CHAPTER 5

When? Continuity of the blow-up time and order of convergence

Order of convergence and regularity are topics strongly related. Usually, rate of convergence results concerning numerical methods need regularity on the data being estimated. This is very reasonable since a numerical approximation is nothing else than a perturbation of the problem (that can be computed!). In this chapter we prove a regularity result for the blow-up time, we find a modulus of continuity with respect to the reaction power, the diffusion coefficient and the initial data. Next we use similar ideas (and in some sense this fact) to obtain order estimates for the convergence of the blow-up times in numerical approximations like the ones considered in the previous chapters. We consider the problem

$$(1)\qquad \begin{array}{ll} u_t = \lambda\Delta u + u^p, & \text{in } \Omega\times(0,T),\\ u = 0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0(x) = \phi(x) + h(x), & x\in\Omega, \end{array}$$

where $p > 1$, $\lambda > 0$ and $h(x)$ are parameters, Ω is a bounded smooth domain in $\mathbb{R}^d$ and u_0 is regular in order to guarantee existence, uniqueness and regularity of the solution. We also assume that p is subcritical.

As was pointed before the solution of (1) may develop singularities in finite time, no matter how smooth u_0 is.

Here we are interested in the study of the dependence of the blow-up time, T, on the parameters that appears in the equation, p, λ and h.

It is known that the blow-up time is continuous with respect to the initial data u_0. This was first proved in **[10]** but other authors obtained similar results under different assumptions and using different techniques, see **[35, 58, 67]**.

In this chapter we want to improve these results showing that there exists a modulus of continuity for T not only with respect to the initial condition but also with respect to the diffusion coefficient λ and the reaction power p. In fact, we find bounds of the form

$$|T - T_h| \leq C|h|^\gamma. \tag{2}$$

The main idea of the proof rely on estimates on the first time where two different solutions spread out. These estimates are based on the rate at which solutions blow up.

The blow-up rate for solutions of (1) with subcritical nonlinearity, $1 < p < (d+2)/(d-2)$, is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-1/(p-1)}$, see **[39, 43]**. In particular in **[24, 55, 56]** it is proved that for p, u_0 and λ near p_0, ϕ and λ_0 there exists a uniform constant C such that

$$u(x,t) \leq C(T-t)^{-\frac{1}{p-1}}. \tag{3}$$

The fact that the constant C is uniform is crucial for our arguments.

We want to remark that the restriction $1 < p < (d+2)/(d-2)$ is not technical, since if this is not the case the blow-up time is not continuous as a function of the initial data, see **[35]**.

The ideas developed to prove estimates like (2) can be applied when one deals with numerical methods to approximate solutions of (1). In fact we find an estimate for the rate of convergence of the numerical blow-up time in a semi-discrete approximation of problem (1).

Along the rest of the chapter we denote by C a constant that do not depends on the relevant parameters involved but may change from one line to another.

1. Dependence of the blow-up time respect to parameters

In this part we prove the continuity of the blow-up time respect to parameters. To begin with we show that the set of parameters $\{h, p, \lambda\}$ that produce blow-up is open. This can be seen using the energy functional introduced in Chapter 2

$$\Phi(u)(t) \equiv \frac{\lambda}{2}\int_\Omega |\nabla u(s,t)|^2\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds.$$

As was pointed there this functional is useful to characterize the solutions that blow up. In **[21, 37, 38, 52]** the authors prove that the fact that u blows up in finite time is equivalent to the existence of a time t_0 such that $\Phi(u)(t_0) < 0$. Now let u be a blowing up solution of (1). Hence there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. As u is continuous with respect to the parameters in $\Omega \times [0, t_0]$, we get that $\Phi(\tilde{u})(t_0) < 0$ for every $(\tilde{h}, \tilde{p}, \tilde{\lambda})$ near (h, p, λ). This shows that the set $\{(h, p, \lambda) : u \text{ blows up}\}$ is open.

To clarify the exposition we deal with perturbations of h, p, λ separately, however the main lines of the arguments are similar.

1.1. Perturbations in the nonlinear source. Let us consider the problem

$$
\begin{array}{lll}
 & u_t = \Delta u + u^{\alpha}, & (x,t) \in \Omega \times (0, T_{\alpha}), \\
(4) & u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, T_{\alpha}), \\
 & u(x,0) = u_0(x), & x \in \Omega.
\end{array}
$$

Our first result is a modulus of continuity of T_{α} as a function of α.

THEOREM 5.1. *Let u and u_h be solutions of problem (4) with $p > 1$, $\alpha = p + h$ (h small), respectively, and let T and T_h their blow-up times. Then there exist positive constants C and γ independent of h such that for every h small enough,*

$$|T - T_h| \leq C|h|^{\gamma}.$$

PROOF. We observe that we can assume $h > 0$. Let us introduce the error function $e(x,t) = u_h(x,t) - u(x,t)$, we have

$$
\begin{gathered}
e_t = \Delta e + u_h^{p+h} - u^p = \Delta e + \frac{u_h^p - u^p}{u_h - u} e + u_h^{p+h} - u_h^p = \\
\Delta e + p\xi(x,t)^{p-1} e + u_h^p (u_h^h - 1) \leq \Delta e + p\xi^{p-1} e + u_h^p h u_h^{h+1}.
\end{gathered}
$$

Now fix $a > 0$ and let t_0 be such that $e(t) \leq a$ in $[0, t_0]$, in that interval we have $\xi \leq u + a$ and as there exists a constant C such that $u(x,t) \leq C(T-t)^{-1/(p-1)}$, for $t \in [0, t_0]$ we have

$$u_h(x,t) \leq u(x,t) + a \leq C(T-t)^{-1/(p-1)}.$$

Therefore,

$$e_t \leq \Delta e + \frac{C}{T-t} e + \frac{Ch}{(T-t)^{\frac{p+h+1}{p-1}}}.$$

Let $\overline{e}$ be the solution of

$$\overline{e}' = \frac{C}{T-t}\overline{e} + \frac{Ch}{(T-t)^{\frac{p+h+1}{p-1}}}, \qquad \overline{e}(0) = 0.$$

Integrating we obtain

$$\overline{e}(t) = Ch(T-t)^{-C} + C_2 h (T-t)^{-\frac{2+h}{p-1}},$$

and by a comparison argument we have

$$e(t) \leq \overline{e}(t) \leq Ch(T-t)^{-C}.$$

This is an important point in the proof, since we obtain a bound on the asymptotic behavior of $e(t)$. In the following perturbations we will see that once we obtain this kind of bounds, the rest of the proof is as follows: we choose $t_1 = t_1(h) \le t_0$ the first time such that

$$Ch(T-t_1)^{-C} = a,$$

so we have

$$T - t_1 = \left(\frac{Ch}{a}\right)^{1/C}.$$

This time t_1 also verifies

$$|T_h - t_1| \le \left(\frac{C}{\|u_h(t_1,\cdot)\|_{L^\infty(\Omega)}}\right)^{p+h-1} \le \left(\frac{C}{\|u(t_1,\cdot)\|_{L^\infty(\Omega)} - a}\right)^{p+h-1} \le$$

$$C(T-t_1)^{\frac{p+h-1}{p-1}} \le C(T-t_1)^{\frac{1}{2}}.$$

We observe here that we must be careful when we choose the constant a in order to guarantee

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} - a \ge C(T-t)^{-\frac{1}{p-1}}, \qquad 0 < t \le t_0.$$

An easy use of the maximum principle shows that $\|u(t,\cdot)\|_{L^\infty(\Omega)} \ge C_1(T-t)^{-\frac{1}{p-1}}$, and this fact ensures that a can be chosen in order to satisfy the above inequality.

Hence we obtain

$$|T - T_h| \le |T - t_1| + |T_h - t_1| \le$$

$$|T - t_1| + C|T-t_1|^{\frac{1}{2}} \le Ch^{\frac{1}{2C}} = Ch^{\gamma}.$$

This completes the proof. □

REMARK 5.1. *Let us observe that the exponent γ and the constant C that appears in Theorem 5.1 are not uniform, they depend on the solution u that we are considering. This can not be improved even for the ODE $u' = u^p$ where solutions are explicit.*

1.2. Perturbations in the initial datum. Now we turn our attention to the dependence of the blow-up time with respect to the initial data. We consider the problem

$$(5) \qquad \begin{array}{ll} (u_h)_t = \Delta u_h + u_h^p, & (x,t) \in \Omega \times (0, T_h), \\ u_h(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T_h), \\ u_h(x,0) = u_0(x) + hf(x), & x \in \Omega. \end{array}$$

We denote by u the solution for $h = 0$ and T its blow-up time.

THEOREM 5.2. *Let u_h, u be solutions of problem* (5) *with $h > 0$ and $h = 0$ respectively, and let T_h and T their blow-up times. Then there exist positive constants C and γ such that for every h small enough,*

$$|T - T_h| \le C|h|^{\gamma}.$$

PROOF. As in Theorem 5.1 we call $e(x,t) = u_h(x,t) - u(x,t)$. Then e verifies,

$$\begin{aligned} e_t = & \quad \Delta e + \frac{u_h^p - u^p}{u_h - u} e \\ = & \quad \Delta e + p\xi(x,t)^{p-1} e \le \Delta e + p\xi^{p-1} e. \end{aligned}$$

Let again t_0 be such that $e(t) \le a$ in $[0, t_0]$ for fixed a. Since Using that $u(x,t) \le C(T-t)^{-\frac{1}{p-1}}$, in that interval $[0, t_0]$, we have

$$e_t \le \Delta e + \frac{C}{T-t} e.$$

Consider $\overline{e}$, the solution of

$$\overline{e}' = \frac{C}{T-t}\overline{e}, \qquad \overline{e}(0) = |h| \|f\|_{L^\infty},$$

that is

$$\overline{e}(t) = C|h|(T-t)^{-C}.$$

Arguing by comparison we obtain

$$e(t) \le \overline{e}(t) \le C|h|(T-t)^{-C}.$$

Now we proceed as in Theorem 5.1 to complete the proof. First, we choose $t_1 = t_1(h) \le t_0$ with the property that is the first time such that

$$C|h|(T-t_1)^{-C} = a,$$

that is

$$T - t_1 = \left(\frac{C|h|}{a}\right)^{1/C}.$$

This time verifies

$$|T_h - t_1| \le \left(\frac{C}{\|u_h(t_1,\cdot)\|_{L^\infty(\Omega)}}\right)^{p-1} \le \left(\frac{C}{\|u(t_1,\cdot)\|_{L^\infty(\Omega)} - a}\right)^{p-1} \le C(T-t_1).$$

Hence

$$\begin{gathered} |T - T_h| \le |T - t_1| + |T_h - t_1| \le \\ C|T - t_1| \le C|h|^{\frac{1}{C}} = C|h|^{\gamma}, \end{gathered}$$

as we wanted to prove. $\square$

1.3. Perturbations in the diffusion coefficient. Next we obtain a similar result for perturbations in the diffusion parameter λ. We consider the problem

$$(6) \qquad \begin{array}{ll} (u_\lambda)_t = \lambda \Delta u_\lambda + u_\lambda^p, & (x,t) \in \Omega \times (0, T_\lambda), \\ u_\lambda(x,t) = 0, & (x,t) \in \partial\Omega \times (0, T_\lambda), \\ u_\lambda(x,0) = u_0(x), & x \in \Omega. \end{array}$$

THEOREM 5.3. *Let u and u_λ be solutions of problem (6) with $\lambda_0 > 0$ and $\lambda > 0$ (close to λ_0), respectively, and let T and T_λ their blow-up times. Then there exist positive constants C and γ such that for every λ close enough from λ_0,*

$$|T - T_\lambda| \le C|\lambda - \lambda_0|^\gamma.$$

PROOF. Once again we call $e(x,t) = u_\lambda(x,t) - u(x,t)$, then

$$\begin{aligned} e_t = & \ \lambda_0 \Delta e + \frac{u_\lambda^p - u^p}{u_\lambda - u} e + (\lambda - \lambda_0)\Delta u_\lambda \\ = & \ \lambda_0 \Delta e + p\xi(x,t)^{p-1} e + (\lambda - \lambda_0)\Delta u_\lambda \\ \le & \ \lambda_0 \Delta e + p\xi^{p-1} e + (\lambda - \lambda_0)\Delta u_\lambda. \end{aligned}$$

Let again t_0 such that $e(t) \le a$ in $[0, t_0]$ for fixed a, in that interval we have

$$e_t \le \lambda_0 \Delta e + \frac{C}{T-t} e + |\lambda - \lambda_0| \frac{C}{(T-t)^\alpha}.$$

Now, let $\overline{e}$ be the solution of

$$\overline{e}' = \frac{C}{T-t}\overline{e} + |\lambda - \lambda_0| \frac{C}{(T-t)^\alpha}, \qquad \overline{e}(0) = 0.$$

Integrating we obtain

$$\overline{e}(t) \le C|\lambda - \lambda_0|(T-t)^{-C},$$

and by a comparison argument we have

$$e(t) \le \overline{e}(t) \le C|\lambda - \lambda_0|(T-t)^{-C}.$$

Now we proceed as in the proof of the previous theorems, Theorem 5.1 and 5.2, to conclude. ☐

1.4. A sharper modulus of continuity. In the previous sections of this chapter we found a modulus of continuity for the blow up time of the form $C|T_h - T|^\gamma$ but we could not obtain any information about γ. If we follow the calculations above we can observe that this constant γ depends basically on the constant C used to bound the blow-up rate, i.e. the constant such that

$$u(\cdot, t) \leq C(T-t)^{-1/(p-1)}$$

holds for every solution of (1) if the parameters are in a neighborhood of (λ_0, p_0, ϕ).

In the case of the initial data this estimate has been refined in **[57, 59]**, where the authors prove that for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon)$ such that for every h small, the solution of problem (1) verifies

$$|u_h(x,t)| \leq C_p(T_h - t)^{-\frac{1}{p-1}} + \left(\frac{dC_p}{2p} + \varepsilon\right) \frac{(T_h - t)^{-\frac{1}{p-1}}}{|\ln(T_h - t)|}, \tag{7}$$

for every $(x,t) \in \Omega \times [T_h - \tau, T_h)$.

This estimate gives us the possibility to improve the modulus of continuity for the blow-up time respect to the initial data. We consider once again problem (5) and state an improvement of Theorem 5.2

THEOREM 5.4. *Let T and T_h be the blow-up times for u and u_h, two solutions of problem* (1) *with initial data u_0 and $u_0 + h$ respectively, then for every $\varepsilon > 0$ there exist a positive constant C such that for every h small enough,*

$$|T - T_h| \leq C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^{\frac{d+2}{2}+\varepsilon}. \tag{8}$$

REMARK 5.2. *We suspect that an estimate of the form $|T - T_h| \leq C\|h\|_{L^\infty}$ is valid, as has been proven in* **[43]** *for the one dimensional case. However, we cannot get rid of the logarithmic term in* (8).

The bound (7) is crucial in getting a sharp estimate on $\|e(t)\|_{L^\infty}$. Given $\varepsilon > 0$, it is easy to see that (7) holds uniformly for u (resp. for u_h), for all $t \in [T-\tau, T)$ (resp. $t \in [T-\tau, T_h)$), where $\tau = \tau(T_0, \varepsilon)$, whenever $\|h\|_{L^\infty}$ is small enough. Since $T - \frac{\tau}{2}$ is independent of h, we prove in the following lemma that the size of e at $t = T - \frac{\tau}{2}$ is comparable to its size at $t = 0$. Therefore, we can make a shift in time to simplify the notation and assume that (7) holds for u (resp. for u_h) for all $t \in [0, T)$ (resp. $t \in [0, T_h)$). We claim the following lemma:

LEMMA 5.1. *Given $\varepsilon > 0$ there exists a constant $C = C(T, \varepsilon) > 0$ such that for every h with $\|h\|_{L^\infty}$ small enough, it holds that u_h, the* *solution with initial datum* $u_0 + h$*, is defined on* $[0, T - \frac{\tau}{2}]$ *where* $\tau = \tau(M_0, T_0, \varepsilon)$*, and moreover*

$$\|u - u_h\|_{L^\infty}(T - \frac{\tau}{2}) \le C\|h\|_{L^\infty}. \tag{9}$$

PROOF. According to (7), there exists a constant $M(T, \varepsilon)$ such that

$$M \ge \|u\|_{L^\infty(\Omega\times[0,T-\frac{\tau}{2}])}.$$

Consider a uniformly Lipschitz function $f(u)$ that coincides with the power nonlinearity $|u|^{p-1}u$ for values of u with $|u| \le M + 1$. Now let v and v_h the solutions of a problem like (5) with the source term $|u|^{p-1}u$ replaced by $f(u)$ and initial datum u_0 and $u_0 + h$ respectively. As $f(u)$ is uniformly Lipschitz we have that both v and v_h are global and moreover there exists a constant C such that

$$\|v - v_h\|_{L^\infty}(T - \frac{\tau}{2}) \le C\|h\|_{L^\infty}.$$

Now we only have to observe that if $\|h\|_{L^\infty}$ is small enough we have that both v and v_h verify that

$$\|v\|_{L^\infty(\Omega\times[0,T-\frac{\tau}{2}])}, \|v_h\|_{L^\infty(\Omega\times[0,T-\frac{\tau}{2}])} \le M + 1.$$

Hence, by uniqueness, $v(T - \frac{\tau}{2})$ and $v_h((T - \frac{\tau}{2})$ coincides with $u(T - \frac{\tau}{2})$ and $u_h((T - \frac{\tau}{2}))$ respectively and the result follows. □

Now we can prove Theorem 5.4,

PROOF OF THEOREM 5.4: From Lemma 5.1, taking as initial datum $u(T - \frac{\tau}{2})$ and $u_h((T - \frac{\tau}{2}))$ if necessary, we can assume that the bound (7) is valid for all t in $[0, T)$ and $[0, T_h)$ respectively.

Let $e(x,t) = u_h(x,t) - u(x,t)$. We have that for every $(x,t) \in \Omega \times [0, \min(T, T_h))$,

$$\begin{aligned} e_t &= \Delta e + u_h^p - u^p = \Delta e + p\xi(x,t)^{p-1}e, \\ \|e(x,0)\|_{L^\infty} &\le C\|h\|_{L^\infty}, \end{aligned} \tag{10}$$

where $\xi(x,t)$ lies between $u(x,t)$ and $u_h(x,t)$. Given $\varepsilon > 0$, we can define for $\|h\|_{L^\infty}$ small enough, a time $t_0 \in (0, T]$ as the maximal time such that

$$\|e(\cdot,t)\|_{L^\infty} \le \varepsilon \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|} \quad \text{for all } t \in [0, t_0). \tag{11}$$

For times $t \in [0, t_0)$ we have that

$$|\xi(x,t)| \le |u(x,t)| + \varepsilon \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|}$$

and from (7) we get that

$$|\xi(x,t)| \le C_p(T-t)^{-\frac{1}{p-1}} + \left(\frac{dC_p}{2p} + 2\varepsilon\right)\frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|},$$

hence,

$$p|\xi(x,t)|^{p-1} \le \frac{p}{p-1}(T-t)^{-1} + \left(\frac{d}{2} + C\varepsilon\right)\frac{(T-t)^{-1}}{|\ln(T-t)|} \equiv \alpha(t).$$

Now, let $\overline{e}(t)$ be the solution of the ODE

$$\overline{e}'(t) = \alpha(t)\overline{e}(t)$$

with initial datum $\overline{e}(0) = C\|h\|_{L^\infty}$. Integrating we obtain

$$\overline{e}(t) = C\|h\|_{L^\infty}(T-t)^{-\frac{p}{p-1}}|\ln(T-t)|^{\frac{d}{2}+C\varepsilon},$$

and, since e is a solution of (10), by a comparison argument we have for every $t \in [0, t_0)$,

$$\|e(\cdot,t)\|_{L^\infty} \le \overline{e}(t) = C\|h\|_{L^\infty}(T-t)^{-\frac{p}{p-1}}|\ln(T-t)|^{\frac{d}{2}+C\varepsilon}. \tag{12}$$

We want to use this to obtain a bound on $|T - t_0|$ in terms of $\|h\|_{L^\infty}$. To this end let us introduce for $\|h\|_{L^\infty}$ small enough, the time $t_1 = t_1(h) \le T$ as the maximal time such that

$$C\|h\|_{L^\infty}(T-t)^{-\frac{p}{p-1}}|\ln(T-t)|^{\frac{d}{2}+C\varepsilon} \le \varepsilon\frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|}$$

for all $t \in [0, t_1)$.

It is clear from (12) that $t_1 \le t_0$, hence $0 \le T - t_0 \le T - t_1$. Note that

$$T - t_1(h) \sim \frac{C}{\varepsilon}\|h\|_{L^\infty}|\ln\|h\|_{L^\infty}|^{\frac{d+2}{2}+C\varepsilon} \qquad \text{as } \|h\|_{L^\infty} \to 0.$$

Therefore,

$$T - t_0 \le T - t_1 \le \frac{C}{\varepsilon}\|h\|_{L^\infty}\left|\ln\|h\|_{L^\infty}\right|^{\frac{d+2}{2}+C\varepsilon}.$$

Now we observe that (7) and the definition of e yield

$$|T_h - t_0| \le C\left(\|u_h(t_0,\cdot)\|_{L^\infty(\Omega)}\right)^{1-p} \le C\left(\|u(t_0,\cdot)\|_{L^\infty} + \|e(t_0,\cdot)\|_{L^\infty}\right)^{1-p}.$$

Using the bound from below for the blow-up rate and (11), we get

$$|T_h - t_0| \le C\left(\|u(t_0,\cdot)\|_{L^\infty(\Omega)} - \varepsilon\frac{(T-t_0)^{-\frac{1}{p-1}}}{\ln(T-t_0)}\right)^{1-p}$$

$$\leq C\left(C_p(T-t_0)^{-\frac{1}{p-1}} - \varepsilon\frac{(T-t_0)^{-\frac{1}{p-1}}}{|\ln(T-t_0)|}\right)^{1-p} \leq C(T-t_0),$$

and so we obtain

$$|T-T_h| \leq |T-t_0| + |T_h - t_0| \leq$$

$$|T-t_0| + C|T-t_0| \leq \frac{C}{\varepsilon}\|h\|_{L^\infty}\,|\ln(\|h\|_{L^\infty})|^{\frac{d+2}{2}+C\varepsilon}.$$

This completes the proof. □

REMARK 5.3. *The above proof gives a constant* $C(M_0, T_0, \varepsilon)$ *for* (8) *with* $C(M_0, T_0, \varepsilon) \to +\infty$ *as* $\varepsilon \to 0$.

2. Order of convergence for the blow-up time

We come back to the numerical method studied in Chapter 3. Our goal here is to use the techniques developed in that chapter to obtain a bound in the order of convergence of the blow-up times. So we consider the problem

$$(13)\qquad \begin{array}{ll} u_t = \Delta u + u^p, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array}$$

and the numerical approximations given by

$$(14)\qquad \begin{array}{rcl} MU'(t) & = & -AU(t) + MU^p(t), \\ U(0) & = & U_0. \end{array}$$

Recall that M and A are $N \times N$ matrices that verify the hypotheses (P1)-(P4) described in Chapter 3. This ODEs system can be written as

$$(15)\qquad \begin{array}{rcll} m_k u_k'(t) & = & -\sum_{j=1}^N a_{kj}u_j(t) + m_k u_k^p(t), & 1 \leq k \leq N, \\ u_k(0) & = & u_{0,k}, & 1 \leq k \leq N. \end{array}$$

In order to have numerical blow-up times that converges to T (and solutions that converges to u!) we need to require a consistency assumption. We introduce here a slightly different definition of consistency than the one in Chapter 3, it is a bit stronger, but it is verified by the finite differences method in a cube, as was shown in the preliminaries chapter.

DEFINITION. We say that the scheme (15) is consistent if for any solution u of (13) holds

$$m_k u_t(x_k,t) = -\sum_{j=1}^{N} a_{kj} u(x_j,t) + m_k u^p(x_k,t) + \rho_k(h,t), \tag{16}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on h and a universal constant θ such that

$$\max_k \left| \frac{\rho_{k,h}(t)}{m_k} \right| \leq \frac{\rho(h)}{(T-t)^\theta}, \qquad \text{for every } t \in (0,T_h),$$

with $\rho(h) \to 0$ if $h \to 0$.

As we mentioned before this consistency hypothesis is valid in the two examples cited in the introduction with $\rho(h) = Ch^2$. The power $C(T-t)^{-\theta}$ is a bound for the spatial derivatives of u.

One of the keys in the calculations performed in the previous sections is that the continuous solution and its perturbations are bounded by the same blow-up rate. So to begin with, we have to prove that numerical solutions satisfy

$$u_h(x,t) \leq \frac{C}{(T_h - t)^{\frac{1}{p-1}}}.$$

This was proved in Chapter 4 in the course of Theorem 4.5 for a system of coupled equations, under the hypothesis $\Delta u_0 + u_0^p > 0$.

THEOREM 5.5. *Let u be a solution of problem* (13) *that verifies $\Delta u_0 + u_0^p > 0$ and blows up at time T, let u_h the numerical solution given by* (15) *and assume that for h small enough u_h blows up. Assume also that the scheme is consistent (according to the above definition) with $\rho(h) \leq Ch^\alpha$, for some $\alpha > 0$. Then there exist positive constants C and γ such that for h small enough,*

$$|T - T_h| \leq Ch^\gamma.$$

REMARK 5.4. *The assumption that u_h blows up for every h small enough can be guaranteed under the hypothesis (H2) of Chapter 3, or if the initial datum verifies $\Phi(u_0) \leq 0$.*

PROOF. Let us start by defining the error functions

$$e_k(t) = u(x_k,t) - u_k(t).$$

By the consistency assumption (16), these functions verify

$$m_k e_k' \;=\; -\sum_{j=1}^{N} a_{kj} e_i + m_k(u^p(x_k,t) - u_k^p) + \rho_k(h).$$

Let $t_0 = \max\{t: \ t < T-\tau, \ \max_k |e_k(t)| \le a\}$, in $[0, t_0]$ we have

$$m_k e_k' \le -\sum_{j=1}^{N} a_{kj} e_i + m_k p \xi_k(t)^{p-1} e_k(t) + \rho_k(h),$$

As $\xi_k(t), u_k(t) \le u(x_k, t) + a \le \frac{C}{(T-t)^{\frac{1}{p-1}}}$, in $[0, t_0]$, $E = (e_1, ..., e_N)$ satisfies

$$ME' \le -AE + \frac{C}{T-t} ME + \frac{\rho(h)}{(T-t)^{\theta}} M(1, ..., 1)^t. \tag{17}$$

Let $\overline{E}(t)$ be the solution of

$$\overline{E}' = \frac{C}{T-t}\overline{E} + \frac{\rho(h)}{(T-t)^{\theta}}, \qquad \overline{E}(0) = E(0) = 0.$$

Integrating, we obtain

$$\overline{E}(t) = C_1 \rho(h)(T-t)^{-\theta+1} + C_2 \rho(h)(T-t)^{-C}.$$

As $\overline{E}(t)$ is a supersolution for (17) and the comparison Lemma 3.1 holds we get

$$E(t) \le \overline{E}(t) \le C\rho(h)(T-t)^{-C}.$$

Now we proceed as in the Theorems of the previous sections, let $t_1 \le t_0$ be the first time such that

$$C\rho(h)(T-t_1)^{-C} = a,$$

that is

$$T - t_1 = \left(\frac{C\rho(h)}{a}\right)^{1/C},$$

this time t_1 also verifies (by the previous Lemma)

$$|T_h - t_1| \le \left(\frac{C}{u_k(t_1)}\right)^{p-1} \le \left(\frac{C}{\|u(\cdot, t_1)\|_{L^\infty(\Omega)} - a}\right)^{p-1}$$

$$C\left(C(T-t_1)^{-\frac{1}{p-1}} - a\right)^{-(p-1)} \le C(T-t_1),$$

and so we obtain

$$|T - T_h| \le |T - t_1| + |T_h - t_1| \le$$

$$|T - t_1| + C|T - t_1| \le (1 + C)\left(\frac{C\rho(h)}{a}\right)^{1/C} = Ch^{\gamma}.$$

This completes the proof. $\square$

We observe that this proof shows, in particular, the convergence of the numerical scheme in sets of the form $\Omega \times [0, T-\tau]$. However, using a different approach it can be obtained a better estimate for the rate of convergence of the solutions of the numerical scheme in sets of the form $\Omega \times [0, T - \tau]$ for a fixed τ. In fact this rate of convergence coincides with the modulus of consistency as was shown in Chapter 3.

CHAPTER 6

Totally discrete schemes

In the previous chapters we analyzed the behavior of semidiscrete schemes when approximating solutions of PDE with blow-up. We replace the PDE by a system of ordinary differential equations, which is simpler to solve and has properties that in some sense approximate the continuous solutions and reproduce its asymptotic behavior. In order to give a complete analysis of the problem of computing solutions of this kind of problems the next step is to investigate how the ODEs system obtained in the space discretization process have to be integrated to keep the asymptotic properties. In this chapter we consider once again the problem of Chapter 3

$$
(1)\qquad \begin{array}{ll} u_t = \Delta u + u^p & \text{in } \Omega \times [0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) > 0 & \text{on } \Omega. \end{array}
$$

We propose an adaptive time step procedure and analyze its behavior.

As always $p > 1$ in order to have solutions with blow-up. We also assume that u_0 is regular and $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain in order to obtain a smooth solution.

Recall that the subcriticality assumption on p is not used in the proofs concerning the numerical scheme, they hold for every $p > 1$. However the asymptotic properties of the continuous solutions are known just for subcritical exponents.

So the first step to obtain a totally discrete method is to get a semidiscrete scheme like the analyzed in the previous chapters. This is not the only way to obtain a fully discrete scheme, for example one can first discretize the time variable. In Chapter 3 we obtained the ODE system

$$
(2)\qquad MU'(t) = -AU(t) + MU^p,
$$

with initial data $u_k(0) = u_0(x_k)$. Recall that $U(t) = (u_1(t), \dots, u_N(t))$ and $u_k(t)$ stands for an approximation of $u(x_k, t)$. This system can also be written as

$$m_k u_k'(t) = -\sum_{i=1}^{N} a_{ki} u_i(t) + m_k u_k^p(t), \qquad 1 \le k \le N,$$

with initial data $u_k(0) = u_0(x_k)$.

We assume, as always, hypotheses (P1)-(P4).

We also need a convergence result for the scheme, we will state the precise hypotheses concerning convergence in each theorem. Finally, in the last section of this chapter we prove an L^∞ convergence theorem under the consistency assumption. Now we state the two convergence hypotheses that we may need. They are the same that appear in Chapter 3.

(H1) For every $\tau > 0$, $\|u - u_h\|_{L^\infty(\Omega\times[0,T-\tau])} \to 0$ as $h \to 0$.

(H2) $\|u - u_h\|_{H_0^1(\Omega)}(t) \to 0$ as $h \to 0$ for a.e. $t \in [0, T)$.

Once obtained the ODE system, the next step is to discretize the time variable t. In [**3**] the authors analyze explicit Euler and Runge-Kutta methods, however all these methods have to deal with restrictions in the time-step due to stability conditions. In this chapter we will first analyze an explicit Euler method and next we introduce an implicit scheme in order to avoid the time-step restrictions. We will use $U^j = (u_1^j, \dots, u_N^j)$ for the values of the numerical approximation at time t_j, and $\tau_j = t_{j+1} - t_j$ for the time step. When we consider the explicit scheme U^j is the solution of

$$\begin{aligned} &MU^{j+1} = MU^j - \tau_j \left(AU^j + M(U^j)^p\right) \\ &U(0) = u_0^I, \end{aligned} \tag{3}$$

or equivalently, if we denote $\partial u_i^{j+1} = \frac{1}{\tau_j}(u_i^{j+1} - u_i^j)$

$$\begin{aligned} &m_i \partial u_i^{j+1} = -\sum_{k=1}^{N} a_{ki} u_i^j + m_i (u_i^j)^p, \qquad 1 \le i \le N \\ &u_i^0 = u_0(x_i), \qquad 1 \le i \le N+1. \end{aligned} \tag{4}$$

While for the implicit scheme U^j is the solution of

$$\begin{aligned} &MU^{j+1} = MU^j - \tau_j \left(AU^{j+1} + M(U^j)^p\right) \\ &U(0) = u_0^I. \end{aligned} \tag{5}$$

Note that the scheme is not totally implicit since the nonlinear source u^p is evaluated at time t^j while the discrete laplacian (A) is evaluated a time t^{j+1}. This mixture makes the scheme free of time-step restrictions while the explicit evaluation of $(U^j)^p$ avoids the problem of solving a nonlinear system at each step.

Now we choose the time steps $\tau_j = t_{j+1} - t_j$ in such a way that the asymptotic behavior of the discrete problem is similar to the continuous one. We will fix λ small and take

$$\tau_j = \frac{\lambda}{(w^j)^p},$$

where

$$w^j = \sum_{k=1}^{N} m_k u_k^j.$$

In **[3]** the authors use an adaptive procedure similar to this. They adapt the time step in a similar way but using the maximum (L^∞-norm) instead of w^j (L^1-norm). The problem they consider is in dimension one and the nonlinearity is at the right boundary as a Neumann condition. This forces the maximum to be fixed at the right boundary node (i.e. $\|U^j\|_\infty = u_N^j$), in this problem the maximum, that is the node k such that $u_k^j = \|U^j\|_\infty$ can move from one node to another as t increases. So it is better to adapt the time step using w^j for which, for example, we can calculate its derivative. Anyway, as we use a fixed mesh for the space discretization we can compare both norms. The motivation for this choice of the time-step is that, as will be shown, the behavior of w^j is given by

$$\partial w^{j+1} \sim (w^j)^p.$$

Hence, with our selection of τ_j we obtain

$$w^{j+1} \sim w^j + \tau_j (w^j)^p = w^j + \lambda \sim w^0 + (j+1)\lambda,$$

and we get the asymptotic behavior of w^j, which is, as we will see, similar to the one for the continuous solution.

When we deal with the explicit scheme we will also require

$$\lambda < \min_{1 \le i \le N} \frac{a_{ii}}{m_i} (w^0)^p. \tag{6}$$

Then we study the asymptotic properties of the numerical schemes. We will say that a solution of (3) (or (5)) blows up if

$$\lim_{j \to \infty} \|U^j\|_\infty = \infty, \qquad \text{and} \qquad T_{h,\lambda} := \sum_{j=1}^{\infty} \tau_j < \infty,$$

we call $T_{h,\lambda}$ the blow-up time. To describe when the blow-up phenomenon occurs in the discrete problem we use the same discrete functional of the previous chapters

$$\Phi_h(U) \equiv \frac{1}{2}\langle A^{1/2}U, A^{1/2}U\rangle - \frac{1}{p+1}\langle MU^p, U\rangle,$$

This functional characterizes the numerical solutions with blow-up as in the semi-discrete case and this allows us to prove that if the continuous solution has finite time blow-up its numerical approximation also blows up when the parameters of the method are small enough.

Next we study the asymptotic behavior for the numerical approximations of solutions with blow-up. We find that this behavior is very similar to the continuous one. In fact we find that if $u_{h,\lambda}$ is a numerical solution with blow-up at time $T_{h,\lambda}$ its behavior is given by

$$\max_{1\le i\le N} u_i^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

We use the notation $f(j) \sim g(j)$ to mean that there exist constants $c, C > 0$ independent of j (but they may depend on h) such that

$$cg(j) \le f(j) \le Cg(j).$$

Moreover, the numerical schemes recover the constant C_p in the sense that

$$\lim_{j\to\infty} \max_{1\le i\le N} u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p.$$

The functional Φ_h is also useful to prove convergence of the blow-up times. Unfortunately we can only prove the convergence of an iterated limit,

$$\lim_{h\to 0} \lim_{\lambda\to 0} T_{h,\lambda} = T.$$

Thanks to the numerical blow-up rate we observe a propagation property for blow-up points. We prove that the number of nodes adjacent to those that blow-up as the maximum that also blow-up is determined only by p and is independent of h and λ.

In other words, if we call $B^*(u_h)$ the set of nodes k such that

$$\lim_{j\to\infty} u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p,$$

the number of blow-up points outside $B^*(u_h)$ depends only on the power p. The blow-up propagates in the same way of Chapter 3 to K nodes near $B^*(u_h)$, where

$$K = \left[\frac{1}{p-1}\right].$$

We split the chapter in two parts: in the first one we deal with the explicit scheme and in the second one we develop the analysis for the implicit method.

1. The explicit scheme

The main tool in our proofs is a comparison argument, so first of all we prove a Lemma which states that this comparison argument is valid. Since we need restrictions in the time-step to prove this Lemma they are essential for every result in this section. That is not the case in the implicit scheme.

DEFINITION. We will say that (Z^j) is a supersolution (resp.: subsolution) for (3) if verifies the equation with an inequality $\geq$ $(\leq)$ instead of an equality.

LEMMA 6.1. *Assume the time step verifies*

$$\tau_j < \min_{1\leq i\leq N} \frac{m_i}{a_{ii}}.$$

Let $(\overline{U}^j), (\underline{U}^j)$ *a super and a subsolution respectively for* (3) *such that* $\underline{U}^0 < \overline{U}^0$, *then* $\underline{U}^j < \overline{U}^j$ *for every* j.

PROOF. Let $Z^j = \overline{U}^j - \underline{U}^j$, by an approximation argument we can assume that we have strict inequalities in (3), then (Z^j) verifies

$$\begin{array}{lcl} M\partial Z^{j+1} & > & -AZ^j + M((\overline{U}^j)^p - (\underline{U}^j)^p, \\ Z^0 & > & 0. \end{array} \tag{7}$$

If the statement of the Lemma is false, then there exists a first time t^{j+1} and a node x_i such that $z_i^{j+1} \leq 0$. Then we have

$$z_i^{j+1} > (1 - \tau_j \frac{a_{ii}}{m_i}) z_i^j + \tau_j \left(-\sum_{k\neq i} a_{ik} z_k^j + (\overline{u}_i^j)^p - (\underline{u}_i^j)^p \right) \geq 0,$$

a contradiction. □

1.1. Blow-up for the numerical scheme. In this section we find conditions to guarantee that the solution of (3) blows up; we begin with some Lemmas.

We observe that as A is symmetric (property (P2)) there exists a basis of eigenvectors for the following eigenvalue problem

$$A\phi_i = \lambda_i M\phi_i.$$

We call $\eta = \eta(h)$ a bound for the greatest eigenvalue,

$$0 \leq \lambda_i \leq \eta(h).$$

LEMMA 6.2. *For every $y \in \mathbb{R}^N$ there holds*

$$\langle Ay, y\rangle \leq \eta(h)\langle My, y\rangle.$$

PROOF. As the matrix M defines a scalar product in $\mathbb{R}^N$, we can assume that the eigenvectors ϕ_i are normalized such that

$$\langle M\phi_i, \phi_j\rangle = \delta_{ij}.$$

Let $y \in \mathbb{R}^N, y = \sum_{i=1}^N \alpha_i\phi_i$, then

$$\begin{aligned}
\langle Ay, y\rangle &= \left\langle \sum_{i=1}^N \alpha_i\lambda_i M\phi_i, \sum_{j=1}^N \alpha_j\phi_j \right\rangle \\
&= \sum_{i=1}^N \alpha_i^2 \lambda_i \langle M\phi_i, \phi_i\rangle \\
&\leq \eta(h)\langle My, y\rangle.
\end{aligned}$$

□

LEMMA 6.3. *If $(U^j)_{j\geq 0}$ is large enough, then it blows up in finite time.*

PROOF. Recall the definition of

$$w^j = \sum_{i=1}^N m_i u_i^j,$$

and observe that there exists constants $c, C > 0$ that depend only on h such that

$$c \max_{1\leq k\leq N} u_i^j \leq w^j \leq C \max_{1\leq i\leq N} u_i^j.$$

Now we observe that we can choose $j \geq j_0$ in order to get w^j sufficiently large to verify

$$\begin{aligned}
w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N}\sum_{i=1}^{N} a_{ik}u_k^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\
&\geq w^j - \frac{1}{2}\tau_j \sum_{i=1}^{N} m_i (u_i^j)^p + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\
&\geq w^j + \frac{\tau_j}{2} \sum_{i=1}^{N} m_i (u_i^j)^p \\
&\geq w^j + c\tau_j (w^j)^p \\
&= w^j + c\lambda.
\end{aligned}$$

Applying this inequality inductively we obtain

$$w^j \geq w^{j_0} + c\lambda(j - j_0) \geq cj,$$

hence $w^j \to \infty$ as $j \to \infty$, it remains to check that $\sum \tau_j < \infty$, to do this we observe

$$\tau_j = \frac{\lambda}{(w^j)^p} \leq \frac{\lambda}{(w^0 + jc\lambda)^p},$$

and

$$\sum_{j=1}^{\infty} \frac{\lambda}{(w^0 + j(c\lambda))^p} \leq \int_0^{\infty} \frac{\lambda}{(w^0 + cs\lambda)^p}\, ds < \infty.$$

This completes the proof. □

REMARK 6.1. *In the course of the proof of the above Lemma we showed not just that w^j blows up, we also proved $w^j \geq cj$.*

Now we are going to prove the reverse inequality to obtain the asymptotic behavior of $\|U^j\|_\infty$.

LEMMA 6.4. *If (U^j) is unbounded then*

$$\|U^j\|_\infty \sim w^j \sim j.$$

PROOF. The relation $\|U^j\|_\infty \sim w^j$ is trivial since they define equivalent norms in $\mathbb{R}^N$. So we just have to prove $w^j \leq Cj$. To do that we observe that

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N}\sum_{i=1}^{N} a_{ik} u_k^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\le w^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\le w^j + C\tau_j (w^j)^p \\ &= w^j + C\lambda. \end{aligned}$$

We apply this inequality inductively again to get

$$w^j \le w^0 + C\lambda j \le Cj,$$

as we wanted to prove. □

THEOREM 6.1. *Assume the time step τ_j verifies the restriction*

$$\tau_j < \frac{2}{p(w^{j+1})^{p-1} + \eta(h)}, \qquad 1 \le i \le N. \tag{8}$$

Then positive solutions of (3) *blow up if there exists j_0 such that $\Phi_h(U^{j_0}) < 0$.*

We remark that the condition $\Phi_h(U^{j_0}) < 0$ is similar to the one for the blow-up phenomena in the continuous problem; in fact, for the continuous problem this is a necessary condition.

PROOF. First we observe that $\Phi_h(U^j)$ decreases with j, to do that we take inner product of (3) with $U^{j+1} - U^j$ to obtain

$$\begin{aligned} 0 &= \langle \frac{1}{\tau_j} M(U^{j+1} - U^j) + AU^j - M(U^j)^p, U^{j+1} - U^j \rangle \\ &= \tau_j \langle M\partial U^{j+1}, \partial U^{j+1} \rangle + \Phi_h(U^{j+1}) - \Phi_h(U^j) - \frac{1}{2}\langle AU^{j+1}, U^{j+1} \rangle \\ &+ \langle AU^j, U^{j+1} \rangle - \frac{1}{2}\langle AU^j, U^j \rangle - \frac{1}{2}\langle Mp(\xi^j)^{p-1}, (U^{j+1} - U^j)^2 \rangle. \end{aligned}$$

Hence we obtain,

$$\begin{aligned} &\Phi_h(U^{j+1}) - \Phi_h(U^j) \\ &\le \tau_j\Big(\tau_j \frac{p(w^{j+1})^{p-1}}{2} - 1\Big)\langle M\partial U^{j+1}, \partial U^{j+1} \rangle + \frac{\tau_j^2}{2}\langle A\partial U^{j+1}, \partial U^{j+1} \rangle \\ &\le \tau_j\Big(\tau_j \frac{p(w^{j+1})^{p-1}}{2} + \frac{\eta(h)\tau_j}{2} - 1\Big)\langle M\partial U^{j+1}, \partial U^{j+1} \rangle \le 0, \end{aligned}$$

due the restriction in the time step τ_j and Lemma 6.2. Actually $\Phi_h(U^{j+1}) < \Phi_h(U^j)$ unless (U^j) is independent of j. So, Φ_h is Lyapunov functional for (3). Next we observe that the steady states of (3) have positive energy. Let $(W^j) = W$ be a stationary solution of (3), then

$$0 = -AW + MW^p.$$

Multiplying by $W/2$ we obtain,

$$\begin{aligned} 0 &= -\frac{1}{2}\langle AW, W\rangle + \frac{p+1}{2}\frac{1}{p+1}\langle MW^p, W\rangle \\ &\geq -\Phi_h(W). \end{aligned}$$

Now, assume (U^j) is a bounded solution of (3), then there exists a convergent subsequence of (U^j) that we still denote (U^j). Its limit W is a steady state with positive energy.

As $\Phi_{h,\lambda}(U^j)$ decreases and there exists j_0 such that $\Phi_{h,\lambda}(U^{j_0}) < 0$ then $\Phi_{h,\lambda}(W) < 0$, a contradiction. We conclude that (U^j) is unbounded and by Lemma 6.3 has finite time blow-up. $\square$

COROLLARY 6.1. *Assume the time-step restriction of the above theorem and the convergence hypotheses (H1), (H2). Let u_0 be an initial data for* (1) *such that u blows up in finite time T. Then $u_{h,\lambda}$ also blows up in finite time $T_{h,\lambda}$ for every h, $\lambda = \lambda(h)$ small enough. Moreover*

$$\lim_{h\to 0}\lim_{\lambda\to 0} T_{h,\lambda} = T.$$

REMARK 6.2. *If the fully-discrete method converges in $H_0^1(\Omega)$ a.e. t then λ can be chosen independent of h.*

PROOF. We observe that if u blows up in finite time T then (see **[21, 38]**)

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\,ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\,ds \to -\infty \qquad (t \nearrow T).$$

This implies that there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. Now we use the convergence of $u_{h,\lambda}$ to u_h in $[0, t_0]$ and the convergence hypothesis (H1) to see that

$$\lim_{h\to 0}\lim_{\lambda\to 0}\Phi_{h,\lambda}(u_{h,\lambda})(t_0) = \Phi(u)(t_0).$$

So for $h, \lambda = \lambda(h)$ small enough we get $\Phi_{h,\lambda}(u_{h,\lambda})(t_0) < 0$ and so, by the above theorem (U^j) blows up.

Now we turn our attention to the blow-up times. In Chapter 3 (Theorem 3.2) we proved that the blow-up time of the semi-discrete solutions (solutions of (2)), that we are going to denote T_h, converges, as $h \to 0$, to T, so we just have to prove that for fixed h

$$\lim_{\lambda \to 0} T_{h,\lambda} = T_h.$$

To do that we observe that from Lemma 6.3 we know that there exists j_0, that does not depend on λ such that for $j \geq j_0$

$$w^j \geq w^{j_0} + c\lambda(j - j_0),$$

hence

$$\begin{aligned} T_{h,\lambda} - t^j &= \sum_{l=j+1}^{\infty} \tau_l = \sum_{l=j+1}^{\infty} \frac{\lambda}{(w^l)^p} \\ &\leq \sum_{k=j+1}^{\infty} \frac{\lambda}{(w^{j_0} + c\lambda(l - j_0))^p} \leq \int_j^{\infty} \frac{\lambda}{(w^{j_0} + c\lambda(s - j_0))^p}\, ds \\ &= \frac{1}{c} \int_{w^{j_0} + c\lambda(j - j_0)}^{\infty} \frac{1}{s^p}\, ds \leq \frac{1}{c} \int_{w^{j_0}}^{\infty} \frac{1}{s^p}\, ds. \end{aligned}$$

This holds for any j_0 large enough and for every $j \geq j_0$. In particular we get

$$T_{h,\lambda} - t^j \leq \frac{1}{c} \int_{w^j}^{\infty} \frac{1}{s^p}\, ds.$$

This inequality has a great meaning. It says that if w^j is large, then t^j is close to blow up (independent of λ). So now, given $\varepsilon > 0$ we can choose K large enough in order to have

$$\frac{1}{c} \int_K^{\infty} \frac{1}{s^p}\, ds \leq \frac{\varepsilon}{3}, \qquad K^{-p} < \frac{\varepsilon}{3}.$$

Next we choose $\tau < \frac{\varepsilon}{3}$ such that $\sum m_k u_k(T_h - 2\tau) \geq 2K$ (remember that $(u_1(t), \ldots, u_N(t))$ is the solution of the semidiscrete scheme). For $\lambda = \lambda(h, \tau)$ small enough we get, from (H2), that $w^j \geq K$ for every j such that $T_h - 2\tau \leq t^j \leq T_h - \tau$. We choose one of those j and compute

$$\begin{aligned} |T_{h,\lambda} - T_h| &\leq |T_{h,\lambda} - t^j| + |t^j - T_h| \\ &\leq \frac{1}{c} \int_K^{\infty} \frac{1}{s^p}\, ds + \tau_j + \tau \\ &\leq \varepsilon. \end{aligned}$$

$\square$

Now we find the blow-up rate of the discrete solutions.

1.2. Blow-up rate.

THEOREM 6.2. *Let $u_{h,\lambda}$ a solution with blow-up at time $T_{h,\lambda}$, then*

$$\max_{1\le i\le N} u_i^j \sim (T_{h,\lambda}-t^j)^{-1/(p-1)}.$$

Moreover

$$\lim_{j\to\infty}\max_{1\le i\le N} u_i^j (T_{h,\lambda}-t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

We want to remark that this is the behavior of the continuous solutions with blow-up.

PROOF. We know from Lemma 6.3 that $w^j = \sum m_i u_i^j$ verifies

$$w^{j+1} \ge w^j + c\tau_j (w^j)^p,$$

so we have

$$\begin{aligned}(T_{h,\lambda}-t^j) &= \sum_{k=j+1}^{\infty} \tau_j = \sum_{k=j+1}^{\infty} \frac{\lambda}{(w^j)^p}\\ &\le \int_j^{\infty} \frac{\lambda}{(w(s))^p}\, ds.\end{aligned}$$

Here $w(s)$ is the linear interpolant of $(w^j = w(j))$, hence for $j\le s\le j+1$ we have $w'(s) = w^{j+1}-w^j \ge c\lambda$, and so

$$\int_j^{\infty} \frac{\lambda}{(w(s))^p}\, ds \le \int_{w^j}^{\infty} \frac{\lambda}{cv^p\lambda}\, dv \le \frac{1}{c(p-1)}\left(\frac{1}{w^j}\right)^{p-1},$$

or equivalently

$$\max_{1\le i\le N} u_i^j \le Cw^j \le C(T_{h,\lambda}-t^j)^{-1/(p-1)}.$$

The inverse inequalities can be handled in a similar way to obtain

$$\max_{1\le i\le N} u_i^j \sim w^j \sim (T_{h,\lambda}-t^j)^{-1/(p-1)}.$$

Next we recover the constant C_p in the asymptotic behavior of the numerical solution, to do that we will change variables but before that we prove a short lemma.

LEMMA 6.5.

$$\lim_{j\to\infty}\frac{T_{h,\lambda}-t^j}{T_{h,\lambda}-t^{j+1}} = 1.$$

PROOF.

$$1 \le \frac{T_{h,\lambda} - t^j}{T_{h,\lambda} - t^{j+1}} = \frac{\sum_{k=j+1}^{\infty} \tau_k}{\sum_{k=j+2}^{\infty} \tau_k} = 1 + \frac{\tau_{j+1}}{\sum_{k=j+2}^{\infty} \tau_k} \le 1 + \frac{\lambda/(w^{j+1})^p}{C/(w^{j+1})^{p-1}} \to 1.$$

□

Now we change variables, in a way inspired by **[39]** and Chapter 2. Let (Y^j) be defined by

$$y_i^j = u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} \qquad 1 \le i \le N. \tag{9}$$

In the sequel of the proof we will use Δy_i^{j+1} to denote

$$\frac{y_i^{j+1} - y_i^j}{\tau_j/(T_{h,\lambda} - t^j)},$$

This can be thought as $\tau_j/(T_{h,\lambda} - t^j)$ to be the time step in the new variables. With this notation the new variables verify

$$\begin{aligned} & m_i \Delta y_i^{j+1} = -\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} (T_{h,\lambda} - t^j) \sum_{i=1}^{N} a_{ki} y_i^j \\ & + m_i \tfrac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} (y_i^j)^p \\ & + \frac{(T_{h,\lambda} - t^j) m_i u_i^j}{\tau_j} \left((T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}} - (T_{h,\lambda} - t^j)^{\frac{1}{p-1}} \right), \\ & \\ & y_i^0 = (T_{h,\lambda})^{1/(p-1)} u_0(x_i), \qquad 1 \le i \le N+1. \end{aligned} \tag{10}$$

Now assume there exists $\varepsilon > 0$ and a subsequence that we still denote (y_i^j) such that $y_i^j > C_p + \varepsilon$ for some $i = i(j)$. Then for those y_i^j we have

$$(y_i^j)^p - \frac{1}{p-1} y_i^j > \frac{3\delta}{m_i}. \tag{11}$$

We also know from the blow-up rate that the new variables y_i^j are bounded and hence, applying Lemma 6.5, we obtain for j large enough

$$\begin{aligned} m_i \Delta y_i^{j+1} & \ge & -\delta + m_i \left((y_i^j)^p - \frac{1}{p-1} y_i^j \right) \\ & + & m_i (y_i^j)^p \left(\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} - 1 \right) \\ & \ge & \delta. \end{aligned}$$

This means that actually $y_i^j > C_p + \varepsilon$ for every j large enough and consequently (11) is verified for all those j. So y_i^j is unbounded, a contradiction.

If we assume $y_i^j < C_p - \varepsilon$ arguing along the same lines as before we obtain that y_i^j verifies

$$\begin{aligned} m_i \Delta y_i^{j+1} &\leq \delta + m_i \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} \left((y_i^j)^p \frac{1}{p-1} y_i^j \right) \\ &+ \frac{m_i}{p-1} y_i^j \left(\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} - \frac{(T_{h,\lambda} - \xi^j)^{\frac{1}{p-1}-1}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}-1}} \right) \\ &\leq 2\delta + C \left((y_i^j)^p - \frac{1}{p-1} y_i^j \right). \end{aligned}$$

This shows that either $y_i^j \to 0$ as $j \to \infty$ or $m_i \Delta y_i^{j+1} < -\delta$, which means that y_i^j is not bounded from below (this is not possible).

We conclude that if y_i^j does not go to zero, then goes to C_p, as the blow-up rate implies that for every j

$$\max_{1 \leq i \leq N} y_i^j \geq c,$$

we have

$$\lim_{j \to \infty} \max_{1 \leq i \leq N} y_i^j = \lim_{j \to \infty} \max_{1 \leq i \leq N} (T_{h,\lambda} - t^j)^{1/(p-1)} u_i^j = C_p,$$

as we wanted to prove. □

1.3. Blow-up set. Now we turn our attention to the blow-up set. In order to do that we consider the set $B^*(u_h)$ composed of those nodes that blow-up as the maximum does, and we will study the behavior of the adjacent nodes.

DEFINITION. We define the graph with vertices in the nodes and say that two different nodes are connected if and only if $a_{ij} \neq 0$. We consider the usual distance between nodes measured as a graph, see [**40**]. Finally, we denote by $d(k)$ the distance of the node x_k to $B^*(u_h)$ also measured as a graph.

We prove that u_k blows up if and only if $d(k) \leq K$ where K depends only on p,

THEOREM 6.3. *Let $B^*(u_h)$ be the set of nodes, $\{x_k\}$, such that*

$$u_k^j \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}}.$$

Then the blow-up propagates in the following way, let $p > 1$ and $K = [1/(p-1)]$. Then the solution of (4) blows up exactly at K nodes near $B^*(u_h)$. More precisely,

$$u_k^j \to +\infty \qquad \Longleftrightarrow \qquad d(k) \le K.$$

Moreover, if $d(k) \le K$, the asymptotic behavior of $(u_k^j)_{j\ge 1}$ is given by

$$u_k^j \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}+d(k)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(k) = K$

$$u_k^j \sim \ln(T_{h,\lambda} - t^j).$$

We want to remark that more than one node can go to infinity, but the asymptotic behavior imposes $\frac{u_k^j}{u_i^j} \to 0$ $(j \to \infty)$ if $d(k) > d(i)$.

Proof. We want to show that the blow-up propagates K nodes around $B^*(u_h)$, we begin with a node x_k such that $d(k) = 1$. We claim that the behavior of u_k^j is given by

$$u_k^j \sim \begin{cases} j^{-p+2} & \text{if} \quad p < 2, \\ \ln j & \text{if} \quad p = 2, \\ C & \text{if} \quad p > 2, \end{cases}$$

to prove this we will show that

$$w_\gamma^j = \gamma \sum_{s=1}^{j} s\tau_s,$$

which has the behavior described above, can be used as super and subsolution for an equation verified by u_k^j choosing γ appropriately.

We observe that u_k^j satisfies

$$m_k \partial u_k^{j+1} = -\sum_{i=1}^{N} a_{ik} u_i^j + m_k (u_k^j)^p \sim C_1(\max_{1\le i\le N} u_i^j) - C_2 u_k^j + C_3 m_k (u_k^j)^p.$$

This means that there exist constants $c_i, C_i > 0$, $i = 1, 2, 3$ such that for j large enough

$$\partial u_k^{j+1} \le C_1 j - C_2 u_k^j + C_3 (u_k^j)^p \tag{12}$$

and

$$\partial u_k^{j+1} \ge c_1 j - c_2 u_k^j + c_3 (u_k^j)^p. \tag{13}$$

Now we observe that if γ and j are large enough w_γ^j verifies

$$\begin{aligned} \partial w_\gamma^j &= \gamma(j+1) \\ &\geq C_1 j - C_2 w_\gamma^j + C_3 (w_\gamma^j)^p, \end{aligned}$$

since $(w_\gamma^j)^p/j \to 0$ as j goes to infinity. Hence w_γ^j is a supersolution for (12) and so

$$u_k^j \leq w_\gamma^j.$$

On the other hand if we choose γ small we get

$$\begin{aligned} \partial w_\gamma^j &= \gamma(j+1) \\ &\leq c_1 j - c_2 w_\gamma^j + c_3 (w_\gamma^j)^p, \end{aligned}$$

So now we can use w_γ^j as a subsolution for (13) to handle the other inequality. So

$$u_k^j \sim w_\gamma^j.$$

We observe that if $p \leq 2$ the node x_k is a blow-up node and we also have the blow-up rate for this node. If $p > 2$ this node is bounded. Next we assume $p < 2$ (if $p > 2$ it is easy to prove that every node k with $d(k) \geq 1$ is bounded, and the case $p = 2$ can be handled in a similar way) and we are going to find the behavior of a node, that we still denote k, such that $d(k) = 2$.That is,it is not adjacent to $B^*(u_h)$ and it is adjacent to a node which has the behavior j^{-p+2}.

Now let

$$w_\gamma^j = \gamma \sum_{s=1}^{j} \tau_s s^{-p+2},$$

and observe that u_k^j verifies

$$m_k \partial u_k^{j+1} = -\sum_{i=1}^{N} a_{ik} u_i^j + m_k (u_k^j)^p \sim C_1 (j^{-p+2}) - C_2 u_k^j + C_3 (u_k^j)^p.$$

This means that there exist constants $c_i, C_i > 0$, $i = 1, 2, 3$ such that for j large enough

$$\partial u_k^{j+1} \leq C_1 j^{-p+2} - C_2 u_k^j + C_3 (u_k^j)^p \tag{14}$$

and

$$\partial u_k^{j+1} \geq c_1 j^{-p+2} - c_2 u_k^j + c_3 (u_k^j)^p. \tag{15}$$

Now we observe that if γ and j are large enough w_γ^j verifies

$$\begin{aligned}\partial w_\gamma^j &= \gamma(j+1)^{-p+2}\\ &\geq C_1 j^{-p+2} - C_2 w_\gamma^j + C_3 (w_\gamma^j)^p,\end{aligned}$$

since $(w_\gamma^j)^p / j^{-p+2} \to 0$ as j goes to infinity. Hence w_γ^j is a supersolution for (14) and so

$$u_k^j \leq w_\gamma^j.$$

On the other hand if we choose γ small we get

$$\begin{aligned}\partial w_\gamma^j &= \gamma(j+1)^{-p+2}\\ &\leq c_1 j^{-p+2} - c_2 w_\gamma^j + c_3 (w_\gamma^j)^p,\end{aligned}$$

Now we can use w_γ^j as a subsolution for (15) to handle the other inequality. So

$$u_k^j \sim w_\gamma^j.$$

We observe that if $p < 3/2$ the node x_k is a blow-up node and we also have the blow-up rate for this node ($u_k^j \sim j^{-2p+3}$). If $p > 3/2$ this node is bounded. In the case $p < 3/2$ we repeat this procedure inductively to obtain the Theorem. □

2. The implicit scheme

In order to avoid the time step restrictions we are going to introduce an implicit scheme and prove that similar properties can be observed. However we have to remark that near the blow-up time the adaptive procedure forces the time step to be much smaller than the space discretization parameter h. This suggest that an adequate method could be to begin at time zero with the implicit scheme in order to avoid time-step restrictions and, as when the solution increases the time step is reduced, if the solution has finite time blow-up there will be a moment such that the time step restriction for the explicit scheme will be verified. From that time one can continue either with the explicit or with the implicit scheme. As in the explicit scheme section we begin with the comparison lemma. Observe that there is no time-step restriction.

LEMMA 6.6. *Let $(\overline{U}^j), (\underline{U}^j)$ a super and a subsolution respectively for* (5) *such that $\underline{U}^0 < \overline{U}^0$, then $\underline{U}^j < \overline{U}^j$ for every j.*

PROOF. Let $Z^j = \overline{U}^j - \underline{U}^j$; we assume that we have strict inequalities in (5), then (Z^j) verifies

$$\begin{array}{lll} M\partial Z^{j+1} & > & -AZ^{j+1} + M((\overline{U}^j)^p - (\underline{U}^j)^p, \\ Z^0 & > & 0. \end{array} \tag{16}$$

If the statement of the Lemma is false, then there exists a first time t^{j+1} and a node x_i such that $z_i^{j+1} \leq 0$. There we have

$$z_i^{j+1} > z_i^j - \tau_j \frac{a_{ii}}{m_i} z_i^{j+1} - \sum_{k \neq i} a_{ik} z_k^j + (\overline{u}_i^j)^p - (\underline{u}_i^j)^p \geq 0,$$

a contradiction. □

2.1. Blow-up for the numerical scheme.

LEMMA 6.7. *If $(U^j)_{j\geq 0}$ is large enough, then blows up in finite time. Moreover*

$$\|U^j\|_\infty \sim w^j \sim j.$$

PROOF.

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N} \sum_{i=1}^{N} a_{ik} u_k^{j+1} + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\leq w^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\leq w^j + C\tau_j (w^j)^p \\ &= w^j + C\lambda. \end{aligned}$$

Hence $w^j \leq Cj$. To prove the inverse inequality we observe that

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N} \sum_{i=1}^{N} a_{ik} u_k^{j+1} + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\geq w^j - \tau_j C_1 w^{j+1} + \tau_j C_2 (w^j)^p \end{aligned}$$

that is,

$$(1 + C_1\tau_j) w^{j+1} \geq w^j + C_2 \tau_j (w^j)^p. \tag{17}$$

Now we look for a subsolution of (17) of the form $z^j = \Gamma j$. This sequence verifies

$$(1 + C_1\tau_j) z^{j+1} = z^j + \Gamma C_1 \tau^j j + \Gamma(1 + C_1\tau_j) \leq z^j + C_2\tau_j (z^j)^p$$

if Γ is small enough. As the discrete maximum principle holds for this equation we obtain

$$w^j \geq z^j = \Gamma j.$$

This completes the proof. $\square$

THEOREM 6.4. *Positive solutions of* (6) *blow up if there exists* j_0 *such that* $\Phi_h(U^{j_0}) < 0$.

PROOF. Again we first observe that $\Phi_h(U^j)$ decreases with j, to do that we take inner product of (5) with $U^{j+1} - U^j$ to obtain

$$\begin{aligned} 0 &= \langle \frac{1}{\tau_j} M(U^{j+1} - U^j) + AU^{j+1} - M(U^j)^p, U^{j+1} - U^j \rangle \\ &= \tau_j \langle M \partial U^{j+1}, \partial U^{j+1} \rangle + \Phi_h(U^{j+1}) - \Phi_h(U^j) + \frac{1}{2} \langle AU^{j+1}, U^{j+1} \rangle \\ &\quad - \langle AU^j, U^{j+1} \rangle + \frac{1}{2} \langle AU^j, U^j \rangle + \frac{p}{2} \langle M(\xi^j)^{p-1}, (U^{j+1} - U^j)^2 \rangle. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Phi_h(U^{j+1}) - \Phi_h(U^j) &= -\tau_j \langle M \partial U^{j+1}, \partial U^{j+1} \rangle - \frac{\tau_j^2}{2} \langle A \partial U^{j+1}, \partial U^{j+1} \rangle \\ &\quad - \frac{p}{2} \langle M(\xi^j)^{p-1}, (U^{j+1} - U^j)^2 \rangle \\ &\leq 0. \end{aligned}$$

This implies that Φ_h is a Lyapunov functional for (5). We observe that the steady states of (5) are the same of (3), so they have positive energy. Now, assume (U^j) is a bounded solution of (5), then there exists a convergent subsequence of (U^j) that we still denote (U^j). Its limit W is a steady state with positive energy. As $\Phi_h(U^j)$ decreases and there exists j_0 such that $\Phi_h(U^{j_0}) < 0$ then $\Phi_{h,\lambda}(W) < 0$, a contradiction. We conclude that (U^j) is unbounded and by Lemma 6.7 has finite time blow-up. $\square$

COROLLARY 6.2. *Assume the convergence hypotheses (H1), (H2). Let* u_0 *an initial data for* (1) *such that* u *blows up in finite time* T. *Then* $u_{h,\lambda}$ *also blows up in finite time* $T_{h,\lambda}$ *for every* h, $\lambda = \lambda(h)$ *small enough. Moreover*

$$\lim_{h \to 0} \lim_{\lambda \to 0} T_{h,\lambda} = T.$$

PROOF. If u blows up in finite time T then (see **[21, 38]**)

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2} \, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1} \, ds \to -\infty \qquad (t \nearrow T).$$

Hence there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. Now we use the convergence hypothesis (H1) and the convergence of $u_{h,\lambda}$ to u_h in $[0, t_0]$ to see that

$$\lim_{h\to 0}\lim_{\lambda\to 0}\Phi_{h,\lambda}(u_{h,\lambda})(t_0) = \Phi(u)(t_0).$$

So for $h, \lambda(h)$ small enough we get $\Phi_{h,\lambda}(u_{h,\lambda})(t_0) < 0$ and so (U^j) blows up. The convergence of the blow-up times is obtained like in the explicit scheme. □

Next we turn our attention to the blow-up rate of the discrete solutions.

2.2. Blow-up rate.

THEOREM 6.5. *Let $u_{h,\lambda}$ a solution with blow-up at time $T_{h,\lambda}$, then*

$$\max_{1\le i\le N} u_i^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Moreover

$$\lim_{j\to\infty}\max_{1\le i\le N} u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

PROOF. The first part of the proof is the same as the one for the explicit scheme, so we assume we have proved

$$\|U^j\|_\infty \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}},$$

and we are going to recover the constant C_p. We change variables as in the explicit scheme; let (Y^j) be defined by

$$y_i^j = u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} \qquad 1 \le i \le N. \tag{18}$$

In the remaining part of the proof we will use Δy_i^{j+1} to denote

$$\frac{y_i^{j+1} - y_i^j}{\tau_j/(T_{h,\lambda} - t^j)}.$$

This can be thought as $\tau_j/(T_{h,\lambda})$ to be the time step in the new variables. With this notation the new variables verify

$$\begin{aligned} m_i \Delta y_i^{j+1} &= -(T_{h,\lambda} - t^j)\sum_{i=1}^N a_{ki} y_i^{j+1} + m_i \frac{(T_{h,\lambda} - t^{j+1})^{1/(p-1)}}{(T_{h,\lambda} - t^j)^{1/(p-1)}} (y_i^j)^p \\ &\quad - m_i u_i^j\big((T_{h,\lambda} - t^j)^{1/(p-1)} - (T_{h,\lambda} - t^{j+1})^{1/(p-1)}\big), \\ y_i^0 &= T_{h,\lambda}^{1/(p-1)} u_0(x_i). \end{aligned}$$

If we assume that there exist $\varepsilon > 0$ and an subsequence such that $y_i^j > C_p + \varepsilon$ for some $i = i(j)$, then for those y_i^j, as they are bounded, if j is large enough we have

$$\begin{aligned}
\Delta y_i^{j+1} &\geq -\delta + m_i \frac{(T_{h,\lambda} - t^{j+1})^{1/(p-1)}}{(T_{h,\lambda} - t^j)^{1/(p-1)}} \left((y_i^j)^p - \frac{1}{p-1} y_i^j \right) \\
&\quad + \frac{1}{p-1} y_i^j \left[1 - \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}-1}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}-1}} \right] \\
&\geq \delta.
\end{aligned} \tag{19}$$

This means that actually $y_i^j > C_p + \varepsilon$ for every j large enough, hence (19) is verified for every j large enough. So y_i^j is unbounded, a contradiction.

The case where there exists an infinite number of y_i^j such that $y_i^j < C_p - \varepsilon$ is very similar and hence we can use the same arguments as the ones used for the explicit scheme. So we can conclude that as $j \to \infty$ either $y_i^j \to 0$ or $y_i^j \to C_p$. Now we use the blow-up rate to obtain

$$\lim_{j\to\infty} \max_{1\leq i\leq N} y_i^j = \lim_{j\to\infty} \max_{1\leq i\leq N} (T_{h,\lambda} - t^j)^{1/(p-1)} u_i^j = C_p,$$

as we wanted to prove. $\square$

2.3. Blow-up set. The propagation property holds for the implicit scheme and its proof is very similar. We do not include the details.

3. Convergence of the method

In this section we prove that if the general method considered for the space discretization is consistent (see below) then the totally discrete method converges in the L^∞ norm. We perform the proofs for the explicit scheme, they can be extended to the implicit one.

DEFINITION. Let w be a regular solution of

$$\begin{array}{ll} w_t = \Delta w + f(x,t) & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T). \end{array}$$

We say that the scheme (2) is consistent if for any $t \in (0, T-\tau)$ it holds

$$m_i w_t(x_i,t) = -\sum_{k=1}^{N} a_{ik} w(x_k,t) + m_i f(x_i,t) + \rho_{i,h}(t), \tag{20}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max_i \frac{|\rho_{i,h}(t)|}{m_i} \leq \rho(h), \qquad \text{for every } t \in (0, T-\tau),$$

with $\rho(h) \to 0$ if $h \to 0$. The function ρ is called the modulus of consistency of the method.

Now we can prove the convergence Theorem

THEOREM 6.6. *Let u be a regular solution of (1) ($u \in C^{2,1}(\overline{\Omega} \times [0, T-\tau])$ and $u_{h,\lambda}$ the numerical approximation given by (3) then there exists a constant C depending on $\|u\|$ in $C^{2,1}(\overline{\Omega} \times [0, T-\tau]$ such that*

$$\|u - u_{h,\lambda}\|_{L^\infty([0,T-\tau],L^2(\Omega))} \leq C(h^2 + \lambda).$$

PROOF. We define the error functions

$$e_i^j = u(x_i, t_j) - u_i^j.$$

By (20), these functions verify

$$m_i \partial e_i^{j+1} = -\sum_{k=1}^{N} a_{ik} e_k^j + m_i(u^p(x_i, t_j) - (u_i^j)^p) + \rho_i(h) + C m_i \lambda,$$

where C is a bound for $\|u_{tt}\|_{L^\infty(\Omega \times [0,T-\tau])}$.

Let

$$t_0 = \max\{t : \ t < T - \tau, \ \max_i \max_{t_j < t} |e_i^j| \leq 1\}.$$

We will see by the end of the proof that $t_0 = T - \tau$ for h, λ small enough.

Hence, in $[0, t_0]$, $E^j = (e_1^j, ..., e_N^j)$ satisfies

$$M\partial E^{j+1} \leq -AE^j + KME^j + (\rho(h) + C\lambda)M1^t, \tag{21}$$

Let us now define $W^j = (w_1^j, \ldots, w_N^j(t))$, which will be used as a supersolution.

$$w_i^j = e^{(2K+1)t_j}(\|e(0)\|_\infty + \rho(h) + C\lambda).$$

It is easy to check that W^j verifies

$$M\partial W^{j+1} > -AW^j + KMW^j + (\rho(h) + C\lambda)M1^t,$$

here K is the Lipchitz constant for $f(u) = u^p$ in $[0, \|u(\cdot, T-\tau)\|_{L^\infty}]$. Hence W^j is a supersolution for (21), and by Lemma 6.1 we get

$$e_i^j \leq e^{(2K+1)t_j}(\|e^0\|_{L^\infty(\Omega)} + \rho(h) + C\lambda), \qquad t_j \in [0, t_0].$$

Arguing along the same lines with $-E^j$, we obtain

$$|e_i^j| \leq e^{(2K+1)T}(\|E(0)\|_\infty + \rho(h) + C\lambda) \leq C(\rho(h) + \lambda), \qquad t_j \in [0, t_0],$$

by our hypotheses on the convergence of the initial data. Using this fact, since $\rho(h)$ goes to zero, we get that $|e_i^j| \leq 1$ for every $t_j \in [0, T-\tau]$ for every h, λ small enough. Therefore $t_0 = T-\tau$ for h, λ small enough. This proves the convergence of the scheme. In fact we have that for every $h < h_0, \lambda < \lambda_0$

$$\max_j \max_{1 \leq i \leq N} |u_i^j - u(x_i, t_j)| \leq C(\rho(h) + \lambda). \quad \square$$

CHAPTER 7

The porous medium equation: regional blow-up

In this chapter we consider the porous medium equation, i.e.

$$(1)\qquad \begin{array}{ll} u_t = (u^m)_{xx} + u^p, & (x,t) \in (-L,L) \times [0,T), \\ u(-L,t) = u(L,t) = 1, & t \in [0,T), \\ u(x,0) = \varphi(x) \geq 1, & x \in (-L,L), \end{array}$$

where $m > 1$ and $p > 0$ are parameters. This problem can be thought as a model for nonlinear propagation (in a porous medium, for example).

Our main interest in this problem is due to the presence of a new phenomenon that is not present in the heat equation: regional and global blow-up, which means that, depending on the values of the parameters, and the initial datum, there exist solutions of (1) that blow up, and their blow-up set can be a subregion of the interval $(-L, L)$ with positive Lebesgue measure (regional blow-up), or even the whole domain (global blow-up). So our main interest here is to investigate how do numerical schemes reproduce this phenomenon.

We assume that φ is smooth and compatible with the boundary conditions in order to obtain a regular solution. Also, for simplicity, we assume that φ is a bell shaped function, that is, φ is symmetric, $\varphi(x) = \varphi(-x)$ for $x > 0$, and decreasing in $[0, L]$. These symmetry properties will be preserved by the numerical scheme and make computations easier.

The nonlinear diffusion that we are considering here, $(u^{m-1}u_x)_x$, degenerates at level $u = 0$, giving only weak solutions if we allow the initial data (or the boundary conditions) to vanish. Since we are interested in blowing up solutions, we avoid this lack of regularity imposing $\varphi \geq 1$ and $u(-L,t) = u(L,t) = 1$. This gives us regular solutions (see [**50**]).

Let us resume what is known for the continuous problem (1) (see [**72**] and references therein for a detailed exposition of the properties of solutions for this problem):

(i) If $p > m$ there exist blowing up solutions for initial data φ large enough. The blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{p-1}}$ and the blow-up set is a single point $x = 0$ (this means that u remains bounded away from the origin).

(ii) If $p = m$ the existence of blowing up solutions depends on the length of the interval. Let $\lambda_1(L) = \pi/2L$ be the first eigenvalue of the Laplacian in $[-L, L]$. Then, every solution blows up if and only if $\lambda_1(L) < 1$. That is, if $L > \pi/2$ every positive solution blows up and the blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{m-1}}$. In this case it is known that the blow-up set is given by

$$B(u) = \begin{cases} [-L, L] & \text{if } \dfrac{\pi}{2} < L \le \dfrac{m\pi}{m-1}, \\ \left(\dfrac{-m\pi}{m-1}, \dfrac{m\pi}{m-1}\right) & \text{if } L > \dfrac{m\pi}{m-1}. \end{cases}$$

Moreover, there exists a self-similar profile, $z(x)$, that gives the asymptotic behavior for u near T in the form,

$$\lim_{t \nearrow T} (T-t)^{\frac{1}{m-1}} u(x,t) = z(x).$$

Let us remark that if $L > m\pi/(m-1)$ the support of z is the interval

$$\left(-\frac{m\pi}{(m-1)}, \frac{m\pi}{(m-1)}\right).$$

Therefore we have regional blow-up for large values of L. On the other hand, if $L \le \pi/2$ every solution is global.

(iii) If $p < m$ every solution is global.

We analyze here the same method proposed in Chapter 2: piecewise linear finite elements with mass lumping in a uniform mesh for the space variable.

We denote with $U(t) = (u_{-N}(t), ..., u_N(t))$ the values of the numerical approximation at the nodes $x_i = ih$ $(h = L/N)$ at time t. Then $U(t)$ verifies the following equation:

$$\begin{aligned} &MU'(t) = -AU^m(t) + MU^p(t), \\ &u_{-N}(t) = u_N(t) = 1, \\ &U(0) = \varphi^I, \end{aligned} \tag{2}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and φ^I is the Lagrange interpolation of the initial datum, φ.

Writing this equation explicitly we obtain the following ODE system,

$$
\begin{aligned}
&u_{-N}(t) = 1,\\
&u_k'(t) = \frac{1}{h^2}(u_{k+1}^m(t) - 2u_k^m(t) + u_{k-1}^m(t)) + u_k^p(t),\\
&u_N(t) = 1,\\
&u_k(0) = \varphi(x_k), \qquad -N+1 \le k \le N-1.
\end{aligned} \tag{3}
$$

First we state a convergence result that says that the method converges uniformly in sets of the form $[-L, L] \times [0, T - \tau]$.

THEOREM 7.1. *Let $u(x,t) \in C^{4,1}([-L,L] \times [0,T-\tau])$ be a positive solution of (1) and $U(t)$ the numerical approximation given by (3). Then, there exists a constant C, that depends on the $C^{4,1}([-L,L] \times [0,T-\tau])$ norm of u, such that for every h small enough it holds*

$$\max_{t\in[0,T-\tau]} \max_{k} |u(x_k,t) - u_k(t)| \le Ch^2.$$

Next we begin with the analysis of the asymptotic behavior of solutions of (3). Next theorem describes when (3) has solutions with blow-up. To do this, we introduce $\lambda_1(L,h)$ that stands for the first discrete eigenvalue for the Laplacian, that is the first eigenvalue of the problem

$$
\begin{aligned}
&A\Psi = \lambda M\Psi,\\
&\Psi_{-N} = \Psi_N = 0.
\end{aligned} \tag{4}
$$

Let us recall that

$$\lambda_1(L,h) = \left(\frac{\pi}{2L}\right)^2 + O(h^2),$$

see **[31]**.

THEOREM 7.2. *There exists blowing up solutions for (3) if and only if $p > m$ or $p = m$ with $\lambda_1(L,h) < 1$. Moreover, in case $p = m$ with $\lambda_1(L,h) < 1$ every nontrivial solution blows up. In case $p > m$ and φ an initial data such that the solution u of problem 1 blows up then U also blows up for every h small enough.*

This theorem shows that the numerical scheme reproduces the blow-up cases in a very accurate way. Let us remark that $\lambda_1(L,h) \to \lambda_1(L)$ as $h \to 0$.

As a consequence of our blow-up results we can get bounds on $T_h - t$ in terms of $U(t)$. This allows us to prove that the numerical blow-up time T_h converges to T when h goes to zero.

COROLLARY 7.1. *Let φ be an initial datum for (1) such that u blows up, if we call T and T_h the blow-up times for u and u_h respectively, we have*

$$\lim_{h\to 0} T_h = T.$$

Now we describe the blow-up rate for the numerical scheme.

THEOREM 7.3. *Let $U(t)$ be a blowing up solution of (3) and T_h its blow-up time, then there exist two positive constants $C_i = C_i(h)$ such that*

$$C_1(T_h - t)^{-1/(p-1)} \le \|U\|_\infty(t) \le C_2(T_h - t)^{-1/(p-1)}.$$

Moreover, if $p > m$ there holds,

$$\lim_{t\nearrow T_h} (T_h - t)^{1/(p-1)} \|U\|_\infty(t) = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

We remark that this rate coincides with the blow-up rate of the continuous problem (1).

With the blow-up rate we can characterize the numerical blow-up set, that is the set of nodes x_k such that $u_k(t) \to +\infty$ when $t \nearrow T_h$.

THEOREM 7.4. *Let $U(t)$ be a blowing up solution of (3) which blows up at time T_h; then*

(a) *For $p > m$ the blow-up set is given by a finite number of nodes, i.e.,*

$$B(U) = [-Kh,\, Kh],$$

where $K \equiv K(p, m)$ is the only integer that verifies the following expression,

$$\frac{\sum_{i=0}^{K+1} m^i}{\sum_{i=0}^{K} m^i} < p \le \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}, \quad \textit{that is,} \quad K = \left[\frac{\ln((p-1)/(p-m))}{\ln(m)}\right].$$

(b) *For the case $p = m$ we obtain global blow-up, i.e.,*

$$B(U) = [-L, L].$$

We observe that in the case $p > m$ the blow-up set of the numerical solution can be larger than a single point, nevertheless, since K does not depend on h, we have that

$$B(U) = [-Kh, Kh] \to \{0\} = B(u), \qquad h \searrow 0.$$

On the other hand, when $p = m$, $B(U) = [-L, L]$ for every h. Therefore we find that regional blow-up is not possible for a numerical scheme with a fixed mesh. However we can recover regional blow-up by looking carefully at $U(t)$ in suitable self-similar variables. To do this we introduce the self-similar variables given by

$$
\begin{aligned}
y_k(s) &= (T_h - t)^{\frac{1}{p-1}} u_k(t), \\
(T_h - t) &= e^{-s}.
\end{aligned} \tag{5}
$$

In these new variables $Y = (y_k(s))$, problem (3) reads,

$$
\begin{aligned}
& y_{-N}(s) = e^{-\frac{1}{p-1}s}, \\
& y_k'(s) = \frac{1}{h^2} e^{\frac{m-p}{p-1}s}(y_{k+1}^m(s) - 2y_k^m(s) + y_{k-1}^m(s)) - \frac{1}{p-1} y_k(s) + y_k^p, \\
& y_N(s) = e^{-\frac{1}{p-1}s}.
\end{aligned} \tag{6}
$$

With this new variables $Y = (y_k(s))$, we have the following result.

THEOREM 7.5. *Let $p = m$ and $Y(s)$ given by (5), then as s goes to infinity we have that*

$$Y(s) \to W_h,$$

The stationary state W_h converges to the continuous profile, $z(x)$, as h goes to zero.

This result implies that W_h goes to zero outside the set $B(u)$ and we recover the blow-up set by looking at the behavior of $Y(s)$ for large s.

ORGANIZATION OF THE CHAPTER: first, in Section 1 we state and prove some auxiliary results that will be used in the rest of the chapter and also we prove the convergence result, Theorem 7.1. For expository reasons we concentrarte in the the case $p = m$ since this is the case where new phenomena appear (Section 2). The case $p > m$ can be handled in a very similar way as the case where $m = 1$ (heat equation) with $p > 1$ that was treated in Chapter 2. We do not include the proofs here because since they are similar to the ones in that chapter. Finally in Section 3 we present some numerical experiments. We leave for Section 4, at the end of the chapter some results about existence and uniqueness of the continuous profile $z(x)$.

1. Properties of the numerical scheme

In this section we collect some preliminary results for our numerical method. In particular we prove convergence for regular solutions.

First, we state a symmetry property for the numerical problem (3). We call a vector symmetric if verifies that $u_{-k} = u_k$.

LEMMA 7.1. *Let $U(0)$ be a symmetric vector then $U(t)$ is also symmetric for all $t \in (0, T_h)$.*

PROOF. The result follows from the uniqueness of problem (3). Indeed, let $\hat{U}(t)$ be defined as the vector such that

$$\hat{u}_i(t) = u_{-i}(t) \qquad i = -N, \cdots, N.$$

This vector is also a solution of (3) and at time $t = 0$ it is equal to $U(0)$. Therefore, by uniqueness, $U(t) = \hat{U}(t)$. □

REMARK 7.1. *Since $u_k(0) = \varphi(x_k)$ and φ is symmetric, then $U(0)$ (and therefore $U(t)$) is symmetric. So we can restrict ourselves to the half interval $[0, L]$ reducing the size of the system of ODEs to be solved.*

Now we want to prove a comparison Lemma. To do this we need the following definition,

DEFINITION. We will call $\overline{U}$ a supersolution if it satisfies

$$
\begin{aligned}
&\overline{u}_{-N}(t) \geq 1, \\
&\overline{u}_k'(t) \geq \frac{1}{h^2}(\overline{u}^m_{k+1}(t) - 2\overline{u}^m_k(t) + \overline{u}^m_{k-1}(t)) + \overline{u}^p_k(t), \\
&\overline{u}_N(t) \geq 1, \\
&\overline{u}_k(0) \geq \varphi(x_k), \qquad -N \leq k \leq N.
\end{aligned}
\tag{7}
$$

Analogously, we say that $\underline{U}$ is a subsolution if it satisfies (7) with the reverse inequalities.

LEMMA 7.2. *Let $\overline{U}$ and $\underline{U}$ be a superolution and a subsolution respectively, then*

$$\overline{U}(t) \geq U(t) \geq \underline{U}(t).$$

PROOF. By an approximation procedure we restrict ourselves to consider strict inequalities in (7). Let us prove that $\overline{U}(t) > U(t)$. We argue by contradiction. Let us assume that there exist a first time t_0 and a node j such that $\overline{u}_j(t_0) = u_j(t_0)$ then $-N+1 < j < N-1$,

$$0 \geq \overline{u}_j'(t_0) - u_j'(t_0) > \frac{1}{h^2}(\overline{u}_{j+1}^m(t_0) - u_{j+1}^m(t_0) + \overline{u}_{j-1}^m(t_0) - u_{j-1}^m(t_0)) \geq 0,$$

a contradiction. This contradiction proves that $\overline{U}(t) > U(t)$.

The inequality $U(t) \geq \underline{U}(t)$ can be handled in a similar way. □

Now we study the monotonicity properties of our numerical scheme in $[0, L]$.

LEMMA 7.3. *Let U be a solution of* (3) *with $u_k(0) > u_{k+1}(0)$, $k = 0, ..., N$ then*

$$u_k(t) > u_{k+1}(t), \qquad 0 \leq k \leq N-1.$$

PROOF. We argue by contradiction, let us assume there exists a first time t_0 and two consecutive nodes where the conclusion of the Lemma fails, let us call them $j, j+1$. So we are assuming that $u_{j+1}(t_0) = u_j(t_0)$. From the equations (3) we get

$$0 \geq u_j'(t_0) - u_{j+1}'(t_0) = \frac{1}{h^2}(u_{j-1}^m(t_0) - u_{j+2}^m(t_0)) \geq 0.$$

We conclude that $u_{j-1}(t_0) = u_{j+2}(t_0) = u_j(t_0)$. Using the same argument, we get that all the nodes must be equal to $u_N(t_0) = 1$ at time t_0 but this is impossible since the initial data verifies $U(0) \geq 1$ and hence the solution verifies $u_i(t) > 1$ for all positive t. Indeed, first we use the comparison Lemma to obtain that $u_i(t) \geq 1$ ($\underline{u}_i \equiv 1$ is a subsolution). Next, if all the nodes attain their minimum 1 at the same time t_0 then all of them have derivative less or equal than zero, but at that time t_0 we have $u_j'(t_0) = u_j^p(t_0) = 1 > 0$, and this is a contradiction. □

REMARK 7.2. *This ensures that the maximum of $U(t)$ is attained at the node $x_0 = 0$, i.e.,*

$$\max_k u_k(t) = u_0(t).$$

We will be use this fact in the following sections.

Now we prove our convergence result for regular solutions up to $t = T - \tau$.

THEOREM. *Let $u(x,t) \in C^{4,1}([-L, L] \times [0, T-\tau])$ be a positive solution of* (1) *and $U(t)$ the numerical approximation given by* (3).

Then, there exists a constant C, that depends on the $C^{4,1}([-L,L]\times[0,T-\tau])$ norm of u, such that for every h small enough it holds

$$\max_{t\in[0,T-\tau]}\max_{k}|u(x_k,t)-u_k(t)|\le Ch^2.$$

PROOF. In the course of this proof we will denote by C_i a constant independent of h which can be different in different occurrences.

If we rewrite the system (3) in terms of $Z=U^m$ we obtain

$$\begin{aligned}
&z_{-N}(t)=1,\\
&(z_k^{1/m})'(t)=\frac{1}{h^2}(z_{k+1}(t)-2z_k(t)+z_{k-1}(t))+z_k^{p/m}(t),\\
&z_N(t)=1,\\
&z_k(0)=\varphi^{1/m}(x_k)\ge 1,\qquad -N+1\le k\le N-1.
\end{aligned}$$

Let $v_k(t)=u^m(x_k,t)$ where u is the solution of the continuous problem (1). We define the error function as

$$e_k=z_k-v_k.$$

Let $t_0=\max\{t\in[0,T-\tau]:\,|e_k|(t)\le 1/2\}$. We perform the following calculations with $t\in[0,t_0]$ and we will prove at the end that $t_0=T-\tau$ for every h small enough.

The error function satisfies that, for $-N+1\le k\le N-1$,

$$\begin{aligned}
\frac{1}{m}z_k^{(1-m)/m}e_k' = {}& \frac{1}{h^2}(e_{k+1}-2e_k+e_{k-1})-\frac{1}{m}(z_k^{(1-m)/m}-v_k^{(1-m)/m})\,v_k'\\
&+z_k^{p/m}-v_k^{p/m}+C_1h^2\\
\le{}& \frac{1}{h^2}(e_{k+1}-2e_k+e_{k-1})+C_2\xi_k^{(1-2m)/m)}|e_k|\\
&+C_3\eta_k^{(p/m)-1}|e_k|+C_1h^2,
\end{aligned}$$

where ξ_k and η_k are intermediate values between z_k and v_k. Taking into account that there exists a constant, C, such that $1\le z_k(t)\le C$ for every $t\in[0,t_0]$ we have

$$e_k'\le\frac{C_1}{h^2}(e_{k+1}-2e_k+e_{k-1})+C_2|e_k|+C_3h^2. \tag{8}$$

The first and the last nodes verify

$$e_{-N}=e_N=0. \tag{9}$$

We remark that the system (8), (9), has a comparison principle, that can be proved as in Lemma 7.2. Let us look for a supersolution of the form

$$w_k(t) = g(t),$$

where $g(t)$ is a solution of

$$\begin{aligned} &g'(t) = C_1 g(t) + C_2 h^2, \\ &g(0) = C_3 h^2. \end{aligned}$$

That is

$$g(t) = h^2 \left(C_1 e^{C_2 t} + C_3 \right).$$

Therefore,

$$e_k(t) \le w_k(t) \le C_1 h^2 e^{C_2 T}.$$

Arguing in the same way with $-e_k$ we obtain

$$|e_k(t)| \le w_k(t) \le C_1 h^2 e^{C_2 T}.$$

From this inequality it is easy to see that $t_0 = T - \tau$ for every h small enough and the Theorem is proved. □

2. Case $p = m$

2.1. Blow-up for the numerical scheme. We consider the first eigenfunction Ψ which is a solution of (4) with $\lambda = \lambda_1(L, h)$, in particular, we choose $\Psi > 0$ such that $\langle M1, \Psi \rangle = 1$. Now, we multiply (3) by Ψ to obtain

$$\begin{aligned} \tfrac{d}{dt}\langle MU, \Psi\rangle = & \; -\langle AU^m, \Psi\rangle + \langle MU^m, \Psi\rangle \\ = & \; -\lambda_1(L,h)\langle MU^m, \Psi\rangle + \langle MU^m, \Psi\rangle \\ \ge & \; (1 - \lambda_1(L,h))\langle MU, \Psi\rangle^m. \end{aligned}$$

If $\lambda_1(L, h) < 1$ the solution U blows up in finite time and if $\lambda_1(L, h) \ge 1$ we have that $\langle MU, \Psi\rangle$ is non-increasing, and therefore bounded, hence U is global. □

Assume that $\lambda_1(L, h) < 1$, so U has finite time blow-up T_h. Integrating the last inequality between t and T_h we obtain a bound for $T_h - t$ in terms of the size of U. In fact we have that

$$(T_h - t) \le C(\langle MU, \Psi\rangle(t))^{-(m-1)}.$$

This allows us to prove the convergence of the blow-up times. Given $\varepsilon > 0$ let R be such that $CRM^{-(m-1)} \le \varepsilon/2$. As blow-up for u is regional we can choose a time t_0 such that

$$T - t_0 \le \varepsilon/2 \qquad \text{and} \qquad \int_{-L}^{L} u(s, t_0)\psi(s)\, ds \ge 2R,$$

where ψ is the first eigenfunction of the laplacian in $[-L, L]$. By our uniform convergence result up to $t = t_0 < T$, we have

$$\langle MU, \Psi\rangle(t_0) \to \int_{-L}^{L} u(s, t_0)\psi(s)\, ds, \qquad h \to 0.$$

Therefore for every h small enough, we have that

$$\langle MU, \Psi\rangle(t_0) > R.$$

Hence

$$|T - T_h| \leq |T - t_0| + |T_h - t_0| \leq \varepsilon/2 + C(\langle MU, \Psi\rangle(t_0))^{-(m-1)} \leq \varepsilon,$$

for every h small enough. This proves that $T_h \to T$ as $h \to 0$.

2.2. Blow-up rate. From our previous computations we have that

$$\langle MU, \Psi\rangle(t) \leq C(T_h - t)^{-1/(m-1)}.$$

Now we observe that there exists a constant $C = C(h)$ such that

$$\max_k u_k(t) = u_0(t) \leq C\langle MU, \Psi\rangle(t)$$

and we conclude that

$$u_0(t) \leq C(T_h - t)^{-1/(m-1)}.$$

On the other hand, as for all t the solution is symmetric and decreasing in $[0, L]$, we have that

$$u_0'(t) \leq u_0^m(t).$$

Therefore, integrating between t and T_h, we get

$$u_0(t) \geq C(T_h - t)^{-1/(m-1)}.$$

Summing up, we have proved that the blow-up rate is given by

$$C_1(T_h - t)^{-1/(m-1)} \leq \max_k u_k(t) = u_0(t) \leq C_2(T_h - t)^{-1/(m-1)}.$$

2.3. Blow-up set.

THEOREM 7.6. *If $p = m$ every node is a blow-up point. Moreover in the self-similar variables,*

$$Y(s) \to W_h, \qquad \textit{as} \quad s \to \infty,$$

where $W_h = (w_{-N}, ..., w_N)$ is a positive solution of the limit problem. Hence the asymptotic behavior of $u_k(t)$ is given by

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k.$$

Proof. In this case, if we write the numerical problem (6) in a form similar to (2), we have a Lyapunov functional. In fact,

$$\Phi(Y) = \frac{1}{2}\langle A^{1/2}Y^m, A^{1/2}Y^m\rangle + \frac{1}{2m}\langle MY^m, Y^m\rangle - \frac{1}{m^2-1}\langle MY^m, Y\rangle$$

satisfies

$$\frac{d}{ds}\Phi(Y)(s) = -\langle MY', Y^{m-1}Y'\rangle(s) \leq 0.$$

Hence, the orbit $Y(s)$ goes to a stationary state, see [**41**]. Therefore we study the limit problem of (6). By Remark 7.1 we can restrict ourselves to $[0, L]$ arriving to

$$\begin{aligned}
&0 = \frac{2}{h^2}(w_1^m - w_0^m) - \tfrac{1}{m-1}w_0 + w_0^m,\\
(10)\qquad &0 = \frac{1}{h^2}(w_{k+1}^m - 2w_k^m + w_{k-1}^m) - \tfrac{1}{m-1}w_k + w_k^m,\\
&0 = w_N.
\end{aligned}$$

Moreover, as we begin with a decreasing data $U(0)$ then for a fixed s the vector $Y(s)$ is positive and decreasing, hence we have to look to nonnegative and nondecreasing stationary solutions.

On the other hand, from the blow-up rate we know that $y_0(s) \geq C(h) > 0$, then W_h is not the trivial solution. We claim that w_k must be positive for all $0 < k < N$. To prove this claim just suppose that there exists some j with $w_j = 0$. Since W_h is non-negative and non-increasing we have that for $k \geq j$, $w_k = 0$ and therefore $w_k = 0$ for all k, but $w_0 > 0$. This contradiction proves the claim. $\square$

Now our goal is to recover regional blow-up by looking carefully at the behavior of the stationary solution as the parameter h goes to zero.

When h goes to zero we expect that $Z = W_h^m$ converges to a solution of the following problem

$$(11)\qquad \begin{cases}
z_{xx} - \tfrac{1}{m-1}z^{\frac{1}{m}} + z = 0, & x \in (0, L),\\
z_x(0) = 0,\\
z(L) = 0.
\end{cases}$$

This is the content of our next Lemma. We remark that since $m > 1$ a non Lipschitz function appears in (11).

Lemma 7.4. *Let W_h be the solution of* (10) *and let $z(x)$ be the unique solution of* (11). *Then*

$$Z = W_h^m \to z(x), \qquad \text{as } h \to 0.$$

PROOF. Multiplying the equation satisfied by Z by

$$\frac{(z_{k+1}-z_k)+(z_k-z_{k-1})}{2}$$

and summing we get

$$0=\frac{(z_N-z_{N-1})^2}{2h^2}-\frac{(z_1-z_0)^2}{2h^2}+\sum_{l=1}^{N}\left(z_l-\frac{1}{m-1}z_l^{\frac{1}{m}}\right)\frac{(z_{l+1}-z_{l-1})}{2}.$$

Hence

$$\begin{aligned}0= & \frac{(z_N-z_{N-1})^2}{2h^2}-\frac{(z_1-z_0)^2}{2h^2}\\ + & \left(\frac{z_N^2}{2}-\frac{m}{m^2-1}z_N^{(1+m)/m}\right)-\left(\frac{z_0^2}{2}-\frac{m}{m^2-1}z_0^{(1+m)/m}\right)+O(h).\end{aligned}$$

Using the first and the last equations of (10) we get that z_{N-1} and z_0 must verify the following equation,

$$0=\frac{z_{N-1}^2}{2h^2}-\frac{m}{m^2-1}z_0^{\frac{1+m}{m}}+\frac{z_0^2}{2}+O(h). \tag{12}$$

On the other hand, we know that z_0 is bounded. Therefore, z_{N-1}/h must be bounded. Then we can take a subsequence of h such that $z_{N-1}/h\to\Gamma\geq 0$.

If $\Gamma>0$ we consider the auxiliary initial value problem

$$\begin{aligned}&z''=\tfrac{1}{m-1}z^{1/m}-z, \qquad x\in[0,L],\\ &z(L)=0,\\ &z'(L)=-\Gamma.\end{aligned} \tag{13}$$

For this problem we define the energy as

$$E_z(x)=\frac{(z'(x))^2}{2}-\frac{m}{m^2-1}z(x)^{(m+1)/m}+\frac{z(x)^2}{2}.$$

Multiplying equation (13) by z' and integrating we obtain that it is conservative, i.e. $E_z(x)=E_z(L)=-\Gamma/2$.

Problem (13) has a unique solution and by classical theory (see [**48**]) in the interval where $z(x)$ is positive

$$Z=W_h^m\to z(x), \qquad \text{as } h\to 0.$$

Notice that $z(x)$ is positive in $[L-\delta,L)$. Since the vector Z is decreasing, we have that $z(x)$ is non-increasing in all interval $[0,L]$. Therefore $z(x)$ is positive in $[0,L)$. So, for all $x\in[0,L)$, $Z\to z(x)$.

Then, as the problem is conservative and (12) holds, we get that $z(0)$ is the only positive root of

$$\frac{\Gamma^2}{2} - \frac{m}{m^2-1} z(0)^{(m+1)/m} + \frac{z(0)^2}{2} = 0,$$

and

$$z'(0) = 0.$$

So, the constant Γ must be the only constant such that z is a solution of

$$\begin{array}{ll} z'' = \frac{1}{m-1} z^{1/m} - z, & x \in [0, L], \\ z'(0) = 0, & \\ z(L) = 0, & \\ z(x) > 0, & x \in [0, L). \end{array}$$

This problem has a solution if and only if $\frac{\pi}{2} < L \leq \frac{m\pi}{m-1}$, we prefer to postpone the proof of this fact until the last section of this chapter. Hence, if $L > \frac{m\pi}{m-1}$ the parameter Γ must be zero.

Now, we assume that $\Gamma = 0$. Then $E_z(L) = 0$ and by (12) we obtain that $z_0 \to A$, where A is the only positive root of

$$0 = \frac{1}{2} A^2 - \frac{m}{m^2-1} A^{\frac{1+m}{m}}$$

and that

$$z'(0) = 0.$$

Hence, we consider the problem

$$\begin{array}{ll} z'' = \frac{1}{m-1} z^{1/m} - z & x \in [0, L], \\ z(0) = A, & \\ z'(0) = 0, & \\ z'(x) \leq 0 & x \in [0, L]. \end{array}$$

This problem has a unique explicit solution given by

$$z(x) = \begin{cases} \left[\dfrac{2m}{m^2-1} \cos^2 \left(\dfrac{(m-1)x}{2m} \right) \right]^{\frac{m}{m-1}} & x < \dfrac{m\pi}{m-1}, \\ 0 & x \geq \dfrac{m\pi}{m-1}. \end{cases}$$

Then if $L < \frac{m\pi}{m-1}$ the parameter Γ must be positive.

On the other hand, by classical theory we have that in the interval where $z(x)$ is positive, $[0, m\pi/(m-1))$,

$$Z = W_h^m \to z(x), \qquad \text{as } h \to 0.$$

Since Z is decreasing, we have that for h small enough,

$$z_k \le z\left(\frac{m\pi}{m-1} - \delta\right) \le \varepsilon, \qquad \forall\, x_k > \frac{m\pi}{m-1} - \delta.$$

Then, $Z = W_h^m \to z(x)$ in all the interval $[0, L]$. □

REMARK 7.3. *As a consequence of Lemma 7.4 if* $L > \frac{m\pi}{m-1}$ *we have that*

$$W_h \to 0, \qquad in \ \left[\frac{m\pi}{m-1}, L\right],$$

and we recover the regional blow-up in the sense that the constants that appear in the blow-up rate for the nodes that lie in $[\frac{m\pi}{m-1}, L]$ *go to zero as* h *goes to zero, i.e.,*

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k$$

with $w_k \to 0$ *as* $h \to 0$ *for every* k *such that* $x_k \in [\frac{m\pi}{m-1}, L]$.

3. Numerical experiments

In this section we present some numerical experiments that illustrate our results. For the numerical experiments we use an adaptive ODE solver. In all cases we take $L = 15$ and the initial data $u_0(x) = L^2 - x^2 + 1$.

First we deal with $p = m = 1.5$ and we find that the numerical blow-up time is given by $T_h = 0.1382$. Our numerical results are shown in figures 1 and 2. Figure 1 shows the evolution of the numerical solution. Figure 2 shows the profile of the solution in self-similar variables near T_h, we also put in the same picture the continuous self-similar profile $z(x)$. From this last picture one can appreciate that the numerical calculations show that the blow-up set is $(-9.4248, 9.4248) \approx (-3\pi, 3\pi)$.

Finally, we consider the case $p = 2$, $m = 1.5$. In this case, $T_h = 0.004432$. In Figure 3 one can see the evolution of the solution. In order to obtain the numerical blow-up rates, in Figure 4 we display $\ln(u_i)$ versus $-\ln(T_h - t)$ for $i = 0, 1, 2$. We can appreciate that the first curve (corresponding to $\max u_k = u_0$) has slope 1, the second (for u_1) has slope $1/2$ (it becomes parallel to the dotted line which has slope $1/2$) and the last one (for u_2) is flat. These behaviors correspond to the expected rates $u_0(t) \sim (T_h - t)^{-1}$, $u_1(t) \sim (T_h - t)^{-1/2}$ and u_2 bounded.

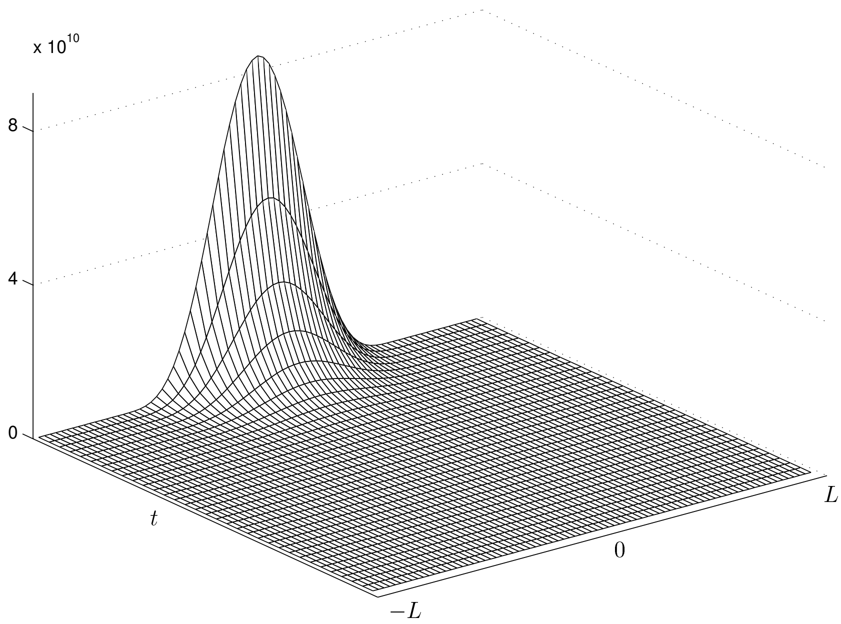

FIGURE 1. CASE $p = m$. EVOLUTION OF THE NUMERICAL SOLUTION.

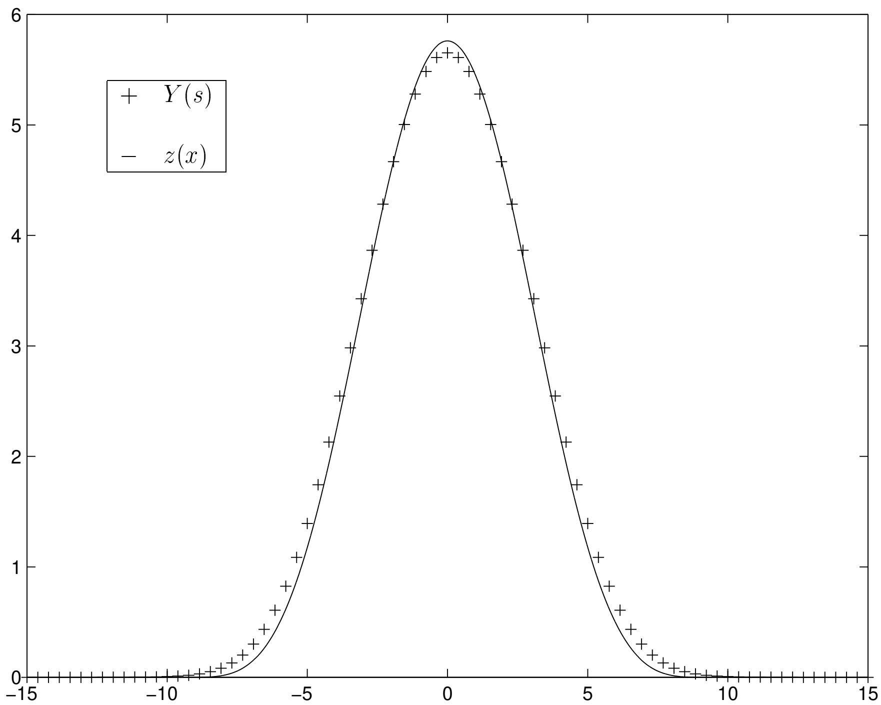

FIGURE 2. CASE $p = m$. SELF-SIMILAR PROFILE.

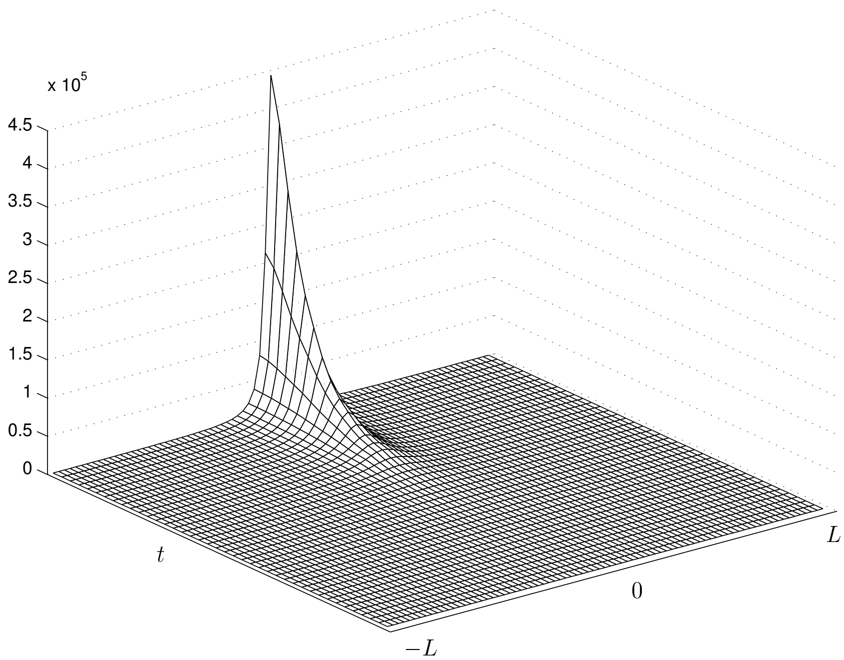

FIGURE 3. EVOLUTION OF THE NUMERICAL SOLUTION $(p > m)$.

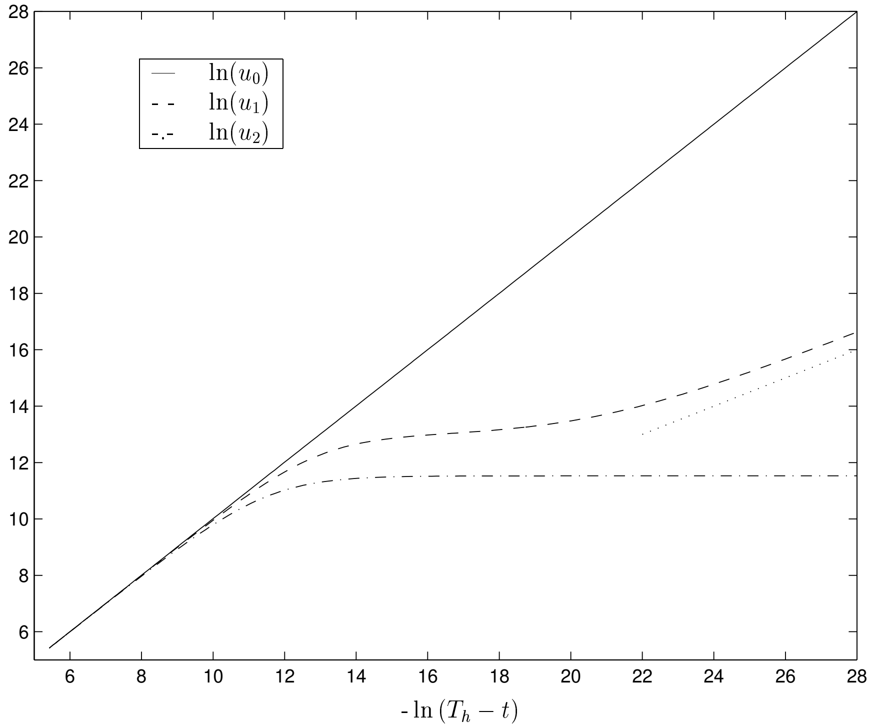

FIGURE 4. BLOW-UP RATES $(p > m)$.

4. A boundary value problem

In order to complete the proof of Lemma 7.4 we look for nonnegative monotone solutions of the following boundary value problem.

$$\begin{array}{ll} (f^m)'' = \frac{1}{m-1} f - f^m, & x \in [0, L], \\ f'(0) = 0, & f(L) = 0. \end{array} \tag{14}$$

We note that if $f(x)$ is a solution of this problem then

$$u(x,t) = (T-t)^{-1/(m-1)} f(x)$$

is a solution of

$$\begin{array}{ll} u_t = (u^m)_{xx} + u^m, & (x,t) \in (0,L) \times [0,T), \\ (u^m)_x(0,t) = 0, & t \in [0,T), \\ u(L,t) = 0, & t \in [0,T), \\ u(0,t) = T^{-1/(m-1)} f(x). & \end{array}$$

This problem has been studied in **[72]** and we know that for all positive initial data

(i) $u(x,t)$ blows up if and only if $L > \pi/2$.

(ii) The blow rate is given by $\|u(x,t)\|_\infty \sim (T-t)^{-1/(m-1)}$.

On the other hand, following a standard technique we introduce the self similar variables

$$v(x,\tau) = (T-t)^{\frac{1}{m-1}} u(x,t), \qquad \tau = -\ln(T-t).$$

This function $v(x,\tau)$ satisfies the following problem,

$$\begin{array}{ll} v_\tau = (v^m)_{xx} - \frac{1}{m-1} v + v^m, & (x,\tau) \in (0,L) \times (0,+\infty), \\ (v^m)_x(0,\tau) = 0, & \tau \in (0,+\infty), \\ v(L,\tau) = 0, & \tau \in (0,+\infty), \\ v(x,0) = T^{\frac{1}{m-1}} u(x,0), & x \in (0,L). \end{array} \tag{15}$$

We remark that the stationary solutions of (15) are solutions of (14).

Now, we prove that $v(x,\tau)$ converges (in terms of ω-limits) to a stationary state. For this we multiply the equation by $(v^m)_\tau$ and integrate with respect to x to obtain

$$\int_0^L (v^m)_\tau v_\tau = -\frac{d}{dt} F(v),$$

where

$$F(v) = \int_0^L \frac{(v^m)_x^2}{2}\, dx + \frac{m}{m^2-1} \int_0^L v^{m+1}\, dx - \frac{1}{2} \int_0^L v^{2m}\, dx.$$

Hence $F(v)$ is a Lyapunov functional for problem (15) and we can conclude that the $\omega-$limit set of $v(x,\tau)$ consists of nontrivial stationary solutions of (15).

LEMMA 7.5. *Let $f(x)$ be a solution of* (14). *Then,*

(i) if $L \leq \pi/2$ no solution exists,

(ii) if $\pi/2 < L < (m\pi)/(m-1)$ there exists a unique positive solution and

(iii) if $L \geq (m\pi)/(m-1)$ the unique monotone solution is given by

$$f(x) = \begin{cases} \left[\frac{2m}{m^2-1}\cos^2\left(\frac{(m-1)x}{2m}\right)\right]^{\frac{1}{m-1}} & x < m\pi/(m-1), \\ 0 & x \geq m\pi/(m-1). \end{cases}$$

PROOF. From the previous argument we have that problem (14) has a solution if and only if $L > \pi/2$.

In order to prove the uniqueness we consider two different solutions of (14), $f_1(x)$ and $f_2(x)$. These profiles must have some intersections. If this is not the case, they are ordered and hence the corresponding solutions in variables (x,t) do not have the same blow-up time. Since the energy

$$E_f(x) = \frac{(f'(x))^2}{2} - \frac{m}{m^2-1}f(x)^{(m+1)/m} + \frac{f(x)^2}{2}$$

is a nonnegative constant function, we obtain that two different profiles have only one intersection point. Moreover, $f_1'(L) < f_2'(L) \leq 0$ if and only if $f_1(0) > f_2(0)$, which is a contradiction with the fact that they have only one intersection point. Finally we remark that, as the energy must be nonnegative, we have

$$f(0) \geq A = \left(\frac{2m}{m^2-1}\right)^{\frac{1}{m-1}},$$

and for each $f(0) \geq A$ there exists a unique $L \in (\pi/2, (m\pi)/(m-1)]$ such that $f(x)$ is a positive solution of (14). For $L > (m\pi)/(m-1)$ we extend by zero for all $x \in [(m\pi)/(m-1), L]$. □