Part 2

Problems with nonlinear boundary conditions

CHAPTER 8

Nonlinearities at the boundary

In this chapter we analyze numerical approximations for linear equations where the blow-up appears due to nonlinear flux boundary conditions.

The heat equation

First we consider semidiscrete approximations for the following parabolic problem,

$$
(1) \qquad \begin{array}{ll} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = u^p & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{on } \Omega. \end{array}
$$

We assume that the domain Ω and the initial datum u_0 are regular in order to guarantee a smooth solution u.

As we pointed out in the introduction, if $p > 1$ solutions of this problem blow up in finite time.

The blow-up set is localized at the boundary of the domain, that is for every subdomain $\Omega' \subset\subset \Omega$ there exists a constant $C = C(d(\Omega', \partial\Omega))$ such that $u(x,t) \le C$ for every $x \in \Omega'$ and for every $0 \le t < T$ (see [**45, 68, 69**]). Also it is known that the blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{2(p-1)}}$ (see [**45**]).

Numerical approximations for this problem have been studied in [**22**], where the authors analyze this problem in one space dimension. They consider a semidiscrete scheme such as ours, find a necessary and sufficient condition for the appearance of blow-up ($p > 1$) and prove the convergence of the blow-up time of the discrete problem to that of the continuous one, when the mesh parameter goes to zero.

Here we extend these results to several space dimensions and prove some new results concerning the asymptotic behavior (blow-up rate) and the localization of blow-up points (blow-up set) for semidiscretetizations in space given by a general method that has the same spirit of those of Part I.

We assume that there exists a set of nodes $\{x_1, \dots, x_N\}$ and that our approximate solution $U(t) = (u_1(t), \dots, u_N(t))$ verifies the following ODE

$$\begin{array}{rcl} MU'(t) & = & -AU(t) + BU^p(t), \\ U(0) & = & U_0. \end{array} \tag{2}$$

The precise assumptions on the matrices involved in the method are:

- (P1) M is a diagonal matrix with positive entries m_k.
- (P2) B is a diagonal matrix with nonnegative entries b_k.
- (P3) A is a nonnegative symmetric matrix, with nonpositive coefficients off the diagonal (that is $a_{ij} \le 0$ if $i \ne j$) and $a_{ii} > 0$.
- (P4) $\sum_{j=1}^{N} a_{ij} = 0$.

This last assumption implies, to begin with, that there is no nontrivial steady state for (2).

Note that the stiffness matrix, A, verifies a slightly different condition here than when we consider homogeneous Dirichlet boundary conditions. This is natural when finite element method or finite difference method are considered.

The matrix B collects the boundary conditions. Hence we assume that the coefficients b_k are zero for every interior node.

In the preliminaries chapter we showed that these requirements are natural since both finite element methods and finite differences methods verify them.

The final hypothesis on the scheme is the following: if we define the graph $\mathcal{G}$ with vertices on the nodes x_k and we say that two nodes x_k, x_j are connected if and only if $a_{kj} \ne 0$, then we assume

- (P5) The graph $\mathcal{G}$ is connected.

It is easy to check that hypotheses (P1)-(P5) imply the maximum principle.

Writing this equation explicitly we obtain the following ODE system,

$$
(3) \qquad \begin{aligned} m_k u_k'(t) &= -\sum_{j=1}^{N} a_{kj} u_j(t) + b_k u_k^p(t), \quad && 1 \le k \le N, \\ u_k(0) &= u_{0,k}, && 1 \le k \le N. \end{aligned}
$$

Observe that although the continuous problem (1) is quite different to the heat equation with a nonlinear source, problem (3.1)[1], the numerical methods have a similar structure. This implies that these numerical approximations and the ones studied in Chapter 3 have similar behaviors, which is undesirable since the continuous solutions behave in a different way.

Since this numerical method is similar to (3.2), the proofs concerning the asymptotic behavior of numerical solutions constitute just a small contribution and we omit various of them. We focus in the differences.

The first important fact is the convergence of the scheme, this holds under a consistency assumption similar to (3.3).

THEOREM 8.1. *Let u be a regular solution of (1) and u_h the numerical approximation given by (3). If the method is consistent then there exists a constant C such that*

$$\|u - u_h\|_{L^\infty(\Omega\times[0,T-\tau])} \le C\rho(h).$$

PROOF. The proof of this theorem is based on comparison principles, so the first step is to prove that these principles hold for the equation satisfied by the error function. The next step is to get an adequate supersolution (that bounds the error from above) with the behavior of $\rho(h)$. The difference between this proof and the one of Theorem 3.1 is that supersolutions for the error function are not so easy to obtain, they must verify an inequlity like

$$MW' \ge -AW + CBW + \rho(h)M(1, ..., 1)^t,$$

where C is a given constant that depends on the L^∞ norm of u (but not on h).

In order to get a function with this property converging to zero we proceed as follows: let $a \in C^2(\overline{\Omega})$ be such that $a(x) \ge \delta > 0$ in Ω, $\partial a/\partial\eta > Ca$ on $\partial\Omega$ and let $b(t) = \exp(Lt)$ where L is to be determined.

[1]In this way we refer to formulae from previous chapters, in this case it is from Chapter 3.

Then, it is easy to check that $w(x,t) = Ca(x)b(t)\rho(h)$ verifies

$$\begin{array}{rcll} w_t & \geq & \Delta w & \text{in } \Omega \\ \frac{\partial w}{\partial \eta} & \geq & Cw & \text{in } \partial\Omega \end{array}$$

Now, by consistency of the scheme, one can verify that

$$W = Cb(t)\rho(h)(a(x_1), ..., a(x_N))$$

is a supersolution for L big enough depending on C but not on h. $\square$

Concerning the blow-up condition for this scheme we have that if $p > 1$ every positive solution blows up and all of them are global if $p \leq 1$. The same thing occurs in the continuous problem.

Next we extend the result of [**22**] on the convergence of numerical blow-up times T_h to the continuous one T when the mesh parameter h goes to zero. To this end, we again assume that the method is consistent and hypotheses on u_0 that imply that the numerical solution u_h is increasing in time.

THEOREM 8.2. *Let u_0 a compatible initial datum for* (1) *such that $\Delta u_0 \geq \alpha > 0$. Assume the method* (3) *is consistent. Let T and T_h be the blow-up times for u and u_h respectively, then*

$$\lim_{h \to 0} T_h = T.$$

The proof of this theorem is similar to the one of Theorem 4.5 (the part concerning convergence of the numerical blow-up times). We do not include it.

Finally, we deal with the asymptotic behavior (blow-up rate) and the localization of blow-up points (blow-up set) of u_h for fixed h.

Concerning the blow-up rate we prove

THEOREM. *Let u_h be a solution of* (3). *Assume that $p > 1$ and that u_h blows up in finite time T_h. Then*

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}.$$

Observe that this is the behavior we obtained in Chapter 3 for the problem with a source, with the exception that in this case we do not have the asymptotic behavior

$$(T_h - t)^{\frac{1}{p-1}} \|U(t)\|_\infty \to C_p.$$

We do not expect this since the continuous problem has not this behavior, opposite to (3.1), where the constant C_p appears in the asymptotic behavior. Let us point out that in this case the blow-up rate for the numerical scheme, $(T_h - t)^{-1/(p-1)}$, is different from the continuous one, $(T - t)^{-1/2(p-1)}$.

This is due to the fact that the ODE system has a structure very similar to the one used for problems with homogeneous Dirichlet boundary conditions and a source u^p in the equation (the problem of Chapter 3), and hence numerical solutions with blow-up behave like solutions of the ODE $u'(t) = u^p$ which do not give the appropriate rate in this case. In Chapter 10 we develop an adaptive in space numerical method that recovers the correct blow-up rate.

We end the analysis of this problem proving a propagation result. We prove that the number of nodes outside $B^*(u_h)$ that go to infinity is finite and determined only by p. We find that u_k blows up if and only if $d(k) \le K$ where $K = [1/(p-1)]$.

In addition, in this problem we can localize $B^*(u_h)$ near the boundary of the domain and hence we have a precise localization of the numerical blow-up set $B(u_h)$ around $\partial\Omega$.

THEOREM. *Let $B^*(u_h)$ be the set of nodes, x_j, such that*

$$u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}},$$

then the blow-up propagates in the following way: let $p > 1$ and $K = [1/(p-1)]$, then the solution of (2), U, blows up exactly at K nodes near $B^*(u_h)$. More precisely,

$$u_k(t) \to +\infty \qquad \Longleftrightarrow \qquad d(k) \le K$$

where $d(k)$ is the distance of x_k to $B^*(u_h)$ in the sense described in Chapter 3. Moreover, if $d(k) \le K$, the asymptotic behavior of u_k is given by

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(k)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(k) = K$

$$u_k(t) \sim \ln(T_h - t).$$

We want to remark that more than one node can go to infinity, but the asymptotic behavior imposes $\frac{u_k(t)}{u_j(t)} \to 0$ $(t \to T_h)$ if $d(k) > d(j)$.

Hence the number of blow-up points outside $B^*(u_h)$ is finite and depends on the power p but is independent of h.

Next we prove that interior nodes can not lie in $B^*(u_h)$ and hence we localize the numerical blow-up set in a small neighborhood of $\partial\Omega$.

THEOREM. *The numerical blow-up set verifies*

$$B(u_h) \subset \partial\Omega + N_{Kh},$$

where N_ε is the ball of radius ε centered at zero in $\mathbb{R}^d$.

This coincides (when $h \to 0$) with the known results for the blow-up set of the continuous problem, namely $B(u) \subset \partial\Omega$ ([**45, 68, 69**]).

Non-simultaneous blow-up at the boundary.

Now we turn our attention to the non-simultaneous blow-up phenomenon, to this end we consider the problem

$$\begin{aligned} u_t &= \Delta u, \quad \text{in } \Omega \times (0,T), \\ v_t &= \Delta v, \quad \text{in } \Omega \times (0,T), \end{aligned} \tag{4}$$

fully coupled by the nonlinear flux boundary condition, given by

$$\frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}}, \qquad \frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}} \qquad \text{on } \partial\Omega \times (0,T), \tag{5}$$

and initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x).$$

We assume that Ω is a bounded smooth domain in $\mathbb{R}^d$, $p_{ij} \geq 0$ and u_0, v_0 are positive, bounded, compatible with the boundary data and smooth enough to guarantee that u, v are regular. Solutions to this problem exist (and are unique) locally in time, [**6**]. The time T is the maximal existence time for the solution, which may be finite or infinite.

In this problem the non-simultaneous blow-up phenomenon appears due to the nonlinear boundary conditions, opposite to (4.1) where the blow-up is due to the source term. Our interest here is to analyze the presence of non-simultaneous blow-up in numerical approximations for this problem.

We consider the general method. The numerical approximation of (u,v) at the nodes x_k is given by

$$U(t) = (u_1(t), \dots, u_N(t)), \qquad V(t) = (v_1(t), \dots, v_N(t)),$$

the solution of the following ODE

$$\begin{aligned} MU'(t) &= -AU(t) + B(U^{p_{11}}(t) V^{p_{12}}(t)), \\ MV'(t) &= -AV(t) + B(U^{p_{21}}(t) V^{p_{22}}(t)), \end{aligned} \tag{6}$$

with initial data given by $u_k(0) = u_0(x_k)$, $v_k(0) = v_0(x_k)$.

The matrices M, A and B satisfy the properties (P1)-(P5).

Since the system (6) is similar to the discrete problem in Chapter 4, the proofs concerning the behavior of the numerical approximation can be handled in a similar way. The main differences arise from the presence of the matrix B replacing M in the nonlinear term. These differences should be treated as in the proofs of the next section. Therefore we do not include the proofs.

The conditions on the exponents p_{ij} that ensure the presence of blow-up and non-simultaneous blow-up in (4)-(5) are exactly the same of (4.1), in Chapter 4 ([**65, 71**]). There we observed that the same holds for the numerical approximations.

For this scheme we have our traditional convergence theorem, its proof can be obtained using the ideas from Theorem 4.1 to handle the system and from Theorem 8.1 to construct the adequate supersolution.

THEOREM. *Let (u, v) be a regular solution of* (4)-(5) *and* (U, V) *the numerical approximation given by* (6)*. If the method is consistent then there exists a constant C, independent of h, such that*

$$\max_{1\le k\le N} \sup_{t\in[0,T-\tau]} |u(x_k,t) - u_k(t)| + |v(x_k,t) - v_k(t)| \le C\rho(h),$$

where ρ is the modulus of consistency of the method.

The following theorems characterize the existence of blowing up solutions for (6) and prove convergence of the blow-up times.

THEOREM. *Every solution (U, V) of* (6) *blows up in finite time if and only if the exponents p_{ij} verify any of the conditions,*

$$p_{11} > 1, \qquad p_{22} > 1 \quad \text{or} \quad (p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}.$$

THEOREM. *Let (u, v) be a solution of* (4)-(5) *with blow-up time T and initial datum (u_0, v_0) satisfying $\Delta u_0, \Delta v_0 \ge \kappa > 0$. Let (U, V) be the corresponding numerical solution. If the scheme is consistent and if one of the following $p_{11} > 1$, $p_{22} > 1$, or $p_{21}, p_{12} > 1$ hold, then the blow-up time T_h of the numerical solution converges to T, i.e.*

$$\lim_{h\to 0} T_h = T.$$

We observe that the hypotheses $\Delta u_0, \Delta v_0 \ge \kappa > 0$, imply monotonicity in time for the solution (u, v), that is $u_t, v_t > 0$. This monotonicity property also holds for our numerical solution (U, V) and is crucial for our arguments. Observe that this monotonicity assumption is the same we required in Chapter 4. It is used in a similar manner.

For certain choices of the parameters p_{ij} there are initial data for which one of the components of the system remains bounded while the other blows up. The next two theorems characterize the range of parameters for which non-simultaneous blow-up occurs in the discrete problem.

THEOREM. *Let (U, V) a solution of (6) such that U blows up at finite time T_h and V remains bounded up to that time. Then $p_{11} > 1$ and $p_{21} < p_{11} - 1$.*

Reciprocally,

THEOREM. *If $p_{11} > 1$ and $p_{21} < p_{11} - 1$, then for every initial datum V_0 for (6) there exists an initial datum U_0 such that U blows up in finite time T_h and V remains bounded up to that time.*

We want to remark that this characterization is the same as the one for the continuous problem, see **[65]**.

Finally we get

THEOREM. *Let u_0, v_0 be initial data for (4)-(5) satisfying*

$$\Delta u_0, \Delta v_0 \geq \kappa > 0,$$

such that u blows up at time T and v remains bounded up to that time. Then, if the scheme is consistent, U blows up while V remains bounded, for every h small enough.

1. The heat equation: ideas of the proofs

We devote this section to give some ideas of the proofs concerning problem (2).

Despite the proofs are similar to those of Chapter 3, in this case we have that every numerical solution blows up if $p > 1$. This is due to the absence of steady states. So we reproduce the proofs here in order to observe why this difference appears.

Let us begin with the following Lemma,

LEMMA 8.1. *Let U be the solution of (2). If $U_0 > 0$, then U is unbounded.*

PROOF. Assume by contradiction that U is uniformly bounded. Then, we observe that

$$\Phi_h(U) = \frac{1}{2}\langle A^{1/2}U, A^{1/2}U\rangle - \frac{1}{p+1}\sum_{k=1}^{N} b_k u_k^{p+1}$$

is a Lyapunov functional for (3). In fact by direct computation we have

$$\frac{d}{dt}\Phi(U)(t) = -\langle MU', U'\rangle.$$

As (2) has only $U \equiv 0$ as a fixed point, it follows that (see [**41**]) $U \to 0$ as $t \to \infty$. As our scheme verifies the minimum principle, $\min_j u_j(t) \geq \min_j u_{0,j} > 0$, a contradiction. □

Now we prove the blow-up condition.

THEOREM 8.3. *Positive solutions of* (3) *blow up in finite time if and only if* $p > 1$.

PROOF. In order to proof this theorem we need the following lemma:

LEMMA 8.2. *Let U be the solution of* (2). *Then*

$$\max_j u_j(t) = u_k(t)$$

where k is such that $b_k \neq 0$ if $\max_j u_j(t) > \max_j u_{0,j}$.

Its proof follows applying the discrete maximum principle that holds for this system thanks to hypotheses (P1)-(P5). Recall that the condition $b_k \neq 0$ means that x_k is a boundary node.

By the previous lemma we have that $U(t)$ is unbounded: using that $p > 1$ and Lemma 8.2 we obtain that there exist a time t_0 and a node x_k with $b_k \neq 0$ such that $-a_{kk}u_k(t_0) + b_k u_k^p(t_0) \geq \frac{b_k}{2} u_k^p(t_0)$. Hence, by our assumptions on A,

$$m_k u_k'(t_0) = -\sum_{j=1}^{N} a_{kj} u_j(t_0) + b_k u_k^p(t_0) \geq \frac{b_k}{2} u_k^p(t_0).$$

As a consequence of this $u_k(t)$ must be increasing for $t \geq t_0$ and verifies

$$m_k u_k'(t) \geq \frac{b_k}{2} u_k^p(t).$$

Again, as $p > 1$, u_k goes to infinity in finite time, and hence $U(t)$ has finite time blow-up as we wanted to show. □

Now we consider positive solutions of (3) with h fixed and we denote by C a positive constant that may depend on h and may vary from one line to another.

THEOREM 8.4. *Let u_h be a solution of* (3)*. Assume that $p > 1$ and that u_h blows up in finite time, T_h. Then*

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}.$$

PROOF. Let us begin by defining

$$w(t) = \sum_{k=1}^{N} u_k(t).$$

As U blows up at time T_h, and by (3) we obtain that there exists t_0 such that for every $t \in [t_0, T_h)$ it holds

$$\begin{aligned} w'(t) &= -\sum_{k=1}^{N}\sum_{j=1}^{N} \frac{a_{kj}}{m_k} u_j(t) + \sum_{k=1}^{N} \frac{b_k}{m_k} u_k^p(t) \le C(\max_k u_k(t))^p \\ &\le C\left(\sum_{k=1}^{N} u_k(t)\right)^p = Cw^p(t). \end{aligned}$$

For $t \in [t_0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h} \frac{w'(s)}{w^p(s)}\, ds \le C(T_h - t).$$

Changing variables we get

$$\int_{w(t)}^{+\infty} \frac{1}{s^p}\, ds \le C(T_h - t),$$

hence

$$w(t) \ge C(T_h - t)^{-\frac{1}{p-1}}.$$

Therefore we obtain

$$\max_j u_j(t) \ge C(T_h - t)^{-\frac{1}{p-1}}.$$

To prove the other inequality we proceed as follows: as $\max_j u_j(t) \to +\infty$ when $t \to T_h$, we have that if $u_k(t) = \max_j u_j(t)$, then $a_{kk}u_k(t) \le \frac{b_k}{2} u_k^p(t)$ for every t close to T_h. In this case we have

$$u_k'(t) = -\sum_{j=1}^{N} \frac{a_{kj}}{m_k} u_j(t) + \frac{b_k}{m_k} u_k^p(t) \ge \frac{b_k}{2m_k} u_k^p(t).$$

Integrating again over $[t, T_h]$ we obtain

$$\int_t^{T_h} \frac{u_k'(s)}{u_k^p(s)}\, ds \ge \frac{b_k}{2m_k}(T_h - t).$$

Changing variables

$$\int_{u_k(t)}^{+\infty} \frac{1}{s^p}\, ds \geq \frac{b_k}{2m_k}(T_h - t),$$

hence

$$u_k(t) \geq C_k (T_h - t)^{-\frac{1}{p-1}}.$$

So $\max_j u_j(t)$ verifies

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$$

in the sense that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}. \quad \square$$

To localize the numerical blow-up set $B(u)$ we apply the same techniques of Chapter 3. We make the same change of variables

$$y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t),$$

$$(T_h - t) = e^{-s},$$

and study the ODE system verified by $Y = (y_k(s))$, which reads as follows

$$m_k y_k'(s) = -e^{-s} \textstyle\sum_{j=1}^{N} a_{kj} y_j(s) - \frac{m_k}{p-1} y_k(s) + b_k y_k^p(s),$$

$$y_k(-\ln(T_h)) = (T_h)^{\frac{1}{p-1}} u_0(x_k), \qquad 1 \leq k \leq N+1.$$

As $\max u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}$, we have that $y_j(s)$ are uniformly bounded.

Observe that we can distinguish in this equation two kinds of terms: the non autonomous term

$$-e^{-s} \sum_{j=1}^{N} a_{kj} y_j(s),$$

and the autonomous one

$$-\frac{m_k}{p-1} y_k(s) + b_k y_k^p(s).$$

The first one vanishes as $s \to \infty$ and so is not very important for the asymptotic behavior. The last one imposes this behavior and, in contrast with Chapter 3 this part depends on the coefficients m_k, b_k and hence on h.

In the course of the proof of Theorem 3.3 we claim that each component y_k converges (as $s \to \infty$) to zero or to C_p, which are the roots of

$$0 = -\frac{1}{p-1} y_k + y_k^p.$$

Applying the same arguments we can prove here that each of these variables $y_k(s)$ converges to one of the roots of

$$0 = -\frac{m_k}{p-1} y_k + b_k y_k^p.$$

This is the reason why the constant C_p is not recovered in this case.

With this in mind one can prove the propagation result as in Theorem 3.4.

We end the chapter by proving the localization of $B(u_h)$.

THEOREM 8.5. *The numerical blow-up set verifies*

$$B(u_h) \subset \partial\Omega + N_{Kh},$$

where N_ε is the ball of radius ε centered at zero in $\mathbb{R}^d$.

PROOF. As we proved $B(u_h) \subset B^*(u_h) + N_{Kh}$ it just remains to prove that the interior nodes can not belong to $B^*(u_h)$. In fact, if x_k is an interior node we have

$$m_k u_k'(t) = -\sum_{j=1}^{N} a_{kj} u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}.$$

Integrating this inequality we obtain the desired result. □

CHAPTER 9

Where? Numerical blow-up sets

In this chapter we address the localization problem of blow-up points for semidiscretizations in space of positive solutions for the following parabolic problem

$$(1)\qquad \begin{array}{rcll} u_t(x,t) &=& u_{xx}(x,t) & (x,t) \in (0,1) \times [0,T), \\ u_x(1,t) &=& f(u(1,t)) & t \in [0,T), \\ u_x(0,t) &=& 0 & t \in [0,T), \\ u(x,0) &=& \varphi(x) & x \in [0,1], \end{array}$$

where $f(s)$ and $\varphi(x)$ are positive and regular enough in order to guarantee existence and uniqueness of solutions. For this type of problems if the initial datum satisfies a compatibility condition the solution is smooth (see [**5, 54**]).

The purpose of this chapter is to show that blow-up sets for semidiscrete approximations of (1) (by finite elements method), either concentrate near the boundary or are the whole $[0,1]$, that is regional blow-up does not exists for these schemes.

When one changes the power-type nonlinear boundary condition for a general function f new phenomena appears, and new differences between the continuous solutions and their numerical approximations.

For example, it is showed in [**76**] that if f is convex and verifies

$$(2)\qquad \int^{+\infty} \frac{1}{f(s)f'(s)}\,ds < +\infty,$$

every solution of (1) blows up. Moreover, in [**68**] it is proved that if f is increasing and verifies

$$(3)\qquad \int^{+\infty} \frac{1}{f(s)}\,ds < +\infty,$$

then the blow-up set is localized at the boundary, in this case $B(u) = \{1\}$, (single point blow-up).

However, when f verifies (2) but not (3), the blow-up set can be larger than a single point. It can be the whole interval $[0,1]$ (global blow-up), or it can be a subinterval $[l, 1]$ with $0 < l < 1$ (regional blow-up). For example, this phenomenon of global blow-up or regional blow-up is obtained when one consider $f(s) = s\log^p(s)$ with $1/2 < p < 1$ or $p = 1$ respectively.

We propose here the same numerical procedure of Chapters 2, 7, namely, a first degree finite element approximation on the variable x keeping t continuous; from a well known fact it coincide with a classical finite difference second order scheme on x. The mass lumping technique simplifies the scheme and preserves the maximum principle, allowing us to use comparison arguments (see Lemma 9.1).

Let $x_i = \frac{i}{N}$, $0 \le i \le N$, be a partition of the interval $[0, 1]$ into subintervals $I_i = [x_i, x_{i+1}]$, of length $h = \frac{1}{N}$. Let V_h the set of continuous functions which are affine on each I_i. We consider the basis functions of V_h taking as usual ψ_i, with $\psi_i(x_j) = \delta_i^j$. Now let

$$u_h(x,t) = \sum_{i=0}^{N} u_i(t)\psi_i(x). \tag{4}$$

For a fixed t, $u_h(x,t)$ belongs to $H^1(0,1)$ so in order to construct an approximate solution of (1) we proceed as follows: replacing (4) in the weak formulation of (1) we get a system of ordinary differential equations for $U = (u_0(t), ..., u_N(t))$,

$$MU' = -AU + f(U), \tag{5}$$

where M is the mass matrix, A is the stiffness matrix, and $f(U) = f(u_N)e_N$ with $e_N = (0, ..., 1)$.

Mass lumping upon the matrix M in (5) gives

$$\begin{aligned} u_0'(t) &= \frac{2}{h^2}(-u_0(t) + u_1(t)), \\ u_i'(t) &= \frac{1}{h^2}(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)), \\ u_N'(t) &= \frac{2}{h^2}(-u_N(t) + u_{N-1}(t)) + \frac{2}{h}f(u_N(t)). \end{aligned} \tag{6}$$

So, from now on, we will assume that M is diagonal with $m_i = h$ if $1 \le i \le N-1$ and $m_0 = m_N = h/2$.

For this scheme it is known that u_h converges uniformly to u provided that u is smooth, see **[22]**. Also it is proved there that solutions blows up if and only if f satisfies (3), so there exists nonlinearities f such that the continuous solution blows up while the numerical solutions does not (for example for $f(s) = s\ln(s)^p$ with $1/2 < p \le 1$).

In Chapter 2 we proved that for the semilinear case the numerical blow-up sets approximate the continuous one but we also saw in Chapter 7 that this does not always hold.

In this chapter we show that regional blow-up can never be reproduced by fixed mesh schemes. This is a big difference in the behavior between solutions of (1) and their numerical approximations.

1. Statement of the results

Let us state the results of this chapter.

THEOREM 9.1. *Assume that f verifies* (3) *and is increasing. Then, the number of blow-up nodes for solutions of* (6) *depends only on f and is independent of h. That is either the numerical blow-up is global or it is localized in a small neighborhood of $\{x = 1\}$ for h small enough. Moreover, the propagation of blow-up is given by the following mechanism:*

Let T_h be the blow-up time of U and let

$$\mathcal{F}(z) = \int_z^{+\infty} \frac{1}{f(y)}\,dy,$$

$$G_0(t) = \mathcal{F}^{-1}(T_h - t), \qquad G_{k+1}(t) = \int_0^t G_k(s)\,ds.$$

Then u_{N-k} blows up if and only if G_k does, and in this case the blow-up rate is given by

$$u_{N-k}(t) \sim \int_0^t u_{N-k+1}(s)\,ds.$$

Also we prove the monotonicity of the blow-up set in terms of the nonlinearity f.

THEOREM 9.2. *Under the hypotheses of Theorem 9.1, the number of blow-up nodes for solutions of* (6) *is monotone decreasing with the nonlinear term f. That is, if $f_1(s) \geq f_2(s)$, then $B_h(U_1) \subset B_h(U_2)$.*

Finally we show that the blow-up rate of the blowing up nodes is strictly decreasing.

THEOREM 9.3. *The blow-up rate for the blowing up nodes x_i is strictly monotone in the sense that*

$$\lim_{t \nearrow T_h} \frac{u_i(t)}{u_{i+1}(t)} = 0.$$

Let us now present some examples.

(I) $f(s) = s^p$.

Numerical solutions blow up if and only if $p > 1$. The blow-up rate for different nodes is given by $u_{N-k}(t) \sim (T_h - t)^{-1/(p-1)+k}$ for $0 \le k < 1/(p-1)$, if $1/(p-1) = k$, an integer, then the node $u_{N-k} \sim -\ln(T_h - t)$. The blow-up set is composed by exactly K nodes, where $K = [1/(p-1)]$.

In this case, the blow-up condition $p > 1$ is the same as in the continuous problem (1), and the blow-up set verifies

$$B(u_h) = [1 - Kh, 1] = B(u) + [-Kh, 0].$$

This case has been considered in Chapter 8.

(II) $f(s) = s(\ln s)^p$.

Numerical solutions blow up if and only if $p > 1$. The blow-up rate is given by

$$\max_i u_i(t) \sim \exp\left(\frac{1}{(T_h - t)^{1/(p-1)}}\right).$$

As a consequence of **(I)** and Theorem 9.2, the blow-up set is the whole interval $[0, 1]$.

In this case, the blow-up condition $p > 1$ is different from the condition in the continuous problem (1), $p > 1/2$. The blow-up set for the continuous problem is a single point, $B(u) = \{1\}$, if $p > 1$; a bounded interval if $p = 1$; and the whole $[0, 1]$ if $1/2 < p < 1$. In this case, we remark that the behavior of the continuous problem is radically different from the numerical one, since not only we have different blow-up cases, but also in the cases where both problems blow up, the numerical solution has global blow-up while the continuous problem has single-point blow-up. This is the first example of such phenomena.

(III) $f(s) = e^s$.

In this case numerical solutions blow up at a single point $B(u_h) = \{1\}$. The blow-up rate is given by $u_N(t) \sim -\ln(T_h - t)$.

For this nonlinearity, the blow-up sets and the blow-up rates of both problems (the continuous and the discrete ones) coincide.

2. Some technical lemmas

In order to study the asymptotic behavior of (6) we need the following results.

LEMMA 9.1. *(Maximum principle) Let $h > 0$ be fixed, and let $U = (u_0, ..., u_N)$ be a solution of*

$$(7)\qquad \begin{aligned} u_0' &\leq \frac{2}{h^2}(-u_0 + u_1), \\ u_i' &\leq \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}), \\ u_N' &\leq \frac{2}{h^2}(-u_N + u_{N-1} + hf(u_N)). \end{aligned}$$

Then

$$\max_{k=0,\dots,N} u_k(t) \leq \max\{\max_{k=0,\dots,N} u_k(0); \sup_{0<\tau<t} u_N(\tau)\}.$$

PROOF. Let us first suppose that $U = (u_0, ..., u_N)$ verifies (7) with strict inequalities. Now, if the maximum is attained in an interior node, say $0 < j < N$, let t_0 be the first time when this happens, then we have

$$u_j'(t_0) \geq 0 \quad \text{and} \quad u_j(t_0) \geq u_k(t_0)\ 0 \leq k \leq N.$$

On the other hand, by our assumption on U we get $u_j'(t_0) < 0$ which leads to a contradiction. Also, it is easy to see that the maximum cannot be reached at u_0 and so the "maximum principle" follows.

To complete the proof, we just observe that if

$$Z = h^2(0, ..., k^2, ..., N^2),$$

then $U_\varepsilon(t) \equiv U(t) + \varepsilon Z$ verifies (7) with strict inequalities whenever U verifies (7). As $\varepsilon > 0$ is arbitrary, the Lemma follows. □

LEMMA 9.2. *Let U be a blowing up solution of* (6)*; then there exists a time $t_0 < T_h$ such that*

$$(8)\qquad \max_{0\leq i\leq N} u_i(t) = u_N(t),$$

for $t_0 < t < T_h$.

PROOF. First, observe that from our assumptions on f, it follows that

$$(9)\qquad \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty.$$

Now, by the maximum principle, it follows that if $\max\{u_i(t)\}$ is large then it coincides with $\sup_{0<s<t} u_N(s)$. Now, from (6) it follows that

$$u'_N \geq -\frac{2}{h^2}u_N + \frac{2}{h}f(u_N) \geq \frac{2}{h}f(u_N)\left(1 - \frac{u_N}{hf(u_N)}\right). \tag{10}$$

As $u_N(t)$ is blowing up and (9) holds, $u_N(t)$ is an increasing function for t near the blow-up time T_h, and hence we conclude that (8) holds for t close to T_h. □

3. Proofs of the theorems

In this section we denote by C a constant that may depend on h but not on t and may change from one line to another. Let us begin with the proof of Theorem 9.1. For the proof, we need the following Lemma,

LEMMA 9.3. *The blow-up rate for $u_N(t)$ (and hence for $U(t)$) is given by*

$$\max_k u_k(t) = u_N(t) \sim \mathcal{F}^{-1}(C(T_h - t)),$$

where $\mathcal{F}(z)$ is defined as

$$\mathcal{F}(z) = \int_z^{+\infty} \frac{1}{f(y)}\,dy.$$

PROOF. Integrating (10), we obtain

$$u_N(t) \leq c_1\mathcal{F}^{-1}(C(T_h - t)).$$

For the other inequality, we observe that for t close to T_h, by Lemma 9.2, we obtain

$$u'_N(t) = \frac{2}{h^2}(u_{N-1} - u_N) + \frac{2}{h}f(u_N) \leq \frac{2}{h}f(u_N),$$

hence integrating this inequality we get

$$u_N(t) \geq \mathcal{F}^{-1}(C(T_h - t)),$$

and the Lemma follows. □

Now, using the preceding lemma we look for the behavior of the node, u_{N-1}.

$$u'_{N-1}(t) = \frac{u_N(t) - 2u_{N-1}(t) + u_{N-2}(t)}{h^2} \geq \frac{1}{h^2}(u_N(t) - 2u_{N-1}(t)).$$

So,

$$(e^{\frac{2t}{h^2}} u_{N-1}(t))' \geq \frac{e^{\frac{2t}{h^2}}}{h^2} u_N(t),$$

and integrating we get

$$u_{N-1}(t) \geq C \int_{t_0}^{t} u_N(s)\, ds + C.$$

On the other hand, by (8) we get

$$u'_{N-1}(t) \leq \frac{2}{h^2}(u_N(t) - u_{N-1}(t)),$$

therefore

$$u_{N-1}(t) \leq C \int_{t_0}^{t} u_N(s)\, ds + C.$$

Using Lemma 9.3 we get

$$u_{N-1}(t) \sim \int_{t_0}^{t} u_N(s)\, ds \sim \int_{t_0}^{t} \mathcal{F}^{-1}(C(T_h - s))\, ds.$$

Recall that $G_0(t) = \mathcal{F}^{-1}(T_h - t)$, and

$$G_1(t) = \int_{t_0}^{t} \mathcal{F}^{-1}(T_h - t)\, ds = \int_{T_h - t}^{T_h - t_0} \mathcal{F}^{-1}(w)\, dw.$$

Since

$$\int_{t_0}^{t} \mathcal{F}^{-1}(C(T_h - s))\, ds = \int_{C(T_h - t)}^{C(T_h - t_0)} \mathcal{F}^{-1}(w)\, dw$$

we have that $u_{N-1}(t)$ blows up if and only if $G_1(t)$ does. Repeating this procedure one can check that u_{N-k} blows up if and only if $G_k(t)$ does, so the number of nodes that blow up depends only on f but not on h. This fact implies the desired result. □

To prove Theorem 9.2 we just have to observe that, if we call $\mathcal{F}_i(z) = \int_z^{+\infty} \frac{1}{f_i(y)}\, dy$, $i = 1, 2$, then we have that

$$f_1(s) \geq f_2(s) \Rightarrow \mathcal{F}_1(z) \leq \mathcal{F}_2(z),$$

and the result follows. □

Finally, we prove Theorem 9.3. First we consider $\frac{u_N(t)}{u_{N-1}(t)}$. From the asymptotic behavior found in Theorem 9.1, we have

$$\frac{u_N(t)}{u_{N-1}(t)} \sim \frac{u_N(t)}{\int_{t_0}^{t} u_N(s)\, ds}.$$

Now by L'Hôspital's rule and (9), we obtain

$$\lim_{t \nearrow T_h} \frac{u_N(t)}{\int_{t_0}^t u_N(s)\, ds} = \lim_{t \nearrow T_h} \frac{u_N'(t)}{u_N(t)} = \lim_{t \nearrow T_h} \frac{Cf(u_N(t))}{u_N(t)} = +\infty$$

and the result follows by induction. $\square$

CHAPTER 10

Adaptive in space numerical schemes

As was shown in previous chapters, fixed mesh methods are not very appropriate to compute blow-up problems. For example, consider the following equation

$$(1)\qquad \begin{array}{ll} u_t = u_{xx} & (x,t) \in (0,1) \times [0,T), \\ u_x(0,t) = 0 & t \in [0,T), \\ u_x(1,t) = u^p(1,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,1). \end{array}$$

Assume as before u_0 positive and compatible in order to have a regular solution. This is the problem considered in Chapters 8, 9. There we observed that if $p > 1$ every positive solution blows-up, and the blow-up rate is given by,

$$\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2(p-1)}}.$$

The blow-up set, $B(u)$ is given by,

$$B(u) = \{1\}.$$

In other words, we have single point blow-up, see [**45**].

It is observed there that significant differences appear between the continuous and the discrete problems. The blow-up rate for the numerical approximation, U, is given by

$$\|U(t)\|_\infty \sim (T_h - t)^{-\frac{1}{p-1}}.$$

Therefore the blow-up rate does not coincide with the expected for the continuous problem. Concerning the blow-up set it holds,

$$B(U) = [1 - Kh, 1] \text{ if } p > 1.$$

The constant K depends only on p, K is the only integer that verifies

$$\frac{K+1}{K} < p \le \frac{K}{K-1}.$$

Hence the numerical blow-up set can be larger than a single point, $x = 1$, but as K does not depend on h,

$$B(U) \to B(u), \qquad \text{as } h \to 0.$$

Collecting all these results, we observe that when computing numerical approximations of a blow-up problem with a fixed mesh significant differences appear concerning the behavior of the numerical and the continuous solutions. For this problem the blow-up rate and the blow-up set can be different. We conclude that the usual methods with a fixed grid are not well suited for the problem under consideration. Therefore an adaptive mesh refinement is necessary. Some references that use adaptive numerical methods are [**3, 11**] and [**14**].

In this chapter we introduce two adaptive methods that give the right blow-up rate and the exact blow-up set. Our main interest is to provide rigorous proofs of these facts.

The first method, that we will call *adding points method*, is based on a semidiscretization in space and adds points near the boundary, $x = 1$, when the numerical solution becomes large. This adaptive procedure leads to a non-uniform mesh that concentrates near the singularity and gives the precise blow-up rate and set.

The second one, that we will call *moving points method*, also uses a semidiscretization in space but this time we move the last K points near the boundary when the numerical solution becomes large. The number of moving points, K, depends only on the exponent p, in fact we will choose $K = [1/(p-1)]$. This procedure is inspired in the moving mesh algorithms developed in [**12, 13, 14, 46**]. In our case we take advantage of the a priori knowledge of the spatial location of the singularity at $x = 1$ and instead of moving the whole mesh continuously as time evolves, we concentrate only the last K points near the boundary, leaving the rest of the mesh fixed. This allows us to use a unified approach to analyze rigorously both schemes simultaneously. One advantage of this moving procedure is keeping the size of the ODE problem to be solved constant in time, while the adding points methods enlarges the number of equations as time evolves.

Both numerical schemes are based on the scale invariance of the heat equation in the half line with the nonlinear boundary condition placed at $x = 0$, $-u_x(0,t) = u^p(0,t)$. If $u(x,t)$ is a solution then $u_\lambda(x,t) = \lambda^{\frac{1}{2(p-1)}} u(\lambda^{\frac{1}{2}} x, \lambda t)$ also is a solution. In [**29**] it is proved that there exists a self-similar blowing up solution in the half-line of the form

$$u_S(x,t) = (T-t)^{-\frac{1}{2(p-1)}} \varphi(\xi), \qquad \xi = x(T-t)^{-\frac{1}{2}}.$$

For an explicit form of the profile φ, see [**29**]. This solution $u_S(x,t)$ gives the behavior near the blow-up time T for solutions of (1) in the following sense

$$u(x,t) \sim (T-t)^{-\frac{1}{2(p-1)}}\varphi(\xi), \tag{2}$$

for $x = 1 - \xi(T-t)^{1/2}$, $|\xi| \le C$. So the behavior near the blow-up point $(1,T)$ is given by the self-similar solution in the half-line. The numerical schemes presented here use this fact to add or move points near $x = 1$ trying to reproduce the scaling invariance in the half-line.

The main goal of this chapter is to present rigorous estimates for both methods simultaneously showing that they give the right blow-up rate and set. Our result reads as follows and holds for the method that adds points as well as for the method that moves the last points.

THEOREM. *Let u be an increasing in x, smooth solution of* (1) *and u_h be the numerical solution obtained by any of the adaptive schemes described in Section 2. Then, for every $\tau > 0$, the numerical solution u_h converges to the continuous one uniformly in $[0,1] \times [0, T-\tau]$, in fact there exists a constant $C = C(\tau)$ such that*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \le Ch^{\frac{3}{2}}.$$

The numerical solution u_h blows up if and only if $p > 1$ in the sense that there exists a finite time T_h with

$$\lim_{t \nearrow T_h} u_h(1,t) = +\infty.$$

This numerical blow-up time converges to the continuous one when $h \to 0$, in fact there exist $\alpha > 0$ and $C > 0$ such that

$$|T_h - T| \le Ch^\alpha.$$

Moreover, the numerical blow-up rate is given by

$$\lim_{t \nearrow T_h} (T_h - t)^{\frac{1}{2(p-1)}} \|u_h(\cdot,t)\|_\infty = \Gamma = \varphi(0)$$

and the numerical blow-up set is

$$B(u_h) = \{1\}.$$

Organization of the chapter: In section 1 we describe the numerical adaptive procedures. In sections 2 to 5 we develop the proofs of the main result. To begin the analysis we prove in section 2 that numerical approximations converge uniformly in sets of the form $[0,1] \times [0, T-\tau]$. In section 3 we prove that the scheme reproduces the blow-up rate. In section 4 we find the numerical blow-up set that coincides with the continuous one. In section 5 we prove that the numerical blow-up time converges to the continuous one. Finally, in section 6 we present some numerical experiments comparing the performance of both methods and make a few comments on possible extensions of the ideas developed here to several space dimensions. We leave for the last section the proof of a technical result needed in section 5.

1. Adaptive numerical schemes

In this section we present the numerical methods. We consider a semidiscrte scheme, that is, we discretize the space variable, keeping t continuous. For spatial discretization we propose piecewise linear finite elements with mass lumping.

Consider a partition (that can be non-uniform), $x_1, ..., x_{N+1}$ of $(0,1)$ of size h ($h = \max(x_i - x_{i-1})$) and its associated standard piecewise linear finite element space V_h. Let $\{\varphi_j\}_{1\le j\le N}$ be the usual Lagrange basis of V_h. The semidiscrete approximation $u_h(t) \in V_h$ obtained by the finite element method with mass lumping is defined by:

$$\int_0^1 (u_h' v)^I dx + \int_0^1 (u_h)_x v_x dx = u^p(1,t)v(1,t), \qquad \forall v \in V_h, \forall t \in (0,T)$$

where the superindex I denotes the Lagrange interpolation. We denote with $U(t) = (u_1(t), \dots, u_{N+1}(t))$ the values of the numerical approximation at the nodes x_k, at time t. Writing

$$u_h(t) = \sum_{j=1}^{N} u_j(t)\varphi_j,$$

a simple computation shows that $U(t)$ satisfies the system of ordinary differential equations (see **[19]**):

$$\begin{aligned} &MU'(t) = -AU(t) + BU^p(t),\\ &U(0) = u_0^I, \end{aligned}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial data, u_0.

1.1. Adding points method. First, let us describe a method that adds points near the boundary. Let us pay special attention to the numerical solution at the point $x = 1$ (at this point is where the continuous solution develops the singularity). Writing the equation satisfied by u_{N+1} explicitly we obtain the following ODE,

$$u_{N+1}'(t) = \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t),$$

where $h_N = 1 - x_N$. Now, let us describe our adaptive procedure. As we want the numerical blow-up rate to be

$$u_h(1,t) \sim (T_h - t)^{-\frac{1}{2(p-1)}}$$

we impose that the last node $u_{N+1}(t) = u_h(1,t)$ satisfies

$$c_1 u_{N+1}^{2p-1}(t) \leq u'_{N+1}(t) \leq c_2 u_{N+1}^{2p-1}(t). \tag{3}$$

Imposing (3) is equivalent to

$$c_1 u_{N+1}^{2p-1}(t) \leq \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t) \leq c_2 u_{N+1}^{2p-1}(t). \tag{4}$$

As it is proved in **[22]** that the scheme with a fixed mesh blows up at the last node, we have that

$$R(t; h_N) = \frac{\frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t)}{u_{N+1}^{2p-1}(t)} \to 0 \qquad \text{as } t \text{ increases.}$$

Let t_1 be the first time such that $R(t; h_N) = c_1$, that is the first time where $R(t; h_N)$ meets the tolerance c_1. At that time t_1 we add a point z between x_N and $x_{N+1} = 1$ to the mesh and give the value of $u_h(z, t_1)$ such that the slope of the line between $(z, u_h(z, t_1))$ and $(1, u_h(1, t_1))$ is the same as the slope between $(x_N, u_h(x_N, t_1))$ and $(1, u_h(1, t_1))$, see Figure 1.

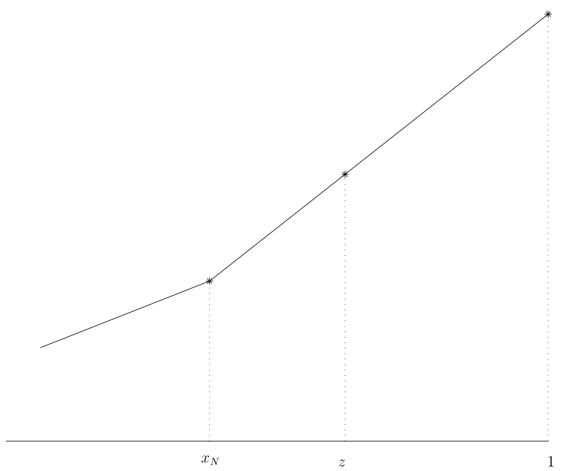

FIGURE 1.

Hence we have a new value for the length of the last interval, $[z, 1]$,

$$h_{N,1} = 1 - z < h_N = 1 - x_N.$$

We remark that $u_h(z,t_1)=u_z(t_1)$ satisfies

$$\frac{1}{h_N}(u_N(t_1)-u_{N+1}(t_1))=\frac{1}{h_{N,1}}(u_z(t_1)-u_{N+1}(t_1)).$$

Let us see what happens with $R(t_1;h_{N,1})$.

$$\begin{aligned} R(t_1;h_{N,1}) &= \frac{\frac{2}{h_{N,1}^2}(u_z(t_1)-u_{N+1}(t_1))+\frac{2}{h_{N,1}}u_{N+1}^p(t_1)}{u_{N+1}^{2p-1}(t_1)} \\ &= \frac{\frac{2}{h_{N,1}}(u_z(t_1)-u_{N+1}(t_1))+2u_{N+1}^p(t_1)}{h_{N,1}\ u_{N+1}^{2p-1}(t_1)} \\ &= \frac{\frac{2}{h_N}(u_N(t_1)-u_{N+1}(t_1))+2u_{N+1}^p(t_1)}{h_{N,1}\ u_{N+1}^{2p-1}(t_1)} \\ &= \frac{h_N}{h_{N,1}}R(t_1;h_N)=\frac{h_N}{h_{N,1}}\ c_1>c_1. \end{aligned}$$

Therefore with this new mesh $x_1,...,x_N,z,x_{N+1}$ we have that

$$R(t_1;h_{N,1})>c_1,$$

and we can apply the method with initial data $u_h(x,t_1)$ in the new mesh to continue. This gives a solution u_h that verifies (4) in a time interval $[t_1,t_2]$ where at time t_2, the function $R(t_2;h_{N,1})$ reaches the tolerance c_1. At that time t_2 we have to add another point in the last interval. As before this increases $R(t_2;h_{N,2})$ and we can continue with initial data $u_h(x,t_2)$ in a new mesh that is the old one plus the point that we have added near the boundary $x=1$. This procedure generates an increasing sequence of times t_i and a decreasing sequence $h_i=h_{N,i}$, at which the tolerance $R(t_i;h_i)=c_1$ is reached, a sequence of added points accumulating at $x=1$ and a numerical solution $u_h(x,t)$.

It remains to be more concrete on the election of the sequence h_i, c_1 and c_2. In fact if we take

$$c_1=\Gamma^{-2(p-1)}(2(p-1))^{-1},$$

and h_i and c_2 such that

$$\frac{h_i}{h_{i+1}}\to 1 \qquad \text{and} \qquad c_2=c_2(t)\sim c_1, \tag{5}$$

integrating (3) we get that

$$\Gamma(T_h-t)^{-\frac{1}{2(p-1)}}\geq u_{N+1}(t)=\max u_h(\cdot,t)$$

$$\geq \left(2(p-1)\int_t^{T_h} c_2(s)\,ds\right)^{-\frac{1}{2(p-1)}},$$

Hence, as $t \nearrow T_h$,

$$u_{N+1}(t) = \max u_h(\cdot,t) \sim \Gamma(T_h - t)^{-\frac{1}{2(p-1)}},$$

and we have recovered the continuous asymptotic behavior (2). So we want that h_i and c_2 verify (5). On the other hand, as we have mentioned in the introduction, the continuous problem verifies that

$$u(x,t) \sim (T-t)^{-\frac{1}{2(p-1)}}\varphi(\xi)$$

for $x = 1 - \xi(T-t)^{1/2} \sim C(u(1,t))^{-(p-1)}$. In order to recover this self-similar scaling we choose h_i such that

$$h_{i+1}(u_{N+1}(t_i))^{p-1} = \frac{2}{c_1} - \frac{A}{u_{N+1}(t_i)},$$

this choice make

$$c_2 \sim c_1\left(\frac{1}{1-\frac{c_1 A}{2u_{N+1}(t_i)}}\right),$$

and hence,

$$1 - x_N = h_{i+1} \sim \frac{2}{c_1}(u_{N+1}(t_i))^{-(p-1)} \sim C(T_h - t)^{1/2}.$$

This procedure gives an approximation of the curve $x = 1-\xi(T-t)^{1/2}$. Notice that if h_{i+1} and c_2 are given as above, relations (5) hold. So we recover the self-similar behavior (2).

1.2. Moving points method. As we mentioned in the introduction we may use a procedure inspired in the moving mesh method. Let us describe briefly the main ingredients of such type of schemes, referring to [**14**] for details. Following [**14**], the numerical solution is defined on a new mesh $x_i(t)$ that is obtained from a differentiable mesh transformation $x = x(\xi,t)$ with $x_i(t) = x(i/N,t)$. For this approach a new partial differential equation for $x(\xi,t)$ is solved numerically simultaneously with the original equation for $u(x,t)$ imposing a self-similar invariance on the resulting problem. In our case we impose that the blowing up end, $x_N = 1$, is fixed and move the rest of the points according to the previously described procedure, obtaining a $2N$ system of ODE to be solved. There are various ways to move the meshes, we refer to [**14**] for the details.

This type of procedure may loose some accuracy as it concentrates the points near $x = 1$ as the solution becomes bigger leaving "holes" near the other end $x = 0$. Our strategy is to take advantage of the fact that we know a priori that the singular set is $x = 1$ and hence to reproduce the main features of the continuous solution: the blow-up rate, the structure of the solution near the blow-up time (approximate self-similar behavior) and the blow-up set. We only need to move the last K points near $x = 1$. In fact we use the ideas developed in the *adding points method* to move the points not in a continuous way but at certain times where $R(t, h_N) = c_1$.

Next we describe our *moving points method* that, as we mentioned before, instead of adding a point when $R(t, h_N) = c_1$, moves the last K points near the boundary $x = 1$.

Let us begin with a mesh composed by two types of nodes, a uniform mesh of size $h = 1/N$ (the fact that this mesh is uniform is not essential for our analysis) and K nodes placed between $x_N = (N-1)h$ and $x_{N+1} = 1$, that we are going to move when appropriate. The number of moving nodes, K, depends only on p, we choose $K = [1/(p-1)]$. The motivation for this choice of K comes from the desired localization of the blow-up set at $x = 1$, see Section 4. Let us call $0 = x_1 < .... < x_N < x_{N+1} = 1$ the fixed mesh and $x_N < y_1 < .... < y_K < x_{N+1} = 1$ the moving nodes. As before, we use a semidiscretization in space, using the mesh composed by the x_i and the y_i together, and arrive to a system of ODE

$$\begin{aligned} &MU'(t) = -AU(t) + BU^p(t), \\ &U(0) = u_0^I. \end{aligned}$$

Again, we pay special attention to the numerical solution at the point $x = 1$. Writing the equation satisfied by u_{N+1} explicitly we obtain the following ODE,

$$u'_{N+1}(t) = \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t),$$

where $h_N = 1 - y_K$. Now we proceed as follows to obtain an adaptive procedure: as before we want the numerical blow-up rate to be

$$u_h(1,t) \sim (T_h - t)^{-\frac{1}{2(p-1)}}.$$

Hence we impose that the last node $u_{N+1}(t) = u_h(1,t)$ satisfies (3), this is equivalent to

$$(6) \qquad c_1 u_{N+1}^{2p-1}(t) \leq \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t) \leq c_2 u_{N+1}^{2p-1}(t).$$

In this case $h_N = 1 - y_K$. We have that

$$R(t; h_N) = \frac{\frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t)}{u_{N+1}^{2p-1}(t)} \to 0 \qquad \text{as } t \text{ increases.}$$

Let t_1 be the first time such that $R(t; h_N) = c_1$, that is the first time where $R(t; h_N)$ meets the tolerance c_1.

We remark that our criteria to stop and modify the mesh is the same as before.

Assume, to simplify the exposition, that $K = 1$, that is, we only have one moving point, y_K between x_{N-1} and $x_N = 1$. At that time t_1 we move the point y_K between x_N and $x_{N+1} = 1$ to the point z and give the value of $u_h(z, t_1)$ such that the slope of the line between $(z, u_h(z, t_1))$ and $(1, u_h(1, t_1))$ is the same as the slope between $(x_N, u_h(x_N, t_1))$ and $(1, u_h(1, t_1))$, see Figure 2.

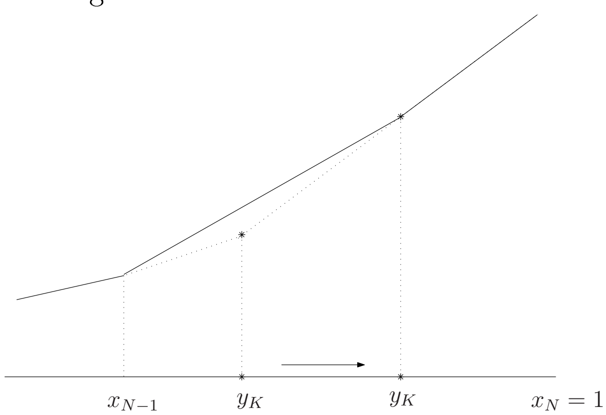

FIGURE 2.

Hence, exactly as before, we have a new value for the length of the last interval, $[z, 1]$,

$$h_{N,1} = 1 - z < h_N = 1 - y_K.$$

As before, $R(t_1; h_{N,1})$ verifies

$$R(t_1; h_{N,1}) = \frac{h_N}{h_{N,1}} R(t_1; h_N) = \frac{h_N}{h_{N,1}}\, c_1 > c_1.$$

Therefore with this new mesh $x_1, ..., x_N, z = y_K, x_{N+1}$ we have that

$$R(t_1; h_{N,1}) > c_1,$$

and we can apply the method with initial data $u_h(x, t_1)$ in the new mesh to continue.

We remark that the number of nodes does not increase. We have only moved the node y_K from its previous position to its new position, z. Hence, the size of the ODE system does not increase.

This gives us a solution u_h that verifies (4) in a time interval $[t_1, t_2]$ where at time t_2, the function $R(t_2; h_{N,1})$ reaches the tolerance c_1. At that time t_2 we have to move again the moving point, y_K, in the last interval. As before this increases $R(t_2; h_{N,2})$ and we can continue with initial data $u_h(x, t_2)$. This procedure generates an increasing sequence of times t_i and a decreasing sequence $h_i = h_{N,i}$, at which the tolerance $R(t_i; h_i) = c_1$ is reached, a sequence of moving points accumulating at $x = 1$ and a numerical solution $u_h(x, t)$. It remains to be more concrete on the election of the sequence h_i, c_1 and c_2. In fact if we take, exactly as before,

$$c_1 = \Gamma^{-2(p-1)}(2(p-1))^{-1},$$

and h_i such that

$$\frac{h_i}{h_{i+1}} \to 1 \qquad \text{and} \qquad c_2 = c_2(t) \sim c_1,$$

we get that

$$u_{N+1}(t) = \max u_h(\cdot, t) \sim \Gamma(T_h - t)^{-\frac{1}{2(p-1)}},$$

and we have recovered the continuous asymptotic behavior (2). On the other hand, to recover the self-similar behavior, we choose h_i as in the adding points method, that is such that

$$h_{i+1}(u_{N+1}(t_i))^{p-1} = \frac{2}{c_1} - \frac{A}{u_{N+1}(t_i)},$$

so

$$c_2 \sim c_1 \left(\frac{1}{1 - \frac{c_1 A}{2u_{N+1}(t_i)}} \right).$$

Hence,

$$1 - y_K = h_{i+1} \sim \frac{2}{c_1}(u_{N+1}(t_i))^{-(p-1)} \sim C(T_h - t)^{1/2}.$$

This procedure gives an approximation of the curve $x = 1 - \xi(T-t)^{1/2}$. Notice that if h_{i+1} and c_2 are given as above, relations (5) hold. So we recover the self-similar behavior (2).

In case we have $K > 1$ we proceed in the same way, but in this case we have to move K points near $x_{N+1} = 1$. Observe that the scaling factor that we are using places y_K at z, we move the rest of the points lying between x_N and 1 such that the mesh remains uniform in the interval $[y_1, 1]$. This imposes on y_j, $j = 1, \dots, K$ the same behavior described above for y_K.

Let us remark that the criteria that we use to modify the mesh is the same in the *adding points method* and in the *moving points method.* This allows us to make a unified approach in the course of the proofs contained in the following sections.

2. Convergence of the numerical schemes

In this section we prove a uniform convergence result. The proof is very similar to the one in **[22]**; we include the arguments here for the sake of completeness. For any $\tau > 0$ we want that $u_h \to u$ (when $h \to 0$) uniformly in $[0,1] \times [0, T - \tau]$. This is a natural requirement since on such an interval the exact solution is regular.

In particular, uniform convergence can be obtained by using standard inverse inequalities. In the following Theorem we give a proof of the L^2 convergence for a problem like (1) considering mass lumping. As a corollary, we will obtain uniform convergence for problem (1).

THEOREM 10.1. *Let u be the solution of* (1) *and let u_h its semidiscrete approximation obtained by any of the adaptive schemes described in Section 2. If $u \in C^{2,1}([0,1] \times [0, T - \tau])$ for some $\tau > 0$ then, there exists a constant C depending on τ such that:*

$$\|u - u_h\|_{L^\infty([0,T-\tau],L^2)} \leq Ch^2.$$

PROOF. We fix our attention on the convergence of the method in a time interval of the form $[t_i, t_{i+1}]$, that is, between two consecutive refinement times. Next we will show that this is enough for our purposes. Let us begin by $t \in (0, t_1)$.

In this proof we use the notation $L^2 = L^2((0,1))$ that refers to the L^2 norm in the x variable for each t (we will use analogous notations for other norms below) and u' for the derivative respect to time, u_t.

As u is a solution of (1) it satisfies

$$\int_0^1 u'v + \int_0^1 u_x v_x = u^p(1,t)v(1,t) \qquad \forall v \in H^1.$$

The numerical scheme is equivalent to

$$\int_0^1 ((u_h)'v)^I + \int_0^1 u_x v_x = u_h^p(1,t)v(1,t) \qquad \forall v \in V_h.$$

Hence we have that $e = u - u_h$ satisfies the following error equation,

$$\int_0^1 (e'v)^I \, dx + \int_0^1 e_x v_x \, dx = (u^p - u_h^p)(1,t)v(1,t) + \int_0^1 ((u'v)^I - u'v) \, dx$$

for all $v \in V_h$. Writing

$$e = u - u_h + u^I - u^I = u - u^I + \eta$$

and using known error estimates for Lagrange interpolation it remains to estimate $\eta = u^I - u_h$.

First, it is easy to see that,

$$\int_0^1 (u - u^I)_x v_x = 0 \qquad \forall v \in V_h,$$

and therefore, replacing in the error equation we have an equation for η,

$$\begin{aligned} \int_0^1 (\eta' v)^I + & \int_0^1 \eta_x v_x = (u^p(1,t) - u_h^p(1,t))v(1,t) \\ + & \int_0^1 ((u'v)^I - u'v) - \int_0^1 ((u' - (u^I)')v)^I \qquad \forall v \in V_h. \end{aligned}$$

In particular if we choose $v = \eta \in V_h$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\int_0^1 (\eta^2)^I) + \int_0^1 (\eta_x)^2 = \; & (u^p - u_h^p)(1,t)\eta(1,t) \\ + & \int_0^1 ((u'\eta)^I - u'\eta) = I + II. \end{aligned}$$

First, let us estimate I.

$$|I| = |(u^p(1,t) - u_h^p(1,t))\eta(1,t)| = \left|((u^I)^p(1,t) - u_h^p(1,t))\eta(1,t)\right|.$$

Hence we get that

$$|I| \le C|\eta(1,t)|^2.$$

Using the well known inequality,

$$|\eta(1,t)|^2 \le (C + \frac{C}{\varepsilon})\|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))},$$

we have that

$$|I| \le C_\varepsilon\|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))}.$$

In order to get a bound for II we decompose it in the following form,

$$II = \int_0^1 ((u'\eta)^I - u'\eta)\,dx = \int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx + \int_0^1 ((u')^I\eta - u'\eta)\,dx.$$

We proceed as before, for each subinterval I_j of the partition we know that,

$$\begin{aligned}\|((u')^I\eta)^I - (u')^I\eta\|_{L^1(I_j)} \leq\ & Ch^2\|((u')^I\eta)_{xx}\|_{L^1(I_j)}\\ \leq\ & Ch^2\|((u')^I)_x\eta_x\|_{L^1(I_j)}\end{aligned}$$

because $(u')^I$ and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|((u')^I)_x\|_{L^2} \leq C\|u'\|_{H^1}$ we obtain,

$$\begin{aligned}\int_0^1 ((u'\eta)^I - & (u')^I\eta)\,dx \leq Ch^2 \int_0^1 |((u')^I)_x||\eta_x|\,dx\\ \leq\ & Ch^2\|((u')^I)_x\|_{L^2((0,1))}\|\eta_x\|_{L^2} \leq Ch^4\|u'\|^2_{H^1} + \tfrac{1}{4}\|\eta_x\|^2_{L^2}.\end{aligned}$$

It remains to estimate the second summand of II. We have,

$$\begin{aligned}\int_0^1 ((u')^I - & u')\eta\,dx \leq \|(u')^I - u'\|_{L^2}\|\eta\|_{L^2}\\ \leq\ & \|(u')^I - u'\|^2_{L^2} + \|\eta\|^2_{L^2} \leq Ch^4\|u'\|^2_{H^2} + \|\eta\|^2_{L^2}.\end{aligned}$$

Collecting all the previous bounds we obtain,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + & \int_0^1 |\eta_x|^2 \leq C_\varepsilon\|\eta\|^2_{L^2((0,1))} + C\varepsilon\|\eta_x\|^2_{L^2((0,1))}\\ + & \ Ch^4\|u'\|^2_{H^2} + \|\eta\|^2_{L^2} + Ch^4\|u'\|^2_{H^1} + \tfrac{1}{4}\|\eta_x\|^2_{L^2}.\end{aligned}$$

We choose ε such that $C\varepsilon = 1/4$ and we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + \frac{1}{2}\int_0^1 |\eta_x|^2 \leq C\|\eta\|^2_{L^2((0,1))} + Ch^4\|u'\|^2_{H^2}.$$

Since $\int_0^1 (\eta^2)^I\,dx \sim \|\eta\|^2_{L^2}$ we can apply Gronwall's Lemma to obtain for $t \in [0, T_1]$,

$$\|\eta(t)\|_{L^2} + \left(\int_0^{t_1} \|\eta_x\|^2_{L^2}\,dt\right)^{1/2} \leq Ce^{C(t_1)}h^2.$$

In particular,

$$\|\eta\|_{L^2} \leq C(u, t_1)h^2.$$

and hence,

$$\|e\|_{L^2} \leq \|u - u^I\|_{L^2} + \|\eta\|_{L^2} \leq C(u, t_1)h^2.$$

This proves the result in $[0, t_1]$. Assume that the result is true in $[0, t_i]$. The same arguments used before prove that the bound holds up to t_{i+1}. The observation that for h small enough there exists i_0 such that $t_{i_0} > T - \tau$ finishes the proof. $\square$

As a corollary of Theorem 10.1 we can prove the following result that gives uniform, L^∞, convergence in an interval of the form $[0, T-\tau]$.

COROLLARY 10.1. *Let u be the solution of* (1) *and u_h its numerical approximation. Given $\tau > 0$, there exists a constant C depending on $\|u\|$ in $C^{2,1}([0,1] \times [0, T-\tau])$ such that, for h small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{\frac{3}{2}}.$$

PROOF. It is known that before the blow up time u is regular, more precisely, $u \in C^{2,1}([0,1] \times [0, T-\tau])$. Let $g(u)$ be a globally Lipschitz function which agrees with $f(u) = u^p$ for $u \leq 2M$ where $M = \|u\|_{L^\infty([0,1]\times[0,T-\tau])}$. Let $\overline{u}$ and $\overline{u}_h$ be the exact and approximate solutions of a problem like (1) with $f(u) = u^p$ replaced by $g(u)$. By uniqueness $u = \overline{u}$ in $[0,1] \times [0, T-\tau]$. A bound for $\|\overline{u} - \overline{u}_h\|_{L^\infty}$ can be obtained from Theorem 10.1. Indeed, it is enough to bound $\|\overline{u}^I - \overline{u}_h\|_{L^\infty}$, and using a standard inverse inequality (see **[19]**) we have,

$$\begin{aligned} \|\overline{u}^I - \overline{u}_h\|_{L^\infty} \leq & \; Ch^{-\frac{1}{2}}\|\overline{u}^I - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)} \\ \leq & \; Ch^{-\frac{1}{2}}\left\{\|\overline{u}^I - \overline{u}\|_{L^\infty([0,T-\tau],L^2)}\right. \\ & \left. +C\|\overline{u} - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}\right\} \leq Ch^{\frac{3}{2}} \end{aligned}$$

with C depending on u and the constant in Theorem 10.1 and so on τ.

Consequently, for h small enough $|\overline{u}_h| \leq 2M$. Therefore $u_h^p = f(\overline{u}_h) = g(\overline{u}_h)$ and so $\overline{u}_h$ is the finite element approximation of u and, by uniqueness $\overline{u}_h = u_h$ which concludes the proof. $\square$

To finish this section, we state a Lemma that says that the maximum of $U(t)$ is attained at the last node, x_{N+1}.

LEMMA 10.1. *If $U(0)$ is increasing, then the numerical solution $U(t)$ is increasing for every time t. Hence it satisfies*

$$\max u_h(\cdot, t) = u_{N+1}(t),$$

for every t.

PROOF. As $U(0)$ is increasing then $u_{i+1}(0) > u_i(0)$, let us see that this holds for every $t > 0$. Assume not, then there exists a first positive time t_0 and an index i such that $u_{i+1}(t_0) = u_i(t_0)$. From the equations satisfied by u_{i+1} and u_i we have that at that time t_0 it holds $u'_{i+1}(t_0) - u'_i(t_0) > 0$ a contradiction that proves the result. $\square$

3. Numerical blow-up

In this section we prove that solutions of the numerical problems blow up if and only if $p > 1$ and we find the blow-up rate in this case.

LEMMA 10.2. *If $p > 1$ then every positive solution of any of the numerical schemes blows up. Moreover, if $U(0)$ is increasing, then*

$$\lim_{t \nearrow T_h} (T_h - t)^{\frac{1}{2(p-1)}} \|U(t)\|_\infty = \Gamma,$$

where T_h is the blow-up time of U.

PROOF. If $U(0)$ is increasing, by Lemma 10.1 $U(t)$ has its maximum at $u_{N+1}(t)$. At a modifying time t_i we have that $R(h_i, t_i) = c_1$ and we modify the mesh to get

$$c_1 < R(h_{i+1}, t_i) = \frac{h_i}{h_{i+1}} R(h_i, t_i) = \frac{h_i}{h_{i+1}} c_1$$

We obtain that

$$\frac{u'_{N+1}}{u_{N+1}^{2p-1}}(t) \geq c_1,$$

for every t. By integration we get that, as $p > 1$, u_{N+1} (and hence u_h) cannot be global. Moreover, if we call T_h the blow-up time, using that $h_i/h_{i+1} \to 1$ $(i \to \infty)$, we have

$$\lim_{t \to T_h} \frac{u'_{N+1}}{u_{N+1}^{2p-1}}(t) = c_1. \tag{7}$$

Integrating (7), we get

$$(c_1 - \varepsilon_2(t))(T_h - t) \leq \int_t^{T_h} \frac{u'_{N+1}}{u_{N+1}^{2p-1}}(s)\, ds \leq (c_1 + \varepsilon_1(t))(T_h - t),$$

where $\varepsilon_i(t) \to 0$ as $t \nearrow T_h$. Changing variables we obtain

$$(c_1 - \varepsilon_2(t))(T_h - t) \leq \int_{u_{N+1}(t)}^{+\infty} \frac{1}{s^{2p-1}}\, ds \leq (c_1 + \varepsilon_1(t))(T_h - t).$$

Hence

$$\lim_{t \to T_h} (T_h - t)^{\frac{1}{2(p-1)}} u_{N+1}(t) = (2(p-1)c_1)^{\frac{-1}{2(p-1)}} = \Gamma,$$

as expected from the description of the method. □

We remark that this rate coincides with the blow-up rate of the continuous problem.

4. Numerical blow-up set

Now we turn our interest to the blow-up set of the numerical solution. We want to look at the set of points, x, such that $u_h(x,t) \to +\infty$ as $t \nearrow T_h$.

In **[3]** and Chapter 8 it is proved that for a fixed mesh the numerical blow-up set is given by $B(U) = [1 - Kh, 1]$, where K depends only on p. Notice that as K does not depend on h, the blow-up set converges to the continuous blow-up set $B(u) = \{1\}$.

We remark that for the methods introduced in this chapter at least K points are collapsed near $x = 1$ when the solution blows up, therefore the parameter h_i goes to zero and formally the blow-up set for the method must be $B(U) = \{1\}$. In order to prove this, we take a point $\bar{x} < 1$ and we claim that the numerical solution is bounded in $[0, \bar{x}]$.

LEMMA 10.3. *Let $\bar{x} < 1$ then for all nodes $x_i < \bar{x}$ the numerical solution is bounded, i.e., there exists $C = C(\bar{x}) > 0$ such that*

$$u_h(x,t) \le C,$$

for all $x \in [0, \bar{x}]$, $t \in [0, T_h)$.

PROOF. Let L be the first integer such that $L - 1/2(p-1) > -1$, i.e. $L = [1/2(p-1)]$; this implies that the function $g(t) = t^{-1/2(p-1)}$ is L times integrable.

From Lemma 10.2 we have that

$$\max u_h(\cdot, t) = u_{N+1}(t) \sim C(T_h - t)^{-\frac{1}{2(p-1)}}.$$

First we focus on the adding points method. The proof for the moving points method is slightly different.

Since we adapt near $x = 1$ collapsing (at least) L points in $\{1\}$ as the solution gets large, we can consider a node $x_{i_L} > \bar{x}$ such that at time t_{i_L} there exits $L + 1$ nodes between $\bar{x}$ and x_{i_L}.

On the other hand, as we modify the location of the nodes only for points between x_N and $x_{N+1} = 1$, we obtain that for the points

x_i, $i = 1, \cdots, N-1$ the mesh is fixed. Therefore from the equation satisfied by these nodes we have that for $t > t_{i_L}$

$$u_i'(t) \leq C \max(u_{i+1}, u_{i-1}) \leq C(T_h - t)^{-\frac{1}{2(p-1)}},$$

and then for $i = 1, \cdots, N-2$ we get

$$u_i(t) \leq C(T_h - t)^{-\frac{1}{2(p-1)}+1}.$$

This bound implies that

$$u_i'(t) \leq C \max(u_{i+1}, u_{i-1}) \leq C(T_h - t)^{-\frac{1}{2(p-1)}+1},$$

for $i = 1, \cdots, N-2$. Integrating again we get

$$u_i(t) \leq C(T_h - t)^{-\frac{1}{2(p-1)}+2},$$

for every $i = 1, \cdots, N-3$. Repeating the same argument L times we have that $u_i(t)$ is bounded for all $i = 1, \cdots, N-L-1$.

Hence, as there are L nodes between $\bar{x}$ and x_{N+1}, for t close to T_h, we get that for all nodes $x_i < \bar{x}$, $u_i(t)$ is bounded.

In the case of the moving points method the idea of the proof is the same but we deal with the K moving points between x_N and $x_{N+1} = 1$. In fact, arguing as before, for u_K the value of the numerical solution at the moving node y_K, we have

$$u_K'(t) \leq C(t)(T_h - t)^{-\frac{1}{2(p-1)}}.$$

The behavior of $C(t)$ depends on our choice of h in the adaptive procedure, hence

$$C(t) \sim (T_h - t)^{-\frac{1}{2}}.$$

Therefore,

$$u_K(t) \leq C(T_h - t)^{-\frac{1}{2(p-1)}+\frac{1}{2}}.$$

For u_{K-1} we get

$$u_{K-1}'(t) \leq C(t)(T_h - t)^{-\frac{1}{2(p-1)}+\frac{1}{2}} \leq C(T_h - t)^{-\frac{1}{2(p-1)}}.$$

Integrating we get

$$u_{K-1}(t) \leq C(T_h - t)^{-\frac{1}{2(p-1)}+1}.$$

Iterating this procedure K times we get that the value of the numerical solution at y_1 is bounded. Since y_1 goes to 1 as $t \to T_h$ and the numerical solution is increasing in space we conclude that $u_h(\bar{x}, t)$ is bounded. $\square$

We remark that this result implies that $B(U) = \{1\}$.

5. Convergence of the numerical blow-up times

In this section we prove that the numerical blow-up times, T_h, converges to the continuous one, T, when h goes to zero.

LEMMA 10.4. *The numerical blow-up times converge to the continuous one when $h \to 0$; in fact there exist $\alpha > 0$ and $C > 0$ such that*

$$|T_h - T| \le Ch^{\alpha}.$$

PROOF. The idea of the proof is as follows, first we get bounds for the first time t_0 such that the error verifies

$$\|(u_h - u)(t)\|_{L^\infty([0,1])} \le 1, \qquad \text{for all } t \in [0, t_0], \tag{8}$$

then we use this bound to prove our convergence result.

To get a bound for t_0 let us look at our scheme between two consecutive refinement times, from section 2 we have,

$$MU' = -AU + BU^p. \tag{9}$$

If we call $z_i(t) = u(x_i, t)$ we get that $Z = (z_1, \dots, z_{N+1})$ satisfies

$$MZ' \le -AZ + BZ^p + Ch(T-t)^{\theta}, \tag{10}$$

where θ depends only on p, in fact $\theta = -2 - 1/2(p-1)$. Let us remark that $C(T-t)^{\theta}$ is a bound for the first four spatial derivatives of $u(x,t)$ a regular solution of (1).

Subtracting (9) from (10) we have, for $E = U - Z$,

$$\begin{aligned} ME' \le & \;\; -AE + B(Z^p - U^p) + Ch(T-t)^{\theta} \\ \le & \;\; -AE + Bp\xi^{p-1}E + Ch(T-t)^{\theta}, \end{aligned} \tag{11}$$

where ξ is a point between u_{N+1} and z_{N+1}. Using that $t \in [0, t_0]$, (8), and the blow-up rate we get

$$\xi^{p-1} \le C(T-t)^{-\frac{1}{2}}.$$

Hence (11) gives

$$ME' \le -AE + BC\frac{E}{(T-t)^{\frac{1}{2}}} + Ch(T-t)^{\theta}, \tag{12}$$

This inequality is a discretization of the problem

$$\begin{array}{ll} E_t \le E_{xx} + C(T-t)^{\theta}h & (x,t) \in (0,1) \times [0,T), \\ E_x(0,t) \le 0 & t \in [0,T), \\ E_x(1,t) \le C(T-t)^{-1/2}E(1,t) & t \in [0,T), \\ E(x,0) \le Ch & x \in (0,1). \end{array} \tag{13}$$

In the last section of this chapter we construct a supersolution for problem (13) such that

$$\overline{E}(x,t) \le Ch(T-t)^{-\gamma} \tag{14}$$

and with the first four spatial derivatives positives in $[0,1]$.

Now we take $\overline{e}_i(t) = \overline{E}(x_i,t)$ and we get a supersolution for (12). We use a comparison argument, see **[22]**, between two consecutive refinement times to obtain that

$$e_i(t) \le \overline{e}_i(t).$$

Therefore, using (14) we conclude that

$$e_i(t) \le C(T-t)^{-\gamma}h.$$

The same arguments applied to $E = Z - U$ gives the lower bound

$$e_i(t) \ge C(T-t)^{-\gamma}h.$$

Therefore

$$|e_i(t)| \le C(T-t)^{-\gamma}h.$$

We get a bound for t_0 as follows, let t_* be the first time such that $C(T-t_*)^{-\gamma}h = 1$. That is, $T - t_* = Ch^{1/\gamma}$. As $t_0 \ge t_*$ we have

$$T - t_0 \le Ch^{1/\gamma}.$$

This means that the first time where the error is of size one is less than a power of the parameter h. With this bound for $T - t_0$ we can get the desired result as follows

$$|T_h - T| \le |T - t_0| + |T_h - t_0| \le Ch^{1/\gamma} + |T_h - t_0|.$$

Now we observe that the blow-up rate for u_h gives that

$$\begin{aligned} |T_h - t_0| \le \;& C(u_{N+1}(t_0))^{-2(p-1)} \le C(z_{N+1}(t_0)+1)^{-2(p-1)} \\ \le \;& C(T-t_0) \le Ch^{\frac{1}{\gamma}}, \end{aligned}$$

and the result follows. □

Let us observe that we can obtain a convergence result using these ideas. However, the methods used in Section 1 provided a better estimate.

6. Numerical experiments

In this section we present numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one perform numerical computations. For the numerical experiments we use an adaptive ODE solver to integrate in time. We take as initial datum $u_0(x) = (x^2 + 1)/2$. In Figures 1, 2 below we compare the adding points scheme with one with a fixed mesh of the same size $h = 1/50$, for $p = 2$ and $p = 1.5$. We present the evolution $\ln(u_h(1, t))$ versus $-\ln(T_h - t)$. The slope of the obtained curves measures the blow-up rate. As expected these slopes for $p = 2$ are $1/2$ with the adding points scheme and 1 with a fixed mesh, while for $p = 1.5$ are 1 and 2 respectively.

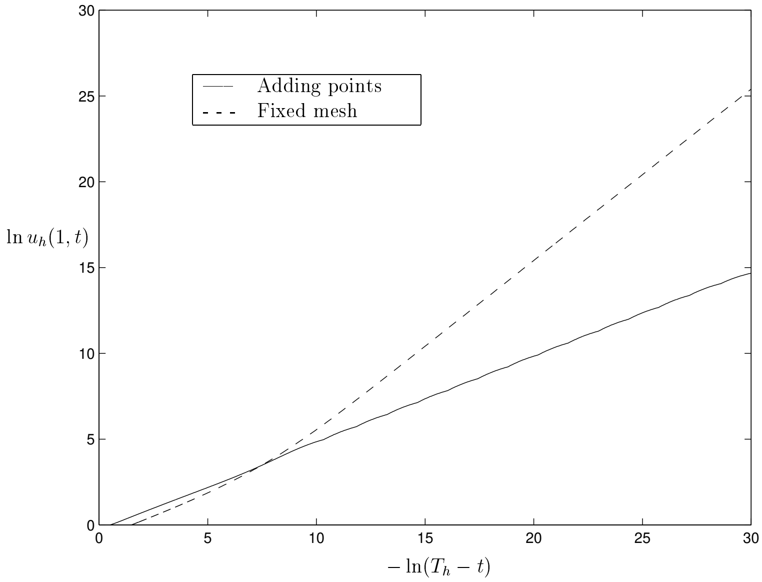

FIGURE 1. BLOW-UP RATES. $p = 2$.

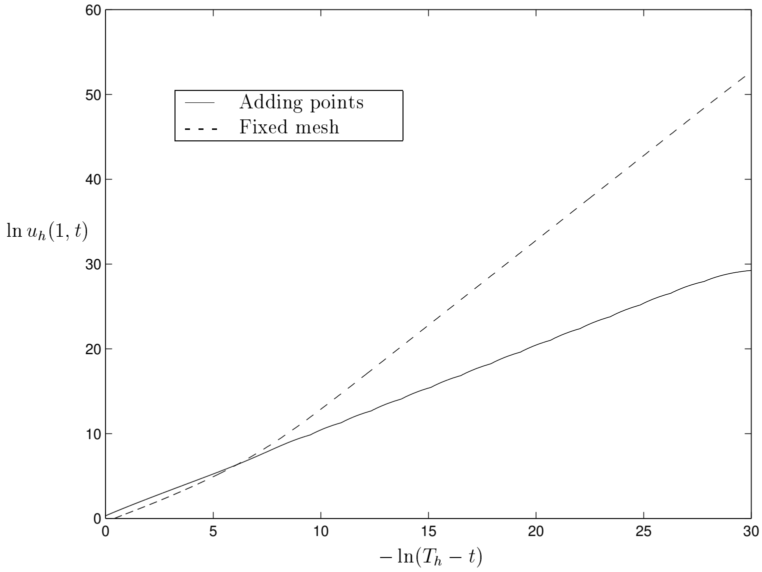

FIGURE 2. BLOW-UP RATES. $p = 1.5$.

In Figures 3, 4 we apply the moving points method, using the same initial datum and the same powers ($p = 2$ and $p = 1.5$) used above. Again we can observe that the slopes of the resulting lines reproduce the expected blow-up rates.

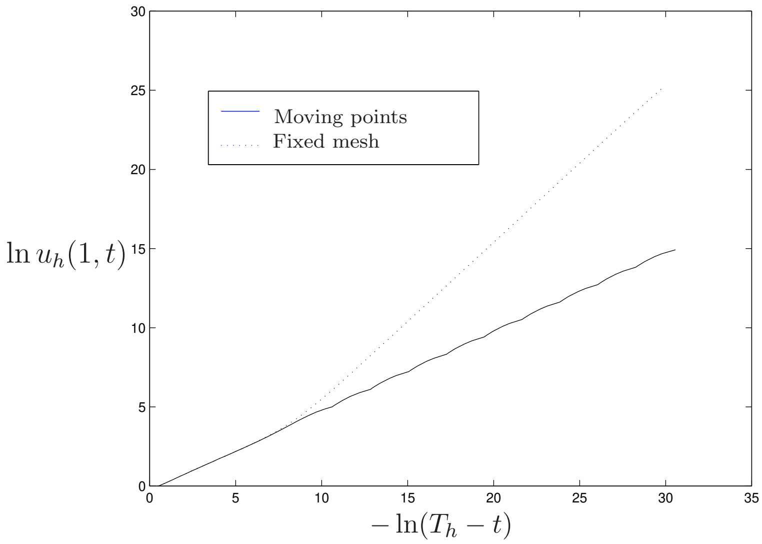

FIGURE 3. BLOW-UP RATES. $p = 2$.

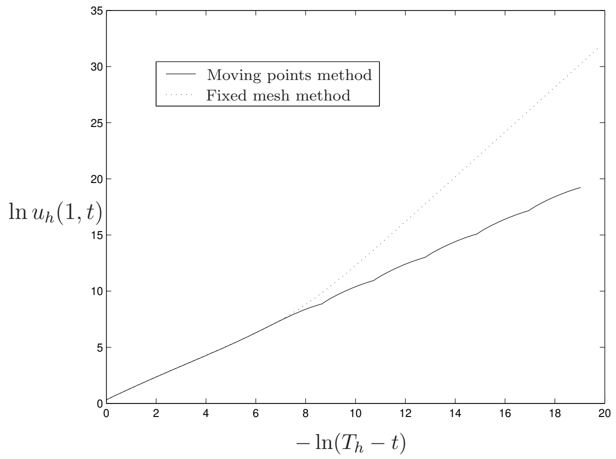

FIGURE 4. BLOW-UP RATES. $p = 1.5$.

Finally Figures 5, 6 show the evolution of the moving nodes in the case of the moving points method for $p = 2$ (just one node is moved) and $p = 1.5$ (two moving nodes) respectively. In these figures it can be observed that the moving nodes satisfy

$$-\ln(T_h - t) \sim 2(-\ln(1 - y_i)) + C_i$$

which means that they obey the scaling invariance of the problem

$$\frac{1 - y_i}{\sqrt{T_h - t}} \sim \xi_i,$$

and so this method recovers the self-similar behavior of the continuous solution as it was proved.

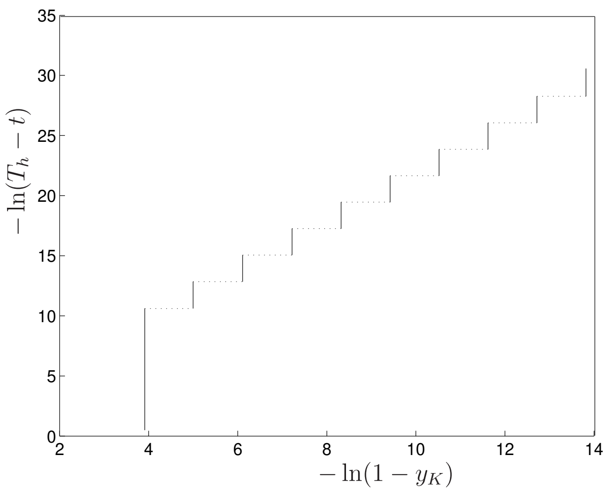

FIGURE 5. POSITION OF THE MOVING POINT. $p = 2$.

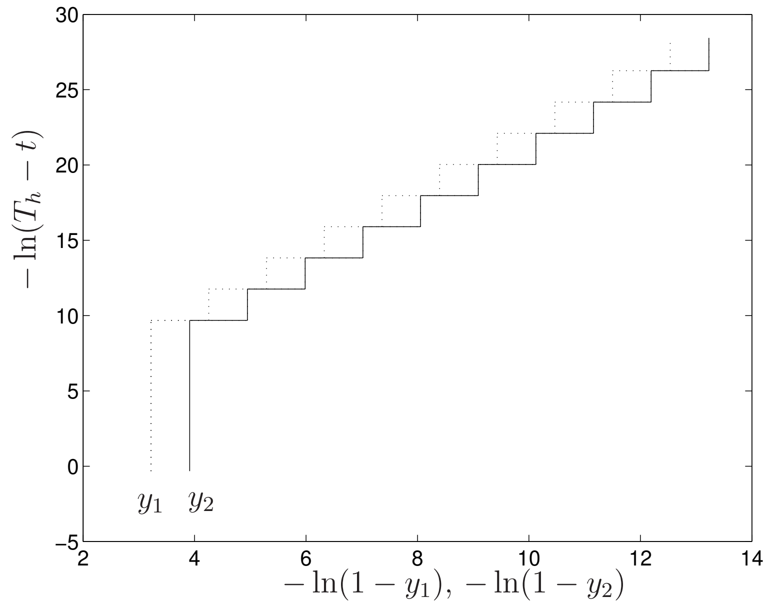

FIGURE 6. POSITION OF THE TWO MOVING POINTS. $p = 1.5$.

We want to comment that moving mesh methods, based in moving mesh partial differential equations are also expected to reproduce these asymptotic behaviors, the evidence is just heuristical and experimental, but rigorous proofs are not available. However, the moving mesh algorithms can be applied to a large family of problems. The idea behind these procedures is to take advantage of the self-similar scaling of the problems under consideration. This seems to us to be the more natural way to face blow-up problems. Our results can be viewed as a preliminary step to obtain rigorous proofs for these kind of schemes.

7. Extensions to higher dimensions

The results contained here are restricted to one space dimension. We may try to extend these ideas to several space dimensions. If Ω is a bounded domain we face the problem $u_t = \Delta u$ in $\Omega \times (0,T)$ with a boundary condition of the form $\frac{\partial u}{\partial \eta} = u^p$ on $\partial\Omega \times (0,T)$ and an initial datum $u_0(x)$. See **[18, 27, 45, 68, 69]** for references that includes the blow-up rate and set. There is a natural extension of the ideas developed here to deal with the higher dimensional case. However, in order to extend any of the two adaptive procedures described in Section 1 we need to have an a priori knowledge of the spatial location of the blow-up set. This is the case for example if the domain under consideration is a square $\Omega = [0,1] \times [0,1]$ for certain initial conditions that forces the blow-up set to be one of the corners (see **[28]**). Assume that we know where a single point blow-up set is located and that the maximum of the solution is placed at that point for every time (as in the case of the square described in [**28**]), then we place a node at that point and look for the asymptotic behavior of that node. It turns out that we get a condition of the form $c_1 \le R(t, h_{N,1}, h_{N,2}, ..., h_{N,J})$. When $R(t,\cdot) = c_1$ we modify the mesh locally near the blow-up point, adding or moving the adjacent nodes, by performing a contraction by a factor $r < 1$. A simple calculation shows that when such a contraction is performed the new value of $R(t,\cdot)$ increases, allowing us to proceed just as described in Section 1.

However, if the maximum of the solution moves from one node to another then we have to perform adaptive procedures at different points that may compensate as time evolves and hence our proofs are no longer valid.

8. A useful supersolution

In this section we find a supersolution for

$$\begin{array}{ll} E_t = E_{xx} + C(T-t)^{\theta} h & (x,t) \in (0,1) \times [0,T), \\ E_x(0,t) = 0 & t \in [0,T), \\ E_x(1,t) = C(T-t)^{-1/2} E(1,t) & t \in [0,T), \\ E(x,0) = Ch & x \in (0,1), \end{array}$$

such that

$$\overline{E}(x,t) \le Ch(T-t)^{-\gamma}$$

and with the first four spatial derivatives positives in $[0,1]$.

Let us look for a supersolution of the form

$$\overline{E}(x,t) = Ch(T-t)^{\theta} a(x,t),$$

with $a(x,t)$ a solution of

$$(15)\qquad \begin{array}{ll} a_t = a_{xx} & (x,t) \in (0,1) \times [0,T), \\ a_x(0,t) = 0 & t \in [0,T), \\ a_x(1,t) = C(T-t)^{-1/2} a(1,t) & t \in [0,T), \\ a(x,0) = a_0(x) & x \in (0,1). \end{array}$$

In order to get a supersolution, $\overline{E}$, with the first four spatial derivatives positives, we impose that $a_0(x)$ is a smooth compatible initial datum with the first four spatial derivatives positives. The positivity of the derivatives is preserved for every $t \in [0,T)$.

Now let us see that there exists r such that

$$a(x,t) \leq \frac{C}{(T-t)^r}. \tag{16}$$

To this end we want to construct a supersolution to (15) such that (16) holds. This can be easily done by the following procedure: take $v(x,t)$ a solution of (1) with boundary condition given by $v_x(1,t) = v^q(1,t)$, with q small, and initial datum v_0 such that $v(x,t)$ blows up exactly at time T, see **[70]** for a proof of the fact that for every time T there exist an initial datum such that $v(x,t)$ blows up at time T. The blow-up rate for solutions of (1), see **[29, 45]**, gives that $v(x,t)$ verifies

$$v(1,t) \sim \frac{L}{(T-t)^{\frac{1}{2(q-1)}}},$$

with $L^{q-1} \to +\infty$ as $q \searrow 1$, see **[29]** for an explicit formula for $L(q)$. Let us fix q such that $L^{q-1}(q) > C$. With this choice, v is a supersolution of (15). In fact,

$$v_x(1,t) = v^q(1,t) = v^{q-1}v(1,t) \geq \frac{L^{q-1}}{(T-t)^{\frac{1}{2}}} v(1,t),$$

as we wanted to prove.

CHAPTER 11

The porous medium equation (II)

In this chapter we deal with a numerical approximation for the following problem,

$$
(1) \qquad \begin{array}{ll}
u_t = (u^m)_{xx} & (x,t) \in (0,L) \times [0,T), \\
(u^m)_x(0,t) = 0 & t \in [0,T), \\
(u^m)_x(L,t) = u^p(L,t) & t \in [0,T), \\
u(x,0) = u_0(x) & x \in (0,L),
\end{array}
$$

where $m > 0$ and $p > 0$ are parameters. We assume that u_0 is positive.

Our main interest in this problem is due to the presence of regional and global blow-up, as was described in Chapter 7. We investigate how do numerical schemes reproduce this phenomenon.

Let us summarize the known results for the solutions of (1).

If $m > 1$ the equation is known as *porous medium equation*. It is proved in **[30]** that for every positive initial data the solution blows up if and only if $p > 1$. Moreover the blow-up rate for increasing initial data is given by,

(a) $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{p-1}}$ if $1 < p \le m$,
(b) $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2p-m-1}}$ if $p > m$.

For this problem the blow-up set, $B(u)$, i.e., the set of points where $u(x,t)$ becomes unbounded, is given by,

(a) $B(u) = [0,L]$ if $1 < p < m$ (global blow-up),
(b) $B(u) = [0,L]$ if $p = m$ and $L \le \frac{2m}{m-1}$ (global blow-up),
(c) $B(u) = [L - \frac{2m}{m-1}, L]$ if $p = m$ and $L > \frac{2m}{m-1}$ (regional blow-up),
(d) $B(u) = \{L\}$ if $p > m$ (single point blow-up).

This is proved in **[30]** for the cases $1 < p < m$ and $p > m$. For the case $p = m$ we perform the analysis in Section 5 at the end of the chapter and give a more precise description of blow-up showing the existence and uniqueness of a profile that gives the asymptotic behavior of blow-up solutions when we consider self-similar variables (**[75]**).

In the case $0 < m \le 1$ (known as *fast diffusion equation* if $0 < m < 1$ or *heat equation* for $m = 1$) the existence of blow up solutions depends on m and p. In fact every positive solution blows up if $p > \frac{m+1}{2}$, and if $p \le \frac{m+1}{2}$ every solution is global (see **[30, 78]**). In this case the blow-up rate is given by

$$\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2p-m-1}} \text{ if } p > \frac{m+1}{2},$$

and the blow-up set is

$$B(u) = \{L\} \text{ if } p > \frac{m+1}{2} \text{ (single point blow-up).}$$

Now we introduce the numerical scheme. We discretize using piecewise linear finite elements with mass lumping in a uniform mesh for the space variable, it is well known that this discretization in space coincides with the classic central finite difference second order scheme.

We denote with $U(t) = (u_1(t), ..., u_{N+1}(t))$ the values of the numerical approximation at the nodes $x_i = (i-1)h$, $h = L/N$, at time t. Then $U(t)$ is a solution of the following problem (see **[19]**):

$$\begin{aligned} MU'(t) &= -AU^m(t) + BU^p(t), \\ U(0) &= u_0^I, \end{aligned} \tag{2}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial data, u_0. Writing this equation explicitly we obtain the following ODE system,

$$\begin{aligned} &u_1'(t) = \frac{2}{h^2}(u_2^m(t) - u_1^m(t)), \\ &u_k'(t) = \frac{1}{h^2}(u_{k+1}^m(t) - 2u_k^m(t) + u_{k-1}^m(t)), \qquad 2 \le k \le N, \\ &u_{N+1}'(t) = \frac{2}{h^2}(u_N^m(t) - u_{N+1}^m(t)) + \frac{2}{h}u_{N+1}^p(t), \\ &u_k(0) = u_0(x_k) > 0, \qquad 1 \le k \le N+1. \end{aligned} \tag{3}$$

To begin our analysis we prove that numerical approximations given by (3) converge uniformly if we consider a regular bounded solution of the continuous problem. Hence our scheme is uniformly convergent in sets of the form $[0, L] \times [0, T-\tau]$.

Our main results concern the behavior of the numerical approximations given by (3). Significant differences appear between the continuous and the discrete problem.

First we prove that positive solutions of the numerical problem blow up if and only if $p > 1$. Hence, in the case $0 < m < 1$ with

$\frac{m+1}{2} < p \leq 1$ the continuos solutions blow up in finite time while the numerical approximations are globally defined.

Next, we turn our attention to the blow-up rate. For increasing in space solutions we find that the blow-up rate for the numerical scheme is given by

(1) $\|U(\cdot,t)\|_\infty \sim (T_h - t)^{-\frac{1}{p-1}}$ if $p > 1$.

Therefore the blow-up rate does not coincide in the range of parameters given by $p > m > 1$ or $p > 1$ with $0 < m \leq 1$, and coincides in the range $m > 1$ with $1 < p \leq m$.

Concerning the blow-up set for the numerical approximations we prove that

(a) $B(U) = [0, L]$ if $1 < p \leq m$ (global blow-up).
(b) $B(U) = [L - Kh, L]$ if $p > m > 1$ or $p > 1$ with $0 < m \leq 1$. Where the constant K depends only on m and p. K is the integer that verifies
$$\frac{\sum_{i=0}^{K+1} m^i}{\sum_{i=0}^{K} m^i} < p \leq \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}.$$

In the range $1 < p < m$ the numerical and the continuous blow-up sets coincide.

In the range $p > m > 1$ or $p > 1$ with $0 < m \leq 1$, the blow-up set can be larger than a single point, $x = L$, but our results show that
$$B(U) = [L - Kh, L] = B(u) + [-Kh, 0],$$
and hence
$$B(U) \to B(u), \qquad \text{as } h \to 0.$$

In the case $p = m > 1$ with L large a mayor difference appears concerning the blow-up sets. The continuous problem blows up in the subinterval $[L - \frac{2m}{m-1}, L]$, strictly contained in $[0, L]$, but the numerical solution blows up in the whole interval $[0, L]$. There is no regional blow-up for the numerical scheme.

However, in this case we can recover regional blow-up for the numerical scheme taking into account the correct self-similar variables and letting h go to zero. We prove that the numerical method blows up at nodes that lies in $[0, L - \frac{2m}{m-1})$ at a blow-up rate that goes to zero as h goes to zero while for nodes that belongs to $[L - \frac{2m}{m-1}, L]$ the blow-up rate does not vanish as h goes to zero. Hence we have recovered the regional blow-up by looking at the different blow-up rates of the solution at different nodes when the mesh parameter goes to zero.

To summarize the results of this chapter: we observe that when computing numerical approximations of a blow-up problem significant differences appear. The continuous problem can blow up while the numerical scheme has global solutions. Moreover, in case that both problems blow up the blow-up rate and the blow-up set can be different. We remark that regional blow-up is impossible for a numerical scheme with a fixed mesh, however if we look at the right variables we can recover regional blow-up when the mesh parameter goes to zero.

Organization of the chapter. In Section 1 we describe some properties of the numerical scheme and prove the convergence of the method. In Section 2 we prove the numerical blow-up results and find the numerical blow-up rates. Section 3 is devoted to the numerical blow-up sets. In Section 4 we present some numerical experiments. Finally we include some results on stability and uniqueness of the asymptotic profile for solutions of the continuous problem (1) in self-similar variables.

1. Properties of the numerical scheme

In this section we collect some preliminary results on our numerical method. In particular we prove convergence for regular solutions.

First, we want to prove a comparison Lemma. To do this we need the following definition,

DEFINITION. We will call $\overline{U}$ a supersolution if it satisfies

$$(4)\qquad \begin{cases} \overline{u}_1'(t) \geq \frac{2}{h^2}(\overline{u}_2^m(t) - \overline{u}_1^m(t)), \\ \overline{u}_k'(t) \geq \frac{1}{h^2}(\overline{u}_{k+1}^m(t) - 2\overline{u}_k^m(t) + \overline{u}_{k-1}^m(t)), \qquad 2 \leq k \leq N, \\ \overline{u}_{N+1}'(t) \geq \frac{2}{h^2}(\overline{u}_N^m(t) - \overline{u}_{N+1}^m(t)) + \frac{2}{h}\overline{u}_{N+1}^p(t), \\ \overline{u}_k(0) \geq u_0(x_k), \qquad 1 \leq k \leq N+1. \end{cases}$$

Analogously, we say that $\underline{U}$ is a subsolution if it satisfies (4) with the reverse inequalities.

LEMMA 11.1. *Let $\overline{U}$ and $\underline{U}$ be a superolution and a subsolution respectively, then*

$$\overline{U}(t) \geq U(t) \geq \underline{U}(t).$$

PROOF. By an approximation procedure we can assume that we have strict inequalities in (4). Let us prove that $\overline{U}(t) > U(t)$. We argue by contradiction. Let us assume that there exists a first time t_0 and a node j such that $\overline{u}_j(t_0) = u_j(t_0)$ then if $1 < j < N+1$,

$$0 \geq \overline{u}_j'(t_0) - u_j'(t_0) > \frac{1}{h^2}(\overline{u}_{j+1}^m(t_0) - u_{j+1}^m(t_0) + \overline{u}_{j-1}^m(t_0) - u_{j-1}^m(t_0)) \geq 0,$$

a contradiction. In case $j = 1$ we get

$$0 \geq \overline{u}_1'(t) - u_1'(t_0) > \frac{2}{h^2}(\overline{u}_2^m(t_0) - u_2^m(t_0)) \geq 0,$$

again a contradiction. Finally, if $j = N+1$ we have

$$0 \geq \overline{u}_{N+1}'(t_0) - u_{N+1}'(t_0) > \frac{2}{h^2}(\overline{u}_N^m(t_0) - u_N^m(t_0)) \geq 0.$$

This contradiction proves that $\overline{U}(t) > U(t)$.

The inequality $U(t) \geq \underline{U}(t)$ can be handled in a similar way. □

Now we prove our convergence result for regular solutions up to $t = T - \tau$.

THEOREM 11.1. *Let $u(x,t) \in C^{4,1}([0,L] \times [0,T-\tau])$ be a positive solution of (1) and $U(t)$ the numerical approximation given by (3). Then, there exists a constant C, that depends on the $C^{4,1}([0,L]\times[0,T-\tau])$ norm of u, such that for every h small enough it holds*

$$\max_{t\in[0,T-\tau]} \max_k |u(x_k,t) - u_k(t)| \leq Ch^2.$$

PROOF. In the course of this proof we will denote by C_i a constant independent on h which can be different in different occurrences.

If we rewrite the system (3) in terms of $Z = U^m$ we obtain

$$\begin{aligned}
&(z_1^{1/m})'(t) = \tfrac{2}{h^2}(z_2(t) - z_1(t)), \\
&(z_k^{1/m})'(t) = \tfrac{1}{h^2}(z_{k+1}(t) - 2z_k(t) + z_{k-1}(t)), \qquad 2 \leq k \leq N, \\
&(z_{N+1}^{1/m})'(t) = \tfrac{2}{h^2}(z_N(t) - z_{N+1}(t)) + \tfrac{2}{h} z_{N+1}^{p/m}(t), \\
&z_k(0) = u_0^{1/m}(x_k) > 0, \qquad 1 \leq k \leq N+1.
\end{aligned}$$

Let $v_k(t) = u^m(x_k,t)$ where u is the solution of the continuous problem (1). We define the error function as

$$e_k = z_k - v_k.$$

Let

$$t_0 = \max\{t \in [0, T-\tau] : |e_k|(t) \le c/2\}$$

where $c = \min\{v_k(t) : 1 \le k \le N+1 \text{ and } 0 \le t \le T-\tau\}$. We perform the following calculations with $t \in [0, t_0]$ and we will prove at the end that $t_0 = T - \tau$ for every h small enough.

The error function satisfies that, for $2 \le k \le N$,

$$\begin{aligned} \frac{1}{m} z_k^{(1-m)/m} e_k' = \; & \frac{1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) \\ & - \frac{1}{m}(z_k^{(1-m)/m} - v_k^{(1-m)/m})\, v_k' + C_1 h^2 \\ \le \; & \frac{1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) + C_1 h^2 + C_2 \xi_k^{(1-2m)/m)} |e_k|, \end{aligned}$$

where ξ_k is an intermediate value between z_k and v_k. Taking into account that there exist constants, c and C, such that $c \le z_k(t) \le C$ for every $t \in [0, t_0]$ we have

$$e_k' \le \frac{C_1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) + C_2|e_k| + C_3 h^2. \tag{5}$$

Making analogous calculations for the first and the last nodes, we get

$$e_1' \le \frac{C_1}{h}(e_2 - e_1) + C_2|e_1| + C_3 h^2, \tag{6}$$

(7)

$$e_{N+1}' \le \frac{C_1}{h^2}(e_N - e_{N+1}) + C_2 h^2 - C_3 \xi^{(1-2m)/m} e_{N+1} + \frac{C_4}{h} \eta^{(p-m)/m} e_{N+1}.$$

Where η is an intermediate value between z_{N+1} and v_{N+1}. Now we use a comparison argument. We remark that the system (5), (6), (7), has a comparison principle, that can be proved as in Lemma 11.1. Let us look for a supersolution of the form

$$w_k(t) = \varphi(t)\left(e^{C x_k} - C\, x_k\right),$$

where $\varphi(t)$ is a solution of

$$\begin{aligned} \varphi'(t) &= C_1 \varphi(t) + C_2 h^2 \\ \varphi(0) &= C_3 h^2. \end{aligned}$$

That is

$$\varphi(t) = h^2\left(C_1 e^{C_2 t} + C_3\right).$$

A direct calculation shows that we can choose the constant C large but independent of h such that, $W(t)$ is a supersolution. Hence,

$$e_k(t) \le w_k(t) \le C_1 h^2 e^{C_2 T}.$$

Arguing in the same way with $-e_k$ we obtain

$$|e_k(t)| \le w_k(t) \le C_1 h^2 e^{C_2 T}.$$

From this inequality it is easy to see that $t_0 = T - \tau$ for every h small enough and the Theorem is proved. $\square$

To finish this section, we state a Lemma that says that an increasing in space initial data gives an increasing solution. This will be used in the following sections to ensure that the maximum of $U(t)$ is attained at the last node, x_{N+1}.

LEMMA 11.2. *Let U be a solution of (3) with $u_k(0) \le u_{k+1}(0)$, $k = 1, ..., N$. Then*

$$u_k(t) < u_{k+1}(t), \qquad\qquad 1 \le k \le N.$$

PROOF. We argue by contradiction, let us assume there exists a first time t_0 and two consecutive nodes where the Lemma fails, let us call them $j, j+1$. We can assume that $u_{j-1}(t_0) < u_j(t_0)$. Indeed, if all the nodes have the same value then all of them have null derivative except the last one, which verifies $u'_{N+1}(t_0) > 0$, but this is not possible. Then we can assume that

$$u_{j-1}(t_0) < u_j(t_0) = u_{j+1}(t_0) \le u_{j+2}(t_0),$$

hence we obtain

$$u'_j(t_0) < 0, \qquad u'_{j+1}(t_0) \ge 0,$$

which is a contradiction. We observe that if at time $t = 0$ we have this situation then instantaneously we have $u_j(t) < u_{j+1}(t)$. $\square$

2. Numerical blow-up

In this section we prove that solutions of the numerical problem blow up if and only if $p > 1$ and we find the blow-up rate in this case.

THEOREM 11.2. *If $p > 1$ then every positive solution of (3) blows up. Moreover, if $U(0)$ is increasing, then there exist two constants, $C_1 = C_1(h)$ and $C_2 = C_2(h)$, such that*

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \le \|U(t)\|_\infty \le \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

where T_h is the blow-up time of U.

PROOF. Let us begin with an increasing initial data $U(0)$. By Lemma 11.2, $U(t)$ is increasing and then its maximum will be $u_{N+1}(t)$.

We define the function

$$w(t) := \frac{1}{2}u_1(t) + \sum_{j=2}^{N} u_j(t) + \frac{1}{2}u_{N+1}(t),$$

which satisfies

$$w'(t) = \frac{1}{h}u_{N+1}^p(t).$$

Since $U(t)$ is a monotone vector it has its maximum at $u_{N+1}(t)$. Hence there exists two positive constants, $c = c(h)$ and $C = C(h)$, such that

$$cw(t) \leq u_{N+1}(t) \leq Cw(t).$$

Therefore, the function $w(t)$ verifies

$$w'(t) \leq Cw^p(t), \qquad w'(t) \geq cw^p(t). \tag{8}$$

From the second inequality we get that w (and hence U) blows up if $p > 1$ at a time T_h, and from the first one, if $p \leq 1$, w (and therefore U) must be global. This blow-up result is valid for every initial data as we can use a comparison argument with an increasing super or subsolution.

Now we find the blow-up rate. By integration of (8) we obtain that

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \leq w(t) \leq \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

In terms of u_{N+1} we get

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \leq u_{N+1}(t) = \|U(t)\|_\infty \leq \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

Note that the blow-up phenomena occurs if and only if $p > 1$ and the blow-up rate is independent of the parameter m. $\square$

We remark that this rate coincides with the blow-up rate of the continuous problem (1) if $1 < p \leq m$, but it is different in the case $p > m > 1$ or $0 < m \leq 1$ and $p > 1$.

3. Numerical blow-up set

Now we turn our interest to the blow-up set of the numerical solution. For a fixed h we want to look at the set of nodes, x_k, such that $u_k(t) \to +\infty$ as $t \nearrow T_h$. To do this we introduce the self-similar variables given by

$$\begin{aligned} &y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t), \\ &(T_h - t) = e^{-s}. \end{aligned} \tag{9}$$

In this new variables $Y = (y_k(s))$, problem (3) reads

(10)

$$y_1'(s) = \tfrac{2}{h^2}e^{\frac{m-p}{p-1}s}(y_2^m(s) - y_1^m(s)) - \tfrac{1}{p-1}y_1(s),$$

$$y_k'(s) = \tfrac{2}{h^2}e^{\frac{m-p}{p-1}s}(y_{k+1}^m(s) - 2y_k^m(s) + y_{k-1}^m(s)) - \tfrac{1}{p-1}y_k(s),$$

$$y_{N+1}'(s) = \tfrac{2}{h^2}e^{\frac{m-p}{p-1}s}(y_N^m(s) - y_{N+1}^m(s)) + \tfrac{2}{h}y_{N+1}^p(s) - \tfrac{1}{p-1}y_{N+1}(s),$$

$$y_k(-\ln(T_h)) = (T_h)^{\frac{1}{p-1}}u_0(x_k), \qquad 1 \le k \le N+1.$$

We observe that, from the blow-up rates proved in Theorem 11.2, the vector Y is bounded and there exists a subsequence such that

$$\lim_{s_j\to\infty} y_{N+1}(s_j) = a_{N+1} \neq 0.$$

THEOREM 11.3. *If $1 < p < m$ then we have uniform global blow-up, that is*

$$u_k(t) \sim \frac{C}{(T_h - t)^{\frac{1}{p-1}}} \qquad\qquad 1 \le k \le N,$$

therefore

$$B(U) = [0, L].$$

PROOF. Since $y_{N+1}(s)$ is bounded we have that $y_{N+1}'(s)$ can not go to infinity as $s \to \infty$. From (10)

$$y_{N+1}'(s) = \frac{2}{h^2}e^{\frac{m-p}{p-1}s}(y_N^m(s) - y_{N+1}^m(s)) + \frac{2}{h}y_{N+1}^p(s) - \frac{1}{p-1}y_{N+1}(s),$$

as $p < m$ there exists a subsequence $\{s_j\}$, such that

$$y_N^m(s_j) - y_{N+1}^m(s_j) \to 0.$$

Now, we can take again a subsequence such that $y_N(s_j) \to a_{N+1}$. Applying the same argument with y_k we obtain that $y_{k-1}(s_j) \to a_{N+1}$ for all $2 \le k < N$. Hence $y_k(s) \to c > 0$ for every node k.

Therefore, in the context of the variable $U(t)$ we have that all nodes blow up in finite time T_h with the same rate. □

THEOREM 11.4. *If $p > \max\{m, 1\}$, then U blows up at exactly K nodes near L, i.e.*

$$B(U) = [L - Kh, L],$$

where K is determined only by p and m in the following way: K is the unique integer such that

$$\frac{\sum_{i=0}^{K+1} m^i}{\sum_{i=0}^{K} m^i} < p \leq \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}.$$

Moreover, the asymptotic behavior of the blow up nodes is given by

$$u_{N+1-i}(t) \sim (T_h - t)^{\gamma_i},$$

$$\gamma_i = -\frac{m^i}{p-1} + \sum_{l=0}^{i} m^l, \qquad \text{if} \quad p \neq \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i} \quad \text{or} \quad i \neq K$$

and by

$$u_K(t) \sim \ln(T_h - t), \qquad \text{if} \quad p = \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}.$$

PROOF. As Y is bounded and $p > m$, we have that for s large enough

$$y'_{N+1}(s) \sim \frac{2}{h} y^p_{N+1}(s) - \frac{1}{p-1} y_{N+1}(s).$$

So, for s large the function $y_{N+1}(s)$ is monotone and it has a positive limit as $s \to \infty$.

On the other hand, if we return to the variable $U(t)$, we have that

$$\begin{aligned} u'_N(t) = &\ \frac{1}{h^2}(u^m_{N+1}(t) - 2u^m_N(t) + u^m_{N-1}(t)) \leq \frac{1}{h^2}(u^m_{N+1}(t) + u^m_{N-1}(t)) \\ \leq &\ C(T_h - t)^{-\frac{m}{p-1}} \end{aligned}$$

and by integration

$$u_N(t) \leq \begin{cases} C(T_h - t)^{\frac{p-1-m}{p-1}} & p < m+1, \\ -C \ln(T_h - t) & p = m+1, \\ C & p > m+1. \end{cases}$$

If we translate this inequality in terms of Y, we get

$$0 \leq y_N(s) \leq \begin{cases} C\, e^{-\frac{p-m}{p-1}s} & p < m+1, \\ C\, s\, e^{-\frac{1}{p-1}s} & p = m+1, \\ C\, e^{-\frac{1}{p-1}s} & p > m+1. \end{cases}$$

Notice that in all cases $y_N(s) \to 0$. Repeating the argument we obtain that

$$y_k(s) \to 0, \qquad \text{for all} \quad k \leq N.$$

Moreover, for s large

$$\begin{aligned} y_N'(s) = &\ \frac{2}{h^2} e^{\frac{m-p}{p-1}s}(y_{N+1}^m(s) - 2y_N^m(s) + y_{N-1}^m(s)) - \frac{1}{p-1} y_N(s) \\ \sim &\ Ce^{\frac{m-p}{p-1}s} - \frac{1}{p-1} y_N(s). \end{aligned}$$

Integrating we obtain the asymptotic behavior of y_N,

$$y_N(s) \sim \begin{cases} C_1 e^{-\frac{1}{p-1}s} + C_2 e^{\frac{m-p}{p-1}s} \\ C_1 e^{-\frac{1}{p-1}s} + C_2 s e^{-\frac{1}{p-1}s} \end{cases} \sim \begin{cases} C_2 e^{\frac{m-p}{p-1}s} & p < m+1, \\ C_2 s e^{-\frac{1}{p-1}s} & p = m+1. \end{cases}$$

Translating this behavior to the context of $U(t)$,

$$u_N(t) \sim \begin{cases} C(T_h - t)^{\frac{p-m-1}{p-1}} & p < m+1, \\ -C \ln(T_h - t) & p = m+1. \end{cases}$$

Therefore, if $p \le 1 + m$ the node $u_N(t)$ blows up with different rate than u_{N+1}, and for $p > m+1$ it is bounded.

Repeating this argument with the nodes $N-1$, $N-2, \dots$ we obtain the asymptotic behavior of each node and the result follows. $\square$

THEOREM 11.5. *Problem* (10) *has a unique nonnegative and non-trivial stationary solution* $W = (w_1, ..., w_{N+1})$, *which is positive. If* $p = m$ *every node is a blow-up point. Moreover in the self-similar variables,*

$$Y(s) \to W, \qquad \text{as} \quad s \to \infty.$$

Hence the asymptotic behavior of u_k *is given by*

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k.$$

PROOF.

Stationary solutions of (10) satisfy

$$\begin{aligned} & 0 = \tfrac{2}{h^2}(w_2^m - w_1^m) - \tfrac{1}{m-1} w_1, \\ (11) \qquad & 0 = \tfrac{1}{h^2}(w_{k+1}^m - 2w_k^m + w_{k-1}^m) - \tfrac{1}{m-1} w_k, \qquad 2 \le k \le N, \\ & 0 = \tfrac{2}{h^2}(w_N^m - w_{N+1}^m) + \tfrac{2}{h} w_{N+1}^m - \tfrac{1}{m-1} w_{N+1}. \end{aligned}$$

In order to prove uniqueness of solutions W to this problem, we observe that the functions w_k are increasing with respect to w_1. Indeed, from (11) we have that w_2 is increasing. Now, let us suppose that w_j

is increasing for $j = 2, \ldots, J$

$$\begin{aligned} w_{J+1}^m - w_J^m = &\ w_J^m - w_{J-1}^m + \frac{h^2}{m-1} w_J \\ = &\ w_2^m - w_1^m + \frac{h^2}{m-1} \sum_{l=2}^{l=J} w_l. \end{aligned}$$

Therefore,

$$w_{J+1}^m = w_J^m + \frac{h^2}{2(m-1)} w_1 + \frac{h^2}{m-1} \sum_{l=2}^{l=J} w_l.$$

So the function w_{J+1} is an increasing function of w_1.

Therefore, if there exists two stationary solutions, $\widetilde{W}$ and W, then we can assume that $\widetilde{W} > W$. But this is imposible since in the original variable t the corresponding solutions $\widetilde{U}(t)$ and $U(t)$ given by

$$\widetilde{U}(t) = (T_h - t)^{-\frac{1}{p-1}} \widetilde{W}, \qquad U(t) = (T_h - t)^{-\frac{1}{p-1}} W$$

verify $\widetilde{U}(t) > U(t)$ and have the same blow-up time.

This proves uniqueness of W. Now assume $p = m$; if we write the numerical problem (10) in matrix form (2), we have a Lyapunov functional. In fact

$$\begin{aligned} \Phi(Y)(s) = -\frac{1}{2}\langle A^{1/2}Y^m, A^{1/2}Y^m\rangle - \frac{1}{2m}\langle BY^m, Y^m\rangle \\ + \frac{1}{(p-1)(m+1)} \langle MY, Y^m\rangle \end{aligned}$$

satisfy

$$\Phi'(Y)(s) = \langle MY', Y^{m-1}Y'\rangle \geq 0.$$

Hence, the orbit $Y(t)$ goes to the only nonnegative stationary state W (see [**41**]). Moreover, if we begin with an increasing data $U(0)$ then for a fixed s the vector $Y(s)$ is positive and increasing (in space), hence W is a nondecreasing stationary solution.

On the other hand, from the blow-up rate we know that $y_{N+1} \geq C(h) > 0$, then W is not the trivial solution. We claim that w_k must be positive for all k. To prove this claim just suppose that there exist some j with $w_j = 0$. Since W is non-negative and non-decreasing we have that $w_1 = 0$, but in this case $w_2 = 0$ and therefore $w_k = 0$ for all k. This contradiction proves the claim. $\square$

Now our goal is to recover regional blow-up by looking carefully at the behavior of the stationary solution as the parameter h goes to zero.

When h goes to zero we expect that $Z = W^m$ converges to a solution of the following problem

$$
(12) \qquad \begin{array}{ll}
w_{xx} - \frac{1}{m-1} w^{\frac{1}{m}} = 0, & x \in (0, L), \\
w_x(0) = 0, & \\
w_x(L) = w(L). &
\end{array}
$$

This is the content of our next Lemma. We remark that since $m > 1$ a non Lipschitz function appears in (12).

Lemma 11.3. *Let W be the solution of* (11) *and let $w(x)$ be the unique stationary solution of* (12). *Then*

$$
Z = W^m \to w(x), \qquad \text{as } h \to 0.
$$

Proof. Multiplying the equation satisfied by Z by

$$
\frac{(z_{k+1} - z_k) + (z_k - z_{k-1})}{2}
$$

and adding the equations we get

$$
0 = \frac{(z_{N+1} - z_N)^2}{h^2} - \frac{(z_2 - z_1)^2}{h^2} - \frac{1}{m-1} \sum_{l=2}^{N} z_l^{\frac{1}{m}} \frac{(z_{l+1} - z_{l-1})}{2}.
$$

Hence

$$
0 = \frac{(z_{N+1} - z_N)^2}{h^2} - \frac{(z_2 - z_1)^2}{h^2} - \frac{m}{m^2-1} (z_{N+1}^{\frac{m+1}{m}} - z_1^{\frac{m+1}{m}}) + O(h).
$$

Using the first and the last equations of (11) we get that z_{N+1} and z_1 must verify the following equation,

$$
(13) \qquad 0 = \frac{1}{2} z_{N+1}^2 - \frac{m}{m^2-1} z_{N+1}^{\frac{1+m}{m}} + \frac{m}{m^2-1} z_1^{\frac{1+m}{m}} + O(h).
$$

Therefore, z_1 must be bounded. Then we can take a subsequence of h such that $z_1 \to \Gamma \geq 0$.

We consider two different cases,

$$
L < \frac{2m}{m-1} \qquad \text{and} \qquad L \geq \frac{2m}{m-1}.
$$

In the first case, if $\Gamma > 0$ we consider the auxiliary initial value problem

$$
\begin{array}{ll}
w'' = \dfrac{1}{m-1} w^{1/m}, & x \in [0, L], \\
w(0) = \Gamma, & \\
w'(0) = 0. &
\end{array}
$$

This problem has a unique solution. Moreover, for $x \in [0, L]$ the function $w(x)$ is increasing. Then, by general theory (see [**48**]) we get that in this case

$$Z = W^m \to w(x), \qquad \text{as } h \to 0.$$

Moreover by (13) we have that

$$0 = \frac{1}{2}w(L)^2 - \frac{m}{m^2-1}w(L)^{\frac{1+m}{m}} + \frac{m}{m^2-1}\Gamma^{\frac{1+m}{m}}.$$

So, the constant Γ must be the only constant such that w is a solution of

$$\begin{aligned} &w'' = \tfrac{1}{m-1}w^{1/m}, \qquad && x \in [0, L], \\ &w'(0) = 0, \\ &w'(L) = w^{p/m}(L). \end{aligned}$$

For existence and uniqueness results for this problem we refer to Section 5.

On the other hand, if we assume that $\Gamma = 0$ we arrive to a contradiction. By (13) we obtain that $z_{N+1} \to A$, where A is the only positive root of

$$0 = \frac{1}{2}w(L)^2 - \frac{m}{m^2-1}w(L)^{\frac{1+m}{m}}.$$

That is

$$A = \left(\frac{2m}{m^2-1}\right)^{\frac{m}{m-1}}. \tag{14}$$

Hence, we consider the problem

$$\begin{aligned} &w'' = \frac{1}{m-1}w^{1/m} \qquad && x \in [0, L], \\ &w(L) = A, \\ &w'(L) = -A^{p/m}. \end{aligned} \tag{15}$$

By classical theory we have that in the interval where w is positive, $(L - \frac{2m}{m-1}, L]$,

$$Z = W^m \to w(x), \qquad \text{as } h \to 0.$$

Since $L < \frac{2m}{m-1}$ we have that $w(x)$ is positive in $[0, L]$ (see Section 5). Hence, $z_1 \to w(0) > 0$, which is a contradiction. This proves that Γ must be positive.

In the second case, $L \geq \frac{2m}{m-1}$. If $\Gamma > 0$ we arrive to a contradiction, as by the previous argument $z_{N+1} \to w(L)$, with $w(L)$ a root of

$$0 = \frac{1}{2}w(L)^2 - \frac{m}{m^2-1}w(L)^{\frac{1+m}{m}} + \frac{m}{m^2-1}\Gamma^{\frac{1+m}{m}}.$$

But this contradicts the uniqueness result given in Section 5. Hence $\Gamma = 0$ and therefore

$$z_{N+1} \to w(L) = A,$$

where A is given by (14). Now we consider the problem (15). When the solution w is positive we can again apply the standard theory to find that

$$Z = W^m \to w(x), \qquad \text{as } h \to 0.$$

On the other hand, since Z is increasing, we have that for h small enough,

$$z_k \le w\left(L - \frac{2m}{m-1} + \delta\right) \le \varepsilon, \qquad \forall\, x_k < L - \frac{2m}{m-1} + \delta.$$

Then, $Z = W^m \to w(x)$ in all the interval $[0, L]$. $\square$

REMARK 11.1. *As a consequence of Lemma 11.3 if $L > \frac{2m}{m-1}$ we have that*

$$W \to 0, \qquad \text{in } \left[0, L - \frac{2m}{m-1}\right],$$

and we recover the regional blow-up in the sense that the constants that appear in the blow-up rate for the nodes that lie in $[0, L - \frac{2m}{m-1}]$ go to zero as h goes to zero, i.e.,

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k$$

with $w_k \to 0$ as $h \to 0$ for every k such that $x_k \in [0, L - \frac{2m}{m-1}]$.

4. Numerical experiments

In this section we present some numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one perform numerical computations. For the numerical experiments we use an adaptive ODE solver to integrate (3).

We start with the case $p < m$, in Figure 1 global blow-up can be appreciated.

Next, in the case $p = m$ we look for regional blow-up. In Figure 2 the numerical solution blows up in every node but the behavior of the stationary solution imposes for $x_i < L_c = L - \frac{2m}{m-1}$ a slower rate.

The next picture, Figure 3, shows the profile of the numerical solution near the numerical blow-up time, T_h, in self similar variables. As it was proved this profile is close to the continuous stationary profile, which is drawn in the same picture.

Figure 4 shows the evolution of the first node (the minimum), which blows up.

Finally, we consider the case $p > m$, Figures 5 and 6. When $p = 2$ and $m = 3$, Theorem 11.4 says that just two nodes blow up. Figure 5 shows the numerical solution, $U(t)$. In Figure 6 it can be appreciated the profile at time close to T_h (left) and the evolution of the nodes $N-1$ and N (right), $u_{N-1}(t)$ is bounded while $u_N(t)$ blows up.

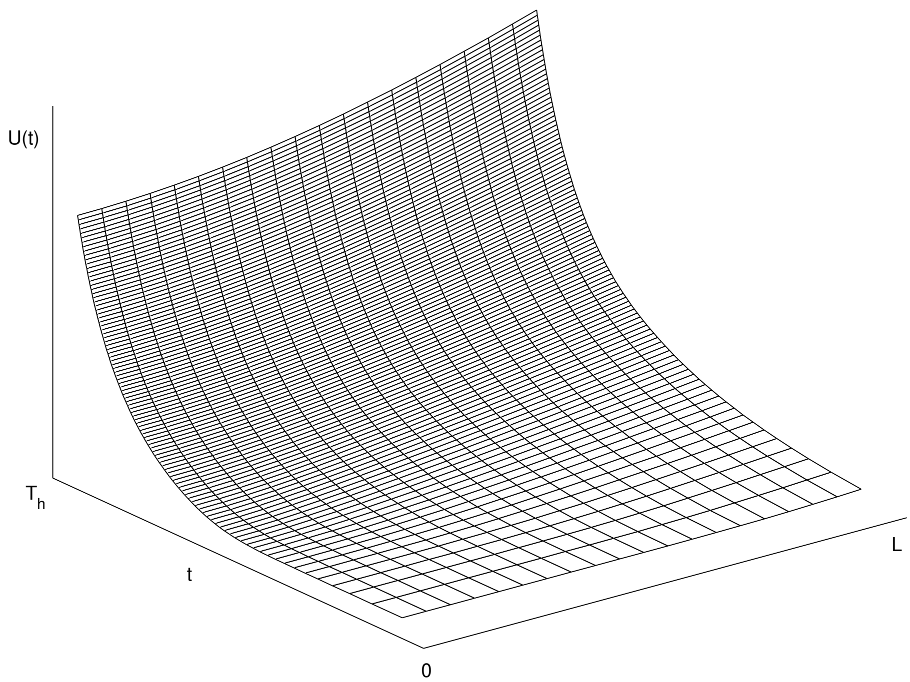

FIGURE 1. $p < m$, GLOBAL BLOW-UP

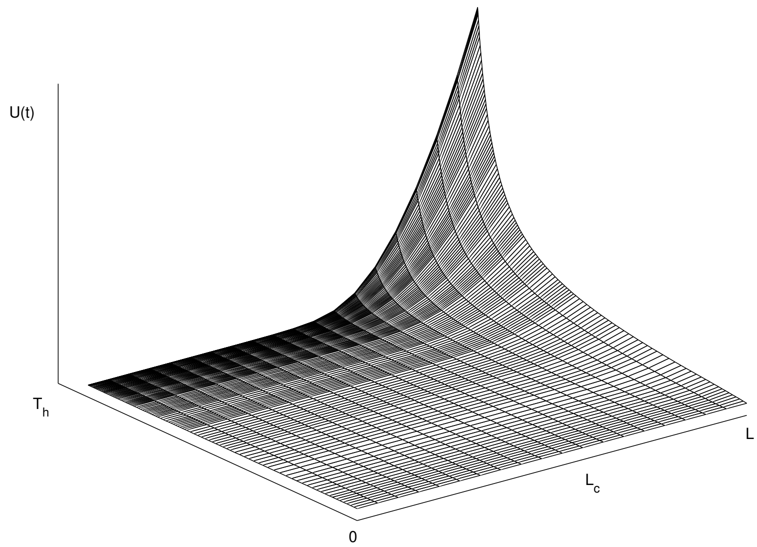

FIGURE 2. CASE $p = m$.

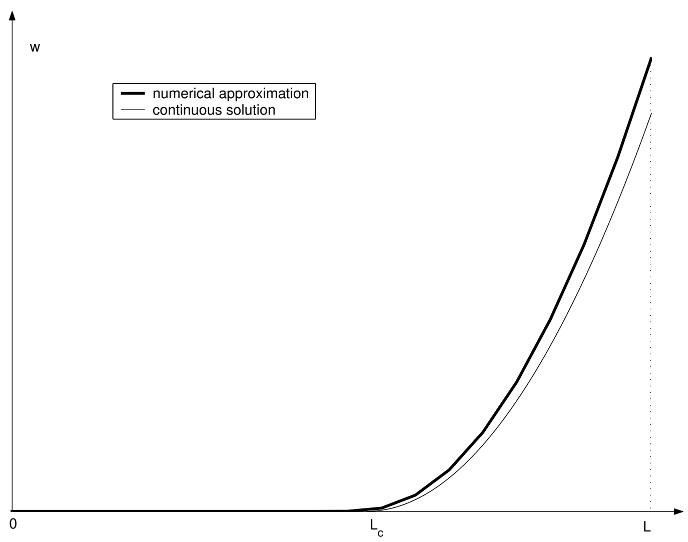

FIGURE 3. PROFILE OF THE NUMERICAL SOLUTION NEAR THE NUMERICAL BLOW-UP TIME IN SELF SIMILAR VARIABLES.

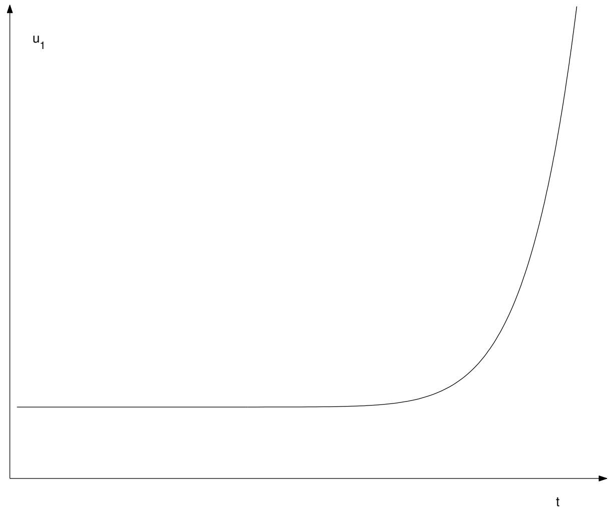

Figure 4. Evolution of the boundary node (the minimum).

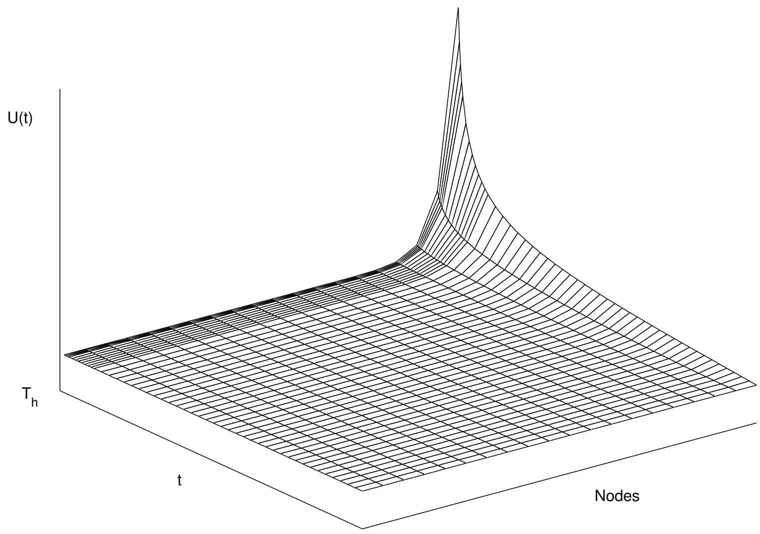

Figure 5. Case $p > m$.

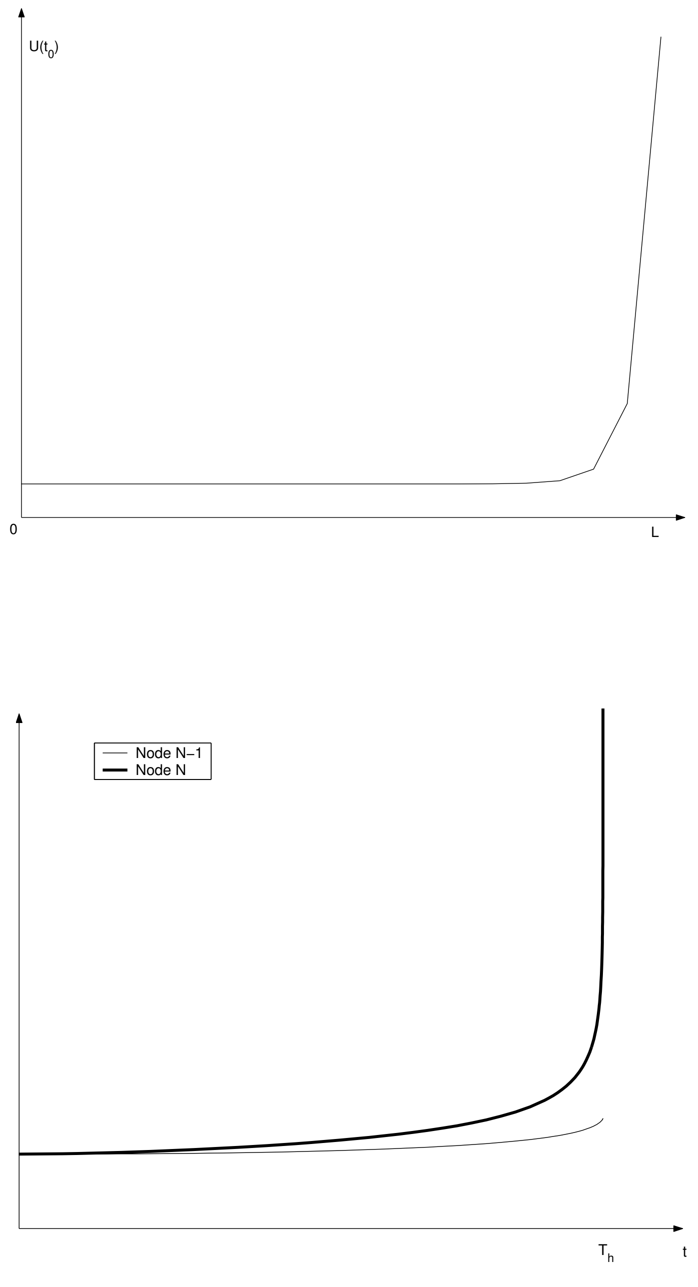

FIGURE 6. CASE $p > m$.

5. Blow-up profile for the continuous problem

In this section we study the continuous problem (1) in the range of parameters $p = m$. In this case the blow-up rate is $(T-t)^{-\frac{1}{p-1}}$, **[30]**. We concentrate in the blow-up set and in addition we find the blow-up profile in self-similar variables.

LEMMA 11.4. *In the region $p = m > 1$ the blow-up set is given by*

$$B(u) = \begin{cases} [0, L] & \text{if } L < \frac{2m}{m-1}, \\ \left[L - \frac{2m}{m-1}, L\right] & \text{if } L \geq \frac{2m}{m-1}. \end{cases}$$

PROOF. The proof of this result is given in several steps.

First, we introduce the self-similar variables

$$v(x,\tau) = (T-t)^{\frac{1}{m-1}} u(x,t), \qquad \tau = -\ln(T-t). \tag{16}$$

This function $v(x,\tau)$ satisfies the following problem,

$$\begin{array}{ll} v_\tau = (v^m)_{xx} - \frac{1}{m-1} v & (x,\tau) \in (0,L) \times (0,+\infty), \\ (v^m)_x(0,\tau) = 0 & \tau \in (0,+\infty), \\ (v^m)_x(L,\tau) = v^m(L,\tau) & \tau \in (0,+\infty), \\ v(x,0) = T^{\frac{1}{m-1}} u_0(x) & x \in (0,L). \end{array} \tag{17}$$

The second step consists in proving that $v(x,\tau)$ converges (in terms of ω-limits) to a stationary state. For this we multiply the equation by $(v^m)_\tau$ and integrate with respect to x to obtain

$$\int_0^L (v^m)_t v_t = -\frac{d}{dt} F(v),$$

where

$$F(v) = \int_0^L \frac{(v^m)_x^2}{2}(s,\tau)\, ds + \frac{m}{m^2-1} \int_0^L v^{m-1}(s,\tau)\, ds - \frac{1}{2} v^{2m}(L,\tau).$$

Hence $F(v)$ is a Lyapunov functional for problem (17).

Therefore, we have that the $\omega-$limit set of $v(x,\tau)$ consists of nontrivial stationary solutions of (17). Now we claim that for each L there exists only one nonnegative and nontrivial stationary solution.

Accepting this claim, we have that the $\omega-$limit set has only one element, $w(x)$, hence

$$v(x,\tau) \to w(x), \qquad \tau \to \infty.$$

The third step is to prove the claim. We consider the stationary problem

$$(18)\qquad \begin{array}{ll} (w^m)_{xx} - \frac{1}{m-1} w = 0, & x \in (0,L), \\ (w^m)_x(0) = 0, & \\ (w^m)_x(L) = w^m(L). & \end{array}$$

If $L \geq \frac{2m}{m-1}$ this problem has a unique compactly supported solution (see [**75**]),

$$(19)\qquad w(x) = \left(\frac{m-1}{2m(m+1)} \right)^{\frac{1}{m-1}} \left(\frac{2m}{m-1} + (x-L) \right)_+^{\frac{2}{m-1}},$$

and if $L < \frac{2m}{m-1}$ there is no compactly supported solution.

To study the existence of other solutions of (18) we use ideas from [**17**]. We set $\widetilde{w} = w^m$ and look for a shooting argument beginning with $\widetilde{w}(0) = \rho$, $\widetilde{w}_x(0) = 0$. Integrating the equation we get

$$\frac{\widetilde{w}_x^2}{2} - \frac{m}{m^2-1} \widetilde{w}^{\frac{1+m}{m}} = - \frac{m}{m^2-1} \rho^{\frac{1+m}{m}}.$$

Evaluating at $x = L$ and using the boundary condition, we observe that $\widetilde{w}(L)$ must be a root of the following polinomial

$$P(z) = \frac{z^2}{2} - \frac{m}{m^2-1} z^{\frac{1+m}{m}} + \frac{m}{m^2-1} \rho^{\frac{1+m}{m}}.$$

As

$$P'(z) = z - \frac{1}{m-1} z^{\frac{1}{m}},$$

P attains a unique minimum at the point

$$z_0 = \left(\frac{1}{m-1} \right)^{\frac{m-1}{m}}.$$

In order to guarantee that $P(z)$ has a root we need that $P(z_0) \leq 0$, this imposes the following restriction on ρ,

$$\rho \leq \rho_c = \left(\frac{1}{m-1} \right)^{\frac{m}{m-1}} \left(1 - \frac{m+1}{2m} \right)^{\frac{m}{m+1}}.$$

For those values of ρ we have two roots $R_1(\rho)$, $R_2(\rho)$. As $P(\rho) \geq 0$ we conclude that

$$\rho \leq R_1(\rho) \leq R_2(\rho).$$

On the other hand, if we integrate again the equation we have, using that $\widetilde{w}_x \geq 0$,

$$\int_\rho^{\widetilde{w}(x)} \frac{ds}{\sqrt{s^{\frac{1+m}{m}} - \rho^{\frac{1+m}{m}}}} = \sqrt{\frac{2m}{m^2-1}}\, x.$$

At $x = L$ this reads

$$\int_\rho^{\widetilde{w}(L)} \frac{ds}{\sqrt{s^{\frac{1+m}{m}} - \rho^{\frac{1+m}{m}}}} = \sqrt{\frac{2m}{m^2-1}}\, L.$$

Hence for each root $R_i(\rho)$ we have a solution of (18) in an interval $[0, L_i(\rho)]$ where $L_i(\rho)$ is given by

$$L_i(\rho) = \sqrt{\frac{m^2-1}{2m}}\, \rho^{\frac{1-m}{2m}} \int_1^{\frac{R_i(\rho)}{\rho}} \frac{ds}{\sqrt{s^{\frac{1+m}{m}} - 1}}.$$

Now we observe that, as $R_1(\rho) \leq R_2(\rho)$ we have that $L_1(\rho) \leq L_2(\rho)$.

In order to prove the uniqueness of the solution of (18) in the interval $[0, L]$ we study the monotonicity of the functions $L_i(\rho)$.

First we observe that $L_1(\rho)$ is increasing with ρ because $R_1(\rho)/\rho$ increases with ρ, in fact a direct calculation shows that

$$\frac{d}{d\rho}\left(\frac{R_i(\rho)}{\rho}\right) = -\left(\frac{m-1}{2m}\right) \frac{R_i^2(\rho)}{\rho^2((m-1)R_i(\rho) - R_i^{\frac{1}{m}}(\rho))}.$$

Notice that $\rho^2((m-1)R_i(\rho) - R_i^{\frac{1}{m}}(\rho)) = (m-1)P'(R_i(\rho))$. To see that $L_2(\rho)$ is decreasing we just observe, after differentiating and simplifying the resulting expression that this is equivalent to

$$\psi(\rho) = \int_1^x \frac{dv}{\sqrt{v^{p+1} - 1}} - \frac{x}{\sqrt{x^{p+1} - 1}((m-1) - R_2^{\frac{1-m}{m}})} \leq 0$$

where we have set $x = R_2(\rho)/\rho$. In fact one can check that

$$\frac{d}{d\rho} L_2(\rho) = \rho^{-\frac{m+1}{2m}} \psi(\rho).$$

We note that $R_2(\rho) \geq (m-1)^{-\frac{m}{m-1}}$ and also $x(\rho)$ is a decreasing function of ρ, so $x(\rho) \geq x(\rho_c) = (\frac{2m}{m-1})^{\frac{m}{m+1}}$. In that range of values it is easy to check that $\psi'(\rho) < 0$ for $\rho > 0$ and also that $\psi(0) = 0$, therefore $\psi(\rho) < 0$ and we conclude that $L_2(\rho)$ is decreasing.

We want to get a bound for $L_2(\rho)$, $0 \le \rho \le \rho_c$. As

$$R_2(\rho) \to \left(\frac{2m}{m^2-1}\right)^{\frac{m}{m-1}}, \qquad \rho \to 0,$$

a direct computation shows that

$$\lim_{\rho \to 0} L_2(\rho) = \frac{2m}{m-1}.$$

Therefore L_2 attains a maximum at $\rho = 0$ that is $L_2(0) = \frac{2m}{m-1}$, which is exactly the support of the explicit solution (19).

Since $R_1(\rho_c) = R_2(\rho_c)$ we have $L_1(\rho_c) = L_2(\rho_c)$. Also, as $R_1(\rho) \to 0$ when $\rho \to 0$, one can easily check that $L_1(0) = 0$.

We conclude that for $L \ge \frac{2m}{m-1}$ we have a unique solution of (18) that is given by (19) and has compact support, and for $L < \frac{2m}{m-1}$, problem (18) has only one positive solution. This proves the claim and finishes our study of the stationary problem.

In the fourth step we show that $v(x,\tau)$ goes to zero exponentially in every compact set included in $[0, \frac{2m}{m-1})$. More precisely:

Let v a solution of (17) that goes to zero in an interval of the form $[a-\delta, b+\delta]$, then there exists a constant C such that

$$v(x,\tau) \le Ce^{-\frac{1}{m-1}\tau} \qquad x \in [a,b].$$

In order to see this we use ideas from **[20, 75]**. We will only sketch the arguments and refer to those works for details.

Since $v(x,\tau)$ goes to zero in $[a-\delta, b+\delta]$, we can use a solution of

$$\begin{array}{ll} p_\tau = (p^m)_{xx} - \frac{1}{m-1}p & (x,\tau) \in [a-\delta, b+\delta] \times (0,+\infty), \\ p(a-\delta,\tau) = \varepsilon & \tau \in (0,+\infty), \\ p(b+\delta,\tau) = \varepsilon & \tau \in (0,+\infty), \\ v(x,0) = \varepsilon & x \in (a-\delta, b+\delta). \end{array} \tag{20}$$

as a supersolution of our problem.

To finish the proof we only have to observe that solutions p of (20) goes exponentially to a stationary solution that has the form

$$h(x) = \begin{cases} C(m)(a-x)^{\frac{2}{m-1}} & x < a, \\ 0 & a \le x \le b, \\ C(m)(x-b)^{\frac{2}{m-1}} & x > b. \end{cases}$$

If ε is small, $h(x)$ has a dead core that includes the interval $[a,b]$ and the result follows.

Finally, if we translate the behavior of $v(x, \tau)$ in terms of the function $u(x, t)$, we obtain the desired result. $\square$

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