Part 1

Problems with a nonlinear source

CHAPTER 2

The heat equation with a source in one space dimension

One of the simplest equations with blow-up is the heat equation with a source u^p, so we begin our analysis of numerical approximations with the following parabolic problem

$$
\begin{array}{rrcll}
(1) & u_t(x,t) & = & u_{xx}(x,t) + u^p(x,t) & \text{in } (0,1)\times[0,T), \\
 & u(1,t) & = & u(0,t) = 0 & \text{on } [0,T), \\
 & u(x,0) & = & u_0(x) \geq 0 & \text{on } [0,1].
\end{array}
$$

We assume that u_0 is nontrivial, smooth and verifies $u_0(0) = u_0(1) = 0$ in order to guarantee that $u \in C^{2,1}$. We also assume $p > 1$.

As we mention in the introduction, the solution may develop singularities in finite time, no matter how smooth u_0 is.

For this problem, the blow up set is localized at single points, that is, there exist $x_1, .., x_k$ such that $u(x,t)$ remains bounded up to T for every $x \neq \{x_1, ..., x_k\}$ (see [**15**] and also [**32, 58, 62, 77**]). The blow-up rate at these blow-up points is given by $u(x_i,t) \sim (T-t)^{-\frac{1}{p-1}}$ (see [**39, 42, 43**]).

For previous work on numerical approximations of blow-up problems like (1) we refer to [**1, 2, 8, 9, 11, 14, 16, 51, 60**] and references therein.

In [**1**] and [**2**] the authors analyze a semidiscrete scheme (keeping t continuous). They find a necessary condition for the appearance of the blow-up phenomenon ($p > 1$ and some assumptions on the initial data) and prove the convergence of the blow up time of the discrete problem to that of the continuous one when the mesh parameter goes to zero (see also [**8**]).

Here we introduce the same semidiscrete scheme analyzed there by using piecewise linear finite elements with mass lumping in a uniform mesh for the space variable; it is known that this discretization coincides with the classic central finite difference second order scheme.

We denote with $U(t) = (u_1(t), ..., u_{N+1}(t))$ the values of the numerical approximation at the nodes $x_i = (i-1)h$ at time t ($h = 1/N$). Then $U(t)$ verifies the following equation:

$$
\begin{aligned}
&MU'(t) = -AU(t) + MU^p(t) \\
&U(0) = u_0^I,
\end{aligned}
\tag{2}
$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial datum, u_0. Writing this equation explicitly we obtain the following ODE system,

$$
\begin{aligned}
&u_1(t) = 0, \\
&u_k'(t) = \frac{1}{h^2}(u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)) + u_k^p(t), \qquad 2 \le k \le N, \\
&u_{N+1}(t) = 0, \\
&u_k(0) = u_0(x_k), \qquad 1 \le k \le N+1.
\end{aligned}
\tag{3}
$$

In **[2]** it is proved that this method converges uniformly under the hypothesis that $u \in C^{4,1}$. Under this assumption the authors find that

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \le Ch^2.$$

We start the analysis of (3) with another convergence theorem for regular solutions that uses different techniques. It only assumes $u \in C^{2,1}$ but the convergence rate is not optimal (we have $h^{\frac{3}{2}}$ and not h^2 like in **[2]**).

THEOREM. *Let u be a regular solution of* (1) *($u \in C^{2,1}([0,1] \times [0,T-\tau])$ and u_h the numerical approximation given by* (3)*, then there exists a constant C depending on $\|u\|$ in $C^{2,1}([0,1]\times[0,T-\tau])$ such that*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \le Ch^{\frac{3}{2}}.$$

For this scheme we say that a solution has finite time blow-up if there exists a finite time T_h with

$$\lim_{t\to T_h} \|U(t)\|_\infty = \lim_{t\to T_h} \max_j u_j(t) = +\infty.$$

Next we describe when the blow-up phenomena occurs for (3).

THEOREM. *Positive solutions of* (3) *blow up in finite time if $p > 1$ and $U(0)$ is large in the following sense; let*

$$\Phi_h(U) \equiv \frac{1}{2}\langle A^{1/2}U, A^{1/2}U\rangle - \frac{1}{p+1}\langle MU^p, U\rangle,$$

then, if there exists t_0 such that $\Phi_h(U(t_0)) < 0$, u_h has finite time blow-up. Moreover, there exists a constant C that does not depends on h such that

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

We remark that the blow-up condition for the numerical scheme, $p > 1$ and $\Phi_h(U(t_0)) < 0$, and for the continuous problem are analogous (see [**7, 21**]). In [**2**] it is proved that if $p > 1$ there exist solutions of (3) that blow up in finite time under different hypotheses on the solution u_h. It is also proved there that, under some assumptions over u_h (symmetry or monotonicity in time), the numerical blow-up time, T_h, converges to the continuous one T when the mesh parameter goes to zero.

As a corollary of the above theorem we show that if u blows up then u_h also blows up for every h small enough and we extend the convergence of the blow-up times to solutions without symmetry or monotonicity assumptions.

In Chapter 5, using a different approach we will obtain a bound for the order of convergence of T_h to T.

Next we arrive at the main point of this chapter, the asymptotic behavior (blow-up rate) and the localization of blow-up points (blow-up set) of u_h for fixed h.

Concerning the blow-up rate for (3) we have the following Theorem,

THEOREM. *Assume that $p > 1$ and that u_h blows up in finite time, T_h, then*

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}},$$

in the sense that there exists two positive constants c, C such that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}.$$

Moreover,

$$\lim_{t \to T_h} \max_j u_j(t)(T_h - t)^{-\frac{1}{p-1}} = C_p,$$

where

$$C_p = (p-1)^{-\frac{1}{p-1}}.$$

We have to remark that the constant C_p that appears in this theorem is the same that appears in the ODE $u'(t) = u^p(t)$ that has solutions of the form $u(t) = C_p(T-t)^{-\frac{1}{p-1}}$. Also we remark that the asymptotic behavior is the same for the continuous problem (1). In fact, it holds $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty = C_p$ (see [**39, 42, 43**]).

Now we turn our attention to the blow-up set for u_h, $B(u_h)$. Let $B^*(u_h)$ be the set of nodes x_j such that $\lim_{t\to T_h}(T_h-t)^{\frac{1}{p-1}}u_j(t) = C_p$. Clearly, $B^*(u_h) \subset B(u_h)$. With the blow-up rate we observe a propagation property of blow-up points, we prove that the number of nodes adjacent to $B^*(u_h)$ that go to infinity (i.e. blow-up points for u_h) is finite and determined by p. We remark that in the continuous case the blow-up set is composed by single points, see [**15**] and also [**32, 58**]).

THEOREM. *Let $B^*(u_h)$ be the set of nodes, x_n, such that*

$$\lim_{t\to T_h} u_n(t)(T_h-t)^{\frac{1}{p-1}} = C_p$$

Then the blow-up propagates in the following way: let $p > 1$ and $K \in \mathbb{N}$ satisfies $\frac{K+2}{K+1} < p \leq \frac{K+1}{K}$. We call $d(i)$ the distance from x_i to $B^*(u_h)$ measured in nodes. Then the solution of (3)*, u_h, blows up exactly at K nodes near $B^*(u_h)$,*

$$u_i(t) \to +\infty \qquad \Longleftrightarrow \qquad d(i) \leq K.$$

Moreover, if $d(i) \leq K$, the asymptotic behavior is given by

$$u_i(t) \sim (T_h-t)^{-\frac{1}{p-1}+d(i)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(i) = K$

$$u_i(t) \sim \ln(T_h - t).$$

We want to remark that more than one node can go to infinity but the asymptotic behavior imposes $\frac{u_j(t)}{u_i(t)} \to 0$ $(t \to T_h)$ if $d(i) < d(j)$. This propagation property to nodes that lie at distance one in the symmetric case was first proved in [**16**] and [**63**].

In the blow-up case, $p > 1$, the number of blow-up points outside $B^*(u_h)$ is finite and depends on the power p but is independent of h. This gives a sort of "numerical localization" of the blow-up set of u_h near the blow-up set of u when the parameter h is small enough.

THEOREM. *Let u_0 be an initial datum for* (1) *such that u blows up, then if we call $B(u)$ and $B(u_h)$ the blow-up sets for u and u_h respectively, we have that, given $\varepsilon > 0$ there exists h_0 such that for every*

$0 < h \leq h_0$,

$$B(u_h) \subset B(u) + (-\varepsilon, \varepsilon) \qquad \forall h \leq h_0(\varepsilon).$$

Moreover, if u_0 is symmetric and increasing in $[0, 1/2]$ we have that

$$B(u_h) = \{1/2\} + [-Kh, Kh] \qquad \forall h \leq h_0.$$

We want to remark that these results show that the numerical scheme (3) has asymptotic properties that are similar to that of the continuous problem (1) when the mesh parameter is small.

1. Convergence of the numerical scheme

In this section we prove the uniform convergence result for regular solutions of the numerical scheme (3).

For any $\tau > 0$ we want that $u_h \to u$ (when $h \to 0$) uniformly in $[0, T - \tau]$. This is a natural requirement since on such an interval the exact solution is regular. Approximations of regular problems like this have been analyzed in [**2**]. In that paper an error estimate of order h^2 in the L^∞ norm is proved under the hypothesis $u \in C^{4,1}$.

In particular, uniform convergence can be obtained by using standard inverse inequalities. In the following theorem we give a proof of the L^2 convergence for a problem like (1) with $f(u) = u^p$ replaced by a globally Lipschitz function $g(u)$ and considering mass lumping. As a corollary, we will obtain uniform convergence for problem (1). We use ideas from [**22**].

THEOREM 2.1. *Let u be the solution of a problem like* (1) *with $f(u) = u^p$ replaced by a globally Lipschitz function $g(u)$ and let u_h its semidiscrete approximation obtained by finite elements with mass lumping. If $u \in C^{2,1}([0,1] \times [0, T_1])$ for some $T_1 > 0$ then, there exists a constant C depending on $\|u\|$ in $C^{2,1}$ and T_1 such that*

$$\|u - u_h\|_{L^\infty([0,T_1],L^2)} \leq Ch^2.$$

PROOF. In this proof we use the notation $L^2 = L^2((0,1))$ that refers to the L^2 norm in the x variable for each t (we will use analogous notations for other norms below) and u' for the derivative respect to time, u_t.

As u is a solution of (1) it satisfies

$$\int_0^1 u'v + \int_0^1 u_x v_x = \int_0^1 g(u)v \qquad \forall v \in H_0^1.$$

The numerical scheme (3) is equivalent to

$$\int_0^1 ((u_h)'v)^I + \int_0^1 u_x v_x = \int_0^1 (g(u_h)v)^I \qquad \forall v \in V_h.$$

Hence we have that $e = u - u_h$ satisfies the following error equation,

$$\int_0^1 (e'v)^I\,dx + \int_0^1 e_x v_x\,dx = \int_0^1 (g(u)v - (g(u_h)v)^I) + \int_0^1 ((u'v)^I - u'v)\,dx$$

for all $v \in V_h$. Writing

$$e = u - u_h + u^I - u^I = u - u^I + \eta$$

and using known error estimates for Lagrange interpolation it rests to estimate $\eta = u^I - u_h$.

First, it is easy to see that,

$$\int_0^1 (u - u^I)_x v_x = 0 \qquad \forall v \in V_h,$$

and therefore, replacing in the error equation we have an equation for η,

$$\int_0^1 (\eta'v)^I + \int_0^1 \eta_x v_x = \int_0^1 (g(u)v - (g(u_h)v)^I) +$$
$$\int_0^1 ((u'v)^I - u'v) - \int_0^1 ((u' - (u^I)')v)^I \qquad \forall v \in V_h.$$

In particular if we choose $v = \eta \in V_h$ we obtain

$$\frac{1}{2}\frac{d}{dt}(\int_0^1 (\eta^2)^I) + \int_0^1 (\eta_x)^2 =$$
$$\int_0^1 (g(u)\eta - (g(u_h)\eta)^I) + \int_0^1 ((u'\eta)^I - u'\eta) =$$
$$= I + II.$$

First, let us estimate I.

$$|I| = \left|\int_0^1 (g(u)\eta - (g(u_h)\eta)^I)\right| =$$
$$\left|\int_0^1 g(u)\eta - (g(u^I)\eta)^I + (g(u^I)\eta)^I - (g(u_h)\eta)^I)\right| =$$
$$\left|\int_0^1 (g(u) - g(u^I))\eta + ((g(u^I) - g(u_h))\eta)^I + g(u^I)\eta - (g(u^I)\eta)^I)\right| \le$$

using that g is Lipschitz,

$$C\int_0^1 |u - u^I||\eta| + C\int_0^1 (\eta^2)^I + \int_0^1 \left|g(u^I)\eta - (g(u^I)\eta)^I\right| \le$$

$$C\|u-u^I\|_{L^2}^2 + C\|\eta\|_{L^2}^2 + \int_0^1 \left|g(u^I)\eta - (g(u^I)\eta)^I\right| \leq$$
$$Ch^4 + C\|\eta\|_{L^2}^2 + \int_0^1 \left|g(u^I)\eta - (g(u^I)\eta)^I\right|.$$

So it rests to estimate

$$\int_0^1 \left|g(u^I)\eta - (g(u^I)\eta)^I\right|.$$

For each subinterval I_j of the partition we know that,

$$\|g(u^I)\eta) - (g(u^I)\eta)^I\|_{L^1(I_j)} \leq Ch^2\|(g(u^I)\eta)_{xx}\|_{L^1(I_j)} \leq$$
$$Ch^2\|(g(u^I))_x\eta_x\|_{L^1(I_j)} + Ch^2\|(g(u^I))_{xx}\eta\|_{L^1(I_j)}$$

because u^I and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|(u^I)_x\|_{L^2} \leq C\|u\|_{H^1}$ we obtain,

$$\int_0^1 \left|g(u^I)\eta - (g(u^I)\eta)^I\right| \leq Ch^2\int_0^1 |(g(u^I))_x||\eta_x| + Ch^2\int_0^1 |(g(u^I))_{xx}||\eta|$$
$$\leq Ch^4 + \varepsilon\|\eta_x\|_{L^2}^2 + C\|\eta\|_{L^2}^2.$$

Since ε will be fixed later on, we write C instead of C_ε. The constant C depends on $\|u\|$ in $C^{2,1}$.

In order to bound II we decompose it in the following form,

$$II = \int_0^1 ((u'\eta)^I - u'\eta)\,dx = \int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx + \int_0^1 ((u')^I\eta - u'\eta)\,dx.$$

We proceed as before, for each subinterval I_j of the partition we know that,

$$\|((u')^I\eta)^I - (u')^I\eta\|_{L^1(I_j)} \leq Ch^2\|((u')^I\eta)_{xx}\|_{L^1(I_j)} \leq \|((u')^I)_x\eta_x\|_{L^1(I_j)}$$

because $(u')^I$ and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|((u')^I)_x\|_{L^2} \leq C\|u'\|_{H^1}$ we obtain,

$$\int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx \leq Ch^2\int_0^1 |((u')^I)_x||\eta_x|\,dx$$
$$\leq Ch^2\|((u')^I)_x\|_{L^2((0,1))}\|\eta_x\|_{L^2} \leq Ch^4 + \varepsilon\|\eta_x\|_{L^2}^2.$$

It rests to estimate the second term of II. We have,

$$\int_0^1 ((u')^I - u')\eta\,dx \leq \|(u')^I - u'\|_{L^2}\|\eta\|_{L^2}$$
$$\leq \|(u')^I - u'\|_{L^2}^2 + \|\eta\|_{L^2}^2 \leq Ch^4 + \|\eta\|_{L^2}^2.$$

Collecting all the bounds we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + \int_0^1 |\eta_x|^2 \leq Ch^4 + C\|\eta\|_{L^2}^2 + 3\varepsilon\|\eta_x\|_{L^2}^2.$$

We choose ε such that $3\varepsilon = 1/2$ and we obtain,

$$\frac{d}{dt}\int_0^1 (\eta^2)^I + \int_0^1 |\eta_x|^2 \leq Ch^4 + C\|\eta\|_{L^2}^2.$$

Since $\int_0^1 (\eta^2)^I\, dx \sim \|\eta\|_{L^2}^2$ we can apply Gronwall's Lemma to obtain for $t \in [0, T_1]$,

$$\|\eta(t)\|_{L^2} + (\int_0^{T_1} \|\eta_x\|_{L^2}^2\, dt)^{1/2} \leq Ce^{C(T_1)}h^2.$$

In particular,

$$\|\eta\|_{L^2} \leq C(u, T_1)h^2.$$

and hence,

$$\|e\|_{L^2} \leq \|u - u^I\|_{L^2} + \|\eta\|_{L^2} \leq C(u, T_1)h^2. \quad \square$$

As a corollary of this theorem we can prove the convergence result.

THEOREM 2.2. *Let u be a regular solution of* (1) *($u \in C^{2,1}([0,1] \times [0, T-\tau])$ and u_h the numerical approximation given by* (3) *then there exists a constant C depending on $\|u\|$ in $C^{2,1}([0,1] \times [0, T-\tau])$ such that*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{\frac{3}{2}}.$$

PROOF. It is known that before the blow up time u is regular, more precisely, $u \in C^{2,1}([0,1] \times [0, T-\tau])$. Let $g(u)$ be a globally Lipschitz function which agrees with $f(u) = u^p$ for $u \leq 2M$ where $M = \|u\|_{L^\infty([0,1]\times[0,T-\tau])}$. Let $\overline{u}$ and $\overline{u}_h$ be the exact and approximate solutions of a problem like (1) with $f(u) = u^p$ replaced by $g(u)$. By uniqueness $u = \overline{u}$ in $[0,1] \times [0, T-\tau]$. A bound for $\|\overline{u} - \overline{u}_h\|_{L^\infty}$ can be obtained from Theorem 2.1. Indeed, it is enough to bound $\|\overline{u}^I - \overline{u}_h\|_{L^\infty}$, and using a standard inverse inequality (see **[19]**) we have,

$$\|\overline{u}^I - \overline{u}_h\|_{L^\infty} \leq Ch^{-\frac{1}{2}}\|\overline{u}^I - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}$$

$$\leq Ch^{-\frac{1}{2}}\{\|\overline{u}^I - \overline{u}\|_{L^\infty([0,T-\tau],L^2)} + C\|\overline{u} - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}\} \leq Ch^{\frac{3}{2}}$$

with C depending on u and the constant in Theorem 2.1 and so on τ.

Consequently, for h small enough $|\overline{u}_h| \leq 2M$. Therefore $u_h^p = f(\overline{u}_h) = g(\overline{u}_h)$ and so $\overline{u}_h$ is the finite element approximation of u and, by uniqueness $\overline{u}_h = u_h$ which concludes the proof. $\square$

2. Blow-up for the numerical scheme

In this section we find a condition that ensures the presence of blow-up in the discrete problem. We begin with a lemma that proves a condition to guarantee unbounded numerical solutions. Next we prove that unbounded solutions must blow up. As a corollary, from the proof of this last fact it can be deduced the convergence of the numerical blow-up times to the continuous one.

LEMMA 2.1. *If $U(t_0)$ verifies that*

$$\Phi_h(U(t_0)) \equiv \frac{1}{2}\langle A^{1/2}U(t_0), A^{1/2}U(t_0)\rangle - \sum_{i=1}^{N+1} m_{ii}\frac{((U(t_0))_i)^{p+1}}{p+1} < 0,$$

then u_h is unbounded and hence $\lim_{t\nearrow T_h} \max u_j(t) = +\infty$. Here

$$T_h = \max\{t \text{ such that } u_h(s) \text{ is defined for } s \in [0,t]\}.$$

Proof: To motivate the proof, let

$$\Phi(u)(t) \equiv \int_0^1 \frac{(u_x(s,t))^2}{2}\, ds - \int_0^1 \frac{(u(s,t))^{p+1}}{p+1}\, ds$$

then, Φ is a Lyapunov functional for (1) and if $\Phi(u(\cdot,t_0)) < 0$ then u blows up in finite time (see [**7**]). The discrete analogous of Φ is

$$\Phi_h(U)(t) \equiv \frac{1}{2}\langle A^{1/2}U(t), A^{1/2}U(t)\rangle - \sum_{i=1}^{N+1} m_{ii}\frac{(U_i)^{p+1}(t)}{p+1}.$$

Now let us compute the derivative of $\Phi_h(U)(t)$.

$$\frac{d}{dt}\Phi_h(U)(t) = \langle A^{1/2}U(t), A^{1/2}U'(t)\rangle - \sum_{i=1}^{N+1} m_{ii}(u_i)^p u_i'(t) =$$

$$\langle AU(t) - MU^p(t), U'(t)\rangle = -\langle MU'(t), U'(t)\rangle$$

Hence, this Φ_h is a Lyapunov functional for (2) in the sense that

$$\frac{d}{dt}\Phi_h(U) \le 0.$$

Moreover $\frac{d}{dt}\Phi_h(U) < 0$ unless U is independent of t.

Now, let us see that the steady states of (3) have positive "energy" (i.e. $\Phi_h(W) \ge 0$). Let $W = (w_1, ..., w_N)$ be a stationary solution of (2), then we have

$$0 = -AW + M(W)^p. \tag{4}$$

Multiplying (4) by W, we obtain

$$0 = -\frac{1}{2}\langle A^{1/2}W, A^{1/2}W\rangle + \frac{p+1}{2}\sum_{i=1}^{N} m_{ii}\frac{(w_i)^p w_i}{p+1} \geq$$

$$\geq -\frac{1}{2}\langle A^{1/2}W, A^{1/2}W\rangle + \sum_{i=1}^{N} m_{ii}\frac{(w_i)^{p+1}}{p+1} = -\Phi_h(W).$$

Then, as every global solution that is bounded must converge to a stationary one (see [**41**]), if $U(t_0)$ satisfies $\Phi_h(U(t_0)) < 0$ it must be unbounded. $\square$

Now we are ready to prove the following result

THEOREM 2.3. *Positive solutions of* (3) *blow up in finite time if* $p > 1$ *and* $U(0)$ *is large in the following sense; let*

$$\Phi_h(U) \equiv \frac{1}{2}\langle A^{1/2}U, A^{1/2}U\rangle - \sum_{i=1}^{N} m_{ii}\frac{U_i^{p+1}}{p+1},$$

then, if there exists t_0 *such that* $\Phi_h(U(t_0)) < 0$*,* u_h *has finite time blow-up. Moreover, there exists a constant* C *that does not depends on* h *such that*

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

PROOF. If $\Phi_h(U(t_0)) < 0$ by the previous lemma we have that $U(t)$ is unbounded, then there exists a time t and a node j such that $-\frac{2}{h^2}u_j(t) + u_j^p(t) \geq \frac{1}{2}u_j^p(t)$. Hence,

$$u_j'(t) = \frac{1}{h^2}(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)) + u_j^p(t) \geq \frac{1}{2}u_j^p(t).$$

As a consequence of this $u_j(s)$ must be increasing for $s \geq t$ and verifies

$$u_j'(s) \geq \frac{1}{2}u_j^p(s).$$

As $p > 1$, u_j goes to infinity in finite time, and hence $U(t)$ has finite time blow-up in the sense that there exists a finite time, T_h, such that $\lim_{t\to T_h} \max u_j(t) = +\infty$. Moreover, we have that, for every $t \in [t_0, T_h)$

$$\frac{d}{dt}\langle MU(t), U(t)\rangle = 2\langle MU'(t), U(t)\rangle =$$

$$2\langle -AU(t), U(t)\rangle + 2\langle MU^p(t), U(t)\rangle =$$

$$-4\Phi_h(U(t)) + \frac{2(p-1)}{p+1}\langle MU^p(t), U(t)\rangle \geq$$

$$4|\Phi_h(U(t))| + \frac{2(p-1)}{p+1}(\langle MU(t), U(t)\rangle)^{\frac{p+1}{2}}.$$

Integrating between t_0 and T_h we obtain

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

where C depends only on p. $\square$

As a consequence of this bound we get

COROLLARY 2.1. *Let u_0 be an initial datum for* (1) *such that u blows up, then u_h blows up for every h small enough, and if we call T and T_h the blow-up times for u and u_h respectively, we have*

$$\lim_{h\to 0} T_h = T.$$

PROOF. First we observe that if u blows up in finite time T then

$$\lim_{t\to T} \Phi(u(\cdot,t)) = -\infty$$

(see [**21**]).

Using the convergence result (Theorem 2.2) one can check that

$$\lim_{h\to 0} \Phi_h(u_h(\cdot,t_0)) = \Phi(u(\cdot,t_0)).$$

Therefore we conclude that if u blows up in finite time then the discrete functional verifies $\Phi_h(u_h(\cdot,t_0)) < 0$ for some t_0 and every h small enough, and hence u_h blows up in finite time, T_h. To prove the convergence of the blow-up times we are going to use the bound,

$$T_h - t_0 \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}. \tag{5}$$

Given $\varepsilon > 0$, we can choose M large enough to ensure that

$$\left(\frac{C}{M^{\frac{p-1}{p+1}}}\right) \leq \frac{\varepsilon}{2}.$$

As u blows up at time T we can choose $\tau < \frac{\varepsilon}{2}$ such that

$$-\Phi(u(\cdot, T-\tau) \geq 2M.$$

If h is small enough,

$$-\Phi_h(U(T-\tau) \geq M,$$

and hence by (5),

$$T_h - (T-\tau) \leq \left(\frac{C}{(-\Phi_h(U(T-\tau)))^{\frac{p-1}{p+1}}}\right) \leq \left(\frac{C}{M^{\frac{p-1}{p+1}}}\right) \leq \frac{\varepsilon}{2}.$$

Therefore,

$$|T_h - T| \leq |T_h - (T-\tau)| + |\tau| < \varepsilon. \quad \square$$

3. Blow-up rate and blow-up set

In this section we find the blow-up rate and the localization of blow-up points (blow-up set).

From now on we consider positive solutions of (3) with h fixed and we denote by C a positive constant that may depend on h but not on t and it may be different in each step of the proofs.

LEMMA 2.2. *Let u_h be a solution of (3) that blows up at time T_h, then there exists two constants c, C depending on h such that*

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}$$

PROOF. First, we observe that, as

$$u_j'(t) = \frac{1}{h^2}(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)) + u_j^p(t)$$

we have that

$$w(t) = \sum_{i=1}^{N} u_i(t)$$

verifies

$$w'(t) = \sum_{i=1}^{N} u_i'(t) = \sum_{i=1}^{N} (u_i)^p(t) - \frac{u_2(t) + u_N(t)}{h^2} \leq$$

$$C\left(\sum_{i=1}^{N} u_i(t)\right)^p = Cw^p(t).$$

For $t \in [0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h} \frac{w'(s)}{w^p(s)}\, ds \leq C(T_h - t),$$

changing variables we get

$$\int_{w(t)}^{+\infty} \frac{1}{s^p}\, ds \leq C(T_h - t).$$

Hence

$$w(t) \geq C(T_h - t)^{-\frac{1}{p-1}}.$$

Using that there exists a constant $C = C(h)$ such that

$$\max_j u_j(t) \geq C \sum_{i=1}^{N} u_i(t)$$

we have

$$\max_j u_j(t) \geq C(T_h - t)^{-\frac{1}{p-1}}.$$

To prove the reverse inequality we proceed as follows, as

$$\lim_{t \nearrow T_h} \max_j u_j(t) = +\infty,$$

we have $\frac{2}{h^2} u_n(t) \leq \frac{1}{2} u_n^p(t)$ for every t close to T_h for some $2 \leq n \leq N$, in this case we have

$$u_n'(t) = \frac{1}{h^2}(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + u_n^p(t) \geq \frac{1}{2} u_n^p(t).$$

Integrating again over $[t, T_h]$ we obtain

$$\int_t^{T_h} \frac{u_n'(s)}{u_n^p(s)}\, ds \geq \frac{1}{2}(T_h - t),$$

changing variables

$$\int_{u_n(t)}^{+\infty} \frac{1}{s^p}\, ds \geq \frac{1}{2}(T_h - t).$$

Hence

$$u_n(t) \leq 2^{\frac{1}{p-1}} C_p (T_h - t)^{-\frac{1}{p-1}}.$$

So we have

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$$

in the sense that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}. \quad \square$$

To conclude the proof of Theorem 2.4 we make the following change of variables (inspired by [**39, 42, 43**]),

$$\begin{aligned} y_k(s) &= (T_h - t)^{\frac{1}{p-1}} u_k(t), \\ (T_h - t) &= e^{-s}. \end{aligned} \tag{6}$$

This kind of transformations are very useful when one looks for the asymptotic behavior of this kind of problems. Observe that the blow-up rate imposes this variable $Y(s)$ to be globally defined and its behavior at infinity provides us the behavior of the numerical solution $U(t)$ near the numerical blow-up time T_h.

The variable, $Y(s) = (y_k(s))$, verifies

$$
(7)\qquad
\begin{aligned}
&y_1(s) = 0,\\
&y_k'(s) = \tfrac{e^{-s}}{h^2}(y_{k+1}(s) - 2y_k(s) + y_{k-1}(s)) - \tfrac{1}{p-1} y_k(s) + y_k^p(s),\\
&y_{N+1}(s) = 0,\\
&y_k(-\ln(T_h)) = (T_h)^{\frac{1}{p-1}} u_0(x_k), \qquad 1 \le k \le N+1.
\end{aligned}
$$

We remark that, as $\max u_j(t) \le C(T_h - t)^{-\frac{1}{p-1}}$, we have that $y_j(s)$ are uniformly bounded,

$$
y_j(s) \le C \qquad \forall s > -\ln(T_h).
$$

We are going to prove that the variables $y_j(s)$ can only go to zero or to C_p as $s \to \infty$.

LEMMA 2.3. *If there exists s_0 such that*

$$
y_j^p(s_0) - \frac{1}{p-1} y_j(s_0) < -Ce^{-s_0}
$$

then

$$
y_j(s) \to 0 \qquad (s \to \infty).
$$

PROOF. We observe that $y_j(s)$ verifies

$$
y_j'(s) = \frac{e^{-s}}{h^2}(y_{j+1}(s) - 2y_j(s) + y_{j-1}(s)) - \frac{1}{p-1} y_j(s) + y_j^p(s) \le
$$

$$
Ce^{-s} - \frac{1}{p-1} y_j(s) + y_j^p(s).
$$

Let $w(s)$ be a solution of

$$
w'(s) = Ce^{-s} - \frac{1}{p-1} w(s) + w^p(s)
$$

with $w(s_0) = y_j(s_0)$. We observe that, $w(s_0) < C_p$ and

$$
w'(s_0) = Ce^{-s_0} - \frac{1}{p-1} y_j(s_0) + y_j^p(s_0) < 0.
$$

We claim that $w'(s) < 0$ for all $s > s_0$. To prove this claim, we argue by contradiction. Assume that there exists a first time s_1 such that $w'(s_1) = 0$. At that time s_1 we have

$$
w''(s_1) = -Ce^{-s_1} - \frac{1}{p-1} w'(s_1) + p w^{p-1}(s_1) w'(s_1) = -Ce^{-s_1}.
$$

Hence $w''(s_1) < 0$. Therefore w' is decreasing at s_1, a contradiction.

So we have proved that $w(s)$ is decreasing for all $s > s_0$, and $w(s) \geq 0$ hence there exists $l = \lim_{s\to\infty} w(s)$. As $\lim_{s\to\infty} w'(s) = 0$ we have that

$$l^p - \frac{1}{p-1} l = 0.$$

We have that $w(s_0) < C_p$ and w is decreasing for $s \geq s_0$, so we conclude that $l \neq C_p$ and hence $l = 0$.

By a comparison argument we have that

$$0 \leq y_j(s) \leq w(s) \to 0 \qquad (s \to \infty),$$

hence $y_j(s) \to 0$ $(s \to \infty)$. $\square$

LEMMA 2.4. *If there exists s_0 such that*

$$y_j^p(s_0) - \frac{1}{p-1} y_j(s_0) > Ce^{-s_0}$$

then $y_j(s)$ blows up in finite time $\tilde{s}$.

PROOF. As before, we observe that $y_j(s)$ verifies

$$y_j'(s) = \frac{e^{-s}}{h^2}(y_{j+1}(s) - 2y_j(s) + y_{j-1}(s)) - \frac{1}{p-1} y_j(s) + y_j^p(s) \geq$$

$$-Ce^{-s} - \frac{1}{p-1} y_j(s) + y_j^p(s).$$

Let $w(s)$ be a solution of

$$w'(s) = -Ce^{-s} - \frac{1}{p-1} w(s) + w^p(s)$$

with $w(s_0) = y_j(s_0)$. We observe that, $w(s_0) > C_p$ and

$$w'(s_0) = -Ce^{-s_0} - \frac{1}{p-1} y_j(s_0) + y_j^p(s_0) > 0.$$

We claim that $w'(s) > 0$ for all $s > s_0$. To prove this claim, we argue by contradiction. Assume that there exists a first time s_1 such that $w'(s_1) = 0$, at that time s_1 we have

$$w''(s_1) = Ce^{-s_1} - \frac{1}{p-1} w'(s_1) + pw^{p-1}(s_1)w'(s_1) = Ce^{-s_1}.$$

Hence $w''(s_1) > 0$. Therefore w' is increasing at s_1, a contradiction.

So we have proved that $w(s)$ is increasing for all $s > s_0$, hence there exists $\varepsilon > 0$ such that

$$w'(s) \geq \varepsilon w^p(s)$$

and then, using that $p > 1$, we have that w blows up in finite time s_2.

As before, we can use a comparison argument to get

$$y_j(s) \geq w(s) \to +\infty \qquad s \to s_2 < \infty,$$

hence $y_j(s)$ blows up in finite time. $\square$

LEMMA 2.5. *Let $y_j(s)$ be a solution of (7) then each y_j verifies*

$$y_j(s) \to 0 \qquad (s \to +\infty),$$
$$or$$
$$y_j(s) \to C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \qquad (s \to +\infty),$$
$$or$$
$$y_j(s) \text{ blows up in finite time.}$$

PROOF. From the previous two lemmas we can conclude that, if $y_j(s)$ does not converge to zero and does not blow up in finite time, then $y_j(s) \to C_p$. To see this fact, we observe that $y_j(s)$ is global and satisfies

$$Ce^{-s} \geq y_j^p(s) - \frac{1}{p-1} y_j(s) \geq -Ce^{-s}.$$

Then

$$y_j^p(s) - \frac{1}{p-1} y_j(s) \to 0 \qquad (s \to +\infty).$$

As y_j is continuous, bounded and does not go to zero, we conclude that $y_j(s) \to C_p$. $\square$

Now we are ready to prove

THEOREM 2.4. *Assume that $p > 1$ and that u_h blows up in finite time, T_h, then*

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}},$$

in the sense that there exists two positive constants c, C such that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}.$$

Moreover,

$$\lim_{t \to T_h} \max_j u_j(t)(T_h - t)^{-\frac{1}{p-1}} = C_p$$

where

$$C_p = (p-1)^{-\frac{1}{p-1}}.$$

PROOF. We just observe that, from Lemma 2.2, $c \leq \max y_j(s) \leq C$, so $y_j(s)$ is global and also $\max y_j(s)$ does not go to zero, hence, using Lemma 2.5, we get that

$$\lim_{s \to \infty} \max_j y_j(s) = C_p.$$

In the original variables $\{u_j,\ t\}$ this is equivalent to

$$\lim_{t\to T_h}(T_h-t)^{\frac{1}{p-1}}\max_j u_j(t)=C_p. \quad \square$$

Now we turn our attention to the blow-up set. We begin by the proof of the propagation result.

THEOREM 2.5. *Let $B^*(u_h)$ be the set of nodes, x_n, such that*

$$\lim_{t\to T_h}u_n(t)(T_h-t)^{\frac{1}{p-1}}=C_p.$$

Then the blow-up propagates in the following way, let $p>1$ and $K\in\mathbb{N}$ satisfies $\frac{K+2}{K+1}<p\le\frac{K+1}{K}$. We call $d(i)$ the distance from x_i to $B^*(u_h)$ measured in nodes. Then the solution of (3)*, u_h, blows up exactly at K nodes near $B^*(u_h)$,*

$$u_i(t)\to+\infty \qquad\Longleftrightarrow\qquad d(i)\le K.$$

Moreover, if $d(i)\le K$, the asymptotic behavior is given by

$$u_i(t)\sim(T_h-t)^{-\frac{1}{p-1}+d(i)},$$

if $p\ne\frac{K+1}{K}$ and if $p=\frac{K+1}{K}$, $d(i)=K$

$$u_i(t)\sim\ln(T_h-t).$$

PROOF. Let $B^*(u_h)=\{x_{j_1},x_{j_2},\dots,x_{j_k}\}$ be the set of nodes such that

$$y_j(s)\to C_p \qquad (s\to\infty).$$

Let K be such that

$$\frac{K+2}{K+1}<p\le\frac{K+1}{K}.$$

We want to see that the blow-up propagates to the K nodes adjacent to $B^*(u_h)$.

To see this let us begin by a considering a node x_{i_1} with $d(i_1)=1$, then there exists $j\in B^*(u_h)$ that is adjacent to i_1. We can assume that i_1 is on the left of j, so $i_1=j-1$.

By Lemma 2.5, as $j-1\notin B^*(u_h)$, we have that $y_{j-1}(s)\to 0$. We want to obtain the asymptotic behavior of $y_{j-1}(s)$. To do this, first we get a bound as follows, $y_{j-1}(s)$ verifies

$$y'_{j-1}(s)=\frac{e^{-s}}{h^2}(y_j(s)-2y_{j-1}(s)+y_{j-2}(s))-\frac{1}{p-1}y_{j-1}(s)+y^p_{j-1}(s)\le$$

$$4Ce^{-s}-\frac{1}{p-1}y_{j-1}(s)+y^p_{j-1}(s).$$

Using that $y_{j-1}(s) \to 0$ we have that, given $\varepsilon > 0$, for every $s > s_0$

$$y'_{j-1}(s) \leq 4Ce^{-s} - \left(\frac{1}{p-1} - \varepsilon\right) y_{j-1}(s).$$

Let $w(s)$ be a solution of

$$w'(s) = 4Ce^{-s} - \left(\frac{1}{p-1} - \varepsilon\right) w(s)$$

with $w(s_0) \geq y_{j-1}(s_0)$. Integrating this equation we get

$$w(s) = C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1} - \varepsilon)s}.$$

By a comparison argument we get that for every $s > s_0$,

$$y_{j-1}(s) \leq w(s) = C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1} - \varepsilon)s}. \tag{8}$$

Now we go back to

$$y'_{j-1}(s) = \frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) - \frac{1}{p-1} y_{j-1}(s) + y^p_{j-1}(s).$$

We have,

$$y'_{j-1}(s) + \frac{1}{p-1} y_{j-1}(s) = \frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) + y^p_{j-1}(s)$$

then,

$$(e^{\frac{1}{p-1}s} y_{j-1}(s))' = e^{\frac{1}{p-1}s}\left(\frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) + y^p_{j-1}(s)\right).$$

Integrating between s_0 and s, we get

$$\begin{aligned}
&y_{j-1}(s) = \\
&= e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{\frac{1}{p-1}\sigma}\left(\frac{e^{-\sigma}}{h^2}(y_j - 2y_{j-1} + y_{j-2}) + y^p_{j-1}\right) d\sigma\right) \\
&= e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}\left(\frac{1}{h^2}(y_j - 2y_{j-1} + y_{j-2}) + e^{\sigma} y^p_{j-1}\right) d\sigma\right).
\end{aligned}$$

Using (8) we have that

$$e^s y^p_{j-1}(s) \leq C_1 e^{-(p-1)s} + C_2 e^{-(\frac{p}{p-1} - p\varepsilon - 1)s} \to 0 \qquad (s \to \infty).$$

Hence, as $y_j(s) \to C_p$, then $y_{j-2}(s) \to 0$ or C_p, $\quad y_{j-1}(s) \to 0$ and $e^s y^p_{j-1}(s) \to 0$, we have,

$$\lim_{s \to +\infty}\left(\frac{(y_j - 2y_{j-1} + y_{j-2})}{h^2} + e^s y^p_{j-1}(s)\right) = C_2 \neq 0.$$

Therefore, the integral behaves like

$$\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}\, d\sigma.$$

If $p \neq 2$, we have

$$y_{j-1}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 e^{-\frac{p-2}{p-1}s}\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 e^{-s}.$$

If $p = 2$ the integral behaves like s, then

$$y_{j-1}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 s\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 s e^{-\frac{1}{p-1}s}.$$

Therefore

$$y_{j-1}(s) \sim \begin{cases} Ce^{-\frac{1}{p-1}s} & \text{if } p > 2, \\ Cse^{-\frac{1}{p-1}s} & \text{if } p = 2, \\ Ce^{-s} & \text{if } p < 2. \end{cases}$$

This implies that $u_{j-1}(t)$ verifies

$$u_{j-1}(t) \sim \begin{cases} C & \text{if } p > 2, \text{ it is bounded}, \\ -C\ln(T_h - t) & \text{if } p = 2, \text{ blows up}, \\ C(T_h - t)^{-\frac{1}{p-1}+1} & \text{if } p < 2, \text{ blows up}. \end{cases}$$

We observe that the same arguments show that if $p > 2$ then u_j is bounded for every j with $d(j) \geq 1$. To continue the proof we assume that $p \leq 2$. We sketch the case $p < 2$ (the case $p = 2$ can be handled in a similar way).

We consider a node i_2 such that $d(i_2) = 2$. We can assume that $i_2 = i_1 - 1 = j - 2$. Since $d(i_2) = 2$ we have that $j-1$ and $j-3 \notin B^*(u_h)$. From similar calculations, we have

$$\begin{aligned} &y_{j-2}(s) = \\ &= e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{\frac{1}{p-1}\sigma}\left(\frac{e^{-\sigma}}{h^2}(y_{j-1} - 2y_{j-2} + y_{j-3}) + y_{j-2}^p\right)\, d\sigma\right) \\ &= e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{-\frac{2p-3}{p-1}\sigma}\left(\frac{e^{\sigma}}{h^2}(y_{j-1} - 2y_{j-2} + y_{j-3}) + e^{2\sigma} y_{j-2}^p\right)\, d\sigma\right) \end{aligned}$$

Using the asymptotic behavior that we have found for y_{j-1}, we can obtain the asymptotic behavior for y_{j-2}. Arguing as before we have that,

$$\lim_{s\to\infty} e^{2s} y_{j-2}^p(s) = 0, \qquad \lim_{s\to\infty} e^{s} y_{j-1}(s) = C \neq 0.$$

Hence,

$$\lim_{s\to+\infty}\left(\frac{e^{s}(y_{j-1} - 2y_{j-2} + y_{j-3})}{h^2} + e^{2s} y_{j-2}^p(s)\right) = C_2 \neq 0.$$

Therefore, the integral behaves like

$$\int_{s_0}^{s} e^{-\frac{2p-3}{p-1}\sigma}\, d\sigma.$$

If $p \neq \frac{3}{2}$, we have

$$y_{j-2}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 e^{-\frac{2p-3}{p-1}s}\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 e^{-2s}.$$

If $p = \frac{3}{2}$ the integral behaves like s, then

$$y_{j-2}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 s\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 s e^{-\frac{1}{p-1}s}.$$

Therefore

$$y_{j-2}(s) \sim \begin{cases} Ce^{-\frac{1}{p-1}s} & \text{if } p > 3/2, \\ Cse^{-\frac{1}{p-1}s} & \text{if } p = 3/2, \\ Ce^{-2s} & \text{if } p < 3/2. \end{cases}$$

This implies that $u_{j-2}(t)$ verifies

$$u_{j-2}(t) \sim \begin{cases} C & \text{if } p > 3/2, \text{ it is bounded}, \\ -C\ln(T_h - t) & \text{if } p = 3/2, \text{ it blows up}, \\ C(T_h - t)^{-\frac{1}{p-1}+2} & \text{if } p < 3/2, \text{ it blows up}. \end{cases}$$

Now we can repeat this procedure with other nodes to find that $u_i(t)$ blows up if $d(i) \leq K$ and $u_i(t)$ is bounded if $d(i) > K$ where $K \in \mathbb{N}$ is determined by p in the following way, K verifies

$$\frac{K+2}{K+1} < p \leq \frac{K+1}{K}.$$

Also we find that the asymptotic behavior of a node i such that $d(i) \leq K$ is given by

$$u_i(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(i)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $i = K$,

$$u_i(t) \sim \ln(T_h - t). \quad \square$$

Now we localize the blow-up set. To do that we need a lemma (proved in [**2**]) that ensures that if the initial datum is symmetric and increasing (in $[0, 1/2]$) then the numerical solution remains in this conditions for every time. Also see a proof of this fact in Chapter 7.

LEMMA 2.6. *([**2**], Proposition 2) Assume that the initial datum verifies $u_{N-j}(0) = u_j(0)$ (symmetry) and that $u_j(0) < u_{j+1}(0)$ for every j such that $x_j \leq 1/2$ (monotonicity in $[0, 1/2]$), then $u_{N-j}(t) = u_j(t)$ and that $u_j(t) < u_{j+1}(t)$ for every j such that $x_j \leq 1/2$ for every $t \in [0, T_h)$.*

THEOREM 2.6. *Let u_0 be an initial datum for* (1) *such that u blows up, then if we call $B(u)$ and $B(u_h)$ the blow-up sets for u and u_h respectively, we have that, given $\varepsilon > 0$ there exists h_0 such that for every $0 < h \leq h_0$,*

$$B(u_h) \subset B(u) + (-\varepsilon, \varepsilon) \qquad \forall h \leq h_0(\varepsilon).$$

Moreover, if u_0 is symmetric and increasing in $[0, 1/2]$ we have that

$$B(u_h) = \{1/2\} + [-Kh, Kh] \qquad \forall h \leq h_0.$$

PROOF. We want to prove that, given $\varepsilon > 0$ there exists h_0 such that for every $0 < h \leq h_0$,

$$B(u_h) \subset B(u) + (-\varepsilon, \varepsilon). \tag{9}$$

We have that the blow-up set of u is composed by a finite number of points, $B(u) = \{b_1, b_2, ...b_k\}$ $(b_1 < b_2 \cdots < b_k)$ (see [**15, 32, 58, 62, 77**]).

We are going to prove that for every interval (b_i, b_{i+1}) and every $\delta > 0$, $(b_i + \delta, b_{i+1} - \delta) \bigcap B^*(u_h) = \emptyset$ if h is small (we recall that $B^*(u_h)$ is the set of nodes j such that $y_j(s) \to C_p$).

So consider an interval (b_i, b_{i+1}) and let t_0 a time close to the blow-up time T, choosing a smaller h and making a small perturbation in b_i, b_{i+1} if necessary we can assume

(a) b_i, b_{i+1} and the middle point $(b_i + b_{i+1})/2$ are nodes of the mesh.
(b) $u_h(x, t_0) < \min\{u_h(b_i, t_0), u_h(b_{i+1}, t_0)\}$ for every $x \in (b_i, b_{i+1})$.

So we call $b_i = \gamma_1 < \cdots < \gamma_{n+1} = b_{i+1}$ the nodes inside (b_i, b_{i+1}) and $z_j(t) = u_h(\gamma_j, t)$ the values of the numerical solution at this nodes. The vector $Z(t) = (z_1(t), \ldots, z_{n+1}(t))$ verifies

$$\begin{aligned}
&z_1(t) \leq \|U(t_0)\|_\infty, \\
&z_j'(t) = \frac{(z_{j+1}(t) - 2z_j(t) + z_{j-1}(t))}{h^2} + z_j^p(t), \qquad 2 \leq j \leq n, \\
&z_{n+1}(t) \leq \|U(t_0)\|_\infty, \\
&z_j(t_0) \leq z_0(\gamma_j), \qquad 1 \leq j \leq n+1,
\end{aligned} \tag{10}$$

where $z_0 : (b_i, b_{i+1}) \to \mathbb{R}$ is a function that verifies

(a) $z_0(b_i) = z_0(b_{i+1}) = \|U(t_0)\|_\infty$.
(1) $z_0(x)$ is symmetric around the point $(b_i + b_{i+1})/2$.
(b) $z_0(x)$ decreaces in $[b_i, (b_i + b_{i+1})/2]$.

So we study $W(t) = (w_1(t), \dots, w_{n+1}(t))$, the solution of (10) with equalities instead of inequalities. As $Z(t)$ is a subsolution for this problem we have $Z(t) \leq W(t)$.

By Lemma 2.6 (the same result can be proved for this problem using the same arguments) $W(t)$ is symmetric and decreasing in space for every $t < T_h$. So we have $w_{n-1}(t) \leq w_n(t) \leq w_{n+1}(t) = \|U(t)\|_\infty$. We want to prove that

$$(T_h - t)^{-\frac{1}{p-1}} w_{n-1}(t) \to 0, \qquad (t \to T_h),$$

to do that we observe that $W(t)$ verifies

$$\begin{aligned} w_n'(t) &= \frac{1}{h^2}(w_{n+1}(t) - 2w_n(t) + w_{n-1}(t)) + w_n^p(t), \\ w_{n-1}'(t) &= \frac{1}{h^2}(w_n(t) - 2w_{n-1}(t) + w_{n-2}(t)) + w_{n-1}^p(t). \end{aligned}$$

Subtracting we obtain

$$\begin{aligned} (w_n - w_{n-1})'(t) &= \frac{(w_{n+1} - w_n - 2(w_n - w_{n-1}) + w_{n-1} - w_{n-2})}{h^2}(t) \\ &\quad + \ w_n^p(t) - w_{n-1}^p(t) \\ &\geq -\frac{2}{h^2}(w_n - w_{n-1}) + \frac{w_n^p - w_{n-1}^p}{w_n - w_{n-1}}(w_n - w_{n-1})(t) \\ &= (-\frac{2}{h^2} + p\xi^{p-1}(t))(w_n - w_{n-1})(t) \end{aligned}$$

Where $w_{n-1}(t) \leq \xi(t) \leq w_n(t)$. Hence

$$(\ln(w_n - w_{n-1}))'(t) \geq (-\frac{2}{h^2} + p\xi^{p-1}(t)),$$

and integrating we have

$$\ln(w_n - w_{n-1})(t) - \ln(w_n - w_{n-1})(t_0) \geq \int_{t_0}^{t} (-\frac{2}{h^2} + p\xi^{p-1}(s))\, ds.$$

Now we argue by contradiction, assume that $(T_h - t)^{\frac{1}{p-1}} w_{n-1}(t) \to C_p$, as $w_{n-1}(t) \leq \xi(t) \leq w_n(t)$ we have that

$$\lim_{t \to T_h} \xi(t)(T - t)^{\frac{1}{p-1}} = C_p$$

and then we just observe that

$$\int_{t_0}^{t} (-\frac{2}{h^2} + p\xi^{p-1}(s))\, ds \geq p \int_{t_0}^{t} \frac{(C_p^{p-1} - \varepsilon)}{(T_h - s)}\, ds - C =$$
$$-p(C_p^{p-1} - \varepsilon)\ln(T_h - t) - C.$$

Hence

$$(w_n - w_{n-1})(t) \geq C(T_h - t)^{-p(C_p^{p-1}-\varepsilon)} = C(T_h - t)^{-\frac{p}{p-1}+p\varepsilon}.$$

Using this fact, we have

$$0 = \lim_{t\to T_h} (T_h - t)^{\frac{1}{p-1}} (w_n - w_{n-1})(t) =$$

$$\lim_{t\to T_h} (T_h - t)^{\frac{1}{p-1}} (T_h - t)^{-\frac{p}{p-1}+p\varepsilon} \frac{(w_n - w_{n-1})(t)}{(T_h - t)^{-\frac{p}{p-1}+p\varepsilon}} \geq$$

$$C \lim_{t\to T_h} (T_h - t)^{\frac{1}{p-1}-\frac{p}{p-1}+p\varepsilon} = +\infty,$$

a contradiction that proves our claim.

As for every $t < T_h$, $W(t)$ is symmetric and decreasing, and $z_i(t) \leq w_i(t)$ the only nodes in (b_i, b_{i+1}) that can (may) lie in $B^*(u_h)$ are $\gamma_1, \gamma_2, \gamma_n$ and γ_{n+1}. This proves that for h small enough

$$B^*(u_h) \subset B(u) + \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right).$$

To finish the first part of the proof of Theorem 2.6 we only have to observe that by the propagation result, Theorem 2.5, we have that, for h small enough,

$$B(u_h) \subset B^*(u_h) + [-Kh, Kh] \subset B(u) + \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) + [-Kh, Kh]$$
$$\subset B(u) + (-\varepsilon, \varepsilon),$$

proving (9).

Now we turn our attention to u_0 symmetric and increasing in the interval $[0, 1/2]$.

In this case the continuous solution u is also symmetric and has only one maximum at $x = 1/2$. In this situation, it it proved that the blow-up set of u consists in a single point, $B(u) = \{1/2\}$, **[15]**.

For the semidiscrete problem (3) the solution must also be symmetric and increasing in $[0, 1/2]$. This result was proved in **[2]**.

We assume that $x = 1/2$ is a node of the mesh. We observe that, by this Lemma, if the initial datum is symmetric, the maximum of u_h is $u_n(t)$ where $x_n = 1/2$. At this point we have

$$u_n(t) = \max u_j(t) \sim (T - t)^{-\frac{1}{p-1}}.$$

Hence, by Theorem 2.4 we have that

$$\lim_{t\to T_h} u_n(t)(T - t)^{\frac{1}{p-1}} = C_p.$$

We claim that $B^*(u_h) = \{n\}$, i. e. for every $j \neq n$ we have $y_j(s) \to 0$. To prove this claim we make similar calculations than above . Observe that u_n and u_{n-1} verify

$$u_n'(t) = \frac{1}{h^2}(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + u_n^p(t),$$

$$u_{n-1}'(t) = \frac{1}{h^2}(u_n(t) - 2u_{n-1}(t) + u_{n-2}(t)) + u_{n-1}^p(t).$$

Subtracting we obtain

$$(u_n - u_{n-1})'(t) = \frac{1}{h^2}(4u_{n-1}(t) - 3u_n(t) - u_{n-2}(t)) + u_n^p(t) - u_{n-1}^p(t) \geq$$

$$-\frac{3}{h^2}(u_n(t) - u_{n-1}(t)) + \frac{u_n^p(t) - u_{n-1}^p(t)}{u_n(t) - u_{n-1}(t)}(u_n(t) - u_{n-1}(t)) =$$

$$(-\frac{3}{h^2} + p\xi^{p-1}(t))(u_n(t) - u_{n-1}(t)).$$

Where $u_{n-1}(t) \leq \xi(t) \leq u_n(t)$. Hence

$$(\ln(u_n - u_{n-1}))'(t) \geq (-\frac{3}{h^2} + p\xi^{p-1}(t)),$$

and integrating we have

$$\ln(u_n - u_{n-1})(t) - \ln(u_n - u_{n-1})(t_0) \geq \int_{t_0}^{t} (-\frac{3}{h^2} + p\xi^{p-1}(s))\, ds.$$

Now we argue by contradiction, assume that $(T_h - t)^{\frac{1}{p-1}} u_{n-1}(t) \to C_p$, as $u_{n-1}(t) \leq \xi(t) \leq u_n(t)$ we have that

$$\lim_{t \to T_h} \xi(t)(T - t)^{\frac{1}{p-1}} = C_p$$

and then we just observe that

$$\int_{t_0}^{t} (-\frac{3}{h^2} + p\xi^{p-1}(s))\, ds \geq p \int_{t_0}^{t} \frac{(C_p^{p-1} - \varepsilon)}{(T_h - s)}\, ds - C =$$

$$-p(C_p^{p-1} - \varepsilon) \ln(T_h - t) - C.$$

Hence

$$(u_n - u_{n-1})(t) \geq C(T_h - t)^{-p(C_p^{p-1} - \varepsilon)} = C(T_h - t)^{-\frac{p}{p-1} + p\varepsilon}.$$

Using this fact, we have

$$0 = \lim_{t \to T_h} (T_h - t)^{\frac{1}{p-1}} (u_n - u_{n-1})(t) =$$

$$\lim_{t \to T_h} (T_h - t)^{\frac{1}{p-1}} (T_h - t)^{-\frac{p}{p-1} + p\varepsilon} \frac{(u_n - u_{n-1})(t)}{(T_h - t)^{-\frac{p}{p-1} + p\varepsilon}} \geq$$

$$C \lim_{t\to T_h} (T_h - t)^{\frac{1}{p-1} - \frac{p}{p-1} + p\varepsilon} = +\infty,$$

a contradiction that proves our claim.

After this we can use the propagation result, Theorem 2.5, to obtain

$$\begin{aligned} B(u_h) &= B^*(u_h) + [-Kh, Kh] = \{1/2\} + [-Kh, Kh] \\ &= B(u) + [-Kh, Kh]. \quad \square \end{aligned}$$

CHAPTER 3

Extensions to higher dimensions

To extend the results obtained in Chapter 2 to higher dimensions, we introduce, in this chapter, a general method for the spatial discretization that is an extension of the scheme considered in the previous one.

We consider the problem

$$(1)\qquad \begin{array}{ll} u_t = \Delta u + u^p, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{on } \Omega. \end{array}$$

We assume that u_0 is regular and $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain in order to guarantee that $u \in C^{2,1}$. The blow-up set, which is defined as the set composed of points $x \in \Omega$ such that there exist sequences $x_n \to x$, $t_n \to T$ with $u(x_n, t_n) \to +\infty$, is localized in small portions of Ω, in [**74**] it is proved that, if p is subcritical ($(d-2)p < d+2$), the $(d-1)$ dimensional Hausdorff measure of the blow-up set is finite, and the blow-up rate at these points is given by $u(x,t) \sim (T-t)^{-\frac{1}{p-1}}$, moreover

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = C_p, \qquad C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$$

(see [**38, 39**] and also [**57, 59**]).

If $p \geq (d+2)/(d-2)$ it is not very well understood the behavior of the solutions near the blow-up time. In [**44**] it is proved that the blow-up rate described above can fail in high dimensions.

We prove here results similar to those obtained in the previous chapter for the same equation in one space dimension. We introduce the numerical method that will be used in some of the next chapters and that was mentioned in the preliminaries chapter.

1. The numerical scheme in higher dimensions

In order to generalize the numerical scheme to several dimensions we consider a general method with adequate assumptions.

As a first step to introduce the method we propose a method of lines, that is: we discretize the space variable, keeping t continuous. At this stage we consider a general method with adequate hypotheses. More precisely, we assume that for every $h > 0$ small (h is the parameter of the method), there exists a set of nodes $\{x_1, \dots, x_N\} \subset \overline{\Omega}$ ($N = N(h)$), such that the numerical approximation of u at the nodes x_k is given by

$$U(t) = (u_1(t), \dots, u_N(t)).$$

That is $u_k(t)$ stands for an approximation of $u(x_k, t)$. We assume that U is the solution of the following ODE

$$MU'(t) = -AU(t) + MU^p, \tag{2}$$

with initial data given by $u_k(0) = u_0(x_k)$. In (2) and hereafter, all operations between vectors are understood coordinate by coordinate.

The precise assumptions on the matrices involved in the method are:

- (P1) M is a diagonal matrix with positive entries m_k.
- (P2) A is a nonnegative symmetric matrix, with nonpositive coefficients off the diagonal (i.e. $a_{ij} \leq 0$ if $i \neq j$) and $a_{ii} > 0$.
- (P3) $\sum_{k=1}^{N} a_{ik} \geq 0$.

If we define the graph $\mathcal{G}$ with vertices on the nodes x_k and we say that two nodes x_k, x_j are connected if and only if $a_{kj} \neq 0$, then we assume

- (P4) The graph $\mathcal{G}$ is connected.

In Chapter 1 we showed that the finite element method in an acute triangulation and finite differences method in a cube $\Omega = (0,1)^d$ verify these properties.

We also need some kind of convergence result for the scheme. The precise hypotheses are stated when needed. We also prove an L^∞ convergence theorem under the consistency assumption. Now we state the two convergence hypotheses that we may need.

- (H1) For every $\tau > 0$, $\|u - u_h\|_{L^\infty(\Omega \times [0, T-\tau])} \to 0$ as $h \to 0$.
- (H2) $\|u - u_h\|_{H_0^1(\Omega)}(t) \to 0$ as $h \to 0$ for a.e. t in $[0, T)$.

We observe that both hypotheses require the numerical scheme to converge in any subregion of $\Omega \times [0,T)$ where the solution is regular. We can not expect the scheme to converge in the whole region $\Omega \times [0,T)$ since little perturbations in the equation make the blow-up time vary (this can be observed even in the ordinary equation $u' = u^p$, where the blow-up time can be explicitly computed). In Chapter 2 we have shown that in dimension one the Finite Element Method with Mass Lumping verifies these hypotheses. The same holds for the Finite Difference Method in a cube $\Omega \subset \mathbb{R}^d$ (see [**61**]).

In the following section we prove that the hypothesis (H1) holds if we require the scheme to be consistent.

2. Convergence of the numerical scheme

DEFINITION. Let w be a regular solution of

$$\begin{array}{rcll} w_t & = & \Delta w + f(x,t) & \text{in } \Omega \times (0,T), \\ w & = & 0 & \text{on } \partial\Omega \times (0,T). \end{array}$$

We say that the scheme (2) is consistent if for any $t \in (0, T-\tau)$ it holds

$$m_i w_t(x_i,t) = -\sum_{k=1}^{N} a_{ik} w(x_k,t) + m_i f(x_i,t) + \rho_{i,h}(t), \tag{3}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max_i \frac{|\rho_{i,h}(t)|}{m_i} \le \rho(h), \qquad \text{for every } t \in (0, T-\tau),$$

with $\rho(h) \to 0$ if $h \to 0$. The function ρ is called the modulus of consistency of the method.

In [**2**] it is proved that this consistency assumption holds when one consider finite element method with mass lumping in one space dimension. In this case we have $\rho(h) = Ch^2$. We also prove this in Chapter 7 for the porous medium equation. It is also valid for finite difference schemes ([**61**]). However this assumption is not valid in general in the case of the finite element method in several dimensions, even for convergent schemes.

Before involving us in the convergence result we give the following definition.

DEFINITION. We say that $\overline{U}$ is a supersolution of (2) if

$$M\overline{U}' \ge -A\overline{U} + M\overline{U}^p.$$

We say that $\underline{U}$ is a subsolution of (2) if

$$M\underline{U}' \le -A\underline{U} + M\underline{U}^p.$$

The inequalities are understood coordinate by coordinate.

We state now a comparison principle, proved in the preliminaries chapter.

LEMMA 3.1. *(Comparison Lemma) Let $\overline{U}$ and $\underline{U}$ be a super and a subsolution of (2) respectively, such that the initial data verify*

$$\overline{U}(0) \ge \underline{U}(0).$$

Then

$$\overline{U}(t) \ge \underline{U}(t).$$

With this in mind we can prove the convergence theorem

THEOREM 3.1. *Let u be a regular solution of (1) and u_h the numerical approximation given by (2). If the scheme is consistent then there exists a positive constant C such that*

$$\|u - u_h\|_{L^\infty([0,T-\tau],L^2(\Omega))} \le C\rho(h).$$

PROOF. We define the error functions

$$e_i(t) = u(x_i, t) - u_i(t).$$

By (3), these functions verify

$$\begin{aligned} m_i e_i'(t) &= -\sum_{k=1}^{N} a_{ik} e_k(t) + m_i(u^p(x_i,t) - u_i(t)^p) + \rho_{i,h}(t) \\ &\le -\sum_{k=1}^{N} a_{ik} e_k(t) + m_i p\ \xi(x,t)^{p-1} e_i(t) + \rho_{i,h}(t) \end{aligned}$$

Let

$$t_0 = \max\{t :\ t < T - \tau,\ \max_i \max_{s<t} |e_i(s)| \le 1\}.$$

We will see by the end of the proof that $t_0 = T - \tau$ for h small enough. In $[0, t_0]$ we have

$$m_i e_i'(t) \le -\sum_{k=1}^{N} a_{ik} e_k(t) + K m_i e_i(t) + \rho_k(h).$$

Hence, in $[0, t_0]$, $E(t) = (e_1(t), ..., e_N(t))$ satisfies

$$ME'(t) \le -AE(t) + KME(t) + M\rho(h)1^t, \tag{4}$$

here K is the Lipchitz constant for $f(u) = u^p$ in $[0, \|u(\cdot, T-\tau)\|_{L^\infty}]$.

Let us now define $W(t) = (w_1(t), \dots, w_N(t))$, which will be used as a supersolution.

$$w_i(t) = e^{(2K+1)t}(\|e(0)\|_\infty + \rho(h)).$$

It is easy to check that W verifies

$$MW'(t) > -AW(t) + KMW(t) + M\rho(h)1^t.$$

Hence W is a supersolution for (4), and by Lemma 3.1 we get

$$e_i(t) \le e^{(2K+1)t}(\|e(0)\|_{L^\infty(\Omega)} + \rho(h)), \qquad t \in [0, t_0].$$

Arguing along the same lines with $-E(t)$, we obtain

$$|e_i(t)| \le e^{(2K+1)t}(\|E(0)\|_\infty + \rho(h)) \le C\rho(h), \qquad t \in [0, t_0],$$

by our hypotheses on the convergence of the initial data. Using this fact, since $\rho(h)$ goes to zero, we get that $|e_i(t)| \le 1$ for every $t \in [0, T-\tau]$ for every h small enough. Therefore $t_0 = T - \tau$ for h small. This proves the convergence of the scheme. In fact we have that for every $h < h_0$,

$$\max_{0 \le t \le T-\tau} \max_{1 \le i \le N} |u_i(t) - u(x_i, t)| \le C\rho(h).$$

□

3. Blow-up for the numerical scheme

In Section 3 of the preceding chapter we proved the existence of solutions with blow-up for the numerical scheme. Moreover we have found that if u_0 is an initial datum that produces blow-up in the solution then the numerical solution also blows up for h small.

The same proofs are also valid for the $d-$dimensional case. Even the discrete functional $\Phi_h(U)$ is the same, generalized to several space dimensions.

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds.$$

In that section we first used the discrete functional to prove that a continuous solution with blow-up (that imposes the existence of a time t_0 with $\Phi(u)(t_0) < 0$ as it was shown in **[21]**) leads to a discrete solution with $\Phi_h(U)(t_0) < 0$ for h small.

In order to prove that one needs that for fixed times $\Phi_h(U) \to \Phi(u)$, $(h \to 0)$. This is true in the one dimensional case for the finite element method, but it does not necessary hold for the general scheme we proposed here. So, to obtain a similar result we have to assume (H1) and (H2). We sketch the proof here, but it is very similar to the one in the previous chapter.

THEOREM 3.2. *Assume (H1) and (H2). Then positive solutions of* (2) *blow up in finite time if* $p > 1$ *and* $U(0)$ *is large in the following sense; let*

$$\Phi_h(U) \equiv \frac{1}{2}\langle A^{1/2}U, A^{1/2}U\rangle - \frac{1}{p+1}\langle MU^p, U\rangle,$$

then, if there exists t_0 *such that* $\Phi_h(U(t_0)) < 0$, u_h *has finite time blow-up. Moreover, there exists a constant* C *that does not depends on* h *such that*

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

And even more, let u_0 *be an initial datum for* (1) *such that* u *blows up, then* u_h *blows up for every* h *small enough, and if we call* T *and* T_h *the blow-up times for* u *and* u_h *respectively, we have*

$$\lim_{h\to 0} T_h = T.$$

PROOF. Arguing as in the previous chapter it can be proved that if for some t_0, $\Phi_h(U)(t_0) < 0$ then the numerical solution is unbounded. So we have to prove that unbounded solutions of (2) blow up. Observe that if U is unbounded then there exist a node j and a time t such that

$$-\sum_{k=1}^{N} a_{kj}u_k(t) + m_j u_j^p(t) \geq \frac{1}{2} m_j u_j^p(t).$$

Hence,

$$m_j u_j'(t) = -\sum_{k=1}^{N} a_{kj}u_k(t) + m_j u_j^p(t) \geq \frac{1}{2} m_j u_j^p(t).$$

As a consequence of this $u_j(s)$ must be increasing for $s \geq t$ and verifies

$$u_j'(s) \geq \frac{1}{2} u_j^p(s).$$

As $p > 1$, u_j can not be global, and hence $U(t)$ has finite time blow-up in the sense that there exists a finite time T_h such that

$$\lim_{t\to T_h} \max_j u_j(t) = +\infty.$$

Now we observe that if u blows up in finite time T then

$$\lim_{t\to T}\Phi(u(\cdot,t)) = -\infty$$

(see [**21**]). Hypotheses (H1) and (H2) imply that

$$\lim_{h\to 0}\Phi_h(u_h(\cdot,t)) = \Phi(u(\cdot,t))$$

for a.e. t. Therefore we conclude that if u blows up in finite time then $\Phi_h(u_h(\cdot,t_0)) < 0$ for some t_0 and every h small enough, and hence u_h blows up in finite time, T_h. It remains to proof the convergence of the blow-up times. This can be done as in the previous chapter (Corollary 2.1). □

4. Blow-up rate

In this section we find the blow-up rate and the localization of blow-up points.

From now on we consider positive solutions of (2) with h fixed and we denote by C a positive constant that may depend on h but not on t and may change at each step of the proofs.

LEMMA 3.2. *Let u_h be a solution of (2) that blows up at time T_h, then there exists two constants c, C depending on h such that*

$$c(T_h - t)^{-\frac{1}{p-1}} \le \max_j u_j(t) \le C(T_h - t)^{-\frac{1}{p-1}}.$$

PROOF. First we observe that, as

$$m_j u_j'(t) = -\sum a_{kj}u_k(t) + m_j u_j^p(t)$$

we have that

$$w(t) = \sum_{j=1}^{N} m_j u_j(t),$$

verifies

$$\begin{aligned} w'(t) &= -\sum_{j=1}^{N}\sum_{k=1}^{N} a_{kj}u_k(t) + \sum_{j=1}^{N} m_j u_j^p(t) \\ &\le -\sum_{k=1}^{N} a_{kj}\left(\sum_{j=1}^{N} u_k(t)\right) + \sum_{j=1}^{N} m_j u_j^p(t) \\ &\le C\left(\sum_{j=1}^{N} m_j u_j(t)\right)^p = Cw^p. \end{aligned}$$

As u_h blows up at time T_h, we have that there exists t_0 such that for every $t \in [t_0, T_h)$ it holds

$$w'(t) \leq C w^p(t).$$

For $t \in [t_0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h} \frac{w'(s)}{w^p(s)}\, ds \leq C(T_h - t),$$

changing variables we get

$$\int_{w(t)}^{+\infty} \frac{1}{s^p}\, ds \leq C(T_h - t).$$

Hence

$$w(t) \geq C(T_h - t)^{-\frac{1}{p-1}}.$$

Using that there exists a constant $C = C(h)$ such that

$$\max_j u_j(t) \geq C \sum_{j=1}^{N} m_j u_j(t)$$

we have

$$\max_j u_j(t) \geq C(T_h - t)^{-\frac{1}{p-1}}.$$

To prove the other inequality we proceed as follows, as $\max u_j(t) \to +\infty$ when $t \to T_h$, we have $\frac{1}{2} a_{jj} u_j(t) \leq \frac{1}{2} m_j u_j^p(t)$ for every t close to T_h for some j, in this case we have

$$\begin{aligned} m_j u_j'(t) &= -\sum a_{kj} u_k(t) + m_j u_j^p(t) \\ &\geq -a_{jj} u_j(t) + m_j u_j^p(t) \\ &\geq \frac{1}{2} m_j u_j^p(t). \end{aligned}$$

Integrating again over $[t, T_h]$ we obtain

$$\int_t^{T_h} \frac{u_n'(s)}{u_n^p(s)}\, ds \geq \frac{1}{2}(T_h - t),$$

changing variables

$$\int_{u_n(t)}^{+\infty} \frac{1}{s^p}\, ds \geq \frac{1}{2}(T_h - t).$$

Hence

$$u_n(t) \leq 2^{\frac{1}{p-1}} C_p (T_h - t)^{-\frac{1}{p-1}}.$$

So we have

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$$

in the sense that

$$c(T_h - t)^{-\frac{1}{p-1}} \le \max_j u_j(t) \le C(T_h - t)^{-\frac{1}{p-1}}. \quad \square$$

Now we obtain a sharper estimate of the blow-up rate of the numerical solutions

THEOREM 3.3. *Assume $p > 1$ and that u_h blow up in finite time T_h. Then*

$$\lim_{t \to T_h} \max_j u_j(t)(T_h - t)^{-\frac{1}{p-1}} = C_p, \qquad C_p = (p-1)^{-\frac{1}{p-1}}.$$

PROOF. We start making the change of variables that was so useful in Chapter 2,

$$y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t)$$

$$(T_h - t) = e^{-s}.$$

As in the previous chapter we use these variables to obtain the behavior of $U(t)$ near the blow-up time by looking at their behavior at infinity.

These new variables, $Y = (y_k(s))$, verify

$$\begin{aligned} m_j y_j'(s) &= -e^{-s} \sum_{k=1}^{N} a_{kj} y_j(s) - \frac{m_j}{p-1} y_j(s) + m_j y_j^p(s), \\ y_j(-\ln(T_h)) &= (T_h)^{\frac{1}{p-1}} u_0(x_k), \qquad 1 \le j \le N. \end{aligned} \tag{5}$$

We observe that, as $\max u_j(t) \le C(T_h - t)^{-\frac{1}{p-1}}$, we have that $y_j(s)$ are uniformly bounded,

$$y_j(s) \le C \qquad \forall s > -\ln(T_h).$$

Actually, each component $y_j(s)$ verifies one of the following

(a) $\lim_{s \to +\infty} y_j(s) = 0$,
(b) $\lim_{s \to +\infty} y_j(s) = C_p$,
(c) $y_j(s)$ blows up in finite time.

This follows from these two claims:

1. If there exists s_0 such that

$$m_j y_j^p(s_0) - \frac{m_j}{p-1} y_j(s_0) < -Ce^{-s_0},$$

then $y_j(s) \to 0$ as $s \to \infty$.

2. If there exists s_0 such that

$$m_j y_j^p(s_0) - \frac{m_j}{p-1} y_j(s_0) > Ce^{-s_0}$$

then $y_j(s)$ blows up in finite time $\tilde{s}$.

The proofs of these claims are analogous to the ones of Lemma 2.3 and Lemma 2.4. The only difference is that the coefficients a_{kj} are not explicit, but can be handled as in the one dimensional case.

Therefore, if $y_j(s)$ does not converge to zero and does not blow up in finite time, then $y_j(s)$ is global and satisfies

$$Ce^{-s} \geq y_j^p(s) - \frac{1}{p-1} y_j(s) \geq -Ce^{-s}.$$

Then

$$y_j^p(s) - \frac{1}{p-1} y_j(s) \to 0 \qquad (s \to +\infty).$$

As y_j is continuous, bounded and does not go to zero, we conclude that $y_j(s) \to C_p$.

From Lemma 3.2, $c \leq \max y_j(s) \leq C$, so $y_j(s)$ is global and also $\max y_j(s)$ does not go to zero, hence, we get that

$$\lim_{s\to\infty} \max_j y_j(s) = C_p.$$

In the original variables $\{u_j,\ t\}$ this is equivalent to

$$\lim_{t\to T_h} (T_h - t)^{\frac{1}{p-1}} \max_j u_j(t) = C_p. \quad \square$$

5. Blow-up set

In the previous sections we have proved the existence of solutions with blow-up for the numerical scheme and we find the asymptotic behavior of the L^∞ norm of the numerical solutions. Now the question goes as follows: Does every blow-up node has the same asymptotic behavior or is it posible to have a different blow-up behavior? The answer is that depends on p as the following theorem shows.

Definition. Let $B^*(u_h)$ be the set of nodes that blow up with the rate given by the maximum, that is $B^*(u_h) = \{x_j : u_j(t) \sim (T_h - t)^{-1/(p-1)}\}$. From the previous section we know that $B^*(u_h)$ is a nonempty set. We say that two different nodes, x_i, x_j, are at distance one if and only if $a_{ij} \neq 0$. Let A_1 be the set of nodes that are at distance one from $B^*(u_h)$. Inductively, we define A_l as the set of nodes that are at distance one from A_{l-1}. We say that a node x_k is at distance $d(k)$ from $B^*(u_h)$ if $x_k \in A_{d(k)} - A_{d(k)-1}$. This is the usual distance from a node x_k to $B^*(u_h)$ measured as a graph.

We prove that u_k blows up if and only if $d(k) \leq K$ where K depends only on p.

THEOREM 3.4. *If U blows up then the blow-up propagates in the following way: let $p > 1$ and $K \in \mathbb{N}$ such that $\frac{K+2}{K+1} < p \leq \frac{K+1}{K}$, then the solution of* (2), *U, blows up exactly at K nodes near $B^*(u_h)$. More precisely,*

$$u_k(t) \to +\infty \qquad \Longleftrightarrow \qquad d(k) \leq K$$

where $d(k)$ is the distance of x_k to $B^*(u_h)$ in the sense described above. Moreover, if $d(k) \leq K$, the asymptotic behavior of u_k is given by

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(k)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(k) = K$

$$u_k(t) \sim \ln(T_h - t).$$

PROOF. Let $B^*(u_h) = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\}$ be the set of nodes such that

$$y_{j_i}(s) \not\to 0 \qquad (s \to \infty).$$

Let K be such that

$$\frac{K+2}{K+1} < p \leq \frac{K+1}{K}, \qquad K = \left[\frac{1}{p-1}\right].$$

We want to see that blow-up propagates to the K nodes adjacent to $B^*(u_h)$, that is, a node x_k blows up if and only if $d(k) \leq K$.

For this purpose let us begin by considering a node x_k such that $d(k) = 1$. As $x_k \notin B^*(u_h)$, we have that $y_k(s) \to 0$. We want to obtain the asymptotic behavior of $y_k(s)$. To this end, first we get a bound as follows, from (5) $y_k(s)$ verifies

$$m_k y_k'(s) \leq Ce^{-s} - \frac{m_k}{p-1} y_k(s) + m_k y_k^p(s).$$

Using that $y_k(s) \to 0$ we have that, given $\varepsilon > 0$ there exists s_0 such that, for every $s > s_0$

$$m_k y_k'(s) \leq Ce^{-s} - \frac{m_k}{p-1} y_k(s) + m_k y_k^p(s) \leq Ce^{-s} - \left(\frac{m_k}{p-1} - \varepsilon\right) y_k(s).$$

Let $w_k(s)$ be a solution of

$$w_k'(s) = Ce^{-s} - \left(\frac{m_k}{p-1} - \varepsilon\right) w_k(s),$$

we have

$$w_k(s) \leq Ce^{-s}.$$

By a comparison argument we obtain that for every $s > s_0$,

$$y_k(s) \leq w_k(s) \leq Ce^{-s}. \tag{6}$$

Again, from (5)

$$m_k y_k'(s) + \frac{m_k}{p-1} y_k(s) = -e^{-s} \sum a_{kj} y_j(s) + m_k y_k^p(s)$$

then,

$$m_k (e^{\frac{1}{p-1}s} y_k(s))' = e^{\frac{1}{p-1}s} \left(-e^{-s} \sum a_{kj} y_j(s) + m_k y_k^p(s)\right).$$

Integrating between s_0 and s, we get

$$m_k y_k(s) = e^{-\frac{1}{p-1}s} \left(C_k + \int_{s_0}^{s} e^{\frac{1}{p-1}\tau} \left(-e^{-\tau} \sum a_{kj} y_j + m_k y_k^p\right) d\tau\right).$$

We need to find the behavior of the last integral. With this in mind let us compute the following limit,

$$\lim_{s \to +\infty} \frac{\int_{s_0}^{s} e^{\frac{1}{p-1}\tau} \left(-e^{-\tau} \sum_{j=1}^{N} a_{kj} y_j(\tau) + m_k y_k^p(\tau)\right) d\tau}{\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\tau}\, d\tau}.$$

If the integral diverges, we can use L'Hôspital's rule to obtain

$$\lim_{s \to +\infty} \frac{e^{(\frac{1}{p-1}-1)s} \left(-\sum_{j=1}^{N} a_{kj} y_j(s) + m_k e^s y_k^p(s)\right)}{e^{-\frac{p-2}{p-1}s}} =$$

$$\lim_{s \to +\infty} -\sum_{j=1}^{N} a_{kj} y_j(s) + m_k e^s y_k^p(s).$$

Using (6) we get

$$e^s y_k^p(s) \leq Ce^{-(p-1)s} \to 0 \qquad (s \to \infty),$$

hence we have,

$$\lim_{s \to +\infty} \frac{\int_{s_0}^{s} e^{\frac{1}{p-1}\tau} \left(-e^{-\tau} \sum_{j=1}^{N} a_{kj} y_j(\tau) + m_k y_k^p(\tau)\right) d\tau}{\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\tau}\, d\tau} =$$

$$\lim_{s\to+\infty} -\sum_{j=1}^{N} a_{kj} y_j(s) = \tilde{C}_k \neq 0.$$

Therefore, the integral behaves like

$$\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\tau}\, d\tau.$$

If $p \neq 2$, we have

$$y_k(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 e^{-\frac{p-2}{p-1}s}\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 e^{-s}.$$

If $p = 2$ we can repeat the above calculations but in this case the integral behaves like s. Therefore

$$y_k(s) \sim \begin{cases} Ce^{-\frac{1}{p-1}s} & \text{if } p > 2, \\ Cse^{-\frac{1}{p-1}s} & \text{if } p = 2, \\ Ce^{-s} & \text{if } p < 2. \end{cases}$$

This implies that $u_k(t)$ verifies

$$u_k(t) \sim \begin{cases} C & \text{if } p > 2, \text{ and hence it is bounded,} \\ -C\ln(T_h - t) & \text{if } p = 2, \text{ and hence it blows up,} \\ C(T_h - t)^{\frac{p-2}{p-1}} & \text{if } p < 2, \text{ and hence it blows up.} \end{cases}$$

Now we can repeat this procedure with a node x_l that is at distance two from $B^*(u_h)$ (using the asymptotic behavior that we have found for y_k) and so on to find that $u_l(t)$ blows up if $d(l) \leq K$ and u_l is bonded if $d(l) > K$. Also we find that the asymptotic behavior of a node x_l is given by

$$u_l(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(l)}, \qquad d(l) = 1, \ldots, K,$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$

$$u_l(t) \sim \ln(T_h - t)$$

if $d(l) = K$. $\square$

REMARK 3.1. *This theorem proves the localization of the numerical blow-up set $B(u_h)$ in a neighborhood of $B^*(u_h)$. If it is posible to localize $B^*(u_h)$ in a neighborhood of $B(u)$ then it holds that given $\varepsilon > 0$ for every h small enough*

$$B(u_h) \subset B(u) + N_\varepsilon,$$

where N_ε is a neighborhood of the origin in $\mathbb{R}^d$. In Chapter 2 we proved that this is posible in one space dimension. We conjecture that it holds also in higher dimensions but we could not prove it.

CHAPTER 4

How? Non-simultaneous blow-up

When considering systems of reaction-diffusion equations new phenomena may appear. One of them is the so called *non-simultaneous blow-up*, which means that the solution has some components that blow up, while the others remain bounded. In this chapter we investigate how this phenomenon is reproduced by standard numerical methods. To this end, we study the behavior of semidiscrete approximations of the following parabolic system

$$
\begin{aligned}
u_t &= \Delta u + u^{p_{11}} v^{p_{12}} \qquad \text{in } \Omega \times [0, T), \\
v_t &= \Delta v + u^{p_{21}} v^{p_{22}} \qquad \text{in } \Omega \times [0, T),
\end{aligned} \tag{1}
$$

with homogeneous Dirichlet boundary data,

$$u = v = 0 \text{ on } \partial\Omega \times [0, T),$$

and initial data

$$
\begin{aligned}
u(x, 0) &= u_0(x) \\
v(x, 0) &= v_0(x).
\end{aligned}
$$

We assume that Ω is a bounded smooth domain, $p_{ij} > 0$ and u_0, v_0 are positive, bounded, compatible with the boundary data and smooth enough to guarantee that u and v are regular. This reaction-diffusion system can be used as a model to describe heat propagation in a two-component combustible mixture. In this case u and v represent the temperatures of the interacting components, thermal conductivity is supposed to be constant and equal for both substances, and a volume energy release given by some powers of u and v is assumed.

Solutions to problem (1) exist locally in time, **[23]**. The time T is the maximal existence time for the solution, which may be finite or infinite. If T is finite, then (u, v) develops a singularity. More precisely,

$$\limsup_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} = +\infty,$$

and we say that (u, v) *blows up* in finite time T. As was mentioned before, blow-up may happen for certain solutions of (1), no matter how smooth the initial data are, see **[23]**.

For this problem there exist solutions (u, v) that blow up in finite time if and only if the exponents p_{ij} verify any of the conditions, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}$, see **[23]**.

A priori there is no reason why both functions u and v should go to infinity simultaneously at time T. In fact, in **[66]** the authors prove under adequate hypotheses that there are initial data such that u blows up while v does not if and only if $p_{11} > 1$ and $p_{21} < p_{11} - 1$. They denote this phenomenon as *non-simultaneous blow-up*.

Our interest here is to analyze how the usual numerical methods behave when applied to (1). We find that the usual numerical approximations have similar blow-up results to those that hold for the continuous problem (1).

We consider a general method that is roughly speaking the same of Chapter 3. We take a set of nodes $\{x_1, \dots, x_N\}$, and denote by $(U, V)(t) = ((u_1(t), \dots, u_N(t)), (v_1(t), \dots, v_N(t)))$ the numerical solution. That is, $(u_k(t), v_k(t))$ stands for the values of the numerical approximation at the node x_k at time t. The boundary conditions impose that some $(u_k(t), v_k(t))$ have to be zero. From now on we focus on the other nodes. In what follows h stands for the parameter of the method. We assume that $(U, V) = (U_h, V_h)$ is the solution of the ODE system

$$
\begin{array}{rcl}
MU'(t) & = & -AU(t) + MU^{p_{11}}V^{p_{12}}(t), \\
MV'(t) & = & -AV(t) + MU^{p_{21}}V^{p_{22}}(t),
\end{array}
\tag{2}
$$

with initial data given by $U(0) = U_0$, $V(0) = V_0$. This system can be written explicitly as

$$
\begin{array}{rcl}
m_k u_k'(t) & = & -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) v_k^{p_{12}}(t), \quad 1 \le k \le N, \\
m_k v_k'(t) & = & -\sum_{j=1}^{N} a_{kj} v_j(t) + m_k u_k^{p_{21}}(t) v_k^{p_{22}}(t), \quad 1 \le k \le N,
\end{array}
\tag{3}
$$

with initial data $u_k(0) = u_{0,k}$, $v_k(0) = v_{0,k}$, for $1 \le k \le N$. We denote the maximal existence time for the system by T_h.

We assume that the matrices M and A satisfy properties (P1)-(P3) of the preceding chapter. We also require the method to be consistent, which means that if we consider (u, v), a solution of (1) and (U, V) its numerical approximation given by (3), then

(4)

$$
m_k u_t(x_k, t) = -\sum_{j=1}^{N} a_{kj} u(x_j, t) + m_k u^{p_{11}}(x_k, t) v^{p_{12}}(x_k, t) + \rho_k^1(h),
$$

$$
m_k v_t(x_k, t) = -\sum_{j=1}^{N} a_{kj} v(x_j, t) + m_k u^{p_{21}}(x_k, t) v^{p_{22}}(x_k, t) + \rho_k^2(h),
$$

for $t \in [0, T-\tau]$ and there exists a positive function $\rho(h)$ such that

$$\max_i \left\{ \frac{|\rho_k^i(h)|}{m_k} \right\} \le \rho(h), \ \max_k |u_0(x_k) - u_k(0)| + |v_0(x_k) - v_k(0)| \le \rho(h),$$

and $\lim_{h\to 0} \rho(h) = 0$.

For the scalar problem, it is easy to see that this consistency assumption holds in $[0,1] \times [0, T-\tau]$ for every $\tau > 0$ in the case of the finite element method with $\rho(h) = Ch^2$.

We begin the study of (3) by proving that under the consistency assumption, this method converges uniformly for times as close to the blow-up time as we want,

THEOREM. *Let (u, v) be a regular solution of* (1) *and (U, V) the numerical approximation given by* (3)*, then given $\tau > 0$ there exists a constant C such that*

$$\max_{1\le k\le N} \sup_{0\le t\le T-\tau} (|u(x_k, t) - u_k(t)| + |v(x_k, t) - v_k(t)|) \le C\rho(h).$$

In the remainder of the chapter we assume that the method converges in the sense described in the above theorem.

As a first step for our analysis of the behavior of solutions of (3) we want to describe when the blow-up phenomenon occurs for the discrete problem. We say that a solution of (3) has finite blow-up time if there exists a time T_h such that

$$\lim_{t\nearrow T_h} \|U(t)\|_\infty + \|V(t)\|_\infty = \lim_{t\nearrow T_h} \max_j u_j(t) + \max_j u_j(t) = +\infty.$$

We prove that the conditions on the structure of the nonlinearity that characterize the presence of blow-up for the discretized system are precisely those obtained in **[23]** for the continuous problem.

THEOREM. *There exist solutions (U, V) of* (3) *that blow up in finite time if and only if the exponents p_{ij} verify any of the conditions, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11} - 1)(p_{22} - 1) - p_{12}p_{21} < 0$.*

For certain choices of the parameters p_{ij} there are initial data for which one of the components of the semidiscrete system remains bounded while the other blows up. We denote this phenomenon as *non-simultaneous numerical blow-up*. We characterize the range of parameters for which this occurs in the next two theorems.

THEOREM. *Let (U, V) a solution of* (3) *such that U blows up at finite time T_h and V remains bounded up to that time. Then $p_{11} > 1$ and $p_{21} < p_{11} - 1$.*

THEOREM. *If $p_{11} > 1$ and $p_{21} < p_{11} - 1$, then for every initial datum $V_0 \neq 0$ for* (3) *there exists an initial datum U_0 such that U blows up in finite time T_h and V remains bounded up to that time.*

We want to remark that the structure of the nonlinearity needed for the non-simultaneous blow-up in the discretized system is identical to what is proved in [**66**] for the continuous system. This fact and the convergence of the discrete problem to the continuous one allow us to prove the following theorem for the non-simultaneous blow-up case.

THEOREM. *Let (u_0, v_0) be initial data for* (1) *such that u and v are strictly increasing in time. Furthermore assume that u blows up at finite time T and v remains bounded up to that time. Then, there exists h_0 such that for every $h < h_0$, U blows up at finite time T_h but V is bounded. Moreover, we have that the blow up rate is given by*

$$\max_i u_k(t) \sim (T_h - t)^{-1/(p_{11}-1)},$$

and the numerical blow-up time converges to the continuous one, i.e.,

$$\lim_{h \to 0} T_h = T.$$

We want to comment on the hypotheses involved in this theorem. They are just technical but we cannot avoid them so far. The first one is the monotonicity assumption. This assumption is usual in problems with blow-up, see [**36**]. It is satisfied if the initial data verify

$$\Delta u_0 + u_0^{p_{11}} v_0^{p_{12}} > 0, \qquad \Delta v_0 + u_0^{p_{21}} v_0^{p_{22}} > 0,$$

see Lemma 4.3 for a discrete version of this fact.

The second hypothesis, the non-simultaneous blow-up, is just to guarantee that $p_{11} > 1$. However, we do not need to use explicitly the fact that u blows up and v remains bounded. It is possible to obtain a similar result if $p_{11} > 1$, $p_{22} > 1$, or $p_{21}, p_{12} > 1$, no matter if the blow-up is simultaneous or not. Since our main concern here is the study of non-simultaneous blow-up we prefer to focus out attention on this case.

We remark that our proofs are based in the use of the maximum principle. Hence similar ideas may be used to study systems where this property holds. The application of energy methods would imply a restriction in the range of the exponents, in order to ensure a variational setting for the system.

The blow-up rate given in Theorem 4.5 is the obtained in [**66**] for the continuous problem.

We conclude that the numerical approximations considered here reflect the behavior of the continuous problem regarding non-simultaneous blow-up. Indeed, the conditions on the structure of the nonlinearity that appear for the continuous problem and for the discrete one are identical.

1. Convergence of the numerical scheme

In this section we prove a uniform convergence result for regular solutions of the numerical scheme (3). Throughout this section, we consider $0 < \tau < T$ fixed.

We assume that the scheme is consistent, that is (4). We want to show that

$$\max_i \{|u_i(t) - u(x_i, t)|,\ |v_i(t) - v(x_i, t)|\} \to 0$$

as $h \to 0$, uniformly in $[0, T - \tau]$, for every $\tau > 0$. This is a natural requirement since before T the exact solution is regular.

Let us begin with a comparison lemma, that will be used throughout the chapter.

Definition. We say that $(\overline{U}, \overline{V})$ is a supersolution of (2) if

$$\begin{aligned} M\overline{U}' &\geq -A\overline{U} + M\overline{U}^{p_{11}}\overline{V}^{p_{12}}, \\ M\overline{V}' &\geq -A\overline{V} + M\overline{V}^{p_{21}}\overline{V}^{p_{22}}. \end{aligned}$$

We say that $\underline{U}$ is a subsolution of (2) if

$$\begin{aligned} M\underline{U}' &\leq -A\underline{U} + M\underline{U}^{p_{11}}\underline{V}^{p_{12}}, \\ M\underline{V}' &\leq -A\underline{V} + M\underline{V}^{p_{21}}\underline{V}^{p_{22}}. \end{aligned}$$

The inequalities are understood coordinate by coordinate.

Lemma 4.1. *Let $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ be a super and a subsolution of (2) respectively such that $(\overline{U}, \overline{V})(0) \geq (\underline{U}, \underline{V})(0)$. Then*

$$(\overline{U}, \overline{V})(t) \geq (\underline{U}, \underline{V})(t).$$

Proof. Let $(W, Z) = (\overline{U} - \underline{U}, \overline{V} - \underline{V})$. We can assume, by an approximation argument, that we have strict inequalities and that $W(0), Z(0) > 0$. We observe that W verifies

$$\begin{aligned} MW' &> -AW + M\left(\overline{U}^{p_{11}}\overline{V}^{p_{12}} - \underline{U}^{p_{11}}\overline{V}^{p_{12}} + \underline{U}^{p_{11}}\overline{V}^{p_{12}} - \underline{U}^{p_{11}}\underline{V}^{p_{12}}\right) \\ &= -AW + M\overline{V}^{p_{12}}\left(\frac{\overline{U}^{p_{11}} - \underline{U}^{p_{11}}}{\overline{U} - \underline{U}}\right)W + M\underline{U}^{p_{11}}\left(\frac{\overline{V}^{p_{12}} - \underline{V}^{p_{12}}}{\overline{V} - \underline{V}}\right)Z. \end{aligned}$$

Now, suppose that the conclusion of the Lemma is false. Thus, let t_0 be the first time such that $\min\{(W,Z)(t)\} = 0$. We can assume that W attains the minimum. At that time, there must be a node j such that $w_j(t_0) = 0$. But on one hand $w_j'(t_0) \le 0$ and, on the other hand, by our hypotheses on A,

$$\begin{aligned} m_j w_j' &> -\sum_{i=1}^{N} a_{ij} w_i \\ &+ m_j \overline{v}_j^{p_{12}} \left(\frac{\overline{u}_j^{p_{11}} - \underline{u}_j^{p_{11}}}{\overline{u}_j - \underline{u}_j} \right) w_j + m_j \underline{u}_j^{p_{11}} \left(\frac{\overline{v}_j^{p_{12}} - \underline{v}_j^{p_{12}}}{\overline{v}_j - \underline{v}_j} \right) z_j \\ &\ge -\sum_{i \ne j} a_{ij} w_i + m_j \overline{v}_j^{p_{12}} p_{11} \underline{u}_j^{p_{11}-1} w_j + m_j \underline{u}_j^{p_{11}} p_{12} \underline{v}_j^{p_{12}-1} z_j \ge 0, \end{aligned}$$

a contradiction that completes the proof. □

We are ready to prove the convergence result.

THEOREM 4.1. *Let (u, v) be a regular solution of (1) and (U, V) the numerical approximation given by (3), then given $\tau > 0$ there exists a constant C such that*

$$\max_{1 \le k \le N} \sup_{0 \le t \le T-\tau} \left(|u(x_k, t) - u_k(t)| + |v(x_k, t) - v_k(t)| \right) \le C\rho(h).$$

PROOF. Using the comparison lemma and the consistency assumption, the proof follows as in Theorem 3.1. □

2. Blow-up for the numerical scheme

In this section we prove the conditions on the exponents for the existence of blowing up solutions of the numerical scheme. To do that we first prove an auxiliary lemma.

LEMMA 4.2. *Let $p_{11} \le 1$, $p_{22} \le 1$ and $(p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}$. Let α_1, α_2 be the solution of*

$$\begin{pmatrix} p_{11} - 1 & p_{12} \\ p_{21} & p_{22} - 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

If $\alpha_1/\alpha_2 \ge 1$ then there exists a constant $C > 0$ such that the solution of (3) satisfies

$$Cu_i(t) \ge v_i^{\alpha_1/\alpha_2}(t).$$

PROOF. Let $z_i(t) = Cu_i(t)$ and $w_i(t) = v_i^{\alpha_1/\alpha_2}(t)$, with C a positive constant. Then $Z = (z_i)$ satisfies,

$$MZ' = -AZ + C^{1-p_{11}} M Z^{p_{11}} W^{p_{12}\alpha_2/\alpha_1}.$$

Using the convexity of the function x^{α_1/α_2} $(\alpha_1/\alpha_2 \geq 1)$ and the properties of the matrix A we have that

$$-\frac{\alpha_1}{\alpha_2} v_i^{(\alpha_1/\alpha_2)-1} \left(\sum_{j=1}^{N} a_{ij} v_j \right) \leq - \left(\sum_{j=1}^{N} a_{ij} v_j^{\alpha_1/\alpha_2} \right),$$

hence,

$$MW' \leq -AW + \frac{\alpha_1}{\alpha_2 C^{p_{21}}} M Z^{p_{21}} W^{(\alpha_2(p_{22}-1)+\alpha_1)/\alpha_1}.$$

We can choose $C > 0$ large enough such that

$$z_i(0) = C u_i(0) > v_i^{\alpha_1/\alpha_2}(0) = w_i(0)$$

for every node. Recall that the nodes such that $u_k(t) = 0$ do not appear in the system. Assume that there exists a first time t_0 and a node x_i such that $z_i(t_0) = w_i(t_0)$. Using that $(C^{1-p_{11}} - (\alpha_1/\alpha_2 C^{p_{21}})) > 0$ (this can be done choosing C larger if necessary), and observing that $p_{11} + p_{12}(\alpha_1/\alpha_2) = (\alpha_2(p_{22}-1)+\alpha_1)/\alpha_1 + p_{21}$, at t_0 we have

$$\begin{aligned} 0 \geq\ & m_i (z_i - w_i)'(t_0) \\ \geq\ & -\sum_{j=1}^{N} a_{ij}(z_j - w_j)(t_0) + m_i \left(C^{1-p_{11}} - \frac{\alpha_1}{\alpha_2 C^{p_{21}}} \right) z_i^{p_{11}+p_{12}(\alpha_2/\alpha_1)} \\ >\ & 0, \end{aligned}$$

a contradiction. □

Now we prove a necessary and sufficient condition for the existence of solutions of the discrete problem with blow-up.

THEOREM 4.2. *There exist solutions (U, V) of (3) that blow up in finite time if and only if the exponents p_{ij} verify any of the conditions, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11}-1)(p_{22}-1) - p_{12}p_{21} < 0$.*

PROOF. Let us define,

$$T_h = \sup\{t \text{ such that } (U, V)(s) \text{ is defined for } s \in [0, t]\}.$$

If T_h is finite, then by a classical result from ODE theory we have

$$\lim_{t \nearrow T_h} \left\{ \max_j u_j(t) + \max_j v_j(t) \right\} = +\infty.$$

As mentioned in the introduction, this means that (U, V) blows up at time T_h.

First, assume that $p_{11} > 1$. As $v_i(t)$ is bounded from below in $[0, t_0]$, we have that

$$u_k'(t) \geq u_k(t) \left(-\frac{a_{kk}}{m_k} + \delta^{p_{12}} u_k^{p_{11}-1}(t) \right).$$

If the initial data u_0 is such that $u_i(0)$ is large enough (depending on h) for some node x_i then we have that $u_i'(t) \geq c u_i^{p_{11}}(t)$ for $t \in [0, t_0]$. Since $p_{11} > 1$, integrating this inequality we obtain that u_i blows up in finite time (smaller than t_0 if $u_i(0)$ is large enough). Hence (U, V) cannot be global.

The same argument applies when $p_{22} > 1$, using V instead of U.

If $p_{11} \leq 1$ and $p_{22} \leq 1$ then we can apply Lemma 4.2 to obtain that, if $v_i(0)$ is large enough, for some node x_i we have

$$\begin{aligned} v_i'(t) &\geq -\frac{a_{ii}}{m_k} v_i(t) + u_i^{p_{21}}(t) v_i^{p_{22}}(t) \\ &\geq -\frac{a_{ii}}{m_k} v_i(t) + C v_i^{p_{22}+p_{21}(\alpha_1/\alpha_2)}(t) \\ &\geq c v_i^{p_{22}+p_{21}(\alpha_1/\alpha_2)}(t). \end{aligned}$$

Our hypotheses on the exponents imply that $p_{22} + p_{21}(\alpha_1/\alpha_2) > 1$ and therefore (U, V) cannot be global. $\square$

3. Non-simultaneous blow-up

In this section we focus on the numerical non-simultaneous blow-up. We consider positive solutions of (3) with h fixed and we denote by C a positive constant that may depend on h and may vary from one line to another. First we find conditions on the structure of the nonlinearity in order to have non-simultaneous blow-up.

THEOREM 4.3. *Let (U, V) a solution of* (3) *such that U blows up at finite time T_h and V remains bounded up to that time. Then $p_{11} > 1$ and $p_{21} < p_{11} - 1$.*

PROOF. First we observe the equations involving U'. We have

$$m_k u_k'(t) = -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) v_k^{p_{12}}(t).$$

Suppose $p_{11} \leq 1$. Since V is bounded we have

$$m_k u_k'(t) \leq -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) C^{p_{12}}.$$

We define $w(t) = \sum_{k=1}^N u_k(t)$. As u blows up at time T_h there exists a time t_0 such that for every $t \in [t_0, T_h)$ we have

$$w'(t) \leq -\sum_{k=1}^N \sum_{j=1}^N \frac{a_{kj}}{m_k} u_j(t) + C \sum_{k=1}^N u_k^{p_{11}} \leq C w(t)^{p_{11}}.$$

Since $p_{11} \leq 1$, w is bounded, which implies that U is bounded. Then p_{11} must be strictly greater than one.

We want to get a bound from below for the blow-up rate. For $t \in [t_0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h} \frac{w'(s)}{w^{p_{11}}(s)}\, ds \leq C(T_h - t).$$

Changing variables we get

$$\int_{w(t)}^{+\infty} \frac{1}{s^{p_{11}}}\, ds \leq C(T_h - t).$$

Hence

$$w(t) \geq C(T_h - t)^{-1/(p_{11}-1)}.$$

Using that there exists a constant $C = C(h)$ such that

$$\max_j u_j(t) \geq C \sum_{i=1}^N u_i(t),$$

we have

$$\max_j u_j(t) \geq C(T_h - t)^{-1/(p_{11}-1)}. \tag{5}$$

We consider now the equations involving V'

$$m_k v_k'(t) = -\sum_{j=1}^N a_{kj} v_j(t) + m_k u_k^{p_{21}}(t) v_k^{p_{22}}(t).$$

Since there exists $\delta > 0$ such that $v_k(t) > \delta$ for $t \in [t_0, T_h)$ and taking into account the bound from below for the blow-up rate we have that for $t \in [t_0, T_h)$

$$m_k v_k'(t) \geq -\sum_{j=1}^N a_{kj} v_j(t) + C(T_h - t)^{-p_{21}/(p_{11}-1)} \delta^{p_{22}}.$$

If $p_{21} \geq p_{11} - 1$ then v_k blows up, a contradiction. Hence we have that $p_{21} < p_{11} - 1$, and this ends the proof. $\square$

Now we prove the converse.

THEOREM 4.4. *If $p_{11} > 1$ and $p_{21} < p_{11} - 1$, then for every initial datum $V_0 \neq 0$ for (3) there exists an initial datum U_0 such that U blows up in finite time T_h and V remains bounded up to that time.*

PROOF. Let V_0 any initial data for V. There exists T, $\delta > 0$ such that for every node we have that $v_k(t) \geq \delta$ for all $t_0 \leq t \leq T$. For u_k we have

$$m_k u_k'(t) \geq -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t)\delta^{p_{22}} \geq C u_k^{p_{11}}(t)$$

if $u_k(0)$ is large enough. As $p_{11} > 1$, $u_k(t)$ cannot be global. Then $U(t)$ blows up in finite time. Actually $U(t)$ exists only until one of its components blows up. Moreover, the blow-up rate is bounded by

$$u_k(t) \leq C(T_h - t)^{-1/(p_{11}-1)}. \tag{6}$$

Then V verifies

$$MV'(t) \leq -AV + MC(T_h - t)^{-p_{12}/(p_{11}-1)} V^{p_{22}}.$$

As $p_{12} < p_{11} - 1$, if T_h is small enough, V remains bounded up to T_h. Now we can choose $U(0)$ such that U blows up at time T_h small enough to ensure that V is bounded and the result follows. $\square$

To prove the last theorem of this chapter, that describes the asymptotic properties of the scheme as the parameter h goes to zero we need a lemma which states that, for solutions that increase in time, if u blows up and v is bounded then (U, V) cannot be global and provides a bound for the blow-up rate independent of h.

LEMMA 4.3. *Let (U, V) be a solution of (2) with $p_{11} > 1$ and such that $u_k'(0) \geq \delta u_k^{p_{11}}(0)$ and $v_k'(0) \geq 0$. Then $u_k'(t) \geq \delta u_k^{p_{11}}(t)$ and $v_k'(t) \geq 0$ for every $t < T_h$.*

PROOF. First, we claim that both $u_k'(t)$ and $v_k'(t)$ are nonnegative. In order to do that, let us define $w_k(t) = u_k'(t)$ and $z_k(t) = v_k'(t)$. Therefore, by a simple computation, (W, Z) verifies

$$MW' = -AW + D_1 W + D_2 Z,$$

$$MZ' = -AZ + D_3 W + D_4 Z,$$

where D_i are time dependent matrices with nonnegative coefficients. By the minimum principle, since $W(0), Z(0) \geq 0$, the claim follows.

Now, let us check that $u_k'(t) \geq \delta u_k^{p_{11}}(t)$. Let $w_k(t) = u_k'(t) - \delta u_k^{p_{11}}(t)$. We want to use the minimum principle to show that $w_k(t)$ is positive.

To this end, we observe that w_k verifies

$$m_k w_k' + \sum_{j=1}^N a_{kj} w_j = m_k(u_k'' - \delta p_{11} u_k^{p_{11}-1} u_k') + \sum_{j=1}^N a_{kj}(u_j' - \delta u_j^{p_{11}})$$

$$= -\delta m_k p_{11} u_k^{p_{11}-1} u_k' + m_k(p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} + p_{12} v_k^{p_{12}-1} v_k' u_k^{p_{11}}) - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}}$$

$$\geq -\delta m_k p_{11} u_k^{p_{11}-1} u_k' + m_k p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}}$$

$$= -\delta p_{11} u_k^{p_{11}-1} \left(\sum_{j=1}^N a_{kj} u_j + m_k u_k^{p_{11}} \right) + m_k p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}}$$

$$= m_k p_{11} u_k^{p_{11}-1} v_k^{p_{12}} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^{p_{11}} - p_{11} u_k^{p_{11}-1} u_j) + a_{kk}(1 - p_{11}) u_k^{p_{11}} \right)$$

$$= m_k p_{11} u_k^{p_{11}-1} v_k^{p_{12}} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^{p_{11}} - p_{11} u_k^{p_{11}-1}(u_j - u_k) - u_k^{p_{11}}) \right.$$

$$\left. + \sum_{j=1}^N a_{kj}(1 - p_{11}) u_k^{p_{11}} \right).$$

As $f(u) = u^{p_{11}}$ is convex ($p_{11} > 1$) and from our hypotheses on the matrix A it follows that $W = (w_1, \dots, w_N)$ verifies

$$MW' \geq -AW + M p_{11} U^{p_{11}-1} V^{p_{12}} W.$$

Since $W(0) > 0$ and the minimum principle holds for this equation, the result follows. $\square$

Now we can prove

THEOREM 4.5. *Let (u_0, v_0) be initial data for (1) such that u and v are strictly increasing in time. Furthermore assume that u blows up at finite time T and v remains bounded up to that time. Then, there exists h_0 such that for every $h < h_0$, U blows up at finite time T_h but V is bounded. Moreover, we have that the blow up rate is given by*

$$\max_i u_k(t) \sim (T_h - t)^{-1/(p_{11}-1)},$$

and the numerical blow-up time converges to the continuous one, i.e.,

$$\lim_{h \to 0} T_h = T.$$

PROOF. In [**66**] the authors prove that if u blows up and v remains bounded then $p_{11} > 1$ and $p_{21} < p_{11} - 1$, so we are going to begin the proof assuming $p_{11} > 1$. The blow-up rate follows from (5) and (6). From the previous lemma we can obtain an upper bound for the blow-up rate of U independent of h. From our hypotheses on the initial data we have that there exists δ independent of h such that the hypothesis of the previous lemma holds. Integrating we obtain that if $u_k(t)$ is blowing up then

$$\int_t^{T_h} \frac{u_k'}{u_k^{p_{11}}}(s)\,ds \geq \delta(T_h - t),$$

so that

$$u_k(t) \leq C(T_h - t)^{-\frac{1}{p_{11}-1}},$$

where C depends only on p_{11} and δ. Moreover,

$$(T_h - t) \leq \frac{1}{\delta} \int_{\max_k u_k(t)}^{+\infty} \frac{1}{x^{p_{11}}}\,dx. \tag{7}$$

This bound says that if $U(t)$ is large then $T_h - t$ is small.

For the remainder of the proof the idea is as follows, as u is blowing up and v is bounded, from the convergence of the method we get that $U(t_0)$ is large and $V(t_0)$ is bounded for t_0 close to T. Using the bound (7) we obtain that $T_h - t_0$ is small and $V(t)$ verifies

$$\begin{aligned} &MV'(t) \leq -AV + MC(T_h - t)^{-p_{12}/(p_{11}-1)} V^{p_{22}}, \\ &V(t_0) \leq \|v\|_\infty + 1. \end{aligned} \tag{8}$$

Hence $V(t)$ is bounded up to T_h. The proof is similar to the one of Theorem 4.4.

Let us write down the details of the proof. Given $\varepsilon > 0$ we can choose K large enough to guarantee

$$\frac{1}{\delta} \int_K^\infty \frac{1}{x^{p_{11}}}\,dx < \varepsilon,$$

and $t_0 < T$ such that

$$\|u(\cdot, t_0)\|_\infty > 2K.$$

As $U(t)$ converges to u in $[0, t_0]$ we can choose h_0 such that for every $h < h_0$

$$\|U(t_0)\|_\infty > K.$$

Hence, from (7) we get

$$T_h - t_0 < \varepsilon,$$

using this fact and that $p_{12} < p_{11} - 1$, we can assert that the solution of

$$\overline{v}_k(t) = C(T_h - t)^{-p_{12}/(p_{11}-1)}\overline{v}_k^{p_{22}}(t),$$

$$\overline{v}_k(t_0) = \|v\|_\infty + 1,$$

is bounded up to T_h.

As $\|V(t_0)\|_\infty \le \|v\|_\infty + 1$ for every h small enough and (8) holds we can use $\overline{V}(t)$ as a supersolution for $V(t)$ to conclude that $V(t)$ is bounded up to T_h. Therefore we have that U blows up and V remains bounded for every h small enough. Hence we get non-simultaneous numerical blow-up for every h small enough.

Now we prove the convergence of the blow-up times. Given $\varepsilon > 0$ we can repeat the arguments used above to obtain that for some $t_0 < T$ such that $T - t_0 < \varepsilon$,

$$|T_h - t_0| \le \frac{1}{\delta}\int_K^{+\infty} \frac{1}{x^{p_{11}}}\,dx < \varepsilon,$$

therefore

$$|T_h - T| \le |T_h - t_0| + |T - t_0| < 2\varepsilon.$$

This ends the proof. □

4. Numerical experiments

We present now some numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one performs numerical computations. For simplicity we restrict ourselves to an interval, $\Omega = (0,1)$ and perform the semidiscrete method in a uniform mesh using finite elements with mass lumping (this coincides with the central differences scheme). As we mentioned in the introduction this method satisfies our general assumptions. To integrate in time we use an adaptive ODE solver.

We begin by considering non-simultaneous blow-up. We perform the calculations with the following parameters, $p_{11} = 2$, $p_{12} = 1$, $p_{21} = 1/2$, $p_{22} = 1/2$, with initial data $u_0(x) = 200x(1-x)$, $v_0(x) = 5x(1-x)$. In this case the numerical blow-up time is $T_h = 0.01902$. In Figure 1 we show the behavior of $\max U(t)$ and of $\max V(t)$. For a better understanding we show the evolution until $t_0 = 0.0187$. We remark that though V does not blow-up, V' does. This numerical experiment suggests that the monotonicity assumption in Theorem 4.5 is not essential.

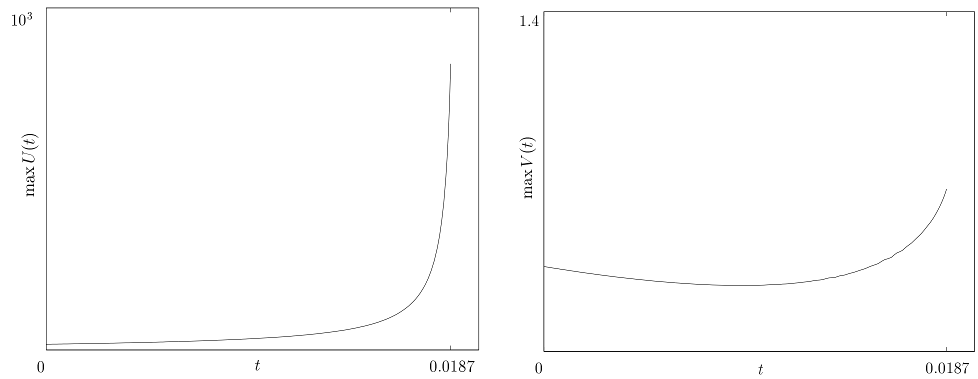

FIGURE 1. $\max U(t)$ BLOWING UP AND $\max V(t)$ BOUNDED.

Now, in Figure 2 we show simultaneous numerical blow-up. For these pictures we have only changed the value of p_{21} from $1/2$ to 2. This forces simultaneous blow-up.

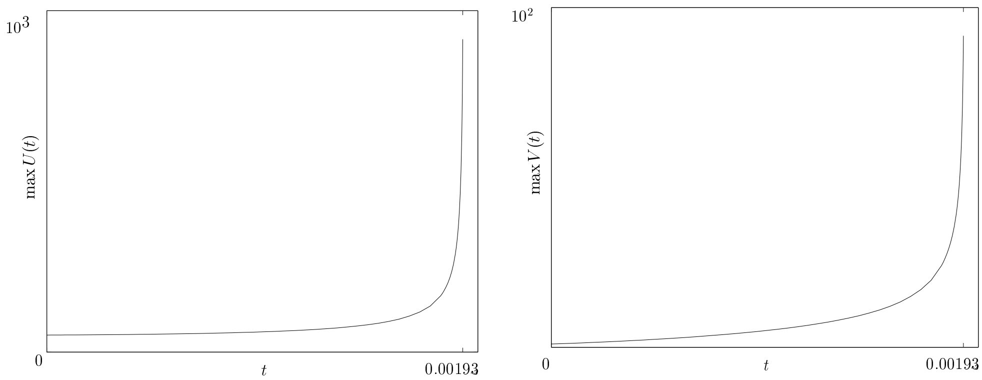

FIGURE 2. $\max U(t)$ AND $\max V(t)$, BOTH BLOWING UP.