A Model for Random Growth with Memory

Averaging Principle and Shape Theorem

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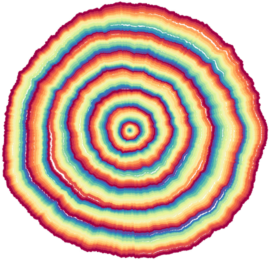
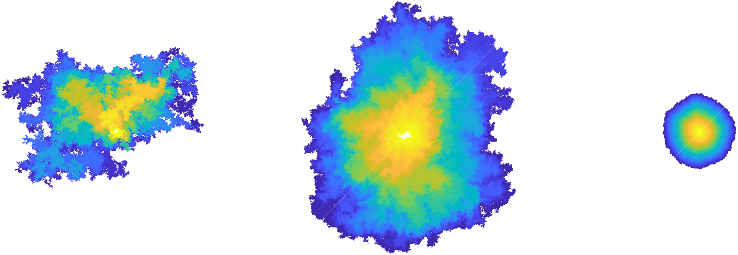


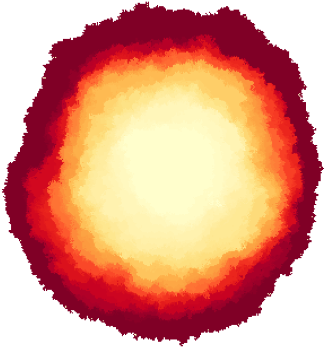
Motivation: some self-interacting random walks



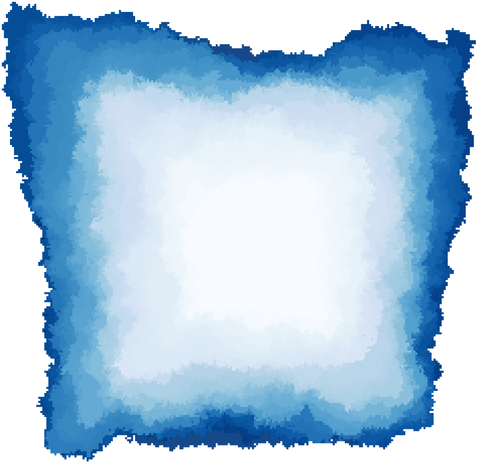
Reinforcement strength $a = 2$ (left), $a = 3$ (middle), $a = 100$ (right) in a box of size 2000. Color proportional to $\sqrt{\cdot}$ of vertex first visit time.

Origin-Excited RW Kozma 06'

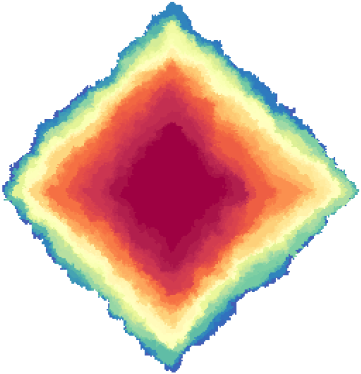
$$\mathbb{E}[X_{t+1} - X_t | \mathcal{F}_t] = -\frac{\delta}{|v|} v \text{ if } X_t = v \text{ is first visit of } v \in \mathbb{Z}^d.$$



Back in the largest direction



Back one unit in both directions



Other models

Eden 61' - First Passage Percolation (Hammersley-Welsh 65')

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Continuous time, $d = 2$, Gravner-Quastel, 2000. General: Levine-Peres, 2010
Logarithmic fluctuations: Asselah-Gaudillière / Jerison-Levine-Sheffield.

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OERW, ORRW: Shape Theorem $\rightarrow$ Open problem.

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- Particle is *fast* and domain is *slow*.
- The process $(R_t^\epsilon, x_t^\epsilon)$ is jointly Markov.
- The jump probabilities are determined by a *hitting probabiliy density* $F(r, x, \cdot)$ on $\mathbb{S}^{d-1}$ and *transportation rule* $H(r, \xi)$

$$F \colon C(\mathbb{S}^{d-1}) \times \mathbb{R}^d \to L^2(\mathbb{S}^{d-1}), \qquad H \colon C(\mathbb{S}^{d-1}) \times \mathbb{R}^d \to \mathbb{R}^d.$$

The model $(d = 2)$

Fix $\epsilon \in (0,1]$. $(R^\epsilon_t, \xi^\epsilon_t)$ is a Markov procsess on $C(\mathbb{S}^1, \mathbb{R}_{\geq 0}) \times \mathbb{R}^2$ that jumps at rate $1/\epsilon$ and has transitions given by

$$
\begin{aligned}
&R^\epsilon_t = R^\epsilon_{t-} + \sqrt{\epsilon} g\left(\tfrac{\cdot - \xi^\epsilon_t}{\sqrt{\epsilon}/y_{R^\epsilon_{t-}, x^\epsilon_{t-}}}\right) \\
&x^\epsilon_t \sim H(R^\epsilon_t, \xi^\epsilon_t, \cdot).
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- $\xi_t^\epsilon \sim F(R_{t-}^\epsilon, x_{t-}^\epsilon, \cdot)$
- g is a "bump" function.
- $y_{r,x} := \int_{\mathbb{S}^1} r(\theta) F(r, x, \theta)\, d\theta$ is a normalizing constant to guarantee that the (expected) volume added is approx. ϵ.

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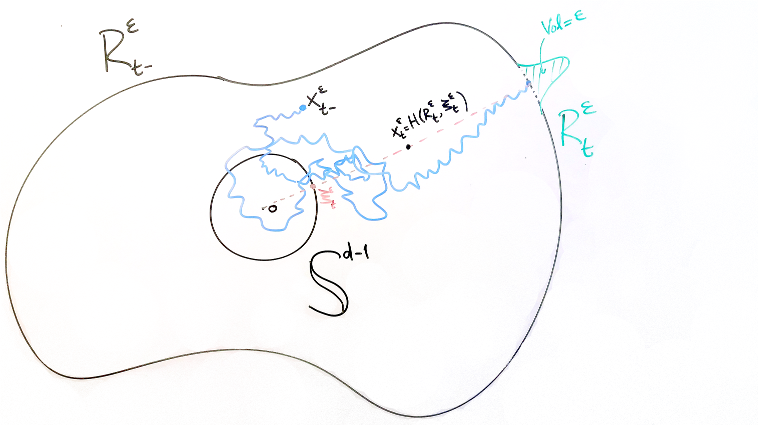

$F(r, x, \cdot) =$ density of harmonic measure of r from x $\qquad H(r, \xi) = \alpha r(\xi).$

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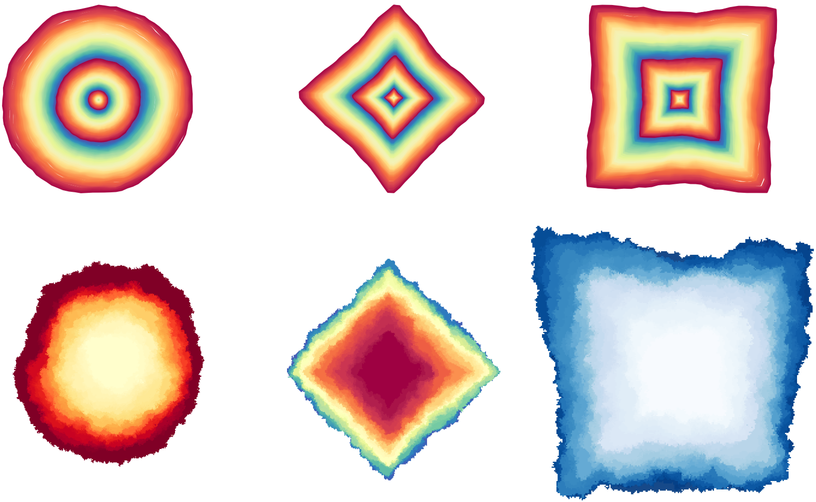
$H(r,\xi) = .99r(\xi)\xi,$ $F =$ **harmonic measure,** $\epsilon = .02$

$H(r,\xi) = (1 - \frac{|\xi|_\infty}{10|\xi|_2}) r(\xi)\xi, \qquad F =$ harmonic measure, $\epsilon = 10^{-6}$

Shapes process vs Origin-Excited Random Walk

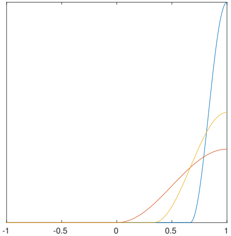


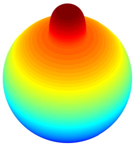
Small bump on $\mathbb{S}^{d-1}$

$g_\eta(s) = c\eta^{-(n-1)}\phi\big(1 - \frac{1-s}{\eta^2}\big)$ for some density $\phi \in C([-1,1], \mathbb{R}_+)$.

$||f \star g_\eta - f||_2 \to 0$ as $\eta \to 0$ ($\star$ denotes spherical convolution).

Add $\epsilon^{1/n}\eta^{n-1}g_\eta(\langle\xi,\cdot\rangle)$ for a bump of height $O(\epsilon^{1/n})$ & support on the spherical cap of Euclidean radius 2η centered at angle ξ.

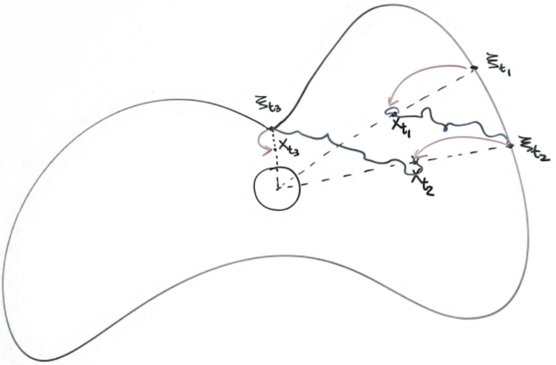




L: $g_\eta(s)$ at different η.

R: Adding $g_\eta(\langle z,\cdot\rangle)$ to $\mathbb{S}^2$; $z = (0,0,1)$.

Frozen domain dynamics



$F(r, X_t, \cdot)$ $\qquad$ $H(r, \xi_t)$

$(X^r_t)_{t \geq 0}$ particle process in frozen domain r

For any $r \in C(S^{n-1})$ *frozen domain* the particle the process $(x_t^r)_{t\geq 0}$ has a unique invariant probability measure ν_r, such that

$$\sup_{r\in\mathcal{A}_1(a)} \sup_{t_0\geq 0} \mathbb{E}\left[\left\| \frac{1}{t}\int_{t_0}^{t_0+t} [b(r, x_s^{1,r}) - \bar{b}(r)] ds \right\|_2^2\right] \leq \lambda(t, a) \to 0$$

as $t \to \infty$, for any fixed $a \in (0, 1)$.

$$b(r, x)(\cdot) = \frac{F(r, x, \cdot)}{y_{r,x}}, \quad \bar{b}(r)(\cdot) = \int_{\mathbb{R}^n} b(r, x)(\cdot)\, d\nu_r(x).$$

$$\mathcal{A}_1(a) := \left\{ r \in C(S^{n-1}) : \inf_\theta \{r(\theta)\} \geq a,\ \|r\|_2 \leq a^{-1} \right\}$$

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Uniform minorization of jump kernel P_r of $\{x_{T_i}^{1,r}\}$:

$$\inf_{\substack{r\in\mathcal{A}(a)\\ x\in \mathrm{Im}H}} \{(\mathsf{P}_r)^{n_0}(x,\cdot)\} \geq m(\cdot) \qquad \Longrightarrow \qquad \text{(E)}\,.$$

Averaging Principle

Heuristics

Simpler case: F independent of x

$$\begin{aligned}\mathbb{E}(R^\epsilon_t(\theta) - R^\epsilon_{t-}(\theta)|\mathcal{F}_t) &= \sqrt{\epsilon}\int_{\mathbb{S}^1} g\Big(\frac{\theta-\xi}{\sqrt{\epsilon}/y_{R^\epsilon_{t-}}}\Big)F(R^\epsilon_{t-},\xi)\,d\xi \\ &= \frac{\epsilon}{y_{R^\epsilon_{t-}}}(g_\epsilon \star F)(\theta) \sim \epsilon\frac{F(R^\epsilon_{t-},\theta)}{y_{R^\epsilon_{t-}}}\end{aligned}$$

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$$R^\epsilon_\cdot \to \bar{r}. \qquad \text{as } \epsilon \to 0.$$

$$\frac{d}{dt}\bar{r}_t(\theta) = b(\bar{r}_t,\theta), \qquad b(r,\cdot) = \frac{F(r,\cdot)}{y_r}.$$

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$$\mathbb{E}(R_t^\epsilon(\theta) - R_{t-}^\epsilon(\theta)|\mathcal{F}_t) = \sqrt{\epsilon}\int_{\mathbb{S}^1} g\Big(\frac{\theta-\xi}{\sqrt{\epsilon}/y_{R_{t-}^\epsilon}}\Big)F(R_{t-}^\epsilon,\xi)\,d\xi$$

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General case:

$$R_\cdot^\epsilon \to \bar{r}_\cdot \qquad \text{as } \epsilon \to 0.$$

$$\frac{d}{dt}\bar{r}_t(\theta) = \int_{\mathbb{R}^d} b(\bar{r}_t,x,\theta)\nu_r(x)\,dx, \qquad b(r,x,\cdot) = \frac{F(r,x,\cdot)}{y_{r,x}}.$$

Problem: prove this for F and H as general as possible.

Consequence: Shape theorem

Shape theorem

Assume F and H are invariant under scaling

$$F(cr, cx, \cdot) = F(r, x, \cdot), \qquad H(cr, x) = cH(r, x)$$

Then

Time-Space scaling

$$R_t^{\epsilon} \stackrel{\mathcal{L}}{=} \epsilon^{1/d} R_{\frac{t}{\epsilon}}^{1}$$

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Take fixed $t > 0$ and $\tau = t/\epsilon$ we get

$$\frac{1}{\sqrt{\tau}} R^1_\tau = \frac{\epsilon^{1/d}}{\sqrt{t}} R^1_{\frac{t}{\epsilon}} = \frac{1}{\sqrt{t}} R_t^{t/\tau} \qquad \to \frac{\bar{r}_t}{\sqrt{t}} \qquad \text{as } \tau \to \infty$$

With

$$\bar{r}_0 = \text{“}\epsilon^{1/d} R^1_0\text{”}$$

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Theorems

Lipschitz property

Assumption (L)

$$
\begin{aligned}
\|F(r,x,\cdot) - F(r',x',\cdot)\|_2 &\le K\big(\|r-r'\|_2 + |x-x'|\big),\\
|H(r,z) - H(r',z')| &\le K\big(\|r-r'\|_2 + |z-z'|\big),\\
\|\overline{b}(r) - \overline{b}(r')\|_2 &\le K\|r-r'\|_2\,.
\end{aligned}
$$

Also, $F(r,x,\cdot) \in C(S^{n-1})$ for every $(r,x) \in \mathcal{D}(F)$.

For any fixed $t \geq 0$ and $a > 0$

$$\lim_{\epsilon \to 0} \|(b \star g_\eta)(R^\epsilon_{t\wedge\tau^\epsilon}, x^\epsilon_{t\wedge\tau^\epsilon}) - b(R^\epsilon_{t\wedge\tau^\epsilon}, x^\epsilon_{t\wedge\tau^\epsilon})\|_2 = 0, \qquad \text{in probability}, \tag{5.1}$$

where $\tau^\epsilon := \inf\{t > 0 : \ \|R^\epsilon_t\|_2 \geq a^{-1}\}$.

Theorem

Under Assumptions (E) and (L) and (C), the solution to the (infinite dimensional) ODE, $\dot{\bar{r}}_t = \bar{b}(\bar{r}_t)$ exists and is unique for every $\bar{r}_0 \in C(\mathbb{S}^{d-1})$. Moreover, if $R_0^\epsilon \to \bar{r}_0$ in $L^2(\mathbb{S}^{d-1})$, then

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{0 \le t \le T} \|R_t^\epsilon - \bar{r}_t\|_2 > \delta \right) = 0$$

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Theorem

If also (I) holds, then for every $\delta > 0$ and $t > 0$

$$\lim_{\tau \to \infty} \mathbb{P}\left(\left\| \frac{R_\tau^1}{\tau^{1/d}} - \bar{r}_t \right\|_2 > \delta \middle| R_0^1 = \bar{r}_0 (\tau/t)^{1/d} \right) = 0$$

Moreover, $\bar{r}_1$ is an Euclidean ball if and only if $\int F(r, z, \cdot) d\nu_r(z)$ is rotationally invariant.

Applications

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  - $F =$ Mean curvature (Richardson's model ?)

Applications and Remarks

- Harmonic measure is not Lipschitz.
- We can get the Lipschitz property if we work in smoothed domains.
- Scaling invariance when drift size is proportional to size.
- We expect the shape theorem to hold even without the scaling invariance property.
- $F(r, x, \theta) = Z^{-1} r(\theta) \Longrightarrow$ infinitely many invariant shapes.
- Other rules:
  - $F =$ Distance to reference point,
  - $F =$ Mean curvature (Richardson's model ?)
  - F independent of R

Smoothed Harmonic measure

- g a (fixed) smoothing kernel, $\tilde{r} = r \star g$.
- $0 \leq \alpha(\ell, z) < \ell$

Consider

$$F(r, x, \cdot) = \text{ Harmonic measure on } \tilde{r} \text{ from } x, \qquad H(r, z) = \alpha(\tilde{r}(z), z)z$$

Theorem

(a) The Averaging Principle holds for this model.
(b) In case $\alpha(\ell, z) = \alpha(z)\ell$, the Shape Theorem also holds. In particular, for $\alpha(\ell, z) = \gamma\ell$ with $\gamma \in [0, 1)$ fixed, the centered Euclidean ball is an invariant shape; and when $\gamma = 0$, it is uniquely attractive.

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- In each interval we can use the ergodic assumption.
- $\mathbb{E}|x_t^\epsilon - \hat{x}_t^\epsilon|^2 \leq C\epsilon, \quad \mathbb{E}|R_t^\epsilon - \hat{R}_t^\epsilon|^2 \to 0, \quad \mathbb{E}|R_t^\epsilon - \bar{r}_t|^2 \to 0.$

A coupling inequality (to bound $\mathbb{E}|x_t^\epsilon - \hat{x}_t^\epsilon|^2$)

Lemma

Let M be a connected Riemannian manifold without boundary compactly embedded in $\mathbb{R}^d$. Let μ, ν be probability distributions on M having densities p, q respectively. Assume $p(x) \geq c > 0$ for all $x \in M$. Then, there exists $C = C(M, c) < \infty$ such that the Wasserstein 2-distance verifes

$$W_2(\mu, \nu) \leq C\|p - q\|_2.$$

Thanks.