Card shuffling, quantum mechanics and representation theory

Gady Kozma

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Cayley graphs

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When G is the finite group S_n, the choice of generators matters dramatically. We are still very far from appreciating the richness that hides in the choice of generators.

Interesting generators

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- A model for high rank linear groups

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- Helfgott-Seress-Żuk: For random generators the diameter is $\leq Cn^2$. (the conjectured diameter in this case is $\approx n \log n$)

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- $\lim_{n\to\infty} d(P_{(\frac{1}{2}-\epsilon)n\log n}, P_\infty) = 1$.

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Generalizations for other conjugacy classes include Roichman (1996), Larsen-Shalev (2008) and Berestycki-Schramm-Zeitouni (2011).

Representation theory

For any function f on S_n we can define its "Fourier transform" $\widehat{f}$ and it still satisfies that $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ so the random walk probabilities satisfy $\widehat{P_t} = \left(\widehat{P}\right)^t$.

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This is behind the analysis of Diaconis and Shahshahani. The actual values of $\widehat{f}$ go back to Frobenius (1901). More general results were obtained by Murnaghan (1937) and Nakayama (1940).

Take home message

Representation theory is great if your generating set is a class function

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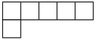
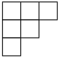
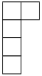
It has analogs of Parseval's formula and of the Fourier inversion formula.

The irreducible representations of S_n

Suppose $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r$ and $\sum \sigma_i = n$. We call σ a partition of n and denote this by $\sigma \vdash n$. Partitions may be represented graphically as *Young diagrams*.

$[5, 1] =$  $[3, 2, 1] =$  $[2, 1^3] =$ 

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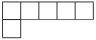
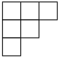
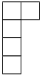
$[5, 1] =$  $[3, 2, 1] =$  $[2, 1^3] =$ 

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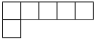
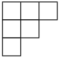
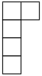
$[5, 1] =$  $[3, 2, 1] =$  $[2, 1^3] =$ 

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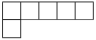
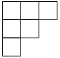
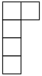
$[5,1] =$  $[3,2,1] =$  $[2,1^3] =$ 

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To relate the random walk on G to the random walk on the Cayley graph S_n we pass to *continuous time*. Define the *(positive) laplacian* $\Delta(g) = |S|(\mathbf{1} - P)$ (here Δ, P and $\mathbf{1}$ are $n! \times n!$ matrices). The probability to move from σ to τ at time t is now given by $e^{-t\Delta}(\sigma, \tau)$ (here this is matrix exponentiation).

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- When the graph $\mathbb{Z}^3$ is replaced by a tree there are results of Angel (2003) and Hammond (2013, preprint).

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- At small time the largest cycle should be logarithmic.
- At large time the largest cycle should be linear in the volume $n = r^3$.

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The conjecture has a natural analog for finite graphs. Take an $r \times r \times r$ cube (possibly identifying the sides cyclically) and examine the cycle structure of the stirring process at time t.

- At small time the largest cycle should be logarithmic.
- At large time the largest cycle should be linear in the volume $n = r^3$.

The finite version was investigated with the cube replaced with the complete graph by Berestycki-Durrett (2006), Schramm (2005), Berestycki-K and Alon-K.

An algebraically attackable version

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Let G be any graph and let $\lambda_1, \ldots, \lambda_{n-1}$ be the non-zero eigenvalues of the laplacian of continuous-time random walk on G. Let q_t be the probability that $X(t)$ is a cycle of length n. Then

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(as $t \to \infty$, the right-hand side converges to $\frac{1}{n}$, as it should. As $t \to 0$ one gets a new proof of the matrix-tree theorem).

Proof skeleton

Theorem

$$q_t = \frac{1}{n} \prod_{i=1}^{n-1} (1 - e^{-t\lambda_i})$$

Denote by Q the set of permutations which are on cycle of length n and write

$$q_t = \langle P_t, \mathbb{1}_Q \rangle = \langle \widehat{P}^t, \widehat{\mathbb{1}_Q} \rangle$$

Using Parseval's identity.

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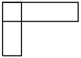
Using Parseval's identity. Since $\mathbb{1}_Q$ is a class function, its Fourier transform consists of scalar matrices. It is possible to calculate it explicitly and it is non-zero *only on the hook-shaped diagrams* $[n-k, 1^k] =$ .

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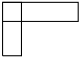
Using Parseval's identity. Since $\mathbb{1}_Q$ is a class function, its Fourier transform consists of scalar matrices. It is possible to calculate it explicitly and it is non-zero *only on the hook-shaped diagrams* $[n-k, 1^k] =$ . The hook-shaped diagrams are very special: the dimension of the representation $[n-k, 1^k]$ is $\binom{n-1}{k}$ and $\widehat{\Delta}([n-k, 1^k])$ may be diagonalized exactly and the eigenvalues are all sums of k-tuples of λ_i (Bacher, 1992).

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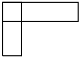
Using Parseval's identity. Since $\mathbb{1}_Q$ is a class function, its Fourier transform consists of scalar matrices. It is possible to calculate it explicitly and it is non-zero *only on the hook-shaped diagrams* $[n-k, 1^k] =$ . The hook-shaped diagrams are very special: the dimension of the representation $[n-k, 1^k]$ is $\binom{n-1}{k}$ and $\widehat{\Delta}([n-k, 1^k])$ may be diagonalized exactly and the eigenvalues are all sums of k-tuples of λ_i (Bacher, 1992). Combining this with the calculation of $\widehat{\mathbb{1}_Q}$ gives the theorem.

Tóth conjecture III

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Failure of the purely algebraic approach

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Consider the event Q' that the permutation is a cycle of length $n-1$ and one fixed point. Then $\widehat{\mathbb{1}_{Q'}}$ may still be calculated, but it is no longer supported on hook-shaped diagrams, but rather on diagrams with two rows and one column.

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Can we estimate $\widehat{\Delta}(\rho)$ analytically? At least for the relevant ρ i.e. with two rows and one column?

Comparison of eigenvalues

Theorem (Caputo-Liggett-Richthammer, 2010)

If $\rho \neq [n]$ *then* $\lambda_1(\rho) \geq \lambda_1([n-1,1])$.

($\lambda_1(\rho)$ stands for the smallest eigenvalue of $\widehat{\Delta}(\rho)$). In other words, the smallest non-zero eigenvalue of Δ is in the representation $[n-1,1]$.

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Take home message (speculative)

Representation theory is useful even when the generating set is not a class function, in combination with analytic methods.

Thank you