Weak universality of the KPZ equation

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KPZ equation


Stochastic partial differential equation:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi \quad (d = 1)$$

Here $\xi$ is space-time white noise: Gaussian generalised random field with $E\xi(s, x)\xi(t, y) = \delta(t - s)\delta(y - x)$.

Model for propagation of nearly flat interfaces.
KPZ equation


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Flat interface propagation

Slope \( \partial_x h \ll 1 \).
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Strong Universality conjecture

At large scales, the fluctuations of every \(1 + 1\)-dimensional model \(\tilde{h}\) of interface propagation exhibits the same fluctuations as the KPZ equation. These fluctuations are self-similar with exponents \(1 - 2 - 3\):

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\lim_{\lambda \to \infty} \lambda^{-1} \tilde{h}(\lambda^2 x, \lambda^3 t) - \tilde{C}_\lambda t = c_1 \lim_{\lambda \to \infty} \lambda^{-1} h(c_2 \lambda^2 x, \lambda^3 t) - C_\lambda t.
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Spectacular recent progress: Amir, Borodin, Corwin, Quastel, Sasamoto, Spohn, etc. Relies on considering models that are “exactly solvable”. Partial characterisation of limiting “KPZ fixed point”: experimental evidence.
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Schematic evolution in “space of models” under rescaling (modulo height shifts):

KPZ equation just one model among many...
Universality for symmetric interface fluctuation models: exponents $1 - 2 - 4$, Gaussian limit. Picture for all interface models:

**KPZ equation:** red line.
Universality for symmetric interface fluctuation models: exponents 1 – 2 – 4, Gaussian limit. Picture for all interface models:

KPZ equation: red line.
**Weak Universality conjecture**

**Conjecture:** the KPZ equation is the *only* model on the “red line”.

**Conjecture:** Let \( \tilde{h}_\varepsilon \) be any “natural” one-parameter family of asymmetric interface models with \( \varepsilon \) denoting the strength of the asymmetry such that propagation speed \( \approx \sqrt{\varepsilon} \).

As \( \varepsilon \to 0 \), there is a choice of \( C_\varepsilon \sim \varepsilon^{-1} \) such that
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\sqrt{\varepsilon} \tilde{h}_\varepsilon(\varepsilon^{-1} x, \varepsilon^{-2} t) - C_\varepsilon t
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converges to solutions \( h \) to the KPZ equation.

Jara-Gonçalves (2010): accumulation points satisfy weak version of KPZ for generalisations of WASEP.
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**Problem:** KPZ equation is ill-posed:

\[ \partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi , \]

Solution behaves like Brownian motion for fixed times: nowhere differentiable!

**Trick:** Write \( Z = e^h \) (Hopf-Cole) and formally derive

\[ \partial_t Z = \partial_x^2 Z + Z\xi , \]

then interpret as Itô equation. WASEP behaves “nicely” under this transformation. Many other models do not...
Difficulties

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Recent progress

Theory of rough paths / regularity structures / paraproducts gives **direct meaning** to nonlinearity in a **robust** way:

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\mathcal{F} \times \mathcal{M} \times \mathcal{C}(S^1) \xrightarrow{S_A} \mathcal{D}_\gamma
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\Psi \quad \mathcal{R}
\]

\[
\mathcal{F} \times \mathcal{R} \times \mathcal{C}(S^1) \xrightarrow{S_C} \mathcal{C}(S^1 \times \mathbb{R}_+)
\]

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\]

**$\mathcal{F}$**: Constant $C$ in $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi_\varepsilon - C$.

**$S_C$**: Classical solution to the PDE with smooth input.

**$S_A$**: Abstract fixed point: locally jointly continuous!
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\[ F \times M \times C(S^1) \xrightarrow{S_A} D^\gamma \]

\[ \Psi \]

\[ \mathcal{F} \times \text{Noise} \times C(S^1) \xrightarrow{S_C} C(S^1 \times \mathbb{R}_+) \]

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\]

Strategy: find $M_\varepsilon \in \mathcal{R}$ such that $M_\varepsilon \Psi(\xi_\varepsilon)$ converges.
Universality result for KPZ

Consider the model

\[ \partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \sqrt{\varepsilon} P(\partial_x h_\varepsilon) + \eta, \]

with \( P \) an even polynomial, \( \eta \) a Gaussian field with compactly supported correlations \( \varrho(t, x) \) s.t. \( \int \varrho = 1 \).

**Theorem (H., Quastel, 2014)** As \( \varepsilon \to 0 \), there is a choice of \( C_\varepsilon \sim \varepsilon^{-1} \) such that \( \sqrt{\varepsilon} h(\varepsilon^{-1} x, \varepsilon^{-2} t) - C_\varepsilon t \) converges to solutions to \((\text{KPZ})_{\lambda}\) with \( \lambda \) depending in a non-trivial way on all coefficients of \( P \).

**Remark:** Convergence to KPZ with \( \lambda \neq 0 \) even if \( P(u) = u^4 \)!!
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Case $P(u) = u^4$

Write $\tilde{h}_\varepsilon(x, t) = \sqrt{\varepsilon} h(\varepsilon^{-1} x, \varepsilon^{-2} t) - C_\varepsilon t$. Satisfies

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 h_\varepsilon + \varepsilon (\partial_x \tilde{h}_\varepsilon)^4 + \xi_\varepsilon - C_\varepsilon,$$

with $\xi_\varepsilon$ an $\varepsilon$-approximation to white noise.

Fact: Derivatives of microscopic model do not converge to 0 as $\varepsilon \to 0$: no small gradients! Heuristic: gradients have $O(1)$ fluctuations but are small on average over large scales... General formula:

$$\lambda = \frac{1}{2} \int P''(u) \mu(du), \quad C_\varepsilon = \frac{1}{\varepsilon} \int P(u) \mu(du) + O(1),$$

with $\mu$ a Gaussian measure, explicitly computable variance.
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Main step in proof

Rewrite general equation in integral form as

\[ H = \mathcal{P}(\mathcal{E}(\mathcal{D}H)^4 + a(\mathcal{D}H)^2 + \Xi), \]

with \( \mathcal{E} \) an abstract integration operator of order 1.

Find two-parameter lift of noise \( \eta \mapsto \Psi_{\alpha,c}(\eta) \) so that \( h = \mathcal{R}H \) solves

\[
\partial_t h = \partial_x^2 h + \alpha H_4(\partial_x h, c) + a H_2(\partial_x h, c) + \eta \\
= \partial_x^2 h + \alpha(\partial_x h)^4 + (a - 6\alpha c)(\partial_x h)^2 + (3\alpha c^2 - ac) + \eta.
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Show that \( \Psi_{\varepsilon,1/\varepsilon}(\xi_{\varepsilon}) \) converges to same limit as \( \Psi_{0,1/\varepsilon}(\xi_{\varepsilon}) \).

(Actually \( \Psi_{\beta\varepsilon,1/\varepsilon}(\xi_{\varepsilon}) \) for every \( \beta \in \mathbb{R} \)...
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Outlook

Some open questions:

1. Strong Universality without exact solvability???
2. Convergence of particle models. (Are “weak solutions” unique??)
3. Convergence on whole space instead of circle (cf. Labbé).
5. Fully nonlinear continuum models.
6. Control over larger scales to see convergence to KPZ fixed point.
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