Spectral universality for a general class of matrices

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INTRODUCTION

Basic question [Wigner]: What can be said about the statistical properties of the eigenvalues of a large random matrix? Do some universal patterns emerge?

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \implies (\lambda_1, \lambda_2, \dots, \lambda_N) \quad \text{eigenvalues?}$$

N = size of the matrix, will go to infinity.

Analogy: Central limit theorem: $\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$

Wigner Ensemble:

$H = (h_{jk})_{1\le j,k\le N}$ complex hermitian $N \times N$ matrix

$h_{jk} = \bar h_{kj}$ (for $j < k$) are complex and h_{kk} are real independent random variables with normalization

$$\mathbb{E} h_{jk} = 0, \qquad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}.$$

The eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ are of order one: (on average)

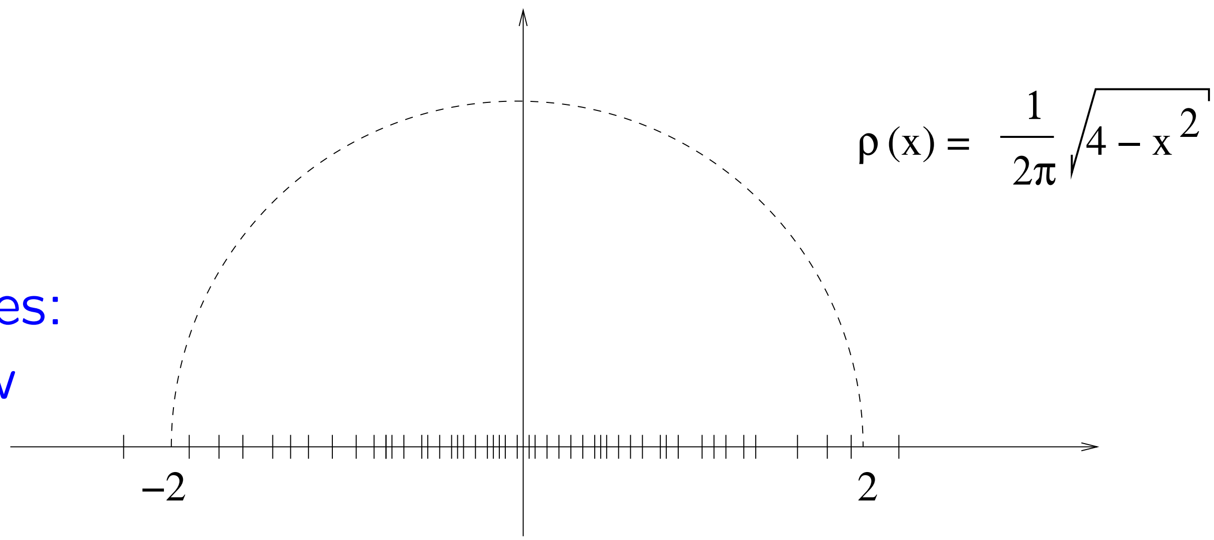
$$\mathbb{E}\frac{1}{N}\sum_i \lambda_i^2 = \mathbb{E}\frac{1}{N}\mathrm{Tr} H^2 = \frac{1}{N}\sum_{ij}\mathbb{E}|h_{ij}|^2 = 1$$

Complex hermitian can be replaced with real symmetric or quaternion self-dual.

If h_{ij} is Gaussian, then GUE, GOE, GSE.

Wigner's observations (holds for all symmetry classes)

i) Density of eigenvalues:
Wigner semicircle law



ii) Level repulsion: Wigner surmise (in the bulk and for GOE)

$$\mathbb{P}\Big(N(\lambda_{i+1}-\lambda_i) = s + \mathrm{d}s\Big) \approx \frac{\pi s}{2}\exp\Big(-\frac{\pi}{4}s^2\Big)\mathrm{d}s$$

Guessed by a 2x2 matrix calculation

SINE KERNEL FOR CORRELATION FUNCTIONS

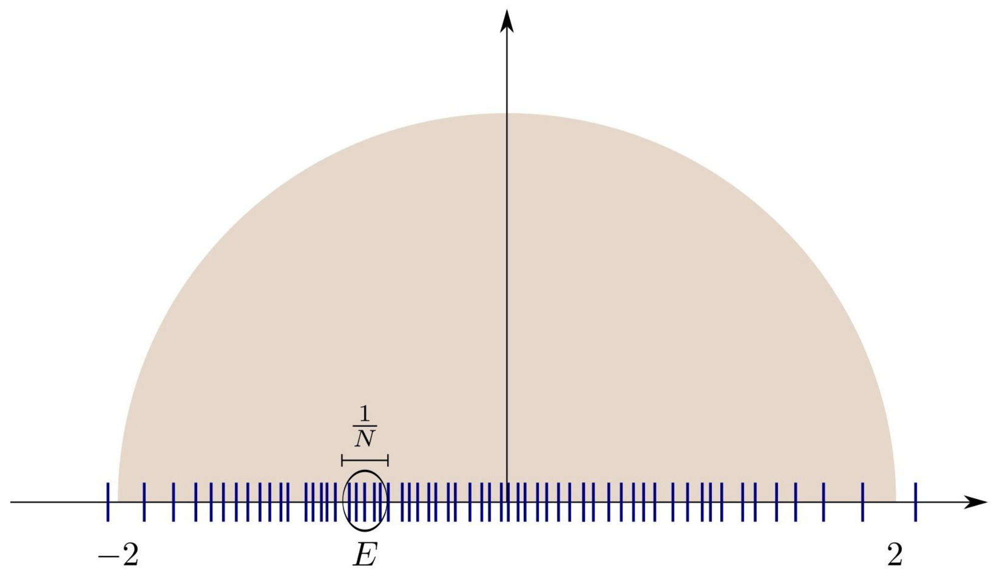
Probability density of the eigenvalues: $p(x_1, x_2, \ldots, x_N)$

The k-point correlation function is given by

$$p_N^{(k)}(x_1, x_2, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} p(x_1, \ldots x_k, x_{k+1}, \ldots, x_N) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N$$

Special case: $k = 1$ (density)

$$\varrho_N(x) := p_N^{(1)}(x) = \int_{\mathbb{R}^{N-1}} p(x, x_2, \ldots, x_N) \mathrm{d}x_2 \ldots \mathrm{d}x_N$$



Rescaled correlation functions at energy E

$$p_E^{(k)}(\mathbf{x}) := \frac{1}{[\varrho(E)]^k} p_N^{(k)}\Big(E + \frac{x_1}{N\varrho(E)}, E + \frac{x_2}{N\varrho(E)}, \ldots, E + \frac{x_k}{N\varrho(E)}\Big)$$

Rescales the gap $\lambda_{i+1} - \lambda_i$ to O(1).

Local level correlation statistics for GUE [Gaudin, Dyson, Mehta]

k-point correlation functions are given by $k \times k$ determinants:

$$\lim_{N\to\infty} p_E^{(k)}(\mathbf{x}) = \det\left\{S(x_i - x_j)\right\}_{i,j=1}^{k}, \qquad S(x) := \frac{\sin \pi x}{\pi x}$$

The limit is independent of E as long as $|E| < 2$ (bulk spectrum)

Gap distribution can be obtained from correlation functions by the exclusion-inclusion formula. Wigner surmise is quite precise.

Main question: going beyond Gaussian towards universality!

Wigner-Dyson-Mehta conjecture: Local statistics is universal in the bulk spectrum for any Wigner matrix; only symmetry type matters.
Solved recently for any symmetry class:

[E-Schlein-Peche-Ramirez-Yau, 2009] – Hermitian case, fixed E

[E-Schlein-Yau-Yin, 2010] – averaged E

[E-Yau, 2012] – fixed gap label

[Bourgade-E-Yau-Yin, 2014] – fixed E

Related results:

[Johansson, 2000] Hermitian case with large Gaussian components

[Tao-Vu, 2009] Hermitian case via moment matching.

(Similar development for the edge and for β-log gases).

Three-step strategy:

1. Local (entry-wise) semicircle law down to scales $\gg 1/N$.

2. Use local equilibration of Dyson Brownian motion to prove universality for matrices with a tiny Gaussian component

3. Use perturbation theory to remove the tiny Gaussian component.

All these results were obtained under the condition that the matrix elements h_{ij} are independent, centered and

$$\sum_j s_{ij} = 1, \qquad s_{ij} := \mathbb{E}|h_{ij}|^2$$

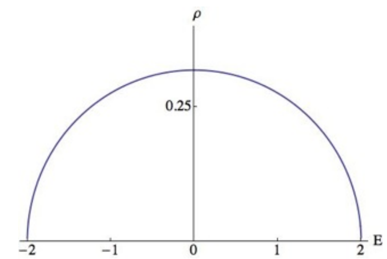
Today's talk is about Wigner matrices without this red condition We'll call them Wigner-type matrices.

Red was used at many places in the previous analysis:

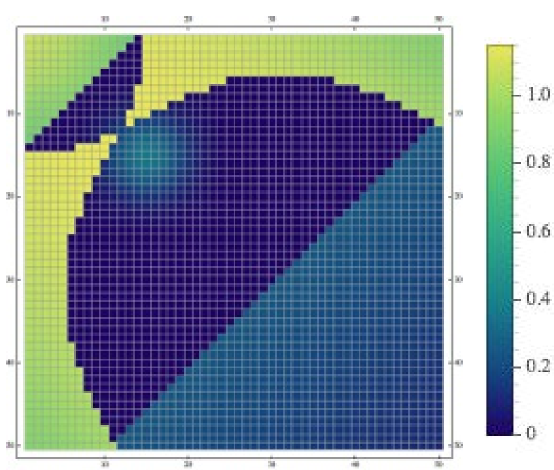
- Limiting density is explicit.
- Homogeneity: $G_{ii} \approx G_{jj}$ for the resolvent matrix elements.
- DBM: Initial data is already close to global equilibrium.

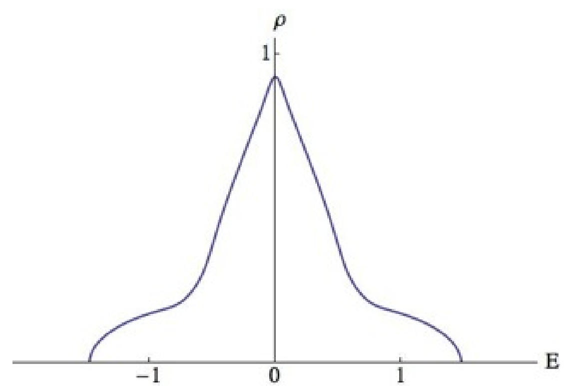
Variance profile and limiting density of states (DOS)

$$\sum_j s_{ij} = 1 \qquad \Longleftrightarrow$$



General variance profile $s_{ij} = \mathbb{E}|h_{ij}|^2$: not the semicircle any more.





$\sum_j s_{ij} \neq$ const $\qquad \Longrightarrow \qquad$ Density of states

Main theorem (informally)

Theorem [Ajanki-E-Krüger] Let $H = H^*$ be a Wigner-type matrix

$$\bar{h}_{ji} = h_{ij} \quad \text{independent, centered}$$

$$\mathbb{E}|h_{ij}|^2 = s_{ij} = \frac{1}{N} S(\frac{i}{N}, \frac{j}{N})$$

with a limiting profile function $S : [0,1]^2 \to \mathbb{R}_+$. Then for the matrix elements of the resolvent $G = (H - z)^{-1}$, we have

$$G_{ij}(z) \approx \delta_{ij} m_{\frac{i}{N}}(z)$$

where $m_x(z)$ solves the self-consistent equation

$$-\frac{1}{m_x(z)} = z + \int_0^1 S(x,y) m_y(z) \mathrm{d}y \qquad (*)$$

Limiting DOS $\qquad \varrho(E) := \frac{1}{\pi} \int_0^1 \mathrm{Im}\, m_x(E + i0) \mathrm{d}x$

Note: The nonlinear vector equation (*) replaces the usual self consistent scalar eq $m^{-1} = -(z + m)$ of the semicircle density.

Constantness of row sums, $\sum_j s_{ij} = 1$, implies semicircle

$$G(z) := (H - z)^{-1}, \quad z = E + i\eta, \;\; \eta > 0$$

Schur formula with the resolvent of the i-th minor

$$\begin{aligned} \frac{1}{G_{ii}} &= h_{ii} - z - \sum_{ab} h_{ia} G^{(i)}_{ab} h_{bi} \\ &\approx \; - z - \mathbb{E}^{(i)} \sum_{ab} h_{ia} G^{(i)}_{ab} h_{bi} \\ &\approx \; - z - \sum_a s_{ia} G_{aa} \end{aligned}$$

Fact: The self-consistent equation

$$\frac{1}{m_i} = -z - \sum_a s_{ia} m_a, \qquad \text{Im } m_a > 0,$$

has a unique solution.

It is constant, $m_a = m$, iff the row sums are constant and then

$$\frac{1}{m} = -z - m \qquad \Longrightarrow \qquad \text{Stieltjes transform of the semicircle}$$

Quadratic vector equation (QVE)

Suppose s_{ij} is given by a limiting profile function $S : [0,1]^2 \to \mathbb{R}_+$:

$$s_{ij} = \frac{1}{N} S(\frac{i}{N}, \frac{j}{N}),$$

Continuum limit of the self-consistent equation for $G_{ii}(z) \approx m_{\frac{i}{N}}(z)$

$$-\frac{1}{m(z)} = z + Sm(z), \qquad (Sf)_x = \int_0^1 S(x,y) f_y \mathrm{d}y \quad (QVE)$$

For any $z \in \mathbb{H}$ (complex upper half plane), we consider solutions under the constraint $\text{Im}\, m > 0$,

Fact: Solution exists and is unique.

Fact: The solution is not constant in general. Semicircle is the "easiest" case.

Two main steps for the proof of the main theorem:

1) Analyse the solution of the continuum QVE, including its stability (no matrix, no N)

2) Prove: the resolvent of the RM is close to the solution of QVE. (Schur, fluctuation averaging, dichotomy becomes trichotomy)

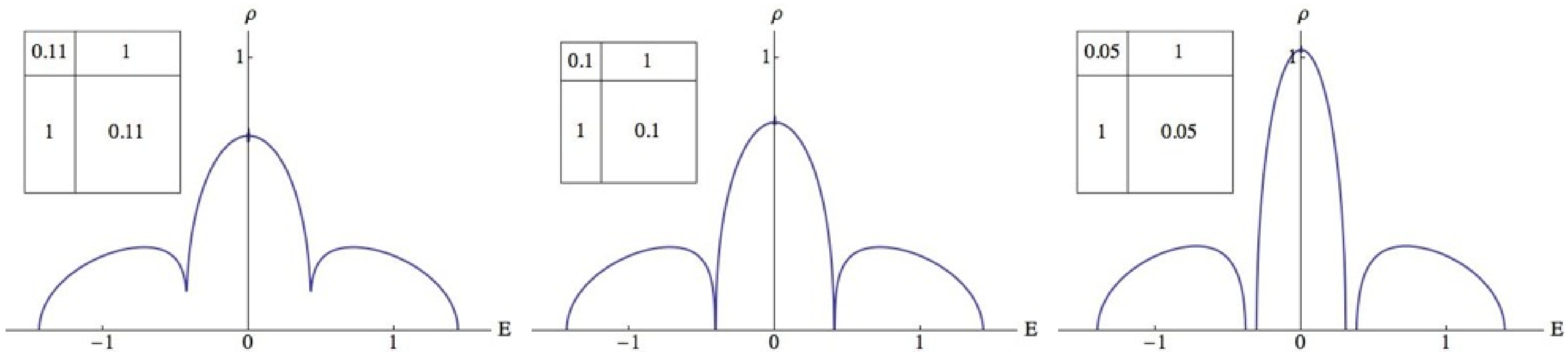
Note: If $\sum_j s_{ij} = 1$, Step 1 is trivial, since the solution $m_x(z) = m(z)$ is given explicitly by a quadratic scalar equation $-m^{-1} = z + m$. So all previous efforts to prove local semicircle law was in Step 2.

If $\sum_j s_{ij} \neq$ const, Step 1 is nontrivial and gives rise to a complex pattern.

Despite its natural form, QVE has not been studied quantitatively.

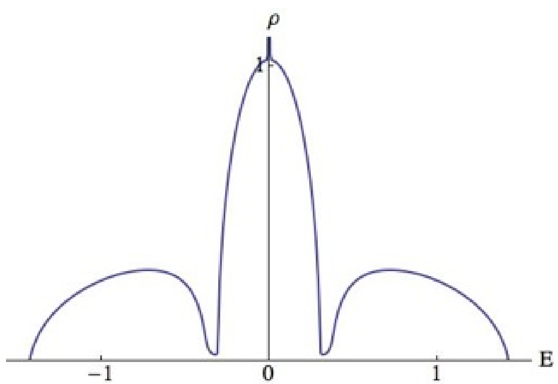
Features of the DOS for Wigner-type matrices

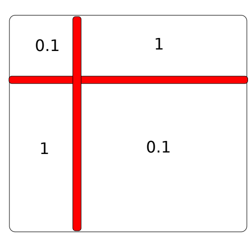
1) Support splits via cusps:



(Matrices in the pictures represent the variance matrix)

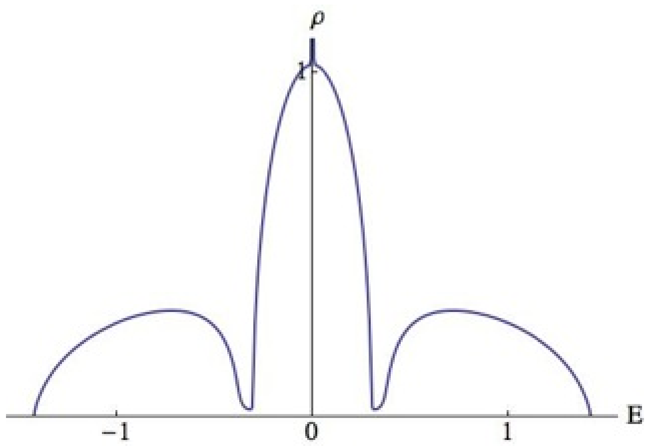
2) Smoothing of the S-profile avoids splitting ($\Rightarrow$ single interval)

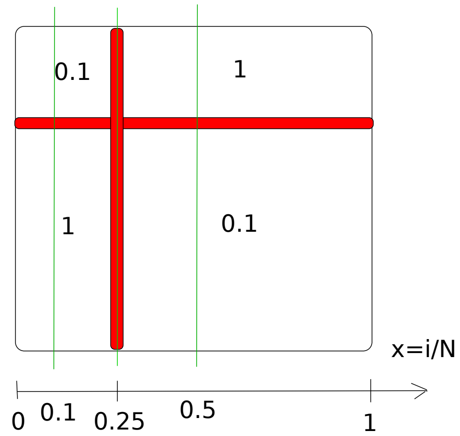




DOS of the same matrix as above but discontinuities in S are regularized

Relation between m_x and $m := \text{Av}_x m_x$

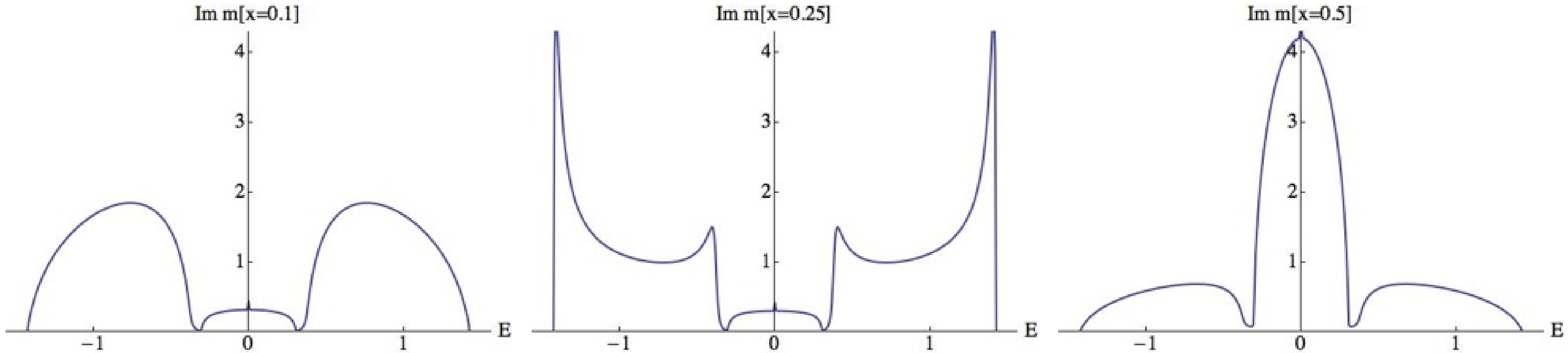




$x = \frac{i}{N}$ continuum coordinates

Red: some interpolation

$\text{Im}\, m_x \not\approx \varrho = \text{Im}\, m$. It may even behave very differently for some x:



Sections of $\text{Im}\, m_x(E)$ at various x's indicated by the green lines.

Natural questions

1) How many intervals are there and what determines them?

2) Blow-up features and instability mechanisms

3) Universality of the singularity patterns?

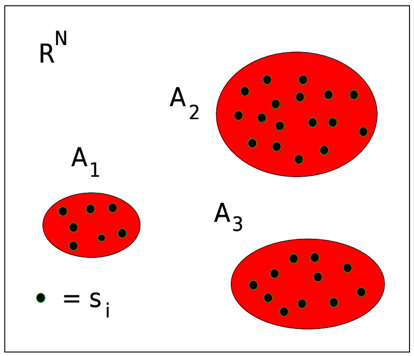
Number of intervals in the support of the DOS

Consider the set of row vectors of S

$$A := \{\mathbf{s}_i \;:\; i = 1, 2, \ldots, N\} \subset \mathbb{R}^N, \qquad (\mathbf{s}_i)_j := s_{ij}$$

Partition

$$A = A_1 \cup A_2 \cup \ldots \cup A_n, \qquad \text{s.t} \quad \mathrm{dist}(A_k, A_\ell) \geq \delta$$



Key object:
n=Number of lumps

Conjecture: # spectral intervals $\leq 2n - 1$. We proved for $n = 1$

E.g. $s_{ij} = \frac{1}{N} S(\frac{i}{N}, \frac{j}{N})$ with $S(x, y)$ smooth $\Rightarrow n = 1$

Theorem [Ajanki-E-Krüger] If all lumps are macroscopic in the sense

$$\inf_x \int_0^1 \frac{1}{\|\mathbf{s}_x - \mathbf{s}_y\|^2} \mathrm{d}y \geq C > 0$$

then the solution $m_x(z)$ of

$$-\frac{1}{m} = z + Sm, \qquad (QVE)$$

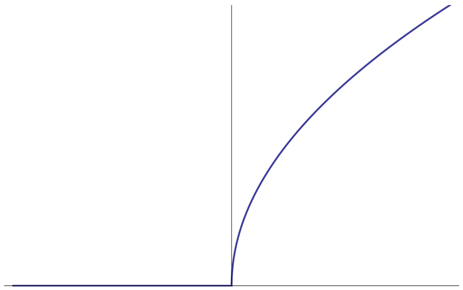
is bounded; $m_x(z)$ is the Stieltjes transform of an a.c. measure

$$m_x(z) = \int_{\mathbb{R}} \frac{v_x(s)}{s - z} \mathrm{d}s.$$

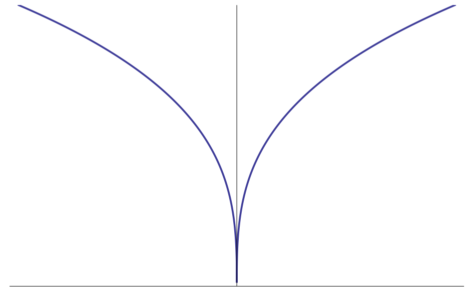
If S is irreducible, then the components are comparable, $\frac{v_x(E)}{v_y(E)} \sim 1$ for all E. The density, $\varrho = \int v_x \mathrm{d}x$, is compactly supported, bounded, and it has a universal shape near the points when it (almost) vanishes.

In particular, blowup can occur only if there is a small lump.
(Discontinuity in S is OK, but isolated row is not)

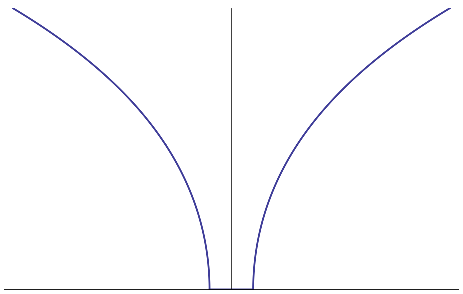
Universality of the singularities in the DOS



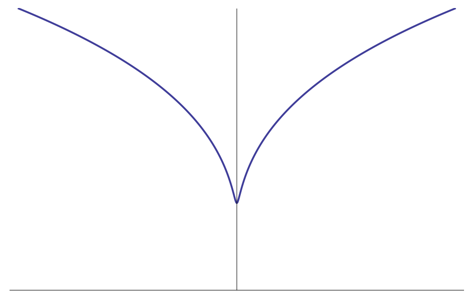
Edge, $\sqrt{E}$ singularity



Cusp, $|E|^{1/3}$ singularity



Small-gap



Smoothed cusp

$$\frac{(2+\tau)\tau}{1+(1+\tau+\sqrt{(2+\tau)\tau})^{2/3}+(1+\tau-\sqrt{(2+\tau)\tau})^{2/3}}$$

$$\frac{\sqrt{1+\tau^2}}{(\sqrt{1+\tau^2}+\tau)^{2/3}+(\sqrt{1+\tau^2}-\tau)^{2/3}-1}-1$$

$$\tau := \frac{|E|}{\text{gap}},$$

$$\tau := \frac{|E|}{(\text{minimum })^{1/3}}$$

Why cubic?

Stability of QVE: (used for both QVE analysis and RM)

$$-\frac{1}{m} = z + Sm, \qquad -\frac{1}{m^{(\varepsilon)}} = z + Sm^{(\varepsilon)} + \varepsilon, \qquad \|\varepsilon\| \ll 1$$

Decompose along the evector $(1 - |m|^2 S)f = 0 \quad (f > 0)$

$$m^{(\varepsilon)} - m = \Theta f + v, \qquad \|v\| \leq \Theta^2 + O(\varepsilon)$$

Decompose the third order perturbation expansion along f and $f^{\perp}$.

$$\tau_3 \Theta^3 + \tau_2 \Theta^2 + \tau_1 \Theta \sim O(\varepsilon)$$

Facts

$$|\tau_3| + |\tau_2| \neq 0 \qquad \Longrightarrow \qquad \text{not more than cubic}$$

$$\tau_2 = \langle (\text{sgnRe}\, m), f^3 \rangle$$

If $\tau_2 = 0$ then cubic (atypical). For the semicircle case, $\text{Re}\, m = const$ and $f = 1$, so $\tau_2 \neq 0$, thus quadratic (typical).

Precision, rigidity

We prove optimal entry-wise local law

$$\Big|G_{jk}(z) - \delta_{jk} m_j(z)\Big| \prec \sqrt{\frac{\varrho(E)}{N\eta}} + \frac{1}{N\eta}, \qquad z = E + i\eta$$

(also for the density and for the "isotropic" version). At the cusps the (current) estimate is slightly weaker than optimal.

In terms of rigidity, i.e. comparing eigenvalues λ_i with the corresponding quantiles γ_i of the limiting density, we have

$$|\lambda_i - \gamma_i| \prec N^{-1} \qquad \text{bulk}$$

$$|\lambda_i - \gamma_i| \prec N^{-2/3} \qquad \text{edges (also internal)}$$

$$|\lambda_i - \gamma_i| \prec N^{-3/5} \qquad \text{cusps}$$

Optimal scale at the cusps should be $N^{-3/4}$.

Application: correlated Gaussian matrices

Consider a hermitian matrix with correlated Gaussian entries:

$$\mathbb{E}h_{jk} = 0, \qquad \mathbb{E}h_{jk}\overline{h_{j'k'}} = \frac{1}{N}\Big(R_{j-j',k-k'} + Q_{j-k',k-j'}\Big)$$

where R, Q have a decay

$$\sum_{j,k}\Big[|R_{jk}| + |Q_{jk}|\Big](|j| + |k|) < \infty$$

For example, such ensemble can be obtained by filtering

$$H = K \star X + c.c. \qquad K_{jk} \quad \text{kernel with} \quad \sum_{jk}|K_{jk}| \sim 1$$

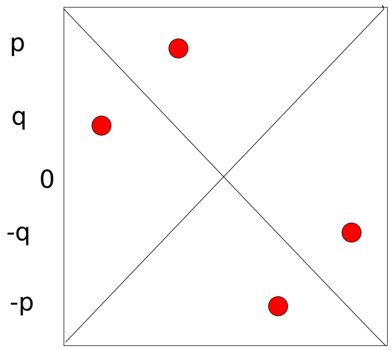
and X has i.i.d. centred Gaussian entries (no symmetry)

$$\mathbb{E}|X_{jk}|^2 = \frac{1}{N}, \quad \mathbb{E}X_{jk}^2 = \frac{\gamma}{N}$$

Then $R = (K \star \tau\bar{K}) + c.c.$ with $(\tau K)_{jk} = K_{-j,-k}$.

Good news: In Fourier space, $\widehat{H}$ has almost independent entries:

$$\widehat{h}_{pq} \perp \widehat{h}_{p'q'} \quad \text{unless} \quad (p',q') \in \Big\{(p,q),(q,p),(-p,-q),(-q,-p)\Big\}$$



Wigner matrix with four-fold symmetry
(hermitian + reflection)
Analysis goes through with this extra symmetry.

Solve the QVE in Fourier space

$$-\frac{1}{m_p(z)} = z + \int_0^1 \widehat{R}_{pq} m_q(z) \mathrm{d}q$$

(only R matters, Q is irrelevant)

Apply the previous theorem for $S = \widehat{R}$, get optimal asymptotics for $(\widehat{H} - z)^{-1}$, then Fourier transform back (using the isotropic law).

Theorem [Ajanki-E-Krüger]: Under a nondegeneracy condition, $\inf_p \sup_q |\widehat{R}_{pq}| > 0$, (holds for generic convolution kernels), we have

$$\max_{jk} \Big| G_{jk}(z) - g_{j-k}(z) \Big| \prec \sqrt{\frac{\varrho(E)}{N\eta}}$$

for the resolvent $G = (H - z)^{-1}$ of the correlated Gaussian RM. Here

$$g_k(z) = \int_0^1 e^{-2\pi i k p} m_p(z) \mathrm{d}p$$

is the Fourier transform of the solution of the QVE. It inherits the decay of R. Note that G_{jk} is not concentrated to $j = k$.
Similar optimal result for the DOS.

Previous results:

[Schenker, Schulz-Baldes, 2005], [Götze, Naumov, Tikhomirov, 2013] Weak dependence, DOS=sc

[Anderson-Zeitouni, 2008] DOS on macro scale with moment method in case finite range correlation.

[Pastur-Shcherbina,2011] DOS on macro scale with resolvents.

Local spectral universality

Theorem [Ajanki-E-Krüger-Schnelli] In all models above, bulk local spectral universality holds (in the sense of fixed label or averaged energy).

There is a more general theorem behind which extends previous analysis of the local equilibration of the DBM flow to arbitrary matrix initial condition.

Previous results applied to:

i) initial matrix follows semicircle [E-Schlein-Yin-Yau]

ii) deformed Wigner matrices with DOS with a single interval support [Lee-Schnelli-Stetler-Yau]

Summary

- Local laws for Wigner-like matrices (independent entries, arbitrary variance matrix).

- Complete analysis of a NL equation $Sm + z = -\frac{1}{m}$.

- Singularities of the DOS are universal.

- Optimal local laws for the translation invariant correlated Gaussian ensemble.

- Bulk universality in all these models