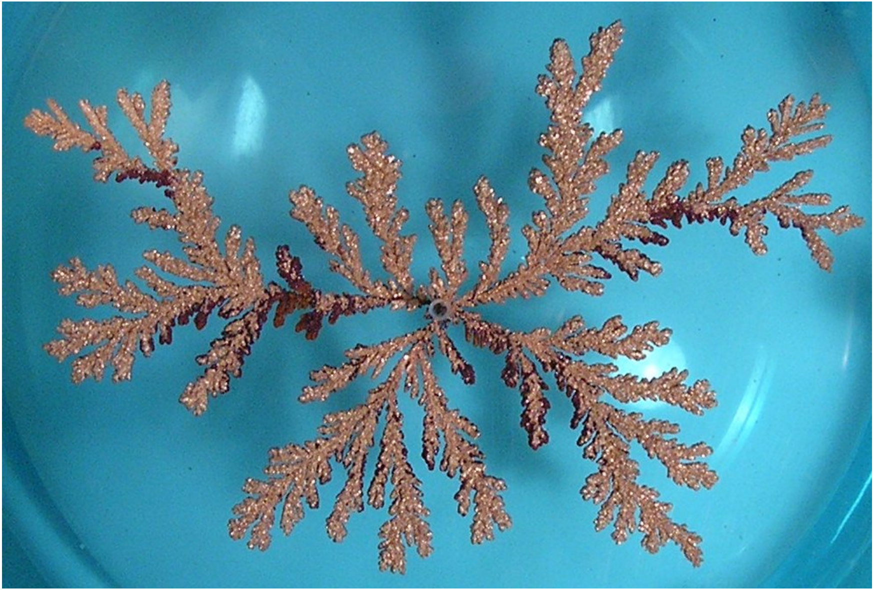
- **η-dieletric breakdown model:** general values of η



Simulation of Euclidean DLA



DLA in nature: "A DLA cluster grown from a copper sulfate solution in an electrodeposition cell" (from Wikipedia)

DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)

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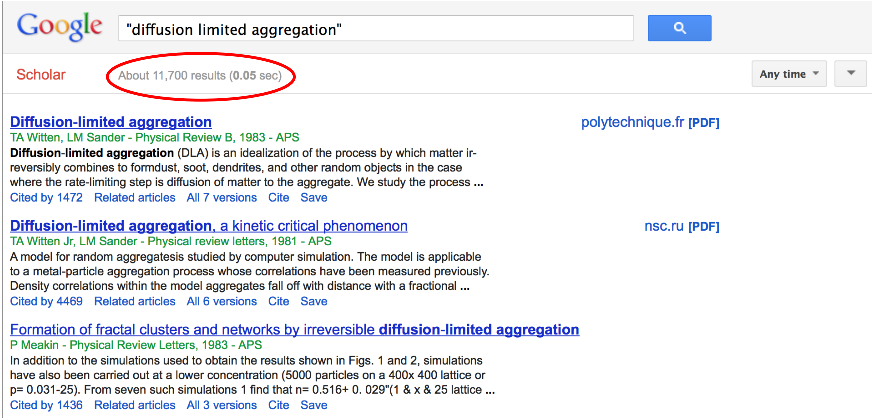
DLA in art: “High-voltage dielectric breakdown within a block of plexiglas” (from Wikipedia)

DLA in physics

Introduced by Witten and Sander in 1981 as a model for crystal growth

An active area of research in physics for the last 33 years:



Google "diffusion limited aggregation"

Scholar About 11,700 results (**0.05** sec) Any time

Diffusion-limited aggregation polytechnique.fr **[PDF]**
TA Witten, LM Sander - Physical Review B, 1983 - APS
Diffusion-limited aggregation (DLA) is an idealization of the process by which matter irreversibly combines to formdust, soot, dendrites, and other random objects in the case where the rate-limiting step is diffusion of matter to the aggregate. We study the process ...
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Diffusion-limited aggregation, a kinetic critical phenomenon nsc.ru **[PDF]**
TA Witten Jr, LM Sander - Physical review letters, 1981 - APS
A model for random aggregatesis studied by computer simulation. The model is applicable to a metal-particle aggregation process whose correlations have been measured previously. Density correlations within the model aggregates fall off with distance with a fractional ...
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Formation of fractal clusters and networks by irreversible **diffusion-limited aggregation**
P Meakin - Physical Review Letters, 1983 - APS
In addition to the simulations used to obtain the results shown in Figs. 1 and 2, simulations have also been carried out at a lower concentration (5000 particles on a 400x 400 lattice or p= 0.031-25). From seven such simulations 1 find that n= 0.516+ 0. 029"(1 & x & 25 lattice ...
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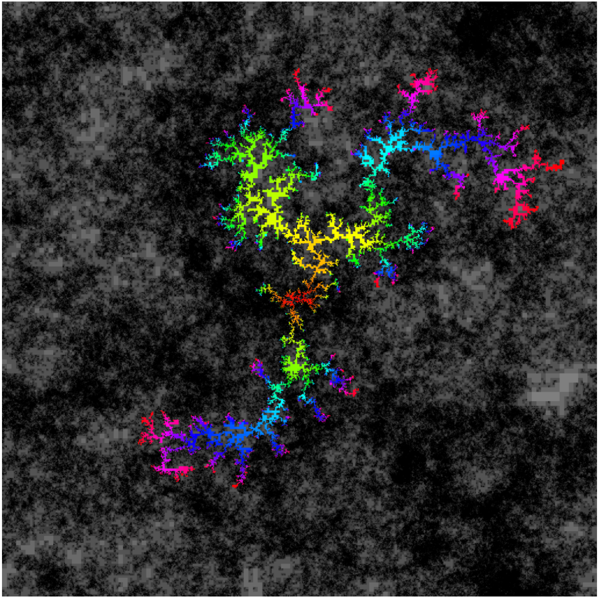
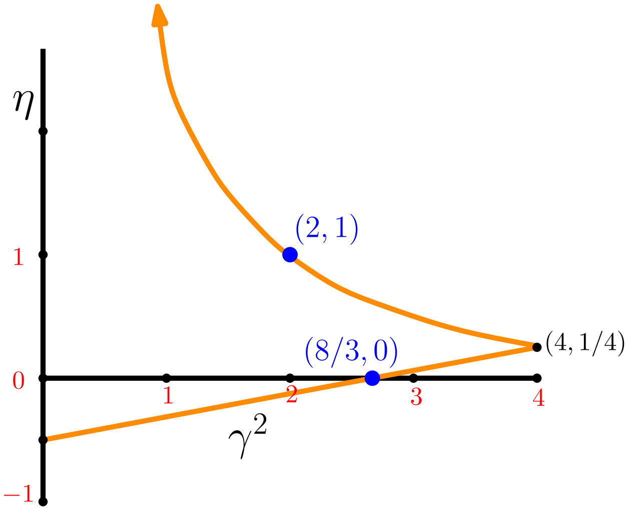
What about DLA on random planar maps and Liouville quantum gravity surfaces?



Discrete approximation of $\mathrm{QLE}(2,1)$. DLA on a $\sqrt{2}$-LQG

$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.

Results

What we can do:

- Existence of $\mathrm{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a γ-LQG surface.
- Derive an SPDE which the measure valued diffusion satisfies
- Continuity of the outer boundary of the growth at a given time
- Phases for sample path behavior: which QLEs are trees, have holes, and fill space

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Work in progress:

- Show that $\mathrm{QLE}(8/3, 0)$ endows $\sqrt{8/3}$-LQG with a distance function
- This metric space is isometric to the Brownian map: LQG $=$ TBM
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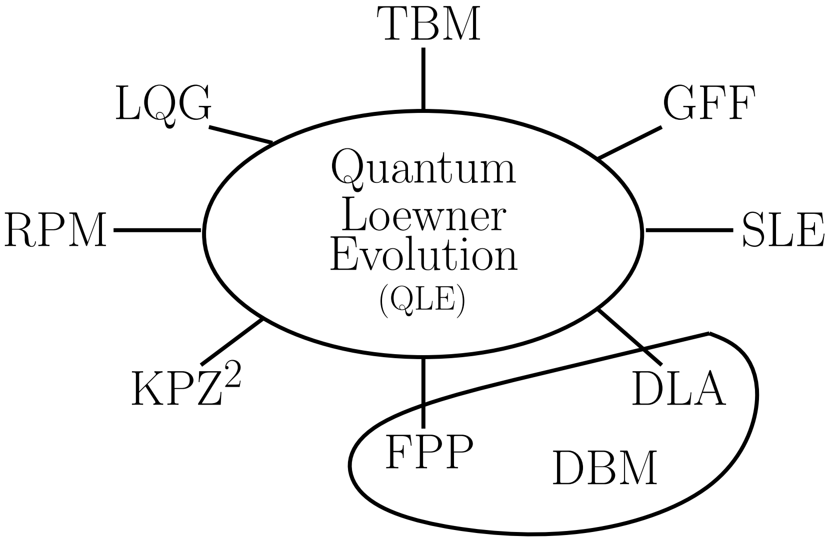
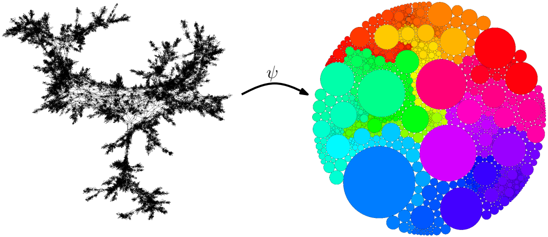
What we would like to do: construct and study $\mathrm{QLE}(\gamma^2, \eta)$ for (γ^2, η) pairs off the orange curves

QLE is connected to other topics in probability





Thanks!