Malliavin calculus and normal approximation

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Malliavin Calculus

- Paul Malliavin (1925-2010) introduced in the 70's a calculus of variations with respect to the trajectories of Brownian motion.
- The purpose of this calculus was to provide a probabilistic proof of Hörmander's hypoellipticity theorem.

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- The main application of this calculus is to show the existence and smoothness of densities of functionals of Gaussian processes.
- In this talk we will present some recent applications of the Mallavin calculus, combined with Stein's method, to normal approximations (Nourdin-Peccati '12 : *Normal Approximations with Malliavin Calculus*).

Multiple stochastic integrals

- H is a separable Hilbert space.
- $\mathcal{H}_1 = \{X(h), h \in H\}$ is a Gaussian family of random variables in $(\Omega, \mathcal{F}, P)$ with zero mean and covariance

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- For $q \geq 2$ we define the qth *Wiener chaos* as

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where $h_q(x)$ is the qth Hermite polynomial.

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  where $h_q(x)$ is the qth Hermite polynomial.
- *Multiple stochastic integral* of order q :

$$I_q : \left(H^{\hat{\otimes} q}, \sqrt{q!}\|\cdot\|_{H^{\otimes q}}\right) \to \mathcal{H}_q$$

  is a linear isometry defined by $I_q(h^{\otimes q}) = h_q(X(h))$, where $H^{\hat{\otimes} q}$ is the qth *symmetric tensor product* of H.

Example :
Let $B = \{B_t, t \in [0,1]\}$ be a Brownian motion.

- Then, $H = L^2([0,1])$ and $X(h) = \int_0^1 h_t dB_t$.
- For any $q \geq 2$, $H^{\hat{\otimes} q} = L^2_{sym}([0,1]^q)$ and I_q is the iterated Itô stochastic integral :

$$I_q(h) = q! \int_0^1 \dots \int_0^{t_2} h(t_1, \dots, t_q) dB_{t_1} \dots dB_{t_q}.$$

Wiener chaos expansion

Assume $\mathcal{F}$ is generated by $\mathcal{H}_1$. We have the orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

where $\mathcal{H}_0 = \mathbb{R}$. Any $F \in L^2(\Omega)$ can be written as

$$F = E(F) + \sum_{q=1}^{\infty} I_q(f_q),$$

where $f_q \in H^{\hat{\otimes} q}$ are determined by F.

Elements of Malliavin Calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(X(h_1), ..., X(h_n)),$$

  where $h_i \in H$ and $f \in C_b^\infty(\mathbb{R}^n)$.
- If $F \in \mathcal{S}$ we define its *derivative* by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), ..., X(h_n))h_i.$$

  DF is a random variable with values in H.
- $\mathbb{D}^{1,2} \subset L^2(\Omega; H)$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_H^2)}.$$

- The adjoint of D is the *divergence* operator δ defined by the duality relationship

$$E(\langle DF, u\rangle_H) = E(F\delta(u))$$

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- Basic formula

$$\boxed{\delta(DF) = -LF,}$$

where L is the generator of the *Ornstein-Uhlenbeck* semigroup defined by

$$LF = -\sum_{q=1}^{\infty} qI_q(f_q)$$

if $F = \sum_{q=0}^{\infty} I_q(f_q)$ and $\sum_{q=1}^{\infty} q^2 q! \|f_q\|^2_{H^{\otimes q}} < \infty$.

Integration-by-parts formula

Let $F \in \mathbb{D}^{1,2}$ with $E(F) = 0$ and $f \in C_b^1(\mathbb{R})$. Using that

$$F = LL^{-1}F = -\delta(DL^{-1}F)$$

yields

$$\begin{aligned} E[f(F)F] &= -E[f(F)\delta(DL^{-1}F)] \\ &= -E[\langle D(f(F)), DL^{-1}F\rangle_H] \\ &= E[f'(F)\langle DF, -DL^{-1}F\rangle_H]. \end{aligned}$$

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- If $F \in \mathcal{H}_q$, with $q \geq 1$, then $DL^{-1}F = -\frac{1}{q}DF$ and

$$\boxed{E[f(F)F] = \frac{1}{q}E[f'(F)\|DF\|_H^2].}$$

Stein's method for normal approximation

- *Stein's lemma :*

$$Z \sim N(0,1) \quad \Leftrightarrow \quad E[f(Z)Z - f'(Z)] = 0 \quad \forall f \in C_b^1(\mathbb{R}).$$

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- Let $Z \sim N(0,1)$, and fix h such that $E(|h(Z)|) < \infty$. The Stein's equation associated with h is

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Proposition

The unique solution to Stein's equation satisfying $\lim_{x\to\pm\infty} e^{-x^2/2} f_h(x) = 0$ *is*

$$f_h(x) = e^{x^2/2} \int_{-\infty}^{x} (h(y) - E[h(Z)]) e^{-y^2/2} dy.$$

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- If $\|h\|_\infty \le 1$, then $\|f_h\|_\infty \le \sqrt{\pi/2}$ and $\|f_h'\|_\infty \le 2$.
- So, for any random variable F, taking $h = \mathbf{1}_B$,

$$\begin{aligned} d_{TV}(F, Z) &= \sup_{B \in \mathcal{B}(\mathbb{R})} |P(F \in B) - P(Z \in B)| \\ &\le \sup_{f \in \mathcal{C}_{TV}} |E[f'(F) - Ff(F)]|, \end{aligned}$$

where $\mathcal{C}_{TV}$ is the class of functions with $\|f\|_\infty \le \sqrt{\pi/2}$ and $\|f'\|_\infty \le 2$.

- If $F \in \mathcal{H}_q$ for some $q \geq 2$ and $E(F^2) = 1$, then

$$\begin{aligned} d_{TV}(F, Z) &\leq \sup_{f \in \mathcal{C}_{TV}} |E[f'(F) - Ff(F)]| \\ &= \sup_{f \in \mathcal{C}_{TV}} \left| E\left[f'(F) \left(1 - \frac{1}{q} \|DF\|_H^2 \right) \right] \right| \\ &\leq \frac{2}{q} \sqrt{\mathrm{Var}\left(\|DF\|_H^2 \right)}, \end{aligned}$$

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  because $E[\|DF\|_H^2] = q$.

- Moreover, using Wiener chaos expansions and product formulas for multiple stochastic integrals

$$\mathrm{Var}\left(\|DF\|_H^2 \right) \leq \frac{(q-1)q}{3}(E(F^4) - 3) \leq (q-1)\mathrm{Var}\left(\|DF\|_H^2 \right).$$

Fourth Moment theorem

Stein's method combined with Malliavin calculus leads to a simple proof of the Fourth Moment theorem :

Theorem (N.-Peccati '05, N.-Ortiz '07)

Fix $q \geq 2$. Let $F_n = I_q(f_n) \in \mathcal{H}_q$, $n \geq 1$ be such that

$$\lim_{n\to\infty} E(F_n^2) = \sigma^2.$$

The following conditions are equivalent :

(i) $F_n \Rightarrow N(0, \sigma^2)$, *as* $n \to \infty$.

(ii) $E(F_n^4) \to 3\sigma^4$, *as* $n \to \infty$.

(iii) *For all* $1 \leq r \leq q-1$, $\|f_n \otimes_r f_n\|_{H^{\otimes(2q-2r)}} \to 0$, *as* $n \to \infty$.

(iv) $\|DF_n\|_H^2 \to q\sigma^2$ *in* $L^2(\Omega)$, *as* $n \to \infty$.

- $f_n \otimes_r f_n$ denotes the contraction of r coordinates.

Remarks :

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- The convergence is in total variation. Nourdin-Peccati '13 proved the following optimal version of the fourth moment theorem (for $\sigma = 1$) :

$$c\mathbf{M}(F_n) \leq d_{TV}(F_n, Z) \leq C\mathbf{M}(F_n),$$

where $\mathbf{M}(F_n) = \max(|E[F_n^3]|, E[F_n^4] - 3)$.

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  where $\mathbf{M}(F_n) = \max(|E[F_n^3]|, E[F_n^4] - 3)$.
- Peccati-Tudor '05 obtained a multidimensional extension, which can also be derived by Stein's method and Malliavin calculus.

Applications

(i) Central limit theorem for the renormalized self-intersection local time of the d-dimensional fractional Brownian motion with Hurst parameter $H \in \left[\frac{3}{2d}, \frac{3}{4}\right)$ (Hu-N. '05).

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(ii) Exact Berry-Esséen asymptotics for functionals of Gaussian processes (Nourdin-Peccati '10) :

$$[P(F_n \le z) - P(Z \le z)] \sim \varphi(n) \frac{\rho}{3q} \Phi^{(3)}(z),$$

as $n \to \infty$, where $F_n \in \mathcal{H}_q$, $E(F_n^2) \to 1$, $\varphi(n) = \sqrt{E\left[\left(1 - \frac{1}{q}\|DF_n\|_H^2\right)^2\right]}$, $\rho = \lim_n E(F_n \|DF_n\|_H^2)$, and $\Phi(z) = P(Z \le z)$.

(iii) Quantitative Breuer-Major theorems for functionals of Gaussian stationary sequences (Nourdin-Peccati '12, Nourdin-Peccati-Podolskij '12, ...).

Theorem (Brauer-Major '73)

Let $f \in L^2(\mathbb{R}, \gamma)$, where $\gamma = N(0,1)$, with Hermite rank d, that is,

$$f(x) = \sum_{q=d}^{\infty} a_q h_q(x),$$

and $a_d \neq 0$. Let $X = \{X_k, k \in \mathbb{Z}\}$ be a centered Gaussian stationary sequence with unit variance. Set $\rho(v) = E[X_0 X_v]$ for $v \in \mathbb{Z}$ and assume $\sum_{v \in \mathbb{Z}} |\rho(v)|^d < \infty$. Then,

$$V_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k) \Rightarrow N(0, \sigma^2),$$

as $n \to \infty$, where $\sigma^2 = \sum_{q=d}^{\infty} q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q$.

Sketch of the proof :

- We reduce the proof to the case $f = a_q h_q$, $q \geq d$.
- $E[V_n^2] \to \sigma^2$.
- Let H be the closure of $\{(b_j, j \in \mathbb{Z})\}$ by the scalar product $\langle b, c \rangle_H = \sum_{i,j \in \mathbb{Z}} b_i c_j \rho(i-j)$, and assume that $X_k = X(e_k)$, with $e_k = (\delta_{kj}, j \in \mathbb{Z})$.
- It suffices to show that

$$\|DF\|_H^2 \to qq! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q$$

in $L^2(\Omega)$.

- We have

$$\|DF\|_H^2 = \frac{a_q^2}{n} \sum_{i,j=1}^{n} h_q'(X_i) h_q'(X_j) \rho(i-j),$$

which has the same limit in L^2 as the sequence

$$B_n := \frac{a_q^2}{n} \sum_{j=1}^{n} h_q'(X_j) \left(\sum_{m=-\infty}^{\infty} h_q'(X_{j+m}) \rho(m) \right).$$

- The sequence

$$\left\{ h_q'(X_j) \left(\sum_{m=-\infty}^{\infty} h_q'(X_{j+m}) \rho(m) \right), j \geq 1 \right\}$$

is strictly stationary and ergodic. By the *Ergodic Theorem*, converges in $L^2(\Omega)$ to its expectation which is equal to

$$q q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q.$$

Convergence in law on a finite sum of Wiener chaos

- Convergence in law is metrizable by the Fortet-Mourier distance :

$$d_{FM}(F,G) = \sup_{\varphi} \left| E[\varphi(F)] - E[\varphi(G)] \right| ,$$

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Theorem (Nourdin-Poly '12)

Let $F_n \in \oplus_{k=1}^{p} \mathcal{H}_k$, $F \Rightarrow F_\infty$, and F_∞ is not identically zero. Then

$$d_{TV}(F_n, F_\infty) \leq c\, d_{FM}(F_n, F_\infty)^{\frac{1}{2p+1}}.$$

A multidimensional extension :

Theorem (Nourdin-N.-Poly '13)

Let $F_n = (F_{1,n}, \dots, F_{d,n})$ *be such that* $F_{i,n} \in \oplus_{k=1}^{p} \mathcal{H}_k$, $F \Rightarrow F_\infty$, *and*

$$E[\det \Gamma_n] \geq \beta > 0, \quad (1)$$

where $\Gamma_n^{i,j} = \langle DF_{i,n}, DF_{j,n} \rangle_H$. *Then*

$$d_{TV}(F_n, F_\infty) \leq c d_{FM}(F_n, F_\infty)^\gamma,$$

for any $\gamma < [(d+1)(4d(q-1)+3)+1]^{-1}$.

Sketch of the proof :
(i) Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $\|\varphi\|_\infty \leq 1$. Then,

$$\begin{aligned} |E[\varphi(F_n) - \varphi(F_m)]| &\leq |E[\varphi * \rho_\alpha(F_n) - \varphi * \rho_\alpha(F_m)]| \\ &\quad + 2 \sup_n |E[\varphi(F_n) - \varphi * \rho_\alpha(F_n)]| \\ &\leq \frac{1}{\alpha} d_{FM}(F_n, F_m) + 2R_\alpha. \end{aligned}$$

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&\leq \frac{1}{\alpha} d_{FM}(F_n, F_m) + 2R_\alpha.
\end{aligned}$$

(ii) Let $h_\alpha = \varphi - \varphi * \rho_\alpha$. For any $\epsilon > 0$,

$$\begin{aligned}
|E[h_\alpha(F_n)]| &= \left| E\left[h_\alpha(F_n) \left(\frac{\epsilon}{\det \Gamma_n + \epsilon} + \frac{\det \Gamma_n}{\det \Gamma_n + \epsilon} \right) \right] \right| \\
&\leq 2\epsilon E[(\det \Gamma_n + \epsilon)^{-1}] + \left| E\left[h_\alpha(F_n) \frac{\det \Gamma_n}{\det \Gamma_n + \epsilon} \right] \right|.
\end{aligned}$$

(iii) For the first term we obtain

$$2\epsilon E[(\det \Gamma_n + \epsilon)^{-1}] \leq c\epsilon^{\frac{1}{2(q-1)d+1}}.$$

This follows from $E[\det \Gamma_n] \geq \beta$ and the Carbery-Wright '01 inequality :

Lemma

For any polynomial Q of degree at most d we have

$$E[Q(X)^{\frac{q}{d}}]^{\frac{1}{q}} P(|Q(X)| \leq \alpha) \leq Cq\alpha^{\frac{1}{d}},$$

where X is a standard Gaussian vector.

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(iv) For the second term, if $|F_n| \leq M$, using Malliavin calculus we obtain

$$\left| E\left[h_\alpha(F_n) \frac{\det \Gamma_n}{\det \Gamma_n + \epsilon} \right] \right| \leq c\epsilon^{-2} \alpha^{\frac{1}{d+1}} M^{\frac{d}{d+1}}.$$

(v) We optimize in ϵ, α and M to get the result.

Sufficient conditions for $E[\det \Gamma_n] \geq \beta > 0$ (assumption (1)) :

- If F_∞ is normal $N_d(0, C)$ with $\det(C) > 0$, then (1) holds, because $\Gamma_n \to C$ in $L^2(\Omega)$ (N.-Ortiz '07).
- If F_∞ has independent and non degenerate components, then (1) holds.
- If $F_n \to F_\infty$ in $L^2(\Omega)$, then $E[\det \Gamma_n] \to E[\det \Gamma_{F_\infty}]$ and (1) holds if

$$E[\det \Gamma_{F_\infty}] > 0.$$

By Kusuoka's '83 theorem, this is equivalent to say that F_∞ has an absolutely continuous law.

Convergence of densities

- The total variation distance is equivalent to L^1-norm of the densities :

$$d_{TV}(F,Z) = \int_{\mathbb{R}} |p_F(x) - \phi(x)| dx,$$

where $Z \sim N(0,1)$ and ϕ is the density of Z.

- Uniform convergence, however, requires stronger hypotheses :

Theorem (Hu-Lu-N. '13)

Let $F \in \mathcal{H}_q$, $q \geq 2$, be such that $E(F^2) = 1$ and $E(\|DF\|_H^{-6}) \leq M$. Then,

$$\sup_{x\in\mathbb{R}} |p_F(x) - \phi(x)| \leq C_{M,q}\sqrt{E(F^4) - 3}.$$

- Using the notion of *Fisher information*, Nourdin and N., provided al alternative proof of this theorem under the weaker assumption $E(\|DF\|_H^{-4-\epsilon}) \leq M$ for some $\epsilon > 0$.

Sketch of the proof :

(i) Formula for the density

$$
\begin{aligned}
p_F(x) &= E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right] \\
&= E\left[\mathbf{1}_{\{F>x\}}\frac{qF}{\|DF\|_H^2}\right] - E[\mathbf{1}_{\{F>x\}}\langle DF, D(\|DF\|_H^{-2})\rangle_H] \\
&= E[\mathbf{1}_{\{F>x\}}F] + E[q\|DF\|_H^{-2} - 1] - E[\mathbf{1}_{\{F>x\}}\langle DF, D(\|DF\|_H^{-2})\rangle_H].
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(ii) The terms $E[|q\|DF\|_H^{-2} - 1|]$ and $E[|\langle DF, D(\|DF\|_H^{-2})\rangle_H|]$ can be estimated by a constant times $\sqrt{E(F^4) - 3}$.

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(iii) Taking into account that

$$\phi(x) = E[\mathbf{1}_{\{Z>x\}}Z],$$

where $Z \sim N(0,1)$, it suffices to estimate the difference

$$E[\mathbf{1}_{\{F>x\}}F] - E[\mathbf{1}_{\{Z>x\}}Z],$$

which can be done by Stein's method and Malliavin calculus.

- Let $q = 2$ and
$$F = \sum_{i=1}^{\infty} \lambda_i (X(e_i)^2 - 1),$$
where $\{e_i, i \geq 1\}$ is a complete orthonormal system in H and λ_i is a decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$. Suppose $E[F^2] = 1$.
- Then, if $\lambda_N \neq 0$ for some $N > 4$, we obtain
$$\sup_{x \in \mathbb{R}} |p_F(x) - \phi(x)| \leq C_{N,\lambda_N} \sqrt{\sum_{i=1}^{\infty} \lambda_i^4}.$$

Example 2 (Brauer-Major theorem revisited)

Fix $q \geq 2$ and consider the sequence

$$V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{j=d}^{q} a_j h_j(X_k), \quad a_d \neq 0,$$

where $X = \{X_k,\ k \in \mathbb{Z}\}$ is a centered Gaussian stationary sequence with unit variance and covariance $\rho(v)$.

Theorem (Hu-N.-Tindel-Xu '14)

Suppose the spectral density of X, f_ρ, satisfies $\log(f_\rho) \in L^1([-\pi, \pi])$. Assume $\sum_{v \in \mathbb{Z}} |\rho(v)|^d < \infty$. Set $\sigma^2 := q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q \in (0, \infty)$. Then for any $p \geq 1$, there exists n_0 such that

$$\sup_{n \geq n_0} E[\|DV_n\|_H^{-p}] < \infty. \qquad (2)$$

Therefore, if $q = d$ and $F_n = V_n / \sqrt{E[V_n^2]}$, we have

$$\sup_{x \in \mathbb{R}} |p_{F_n}(x) - \phi(x)| \leq c \sqrt{E[F_n^4] - 3}.$$

Sketch of the proof :

- From the non-causal representation $X_k = \sum_{j=0}^{\infty} \psi_j w_{k-j}$, where $\{w_k, k \in \mathbb{Z}\}$ is a discrete Gaussian white noise, it follows that

$$\|DV_n\|_H^2 \geq \frac{1}{n} \sum_{m=1}^{n} \left(\sum_{k=m}^{n} \sum_{j=d}^{q} a_j h_j'(X_k) \psi_{k-m} \right)^2 := B_n.$$

- Fix N and consider a block decomposition $B_n = \sum_{i=1}^{N} B_n^i$, where B_n^i is the sum of $\frac{n}{N}$ squares.
- We use the estimate

$$B_n^{-\frac{p}{2}} \leq \prod_{i=1}^{N} (B_n^i)^{-\frac{p}{2N}}$$

and we can apply the Carbery-Wright inequality to control the expectation of $(B_n^i)^{-\frac{p}{2N}}$ if $\frac{p}{2N}$ is small enough.

Particular case :

- Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0,1)$:

$$E(B_t^H B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- Set $\{X_k = B_k^H - B_{k-1}^H \,, k \geq 1\}$. In this case,

$$\rho_H(v) = \frac{1}{2}(|v+1|^{2H} + |v-1|^{2H} - 2|v|^{2H}),$$

  and the spectral density satisfies $\log(f_{\rho_H}) \in L^1([-\pi,\pi])$.

- As a consequence, we obtain the uniform convergence of densities to ϕ for the sequence of Hermite variations $F_n = V_n/E[V_n^2]$, where

$$V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} h_q(n^H \Delta_{k/n} B^H), \quad q \geq 2,$$

  for $0 < H < 1 - \frac{1}{2q}$, where $\Delta_{k/n} B^H = B_{k/n}^H - B_{(k-1)/n}^H$.

- For $q = 2$ we have

$$V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} [(n^H \Delta_{k/n} B^H)^2 - 1].$$

- If $H \in (0, \frac{3}{4})$ and $F_n = V_n / E[V_n^2]$ we have (Biermé-Bonami-León '11)

$$\sup_{x \in \mathbb{R}} |p_{F_n}(x) - \phi(x)| \leq c\sqrt{E(F_n^4) - 3} \leq c_H \begin{cases} n^{-\frac{1}{2}} & \text{if } H \in (0, \frac{5}{8}) \\ n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} & \text{if } H = \frac{5}{8} \\ n^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}) \end{cases}$$

Generalizations

(i) One can show the uniform approximation of the mth derivative of p_F by the corresponding mth derivative of the Gaussian density $\phi^{(m)}$ under the stronger assumption $E(\|DF\|_H^{-\beta}) < \infty$ for some $\beta > 6m + 6\left(\lfloor \frac{m}{2} \rfloor \vee 1\right)$.

Generalizations

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(ii) Consider a d-dimensional vector F, whose components are in a fixed chaos, and such that $E[(\det \Gamma_F)^{-p}] < \infty$ for all p, where Γ_F denotes the Malliavin matrix of F. In this case for any multi-index $\beta = (\beta_1, \dots, \beta_k)$, $1 \le \beta_i \le d$, one can show

$$\sup_{x \in \mathbb{R}^d} |\partial_\beta p_F(x) - \partial_\beta \phi_d(x)| \le c\Big(|C - I|^{\frac{1}{2}} + \sum_{j=1}^{d} \sqrt{E[F_j^4] - 3(E[F_j^2])^2}\Big)$$

where C is the covariance matrix of F, ϕ_d is the standard d-dimensional normal density, and $\partial_\beta = \frac{\partial^k}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}}$.

Further developments

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- *Examples* :

  (i) Fluctuations of the error in approximation schemes for SDE.
  (ii) Weighted Hermite variations of stationary Gaussian processes.
  (iv) Central limit theorems for the second and third moments in space of Brownian local time increments.

Further developments

- Rate of convergence in stable limit theorems when the limit is a mixture of Gaussian distributions ?
- *Examples* :
  - (i) Fluctuations of the error in approximation schemes for SDE.
  - (ii) Weighted Hermite variations of stationary Gaussian processes.
  - (iv) Central limit theorems for the second and third moments in space of Brownian local time increments.
- General results on the rate of convergence have been obtained by Nourdin-N.-Peccati '14 using an interpolation method and Malliavin calculus.

Example (Weighted quadratic variation of the fBm) :

$$F_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(B^H_{(k-1)/n}) \left[(n^H \Delta_{k/n} B^H)^2 - 1 \right].$$

Theorem (Nourdin-N.-Peccati '14)

Let $H \in (\frac{1}{4}, \frac{3}{4})$ *and* $f \in C^4(\mathbb{R})$ *such that* $|f^{(i)}(x)| \leq c_1 \exp(c_2 |x|^\beta)$ *for some* $\beta < 2$ *and for* $i = 0, \ldots, 4$. *Let*

$$S = \sqrt{\sigma_H \int_0^1 f^2(B^H_s) ds},$$

with $\sigma_H^2 = \sum_{k=-\infty}^{\infty} \rho_H(k)^2$. *Suppose* $E[S^{-\alpha}] < \infty$ *for some* $\alpha > 2$. *Then,*

$$|E[\varphi(F_n)] - E[\varphi(S\eta)]| \leq C_{f,H} \max_{1 \leq i \leq 5} \|\varphi^{(i)}\|_\infty \; n^{-(|2H - \frac{1}{2}| \wedge |2H - \frac{3}{2}|)},$$

where η *is a* $N(0,1)$ *random variable independent of* B^H.