Singular Fluctuations of Interacting Particle Systems

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Occupation times of conservative systems

- $\{\eta_t(x); t \geq 0, x \in \mathbb{Z}\}$: conservative, one-dimensional stochastic system
- η_0: stationary state of density ρ
- Occupation-time problem: scaling limit of

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Theorem (Gonçalves, J. '13)

For diffusive systems,

$$\lim_{n\to\infty} \sqrt{n} \int_0^{tn^2} \big(\eta_s(0) - \rho\big)\, ds = \mathcal{Z}_t,$$

where $\{\mathcal{Z}_t; t \geq 0\}$ *is a fractional Brownian motion of Hurst index* $H = 3/4$*.*

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- Solutions "look like" Brownian motion of variance χ
- D is the mobility, χ is the static compressibility of the system
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Conservative version: Ornstein-Uhlenbeck equation

(OUE) $\mathcal{Y}_t = \partial_x X_t; \qquad \partial_t \mathcal{Y}_t = D\Delta \mathcal{Y}_t + \sqrt{2\chi D}\partial_x \dot{\mathcal{W}}_t$

- Solutions “look like” white noise of variance χ
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- Formally, we have

$$\mathcal{Z}_t = \int_0^t \mathcal{Y}_s(0)ds,$$

but the latter is not well defined. How do we define it?

$\rightarrow$ naïve way: ι_ϵ approximation of the identity,

$$\mathcal{Z}_t = \lim_{\epsilon \to 0} \int_0^t \mathcal{Y}_s(\iota_\epsilon)ds =: \mathcal{Z}_t^\epsilon$$

- How to get convergence? $\rightarrow$ Energy condition:

(EC) $$\mathbb{E}\Big[\big(\mathcal{Z}_t^\epsilon - \mathcal{Z}_t^\delta\big)^2\Big] \le Ct \min\{\epsilon, \delta\}$$

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Occupation times and singular functionals of OUE

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Theorem (Gonçalves, J. '13)

EC+ stationarity $\implies$ $\mathcal{Z}_t$ *well defined*

- EC is very easy for OUE $\to$ correlation computation
- EC is far from trivial for SBE $\to$ Hairer's Taylor expansion or GJ second-order Boltzmann-Gibbs principle.
- Does not depend on the choice of ι_ϵ
- (EC) holds uniformly for conservative systems $\implies$ convergence of occupation times

Theorem (Assing'07)

For the simple, symmetric exclusion process,

$$\mathcal{A}_t(f) = \lim_{n\to\infty} \frac{1}{n^2} \int_0^{tn^2} \sum_{x\in\mathbb{Z}} \big(\eta_s(x)-\rho\big)\big(\eta_s(x+1)-\rho\big) f\left(\tfrac{x}{n}\right) ds$$

exists for any smooth function f and it is equal to

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Theorem (Gonçalves, J.'14)

Assing's Theorem holds for general conservative systems.

At a formal level,

$$\mathcal{A}_t(f) = \int_0^t \int_{\mathbb{R}} \mathcal{Y}_s(x)^2 f(x) dx ds,$$

but again this *quadratic functional* is not well defined

$\to$ naïve interpretation:

$$\mathcal{Y}_t(x)^2 = \lim_{\epsilon \to 0} \mathcal{Y}_t * \iota_\epsilon(x)^2 - \frac{C(D,\chi)}{\epsilon}$$

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Quadratic fluctuations of conservative systems

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Energy condition:

$$\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}} \mathcal{Y}_s * (\iota_\epsilon - \iota_\delta)(x)^2 f(x) dx ds\Big)^2\Big] \leq Ct \min\{\epsilon, \delta\} \int f(x)^2 dx.$$

- This energy condition implies the existence of $\mathcal{A}_t$
- Easy to verify for OUE, harder for SBE
- Holds for conservative systems $\Longrightarrow$ convergence of quadratic functionals

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- Start with $\mathcal{Y}_t$, solution of OUE
- Define $\mathcal{Q}_t(x,y) = \mathcal{Y}_t(x) \otimes \mathcal{Y}_t(y)$

  $\rightarrow$ well defined, two-dimensional distribution-valued process
- $\mathcal{Q}_t$ solves

$$\partial_t Q_t = D\Delta \mathcal{Q}_t + \dot{\mathcal{M}}_t,$$

where $\dot{\mathcal{M}}_t$ is a noise satisfying

$$\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}^2} F_s(x,y) d\mathcal{M}_s\Big)^2\Big] = 2D\chi \int_0^t \int_{\mathbb{R}^2} \|\nabla F_s\|^2 dxdyds.$$

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- For $g : \mathbb{R}^2 \to \mathbb{R}$ regular enough, solve the Poisson equation

$$\Delta \psi_g = g$$

- We have the energy estimate

$$\mathbb{E}\Big[\Big(\int_0^t Q_s(g)ds\Big)^2\Big] \leq C\Big(\|\psi_g\|^2 + t\|\nabla\psi_g\|^2\Big)$$

- This estimate is all we need to make sense of the diagonal process

$$\mathcal{A}_t(f) = \int_0^t \int_{\mathbb{R}} Q_s(x,x)f'(x)dxds.$$

for regular functions $f : \mathbb{R} \to \mathbb{R}$.

How to square a distribution-valued process

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- Fine properties of $\mathcal{A}_t$:
- At small scales, it looks like fractional Brownian motion:

$$\epsilon^{-3/4}\mathcal{A}_{\epsilon t}(f) \xrightarrow{\epsilon\to 0} c\|f'\|^2 \mathcal{Z}_t$$

- At large scales, it looks like standard Brownian motion

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