Asymptotic behaviour for a numerical approximation of a parabolic problem with blowing up solutions

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Abstract

In this paper, we study the asymptotic behaviour of a semidiscrete numerical approximation for $u_t = u_{xx} + u^p$ in a bounded interval, $(0,1)$, with Dirichlet boundary conditions. We focus in the behaviour of blowing up solutions. We find that the blow-up rate for the numerical scheme is the same as for the continuous problem. Also we find the blow-up set for the numerical approximations and prove that it is contained in a neighbourhood of the blow-up set of the continuous problem when the mesh parameter is small enough.

1 Introduction.

In this paper, we study the behavior of a semidiscrete approximation of the following parabolic problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + u^p(x,t) & \text{in } (0,1)\times[0,T), \\ u(1,t) = u(0,t) = 0 & \text{on } [0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{on } [0,1]. \end{cases} \tag{1.1}$$

We assume that u_0 is nontrivial, smooth and verifies $u_0(0) = u_0(1) = 0$ in order to guarantee that $u \in C^{2,1}$.

A remarkable (and well known) fact is that the solution may develop singularities in finite time, no matter how smooth u_0 is. For many differential

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equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow-up). Typical examples where this happens are problems involving reaction terms in the equation like (1.1) (see [SGKM], [P] and the references therein).

In our problem one has a reaction term in the equation of power type and if $p > 1$ this blow up phenomenum occurs in the sense that there exists a finite time T such that $\lim_{t\to T} \|u(\cdot,t)\|_\infty = +\infty$ for initial data large enough (see [SGKM]). The blow up set is localized at single points, that is, there exists $x_1, .., x_k$ such that $u(x,t)$ remains bounded up to T for every $x \neq \{x_1, ..., x_k\}$ (see [CM] and also [FMc], [M], [MW], [W]). The blow-up rate at these blow-up points is given by $u(x_i,t) \sim (T-t)^{-\frac{1}{p-1}}$ (see [GK], [HV1], [HV2]).

In this paper we are interested in numerical approximations of (1.1).

Since the solution u develops a singularity in finite time, it is an interesting question what can be said about numerical approximations of this kind of problems. For previous work on numerical approximations of blowing up solutions of (1.1) we refer to [ALM1], [ALM2], [BB2], [BK], [BHR], [C], [LR], [NU] the survey [BB] and references therein.

In [ALM1] and [ALM2] the authors analyze a semidiscrete scheme (keeping t continuous). They find a necessary condition for the appearance of the blow-up phenomenum ($p > 1$ and some assumptions on the initial data) and prove the convergence of the blow up time of the discrete problem to that of the continuous one when the mesh parameter goes to zero (see also [BB2]).

Here we introduce the same semidiscrete scheme analyzed there by using piecewise linear finite elements with mass lumping in a uniform mesh for the space variable (it is well known that this discretization in space coincides with the classic central finite difference second order scheme).

We denote with $U(t) = (u_1(t), ..., u_{N+1}(t))$ the values of the numerical approximation at the nodes $x_i = (i-1)h$ at time t ($h = 1/N$). Then $U(t)$ verifies the following equation:

$$\begin{aligned} &MU'(t) = -AU(t) + MU^p(t) \\ &U(0) = u_0^I \end{aligned} \tag{1.2}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial datum, u_0. Writing this equation explicitly we obtain the following ODE system,

$$\begin{cases} u_1(t) = 0, \\ u_k'(t) = \frac{1}{h^2}(u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)) + u_k^p(t), \qquad 2 \le k \le N, \\ u_{N+1}(t) = 0, \\ u_k(0) = u_0(x_k), \qquad 1 \le k \le N+1. \end{cases} \tag{1.3}$$

In [ALM2] it was proved that this method converges uniformly under the hypothesis that $u \in C^{4,1}$. Under this assumption the authors find that

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^2$$

In Section 3, we start our analysis of (1.3) and prove the following convergence theorem for regular solutions.

Theorem 1.1 *Let u be a regular solution of* (1.1) *($u \in C^{2,1}([0,1] \times [0,T-\tau])$ and u_h the numerical approximation given by* (1.3) *then there exists a constant C depending on $\|u\|$ in $C^{2,1}([0,1] \times [0,T-\tau]$ such that*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{\frac{3}{2}}.$$

We remark that we are only assuming that $u \in C^{2,1}$ but our convergence rate is not optimal (we have $h^{\frac{3}{2}}$ and not h^2 like in [ALM2]).

For this scheme we say that a solution has finite time blow-up if there exists a finite time T_h with

$$\lim_{t\to T_h} \|U(t)\|_\infty = \lim_{t\to T_h} \max_j u_j(t) = +\infty.$$

We want to describe when the blow-up phenomena occurs for (1.3). In Section 3 we prove the following Theorem,

Theorem 1.2 *Positive solutions of* (1.3) *blow up in finite time if $p > 1$ and $U(0)$ is large in the following sense; let*

$$\Phi_h(U) \equiv \frac{1}{2}\langle A^{1/2}U; A^{1/2}U\rangle - \sum_{i=1}^{N} m_{ii}\frac{U_i^{p+1}}{p+1},$$

then, if there exists t_0 such that $\Phi_h(U(t_0)) < 0$, u_h has finite time blow-up. Moreover, there exists a constant C that does not depends on h such that

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

We want to remark that the blow-up condition, $p > 1$ and $\Phi_h(U(t_0)) < 0$, is analogous to that of the continuous problem, see [B]. In [ALM2] it is proved that if $p > 1$ there exists solutions of (1.3) that blow up in finite time under different assumptions on the solution u_h.

In [ALM2] under some assumptions over u_h (symmetry or monotonicity in time) it is proved the convergence of the numerical blow-up time, T_h, to the continuous one, T, when the mesh parameter goes to zero.

As a corollary of Theorem 1.2 we show that if u blows up then u_h also blows up for every h small enough and we extend the convergence of the blow-up times to solutions without symmetry nor monotonicity assumptions.

Corollary 1.1 *Let u_0 be an initial datum for* (1.1) *such that u blows up, then u_h blows up for every h small enough, and if we call T and T_h the blow-up times for u and u_h respectively, we have*

$$\lim_{h\to 0} T_h = T.$$

In Section 4 we arrive at the main points of this article, the asymptotic behaviour (blow-up rate) and the localization of blow-up points (blow-up set) of u_h for fixed h.

Concerning the blow-up rate for (1.3) we have the following Theorem,

Theorem 1.3 *Assume that $p > 1$ and that u_h blows up in finite time, T_h, then*

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}},$$

in the sense that there exists two positive constants c, C such that

$$c(T_h - t)^{-\frac{1}{p-1}} \le \max_j u_j(t) \le C(T_h - t)^{-\frac{1}{p-1}}.$$

Moreover,

$$\lim_{t\to T_h} \max_j u_j(t)(T_h - t)^{-\frac{1}{p-1}} = C_p$$

where

$$C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}.$$

We have to remark that the constant C_p that appears in Theorem 1.3 is the same that appears in the ODE $u'(t) = u^p(t)$ that has solutions of the form $u(t) = C_p(T-t)^{-\frac{1}{p-1}}$. Also we remark that the asymptotic behaviour is the same for the continuous problem (1.1). In fact, it holds $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty = C_p$ (see [GK], [HV1], [HV2]).

Now we turn our attention to the blow-up set for u_h, $B(u_h)$. Let F be the set of indices j such that $\lim_{t\to T_h}(T_h - t)^{\frac{1}{p-1}} u_j(t) = C_p$. By Theorem 1.3, $F \ne \emptyset$. Clearly, $F \subset B(u_h)$. With the blow-up rate given by Theorem 1.3 we observe a propagation property of blow-up points, we prove that the number of nodes adjacent to F that go to infinity (i.e. blow-up points for u_h) is finite and determined by p. We remark that in the continuous case the blow-up set is composed by single points, see [CM] and also [FMc], [M]).

Theorem 1.4 *Let F be the set of nodes, n, such that*

$$\lim_{t\to T_h} u_n(t)(T_h - t)^{\frac{1}{p-1}} = C_p$$

Then the blow-up propagates in the following way, let $p > 1$ and $K \in \mathbb{N}$ satisfies $\frac{K+2}{K+1} < p \le \frac{K+1}{K}$. We call $d(i)$ the distance from x_i to F measured in nodes. Then the solution of (1.3), *u_h, blows up exactly at K nodes near F,*

$$u_i(t) \to +\infty \qquad \Longleftrightarrow \qquad d(i) \le K.$$

Moreover, if $d(i) \le K$, the asymptotic behaviour is given by

$$u_i(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(i)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(i) = K$

$$u_i(t) \sim \ln(T_h - t).$$

We want to remark that more than one node can go to infinity but the asymptotic behavior imposes $\frac{u_j(t)}{u_i(t)} \to 0$ $(t \to T_h)$ if $d(i) < d(j)$. This propagation property to nodes that lies at distance one in the symmetric case was first proved in [C] and in [N].

In the blow-up case, $p > 1$ and the number of blow-up points outside F is finite and depends on the power p but is independent of h. This gives a sort of "numerical localization" of the blow-up set of u_h near the blow-up set of u when the parameter h is small enough.

Theorem 1.5 *Let u_0 an initial datum for* (1.1) *such that u and u_h blows up for every h small enough, then if we call $B(u)$ and $B(u_h)$ the blow-up sets for u and u_h respectively, we have that, given $\varepsilon > 0$ there exists h_0 such that for every $0 < h \le h_0$,*

$$B(u_h) \subset B(u) + (-\varepsilon, \varepsilon) \qquad \forall h \le h_0(\varepsilon).$$

Moreover, if u_0 is symmetric and increasing in $[0, 1/2]$ we have that

$$B(u_h) \subset \{1/2\} + [-Kh, Kh] \qquad \forall h \le h_0.$$

We want to remark that Corollary 1.1 and Theorems 1.2, 1.3 and 1.5 shows that the numerical scheme (1.3) has asymptotic properties that are similar to that of the continuous problem (1.1) when the mesh parameter is small.

The paper is organized as follows: in Section 2 we prove our convergence result (Theorem 1.1), in Section 3 our blow-up result (Theorem 1.2 and Corollary 1.1) and finally in Section 4 we prove our main results, the blow-up rate and localization of blow-up points for u_h (Theorems 1.3, 1.4 and 1.5).

2 Convergence of the numerical scheme.

In this Section we prove a uniform convergence result for regular solutions of the numerical scheme (1.3).

For any $\tau > 0$ we want that $u_h \to u$ (when $h \to 0$) uniformly in $[0, T - \tau]$. This is a natural requirement since on such an interval the exact solution is regular. Approximations of regular problems like ours have been analyzed in [ALM2]. In that paper an error estimate of order h^2 in the L^∞ norm is proved under the hypothesis $u \in C^{4,1}$.

In particular, uniform convergence can be obtained by using standard inverse inequalities. In the following Theorem we give a proof of the L^2 convergence for a problem like (1.1) with $f(u) = u^p$ replaced by a globally Lipschitz function $g(u)$ and considering mass lumping. As a corollary, we will obtain uniform convergence for problem (1.1).

Theorem 2.1 *Let u be the solution of a problem like* (1.1) *with $f(u) = u^p$ replaced by a globally Lipschitz function $g(u)$ and let u_h its semidiscrete approximation obtained by finite elements with mass lumping. If $u \in C^{2,1}([0,1]\times[0,T_1])$ for some $T_1 > 0$ then, there exists a constant C depending on $\|u\|$ in $C^{2,1}$ and T_1 such that:*

$$\|u - u_h\|_{L^\infty([0,T_1],L^2)} \le Ch^2$$

Proof: In this proof we use the notation $L^2 = L^2((0,1))$ that refers to the L^2 norm in the x variable for each t (we will use analogous notations for other norms below) and u' for the derivative respect to time, u_t.

As u is a solution of (1.1) it satisfies

$$\int_0^1 u'v + \int_0^1 u_x v_x = \int_0^1 g(u)v \qquad \forall v \in H_0^1$$

The numerical scheme (1.3) is equivalent to

$$\int_0^1 ((u_h)'v)^I + \int_0^1 u_x v_x = \int_0^1 (g(u_h)v)^I \qquad \forall v \in V_h$$

Hence we have that $e = u - u_h$ satisfies the following error equation,

$$\int_0^1 (e'v)^I\,dx + \int_0^1 e_x v_x\,dx = \int_0^1 (g(u)v - (g(u_h)v)^I) + \int_0^1 ((u'v)^I - u'v)\,dx$$

for all $v \in V_h$. Writing

$$e = u - u_h + u^I - u^I = u - u^I + \eta$$

and using known error estimates for Lagrange interpolation it rest to estimate $\eta = u^I - u_h$.

First, it is easy to see that,

$$\int_0^1 (u-u^I)_x v_x = 0 \qquad \forall v \in V_h,$$

and therefore, replacing in the error equation we have an equation for η,

$$\int_0^1 (\eta' v)^I + \int_0^1 \eta_x v_x = \int_0^1 (g(u)v - (g(u_h)v)^I) +$$

$$\int_0^1 ((u'v)^I - u'v) - \int_0^1 ((u' - (u^I)')v)^I \qquad \forall v \in V_h.$$

In particular if we choose $v = \eta \in V_h$ we obtain

$$\frac{1}{2}\frac{d}{dt}(\int_0^1 (\eta^2)^I) + \int_0^1 (\eta_x)^2 =$$

$$\int_0^1 (g(u)\eta - (g(u_h)\eta)^I) + \int_0^1 ((u'\eta)^I - u'\eta) =$$

$$= I + II.$$

First, let us estimate I.

$$|I| = \left| \int_0^1 (g(u)\eta - (g(u_h)\eta)^I) \right| =$$

$$\left| \int_0^1 g(u)\eta - (g(u^I)\eta)^I + (g(u^I)\eta)^I - (g(u_h)\eta)^I) \right| =$$

$$\left| \int_0^1 (g(u) - g(u^I))\eta + ((g(u^I) - g(u_h))\eta)^I + g(u^I)\eta - (g(u^I)\eta)^I) \right| \leq$$

using that g is Lipschitz,

$$C\int_0^1 |u-u^I||\eta| + C\int_0^1 (\eta^2)^I + \int_0^1 \left| g(u^I)\eta - (g(u^I)\eta)^I \right| \leq$$

$$C\|u-u^I\|_{L^2}^2 + C\|\eta\|_{L^2}^2 + \int_0^1 \left| g(u^I)\eta - (g(u^I)\eta)^I \right| \leq$$

$$Ch^4 + C\|\eta\|_{L^2}^2 + \int_0^1 \left| g(u^I)\eta - (g(u^I)\eta)^I \right|.$$

So it rest to estimate

$$\int_0^1 \left| g(u^I)\eta - (g(u^I)\eta)^I \right|.$$

For each subinterval I_j of the partition we know that,

$$\|g(u^I)\eta) - (g(u^I)\eta)^I\|_{L^1(I_j)} \leq Ch^2\|(g(u^I)\eta)_{xx}\|_{L^1(I_j)} \leq$$

$$Ch^2\|(g(u^I))_x\eta_x\|_{L^1(I_j)} + Ch^2\|(g(u^I))_{xx}\eta\|_{L^1(I_j)}$$

because u^I and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|(u^I)_x\|_{L^2} \leq C\|u\|_{H^1}$ we obtain,

$$\int_0^1 \left|g(u^I)\eta - (g(u^I)\eta)^I\right| \leq Ch^2 \int_0^1 |(g(u^I))_x||\eta_x|\, dx + Ch^2 \int_0^1 |(g(u^I))_{xx}||\eta|\, dx$$

$$\leq Ch^4 + \varepsilon\|\eta_x\|_{L^2}^2 + C\|\eta\|_{L^2}^2.$$

Since ε will be fixed later on, we write C instead of C_ε. The constant C depends on $\|u\|$ in $C^{2,1}$.

In order to bound II we decompose it in the following form,

$$II = \int_0^1 ((u'\eta)^I - u'\eta)\, dx = \int_0^1 ((u'\eta)^I - (u')^I\eta)\, dx + \int_0^1 ((u')^I\eta - u'\eta)\, dx$$

We proceed as before, for each subinterval I_j of the partition we know that,

$$\|((u')^I\eta)^I - (u')^I\eta\|_{L^1(I_j)} \leq Ch^2\|((u')^I\eta)_{xx}\|_{L^1(I_j)} \leq \|((u')^I)_x\eta_x\|_{L^1(I_j)}$$

because $(u')^I$ and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|((u')^I)_x\|_{L^2} \leq C\|u'\|_{H^1}$ we obtain,

$$\int_0^1 ((u'\eta)^I - (u')^I\eta)\, dx \leq Ch^2 \int_0^1 |((u')^I)_x||\eta_x|\, dx$$

$$\leq Ch^2\|((u')^I)_x\|_{L^2((0,1))}\|\eta_x\|_{L^2} \leq Ch^4 + \varepsilon\|\eta_x\|_{L^2}^2$$

It rests to estimate the second term of II. We have,

$$\int_0^1 ((u')^I - u')\eta\, dx \leq \|(u')^I - u'\|_{L^2}\|\eta\|_{L^2}$$

$$\leq \|(u')^I - u'\|_{L^2}^2 + \|\eta\|_{L^2}^2 \leq Ch^4 + \|\eta\|_{L^2}^2$$

Collecting all the bounds we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + \int_0^1 |\eta_x|^2$$

$$\leq Ch^4 + C\|\eta\|_{L^2}^2 + 3\varepsilon\|\eta_x\|_{L^2}^2$$

We choose ε such that $3\varepsilon = 1/2$ and we obtain,

$$\frac{d}{dt}\int_0^1 (\eta^2)^I + \int_0^1 |\eta_x|^2 \leq Ch^4 + C\|\eta\|_{L^2}^2$$

Since $\int_0^1 (\eta^2)^I \, dx \sim \|\eta\|_{L^2}^2$ we can apply Gronwall's Lemma to obtain for $t \in [0, T_1]$,

$$\|\eta(t)\|_{L^2} + (\int_0^{T_1} \|\eta_x\|_{L^2}^2 \, dt)^{1/2} \leq Ce^{C(T_1)}h^2.$$

In particular,

$$\|\eta\|_{L^2} \leq C(u, T_1)h^2.$$

and hence,

$$\|e\|_{L^2} \leq \|u - u^I\|_{L^2} + \|\eta\|_{L^2} \leq C(u, T_1)h^2. \square$$

As a corollary of Theorem 2.1 we can prove Theorem 1.1.

Corollary 2.1 *Let u be the solution of* (1.1) *and u_h its approximation defined by* (1.3)*. Given $\tau > 0$ there exists a constant C depending on τ and $\|u\|$ in $C^{2,1}([0,1] \times [0, T - \tau]$ such that, for h small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{\frac{3}{2}}$$

Proof: It is known that before the blow up time u is regular, more precisely, $u \in C^{2,1}([0,1] \times [0, T - \tau])$. Let $g(u)$ be a globally Lipschitz function which agrees with $f(u) = u^p$ for $u \leq 2M$ where $M = \|u\|_{L^\infty([0,1]\times[0,T-\tau])}$. Let $\overline{u}$ and $\overline{u}_h$ be the exact and approximate solutions of a problem like (1.1) with $f(u) = u^p$ replaced by $g(u)$. By uniqueness $u = \overline{u}$ in $[0,1] \times [0, T - \tau]$. A bound for $\|\overline{u} - \overline{u}_h\|_{L^\infty}$ can be obtained from Theorem 2.1. Indeed, it is enough to bound $\|\overline{u}^I - \overline{u}_h\|_{L^\infty}$, and using a standard inverse inequality (see [Ci]) we have,

$$\|\overline{u}^I - \overline{u}_h\|_{L^\infty} \leq Ch^{-\frac{1}{2}}\|\overline{u}^I - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}$$

$$\leq Ch^{-\frac{1}{2}}\{\|\overline{u}^I - \overline{u}\|_{L^\infty([0,T-\tau],L^2)} + C\|\overline{u} - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}\} \leq Ch^{\frac{3}{2}}$$

with C depending on u and the constant in Theorem 2.1 and so on τ.

Consequently, for h small enough $|\overline{u}_h| \leq 2M$. Therefore $u_h^p = f(\overline{u}_h) = g(\overline{u}_h)$ and so $\overline{u}_h$ is the finite element approximation of u and, by uniqueness $\overline{u}_h = u_h$ which concludes the proof. $\square$

3 Blow-up for the numerical scheme.

In this section we prove Theorem 1.2 which states a condition for the existence of blow-up of the discrete solution.

Let us begin by the following Lemma,

Lemma 3.1 *If $U(t_0)$ verifies that*

$$\Phi_h(U(t_0)) \equiv \frac{1}{2}\langle A^{1/2}U(t_0); A^{1/2}U(t_0)\rangle - \sum_{i=1}^{N+1} m_{ii}\frac{((U(t_0))_i)^{p+1}}{p+1} < 0,$$

then u_h is unbounded and hence $\lim_{t\nearrow T_h} \max u_j(t) = +\infty$. Here

$$T_h = \max\{t \text{ such that } u_h(s) \text{ is defined for } s \in [0,t]\}.$$

Proof: To motivate the proof, let

$$\Phi(u)(t) \equiv \int_0^1 \frac{(u_x(s,t))^2}{2}\, ds - \int_0^1 \frac{(u(s,t))^{p+1}}{p+1}\, ds$$

then, Φ is a Lyapunov functional for (1.1) and if $\Phi(u(\cdot, t_0)) < 0$ then u blows up in finite time (see [B]). The discrete analogous of Φ is

$$\Phi_h(U)(t) \equiv \frac{1}{2}\langle A^{1/2}U(t); A^{1/2}U(t)\rangle - \sum_{i=1}^{N+1} m_{ii}\frac{(U_i)^{p+1}(t)}{p+1}.$$

Now let us compute the derivative of $\Phi_h(U)(t)$.

$$\frac{d}{dt}\Phi_h(U)(t) = \langle A^{1/2}U(t); A^{1/2}U'(t)\rangle - \sum_{i=1}^{N+1} m_{ii}(u_i)^p u_i'(t) =$$

$$\langle AU(t) - MU^p(t); U'(t)\rangle = -\langle MU'(t); U'(t)\rangle$$

Hence, this Φ_h is a Lyapunov functional for (1.2) in the sense that

$$\frac{d}{dt}\Phi_h(U) \le 0.$$

Moreover $\frac{d}{dt}\Phi_h(U) < 0$ unless U is independent of t.

Now, let us see that the steady states of (1.3) have positive "energy" (i.e. $\Phi_h(W) \ge 0$). Let $W = (w_1, ..., w_N)$ be a stationary solution of (1.2), then we have

$$0 = -AW + M(W)^p. \tag{3.1}$$

Multiplying (3.1) by W, we obtain

$$0 = -\frac{1}{2}\langle A^{1/2}W; A^{1/2}W\rangle + \frac{p+1}{2}\sum_{i=1}^{N} m_{ii}\frac{(w_i)^p w_i}{p+1} \ge$$

$$\geq -\frac{1}{2}\langle A^{1/2}W; A^{1/2}W\rangle + \sum_{i=1}^{N} m_{ii}\frac{(w_i)^{p+1}}{p+1} = -\Phi_h(W).$$

Then, as every global solution that is bounded must converge to a stationary one (see [H]), if $U(t_0)$ satisfies $\Phi_h(U(t_0)) < 0$ it must be unbounded. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2: If $\Phi_h(U(t_0)) < 0$ by the previous lemma we have that $U(t)$ is unbounded, then there exists a time t and a node j such that $-\frac{2}{h^2}u_j(t) + u_j^p(t) \geq \frac{1}{2}u_j^p(t)$. Hence,

$$u_j'(t) = \frac{1}{h^2}(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)) + u_j^p(t) \geq \frac{1}{2}u_j^p(t)$$

As a consequence of this $u_j(s)$ must be increasing for $s \geq t$ and verifies

$$u_j'(s) \geq \frac{1}{2}u_j^p(s).$$

As $p > 1$, u_j goes to infinity in finite time, and hence $U(t)$ has finite time blow-up in the sense that there exists a finite time, T_h, such that $\lim_{t\to T_h}\max u_j(t) = +\infty$. Moreover, we have that, for every $t \in [t_0, T_h)$

$$\frac{d}{dt}\langle MU(t), U(t)\rangle = 2\langle MU'(t), U(t)\rangle =$$

$$2\langle -AU(t), U(t)\rangle + 2\langle MU^p(t), U(t)\rangle =$$

$$-4\Phi_h(U(t)) + \frac{2(p-1)}{p+1}\langle MU^p(t), U(t)\rangle \geq$$

$$4|\Phi_h(U(t))| + \frac{2(p-1)}{p+1}(\langle MU(t), U(t)\rangle)^{\frac{p+1}{2}}.$$

Integrating between t_0 and T_h we obtain

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

where C depends only on p. □

As a consequence of this bound we get Corollary 1.1.

Proof of Corollary 1.1: First we observe that if u blows up in finite time T then

$$\lim_{t\to T}\Phi(u(\cdot,t)) = -\infty$$

(see [CPE]).

Using the convergence result (Theorem 1.1) one can check that

$$\lim_{h\to 0}\Phi_h(u_h(\cdot,t_0))=\Phi(u(\cdot,t_0)).$$

Therefore we conclude that if u blows up in finite time then $\Phi_h(u_h(\cdot,t_0))<0$ for some t_0 and every h small enough, and hence u_h blows up in finite time, T_h. To prove the convergence of the blow-up times we are going to use the bound,

$$T_h - t_0 \le \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}. \tag{3.2}$$

Given $\varepsilon>0$, we can choose M large enough to ensure that

$$\left(\frac{C}{M^{\frac{p-1}{p+1}}}\right)\le \frac{\varepsilon}{2}.$$

As u blows up at time T we can choose $\tau<\frac{\varepsilon}{2}$ such that

$$-\Phi(u(\cdot,T-\tau)\ge 2M.$$

If h is small enough,

$$-\Phi_h(U(T-\tau)\ge M,$$

and hence by (3.2),

$$T_h-(T-\tau)\le\left(\frac{C}{(-\Phi_h(U(T-\tau)))^{\frac{p-1}{p+1}}}\right)\le\left(\frac{C}{M^{\frac{p-1}{p+1}}}\right)\le\frac{\varepsilon}{2}.$$

Therefore,

$$|T_h-T|\le|T_h-(T-\tau)|+|\tau|<\varepsilon.\Box$$

4 Blow-up rate and blow-up set.

In this Section we prove the converge of the blow-up times (Corollary 1.1) and we find the blow-up rate (Theorem 1.3) and the localization of blow-up points (Theorems 1.4 and 1.5).

From now on we consider positive solutions of (1.3) with h fixed and we denote by C a positive constant that may depend on h but not on t and it is different in each step of the proofs.

Lemma 4.1 *Let u_h be a solution of* (1.3) *that blows up at time T_h, then there exists two constants c, C depending on h such that*

$$c(T_h-t)^{-\frac{1}{p-1}}\le\max_j u_j(t)\le C(T_h-t)^{-\frac{1}{p-1}}$$

Proof: First, we observe that, as

$$u_j'(t) = \frac{1}{h^2}(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)) + u_j^p(t)$$

we have that

$$w(t) = \sum_{i=1}^{N} u_i(t)$$

verifies

$$w'(t) = \sum_{i=1}^{N} u_i(t) = \sum_{i=1}^{N} (u_i)^p(t) + \frac{u_N(t) - u_1(t)}{h^2} \le$$

$$C\left(\sum_{i=1}^{N} u_i(t)\right)^p + \frac{u_N(t) - u_1(t)}{h^2}$$

As u_h blows up at time T_h, we have that there exists t_0 such that for every $t \in [t_0, T_h)$ it holds

$$w'(t) \le C w^p(t).$$

For $t \in [t_0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h} \frac{w'(s)}{w^p(s)}\, ds \le C(T_h - t),$$

changing variables we get

$$\int_{w(t)}^{+\infty} \frac{1}{s^p}\, ds \le C(T_h - t).$$

Hence

$$w(t) \ge C(T_h - t)^{-\frac{1}{p-1}}.$$

Using that there exists a constant $C = C(h)$ such that

$$\max_j u_j(t) \ge C \sum_{i=1}^{N} u_i(t)$$

we have

$$\max_j u_j(t) \ge C(T_h - t)^{-\frac{1}{p-1}}$$

To prove the other inequality we proceed as follows, as $\max u_j(t) \to +\infty$ when $t \to T_h$, we have $\frac{2}{h^2} u_n(t) \le \frac{1}{2} u_n^p(t)$ for every t close to T_h for some $n \in \{2, 3, ...N\}$ in this case we have

$$u_n'(t) = \frac{1}{h^2}(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + u_n^p(t) \ge \frac{1}{2} u_n^p(t).$$

Integrating again over $[t, T_h]$ we obtain

$$\int_t^{T_h} \frac{u_n'(s)}{u_n^p(s)}\, ds \geq \frac{1}{2}(T_h - t),$$

changing variables

$$\int_{u_n(t)}^{+\infty} \frac{1}{s^p}\, ds \geq \frac{1}{2}(T_h - t).$$

Hence

$$u_n(t) \leq 2^{\frac{1}{p-1}} C_p (T_h - t)^{-\frac{1}{p-1}}.$$

So $\max_j u_j(t)$ verifies

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$$

in the sense that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}.\square$$

To conclude the proof of theorem 1.3 we make the following change of variables (inspired by [GK], [HV1], [HV2]),

$$\begin{cases} y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t) \\ \\ (T_h - t) = e^{-s} \end{cases} \tag{4.1}$$

These new variables, $Y = (y_k(s))$, verify

$$\begin{cases} y_1(s) = 0, \\ \\ y_k'(s) = \frac{e^{-s}}{h^2}(y_{k+1}(s) - 2y_k(s) + y_{k-1}(s)) - \frac{1}{p-1} y_k(s) + y_k^p(s), \\ \\ y_{N+1}(s) = 0, \\ \\ y_k(-\ln(T_h)) = (T_h)^{\frac{1}{p-1}} u_0(x_k), \qquad 1 \leq k \leq N+1. \end{cases} \tag{4.2}$$

We observe that, as $\max u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}$, we have that $y_j(s)$ are uniformly bounded,

$$y_j(s) \leq C \qquad \forall s > -\ln(T_h).$$

Lemma 4.2 *If there exists s_0 such that*

$$y_j^p(s_0) - \frac{1}{p-1} y_j(s_0) < -Ce^{-s_0}$$

then

$$y_j(s) \to 0 \qquad (s \to \infty).$$

Proof: We observe that $y_j(s)$ verifies

$$y_j'(s) = \frac{e^{-s}}{h^2}(y_{j+1}(s) - 2y_j(s) + y_{j-1}(s)) - \frac{1}{p-1}y_j(s) + y_j^p(s) \leq$$

$$Ce^{-s} - \frac{1}{p-1}y_j(s) + y_j^p(s)$$

Let $w(s)$ be a solution of

$$w'(s) = Ce^{-s} - \frac{1}{p-1}w(s) + w^p(s)$$

with $w(s_0) = y_j(s_0)$. We observe that, $w(s_0) < C_p$ and

$$w'(s_0) = Ce^{-s_0} - \frac{1}{p-1}y_j(s_0) + y_j^p(s_0) < 0.$$

We claim that $w'(s) < 0$ for all $s > s_0$. To prove this claim, we argue by contradiction. Assume that there exists a first time s_1 such that $w'(s_1) = 0$. At that time s_1 we have

$$w''(s_1) = -Ce^{-s_1} - \frac{1}{p-1}w'(s_1) + pw^{p-1}(s_1)w'(s_1) = -Ce^{-s_1}.$$

Hence $w''(s_1) < 0$. Therefore w' is decreasing at s_1, a contradiction.

So we have proved that $w(s)$ is decreasing for all $s > s_0$, and $w(s) \geq 0$ hence there exists $l = \lim_{s\to\infty} w(s)$. As $\lim_{s\to\infty} w'(s) = 0$ we have that

$$l^p - \frac{1}{p-1}l = 0.$$

We have that $w(s_0) < C_p$ and w is decreasing for $s \geq s_0$, so we conclude that $l \neq C_p$ and hence $l = 0$.

By a comparison argument we have that

$$0 \leq y_j(s) \leq w(s) \to 0 \qquad (s \to \infty),$$

hence $y_j(s) \to 0$ $(s \to \infty)$. □

Lemma 4.3 *If there exists s_0 such that*

$$y_j^p(s_0) - \frac{1}{p-1}y_j(s_0) > Ce^{-s_0}$$

then $y_j(s)$ blows up in finite time $\tilde{s}$.

Proof: As before, we observe that $y_j(s)$ verifies

$$y_j'(s) = \frac{e^{-s}}{h^2}(y_{j+1}(s) - 2y_j(s) + y_{j-1}(s)) - \frac{1}{p-1}y_j(s) + y_j^p(s) \geq$$

$$-Ce^{-s} - \frac{1}{p-1}y_j(s) + y_j^p(s)$$

Let $w(s)$ be a solution of

$$w'(s) = -Ce^{-s} - \frac{1}{p-1}w(s) + w^p(s)$$

with $w(s_0) = y_j(s_0)$. We observe that, $w(s_0) > C_p$ and

$$w'(s_0) = -Ce^{-s_0} - \frac{1}{p-1}y_j(s_0) + y_j^p(s_0) > 0.$$

We claim that $w'(s) > 0$ for all $s > s_0$. To prove this claim, we argue by contradiction. Assume that there exists a first time s_1 such that $w'(s_1) = 0$, at that time s_1 we have

$$w''(s_1) = Ce^{-s_1} - \frac{1}{p-1}w'(s_1) + pw^{p-1}(s_1)w'(s_1) = Ce^{-s_1}.$$

Hence $w''(s_1) > 0$. Therefore w' is increasing at s_1, a contradiction.

So we have proved that $w(s)$ is increasing for all $s > s_0$, hence there exists $\varepsilon > 0$ such that

$$w'(s) \geq \varepsilon w^p(s)$$

and then, using that $p > 1$, we have that w blows up in finite time s_2.

As before, we can use a comparison argument to get

$$y_j(s) \geq w(s) \to +\infty \qquad s \to s_2 < \infty$$

hence $y_j(s)$ blows up in finite time. □

Lemma 4.4 *Let $y_j(s)$ be a solution of* (4.2) *then each y_j verifies*

$$\begin{cases} y_j(s) \to 0 \qquad (s \to +\infty), \\ \text{or} \\ y_j(s) \to C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \qquad (s \to +\infty), \\ \text{or} \\ y_j(s) \text{ blows up in finite time.} \end{cases} \tag{4.3}$$

Proof: From the previous two lemmas we can conclude that, if $y_j(s)$ does not converge to zero and does not blow up in finite time, then $y_j(s) \to C_p$. To see this fact, we observe that $y_j(s)$ is global and satisfies

$$Ce^{-s} \geq y_j^p(s) - \frac{1}{p-1} y_j(s) \geq -Ce^{-s}.$$

Then

$$y_j^p(s) - \frac{1}{p-1} y_j(s) \to 0 \qquad (s \to +\infty).$$

As y_j is continuous, bounded and does not go to zero, we conclude that $y_j(s) \to C_p$. $\square$

Proof of Theorem 1.3 We just observe that, from Lemma 4.1, $c \leq \max y_j(s) \leq C$, so $y_j(s)$ is global and also $\max y_j(s)$ does not go to zero, hence, using Lemma 4.4, we get that

$$\lim_{s\to\infty} \max_j y_j(s) = C_p.$$

In the original variables $\{u_j,\ t\}$ this is equivalent to

$$\lim_{t\to T_h} (T_h - t)^{\frac{1}{p-1}} \max_j u_j(t) = C_p. \square$$

Now we turn our attention to the blow-up set. We begin by the proof of the propagation result, Theorem 1.4.

Proof of Theorem 1.4: Let $F = \{j_1, j_2, , , j_k\}$ be the set of nodes such that

$$y_j(s) \to C_p \qquad (s \to \infty).$$

Let K be such that

$$\frac{K+2}{K+1} < p \leq \frac{K+1}{K}.$$

We want to see that the blow-up propagates to the K nodes adjacent to F.

To see this let us begin by a considering a node i_1 with $d(i_1) = 1$, then there exists $j \in F$ that is adjacent to i_1. We can assume that i_1 is on the left of j, so $i_1 = j - 1$.

By Lemma 4.4, as $j - 1 \notin F$, we have that $y_{j-1}(s) \to 0$. We want to obtain the asymptotic behaviour of $y_{j-1}(s)$. To do this, first we get a bound as follows, $y_{j-1}(s)$ verifies

$$y'_{j-1}(s) = \frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) - \frac{1}{p-1} y_{j-1}(s) + y_{j-1}^p(s) \leq$$

$$4Ce^{-s} - \frac{1}{p-1} y_{j-1}(s) + y_{j-1}^p(s).$$

Using that $y_{j-1}(s) \to 0$ we have that, given $\varepsilon > 0$, for every $s > s_0$

$$y'_{j-1}(s) \le 4Ce^{-s} - \left(\frac{1}{p-1} - \varepsilon\right) y_{j-1}(s)$$

Let $w(s)$ be a solution of

$$w'(s) = 4Ce^{-s} - \left(\frac{1}{p-1} - \varepsilon\right) w(s)$$

with $w(s_0) \ge y_{j-1}(s_0)$. Integrating this equation we get

$$w(s) = C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1} - \varepsilon)s}$$

By a comparison argument we get that for every $s > s_0$,

$$y_{j-1}(s) \le w(s) = C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1} - \varepsilon)s} \tag{4.4}$$

Now we go back to

$$y'_{j-1}(s) = \frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) - \frac{1}{p-1} y_{j-1}(s) + y^p_{j-1}(s).$$

We have,

$$y'_{j-1}(s) + \frac{1}{p-1} y_{j-1}(s) = \frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) + y^p_{j-1}(s)$$

then,

$$(e^{\frac{1}{p-1}s} y_{j-1}(s))' = e^{\frac{1}{p-1}s} \left(\frac{e^{-s}}{h^2}(y_j(s) - 2y_{j-1}(s) + y_{j-2}(s)) + y^p_{j-1}(s) \right).$$

Integrating between s_0 and s, we get

$$y_{j-1}(s) = e^{-\frac{1}{p-1}s} \left(C_1 + \int_{s_0}^{s} e^{\frac{1}{p-1}\sigma} \left(\frac{e^{-\sigma}}{h^2}(y_j - 2y_{j-1} + y_{j-2}) + y^p_{j-1} \right) d\sigma \right) =$$

$$e^{-\frac{1}{p-1}s} \left(C_1 + \int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma} \left(\frac{1}{h^2}(y_j - 2y_{j-1} + y_{j-2}) + e^{\sigma} y^p_{j-1} \right) (\sigma)\, d\sigma \right).$$

Using (4.4) we have that

$$e^s y^p_{j-1}(s) \le C_1 e^{-(p-1)s} + C_2 e^{-(\frac{p}{p-1} - p\varepsilon - 1)s} \to 0 \qquad (s \to \infty).$$

Hence, as $y_j(s) \to C_p$, $\;y_{j-2}(s) \to 0$ or C_p, $\;y_{j-1}(s) \to 0$ and $e^s y^p_{j-1}(s) \to 0$, we have,

$$\lim_{s \to +\infty} \left(\frac{(y_j - 2y_{j-1} + y_{j-2})}{h^2} + e^s y^p_{j-1}(s) \right) = C_2 \neq 0.$$

Therefore, the integral behaves like

$$\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}\, d\sigma.$$

If $p \neq 2$, we have

$$y_{j-1}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 e^{-\frac{p-2}{p-1}s}\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 e^{-s}.$$

If $p = 2$ the integral behaves like s, then

$$y_{j-1}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 s\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 s e^{-\frac{1}{p-1}s}.$$

Therefore

$$y_{j-1}(s) \sim \begin{cases} Ce^{-\frac{1}{p-1}s} & \text{if } p > 2, \\ Cse^{-\frac{1}{p-1}s} & \text{if } p = 2, \\ Ce^{-s} & \text{if } p < 2. \end{cases}$$

This implies that $u_{j-1}(t)$ verifies

$$u_{j-1}(t) \sim \begin{cases} C & \text{if } p > 2, \text{ and hence it is bounded,} \\ -C\ln(T_h - t) & \text{if } p = 2, \text{ and hence it blows up,} \\ C(T_h - t)^{-\frac{1}{p-1}+1} & \text{if } p < 2, \text{ and hence it blows up.} \end{cases}$$

We observe that the same arguments show that if $p > 2$ then u_j is bounded for every j with $d(j) \geq 1$. To continue the proof we assume that $p \leq 2$. We sketch the case $p < 2$ (the case $p = 2$ can be handled in a similar way).

We consider a node i_2 such that $d(i_2) = 2$. We can assume that $i_2 = i_1 - 1 = j-2$. Since $d(i_2) = 2$ we have that $j-1$ and $j-3 \notin F$. From similar calculations, we have

$$y_{j-2}(s) = e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{\frac{1}{p-1}\sigma}\left(\frac{e^{-\sigma}}{h^2}(y_{j-1} - 2y_{j-2} + y_{j-3}) + y_{j-2}^{p}\right) d\sigma\right) =$$

$$e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{-\frac{2p-3}{p-1}\sigma}\left(\frac{e^{\sigma}}{h^2}(y_{j-1} - 2y_{j-2} + y_{j-3}) + e^{2\sigma} y_{j-2}^{p}\right)(\sigma)\, d\sigma\right).$$

Using the asymptotic behaviour that we have found for y_{j-1}, we can obtain the asymptotic behaviour for y_{j-2}. Arguing as before we have that,

$$\lim_{s\to\infty} e^{2s} y_{j-2}^{p}(s) = 0, \qquad \lim_{s\to\infty} e^{s} y_{j-1}(s) = C \neq 0.$$

Hence,

$$\lim_{s\to+\infty}\left(\frac{e^{s}(y_{j-1} - 2y_{j-2} + y_{j-3})}{h^2} + e^{2s} y_{j-2}^{p}(s)\right) = C_2 \neq 0.$$

Therefore, the integral behaves like

$$\int_{s_0}^{s} e^{-\frac{2p-3}{p-1}\sigma}\, d\sigma.$$

If $p \neq \frac{3}{2}$, we have

$$y_{j-2}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 e^{-\frac{2p-3}{p-1}s}\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 e^{-2s}.$$

If $p = \frac{3}{2}$ the integral behaves like s, then

$$y_{j-2}(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 s\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 s e^{-\frac{1}{p-1}s}.$$

Therefore

$$y_{j-2}(s) \sim \begin{cases} Ce^{-\frac{1}{p-1}s} & \text{if } p > \frac{3}{2}, \\ Cse^{-\frac{1}{p-1}s} & \text{if } p = \frac{3}{2}, \\ Ce^{-2s} & \text{if } p < \frac{3}{2}. \end{cases}$$

This implies that $u_{j-2}(t)$ verifies

$$u_{j-2}(t) \sim \begin{cases} C & \text{if } p > \frac{3}{2}, \text{ and hence it is bounded,} \\ -C\ln(T_h - t) & \text{if } p = \frac{3}{2}, \text{ and hence it blows up,} \\ C(T_h - t)^{-\frac{1}{p-1}+2} & \text{if } p < \frac{3}{2}, \text{ and hence it blows up.} \end{cases}$$

Now we can repeat this procedure with other nodes to find that $u_i(t)$ blows up if $d(i) \leq K$ and $u_i(t)$ is bounded if $d(i) > K$ where $K \in \mathbb{N}$ is determined by p in the following way, K verifies

$$\frac{K+2}{K+1} < p \leq \frac{K+1}{K}.$$

Also we find that the asymptotic behaviour of a node i such that $d(i) \leq K$ is given by

$$u_i(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(i)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $i = K$,

$$u_i(t) \sim \ln(T_h - t). \square$$

Now we localize the blow-up set.

Proof of Theorem 1.5: We want to prove that, given $\varepsilon > 0$ there exists h_0 such that for every $0 < h \leq h_0$,

$$B(u_h) \subset B(u) + (-\varepsilon, \varepsilon). \tag{4.5}$$

We have that the blow-up set of u is composed by a finite number of points, $B(u) = \{x_1, x_2, ...x_k\}$ (see [CM], [FMc], [M], [MW], [W]). Let us call $A = B(u) + (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. First we claim that, for every h small enough, we have $F \subset A$ (we recall that F is the set of nodes j such that $y_j(s) \to C_p$). To prove this claim we observe that there exists a constant L such that

$$|u(x,t)| < L \qquad \forall x \in [0,1] \setminus A, \ \forall t \in [0,T).$$

Now, Theorem 1.1 implies that

$$\|(u_h - u)(\cdot, T - \tau)\|_{L^\infty} \leq Ch^{\frac{3}{2}}.$$

Hence, given τ, for every h small enough

$$|u_h(x, T - \tau)| \leq L \qquad \forall x \in [0,1] \setminus A.$$

Let j be a node in $[0,1] \setminus A$, then it holds

$$(T_h - (T - \tau))^{\frac{1}{p-1}} u_j(T - \tau) \leq L(T_h - (T - \tau))^{\frac{1}{p-1}}$$

and then

$$y_j(s_0) \leq L(T_h - (T - \tau))^{\frac{1}{p-1}},$$

where $s_0 = -\ln(T_h - (T - \tau))$. By Corollary 1.1 we have that $T_h \to T$. Therefore, choosing τ and h small enough we can make $y_j(s_0)$ small and fall into the hypothesis of Lemma 4.2. We conclude that $y_j(s) \to 0$, proving our claim.

To finish the first part of the proof of Theorem 1.5 we only have to observe that by our propagation result, Theorem 1.4, we have that, for h small enough,

$$B(u_h) \subset F + [-Kh, Kh] \subset A + [-Kh, Kh] \subset B(u) + (-\varepsilon, \varepsilon),$$

proving (4.5).

Now we turn our attention to u_0 symmetric and increasing in $[0, 1/2]$.

In this case the continuous solution u is also symmetric and has only one maximum at $x = 1/2$. In this situation, it it proved that the blow-up set of u consists in a single point, $B(u) = \{1/2\}$, [CM].

For the semidiscrete problem (1.3) the solution must also be symmetric and increasing in $[0, 1/2]$. This result was proved in [ALM2].

Lemma 4.5 *([ALM2], Proposition 2) Assume that $u_h(0)$ verifies $u_{N-j}(0) = u_j(0)$ (symmetry) and that $u_j(0) < u_{j+1}(0)$ for every j such that $x_j \leq 1/2$ (monotonicity in $[0, 1/2]$), then $u_{N-j}(t) = u_j(t)$ and that $u_j(t) < u_{j+1}(t)$ for every j such that $x_j \leq 1/2$ for every $t \in [0, T_h)$.*

We assume that $x = 1/2$ is a node of the mesh. We observe that, by this Lemma, if the initial datum is symmetric, the maximum of u_h is $u_n(t)$ where $x_n = 1/2$. At this point we have

$$u_n(t) = \max u_j(t) \sim (T-t)^{-\frac{1}{p-1}}$$

Hence, by our Theorem 1.3 we have that

$$\lim_{t \to T_h} u_n(t)(T-t)^{\frac{1}{p-1}} = C_p.$$

We claim that $F = \{n\}$, i. e. for every $j \neq n$ we have $y_j(s) \to 0$. To see this claim we observe that u_n and u_{n-1} verify

$$u_n'(t) = \frac{1}{h^2}(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + u_n^p(t),$$

$$u_{n-1}'(t) = \frac{1}{h^2}(u_n(t) - 2u_{n-1}(t) + u_{n-2}(t)) + u_{n-1}^p(t).$$

Subtracting we obtain

$$(u_n - u_{n-1})'(t) = \frac{1}{h^2}(4u_{n-1}(t) - 3u_n(t) - u_{n-2}(t)) + u_n^p(t) - u_{n-1}^p(t) \geq$$

$$-\frac{3}{h^2}(u_n(t) - u_{n-1}(t)) + \frac{u_n^p(t) - u_{n-1}^p(t)}{u_n(t) - u_{n-1}(t)}(u_n(t) - u_{n-1}(t)) =$$

$$(-\frac{3}{h^2} + p\xi^{p-1}(t))(u_n(t) - u_{n-1}(t))$$

Where $u_{n-1}(t) \leq \xi(t) \leq u_n(t)$. Hence

$$(\ln(u_n - u_{n-1}))'(t) \geq (-\frac{3}{h^2} + p\xi^{p-1}(t)),$$

and integrating we have

$$\ln(u_n - u_{n-1})(t) - \ln(u_n - u_{n-1})(t_0) \geq \int_{t_0}^{t} (-\frac{3}{h^2} + p\xi^{p-1}(s))\, ds$$

Now we argue by contradiction, assume that $(T_h - t)^{\frac{1}{p-1}} u_{n-1}(t) \to C_p$, as $u_{n-1}(t) \leq \xi(t) \leq u_n(t)$ we have that

$$\lim_{t \to T_h} \xi(t)(T-t)^{\frac{1}{p-1}} = C_p$$

and then we just observe that

$$\int_{t_0}^{t} (-\frac{3}{h^2} + p\xi^{p-1}(s))\, ds \geq p \int_{t_0}^{t} \frac{(C_p^{p-1} - \varepsilon)}{(T_h - s)}\, ds - C =$$

$$-p(C_p^{p-1}-\varepsilon)\ln(T_h-t)-C.$$

Hence

$$(u_n-u_{n-1})(t)\geq C(T_h-t)^{-p(C_p^{p-1}-\varepsilon)}=C(T_h-t)^{-\frac{p}{p-1}+p\varepsilon}.$$

Using this fact, we have

$$0=\lim_{t\to T_h}(T_h-t)^{\frac{1}{p-1}}(u_n-u_{n-1})(t)=$$

$$\lim_{t\to T_h}(T_h-t)^{\frac{1}{p-1}}(T_h-t)^{-\frac{p}{p-1}+p\varepsilon}\frac{(u_n-u_{n-1})(t)}{(T_h-t)^{-\frac{p}{p-1}+p\varepsilon}}\geq$$

$$C\lim_{t\to T_h}(T_h-t)^{\frac{1}{p-1}-\frac{p}{p-1}+p\varepsilon}=+\infty,$$

a contradiction that proves our claim.

After this we can use our propagation result, Theorem 1.4, to obtain

$$B(u_h)\subset F+[-Kh,Kh]=\{1/2\}+[-Kh,Kh]=B(u)+[-Kh,Kh].\Box$$

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References

[ALM1] L. M. Abia, J. C. Lopez-Marcos, J. Martinez. *Blow-up for semidiscretizations of reaction diffusion equations.* Appl. Numer. Math. Vol 20, (1996), 145-156.

[ALM2] L. M. Abia, J. C. Lopez-Marcos, J. Martinez. *On the blow-up time convergence of semidiscretizations of reaction diffusion equations.* Appl. Numer. Math. Vol 26, (1998), 399-414.

[B] J. Ball. *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations.* Quart. J. Math. Oxford, Vol. 28, (1977), 473-486.

[BB] C. Bandle and H. Brunner. *Blow-up in diffusion equations: a survey.* J. Comp. Appl. Math. Vol. 97, (1998), 3-22.

[BB2] C. Bandle and H. Brunner. *Numerical analysis of semilinear parabolic problems with blow-up solutions.* Rev. Real Acad. Cienc. Exact. Fis. Natur. Madrid. 88, (1994), 203-222.

[BK] M. Berger and R. V. Kohn. *A rescaling algorithm for the numerical calculation of blowing up solutions.* Comm. Pure Appl. Math. Vol 41, (1988), 841-863.

[BHR] C. J. Budd, W. Huang and R. D. Russell. *Moving mesh methods for problems with blow-up.* SIAM Jour. Sci. Comput. Vol 17(2), (1996), 305-327.

[C] X. Y. Chen. *Asymptotic behaviours of blowing up solutions for finite difference analogue of* $u_t = u_{xx} + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo, Sec IA, Math. Vol. 33, (1986), 541-574.

[Ci] P. Ciarlet. The finite element method for elliptic problems. North Holland, (1978).

[CM] X. Y. Chen and H. Matano. *Convergence, asymptotic periodicity and finite point blow up in one-dimensional semilinear heat equations.* J. Differential Equations, Vol. 78, (1989), 160-190.

[CPE] C. Cortázar, M. del Pino and M. Elgueta. *The problem of uniqueness of the limit in a semilinear heat equation.* Comm. Part. Diff. Eq., 24 (11&12), (1999), 2147-2172.

[DER] R. G. Duran, J. I. Etcheverry and J. D. Rossi. *Numerical approximation of a parabolic problem with a nonlinear boundary condition.* Discr. and Cont. Dyn. Sys. Vol. 4 (3), (1998), 497-506.

[FMc] A. Friedman and J. B. Mc Leod. *Blow up of positive solutions of semilinear heat equations.* Indiana Univ. Math. J., Vol. 34, (1985), 425-447.

[GK] Y. Giga and R. V. Kohn. *Nondegeneracy of blow up for semilinear heat equations.* Comm. Pure Appl. Math. Vol. 42, (1989), 845-884.

[H] Henry. Geometric theory of semilinear parabolic equation. Lecture Notes in Math. vol 840. Springer Verlag. 1981.

[HV1] M. A. Herrero and J. J. L. Velazquez. *Flat blow up in one-dimensional, semilinear parabolic problems*, Differential Integral Equations. Vol. 5(5), (1992), 973-997.

[HV2] M. A. Herrero and J. J. L. Velazquez. *Generic behaviour of one-dimensional blow up patterns.* Ann. Scuola Norm. Sup. di Pisa, Vol. XIX (3), (1992), 381-950.

[LR] M. N. Le Roux. *Semidiscretizations in time of nonlinear parabolic equations with blow-up of the solutions.* SIAM J. Numer. Anal. Vol. 31, (1994), 170-195.

[M] F. Merle. *Solution of a nonlinear heat equation with arbitrarily given blow-up points.* Comm. Pure Appl. Math. Vol. XLV, (1992), 263-300.

[MW] C. E. Muller and F. B. Weissler. *Single point blow up for a general semilinear heat equation.* Indiana Univ. math. J., Vol. 34, (1983), 881-913.

[N] T. Nakagawa. *Blowing up of a finite difference solution to* $u_t = u_{xx} + u^2$. Appl. Math. Optim., 2, (1976), 337-350.

[NU] T. Nakagawa and T. Ushijima. *Finite element analysis of the semilinear heat equation of blow-up type.* Topics in Numerical Analysis III (J.J.H Miller, ed.), Academic Press, London, (1977), 275-291.

[P] C. V. Pao. Nonlinear parabolic and elliptic equations. Plenum Press 1992.

[SGKM] A. Samarski, V. A. Galacktionov, S. P. Kurdyunov and A. P. Mikailov. Blow-up in quasilinear parabolic equations. Walter de Gruyter, Berlin, (1995).

[W] F. B. Weissler. *Single point blow up of semilinear initial boundary value problems.* J. Differential Equations, Vol. 55, (1984), 204-224