Stochastic Differential Equations with explosions

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Joint work with

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The Problem

$$dx = b(x)\,dt + \sigma(x)\,dW$$
$$x(0) = z \in \mathbb{R}_{>0}$$

- W is a one dimensional Wiener process (dW is "white noise")
- b, σ are smooth and positive.

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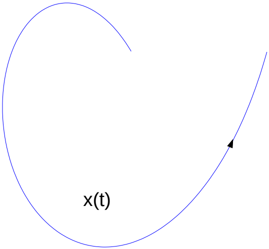
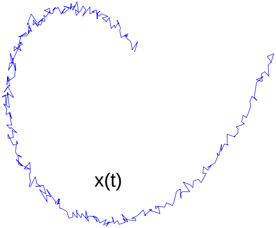
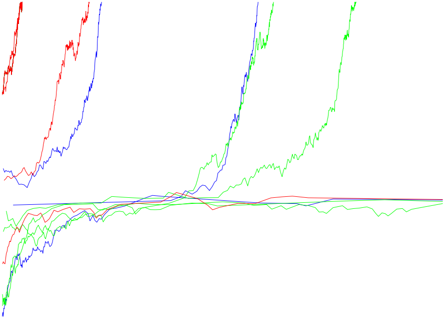
If b is not globally Lipschitz, solutions to this problem may explode in finite time.

ODE vs. SDE





Explosions



There exist a stopping time T such that $x(\omega, t)$ is defined in $[0, T(\omega))$, but

$$x(\omega, t) \nearrow +\infty \qquad \text{as } t \nearrow T(\omega).$$

Fatigue Cracking

- This kind of SDE are used, for example, to model fatigue cracking (fatigue failures in solid materials)
- $x(t)$ represents the evolution of the length of the largest crack.
- The explosion time corresponds to the time of ultimate damage or fatigue failure in the material.

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   - No closed criteria to decide if blow-up will occur.
   - No explicit formula for the blow-up time.
   - The phenomenon is very well understood (blow-up times, blow-up sets, blow-up rates, numerical computation of solutions, etc.)

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   - $b(s)$ is nondecreasing for $s > s_0$ and $\displaystyle\int^{\infty} \frac{1}{b(s)}\, ds < \infty$

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Both for SDE and SPDE: No further results on (for example) the behavior of the explosion time, the set where explosions occur, numerical computation of the solutions, etc.

Our work...

- Theoretical study of the regularity of the explosion time

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- Numerical approximations for this kind of problems

Numerical approximations

We assume

- $0 < C_1 \leq \sigma^2(s) \leq C_2 b(s)$.
- $b(s)$ is nondecreasing for $s > s_0$ and $\int^{\infty} \frac{1}{b(s)}\, ds < \infty$.

Under these conditions, explosions occur with probability one.
Example:

$$dx = (1 + x^2)\, dt + dW$$

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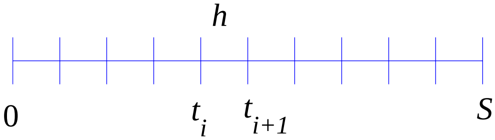
- Explosions in the numerical solutions for small choices of the parameter of the method.
- Convergence of the numerical explosion times to the continuous one.

The Euler-Maruyama scheme (for bounded solutions)

$$dx = b(x)\,dt + \sigma(x)\,dW$$



$$X_i \approx x(t_i)$$

$$h = t_{i+1} - t_i, \qquad \Delta W_i = W(t_{i+1}) - W(t_i).$$

$$X_{i+1} = X_i + hb(X_i) + \sigma(X_i)\Delta W_i, \qquad X_0 = x(0) = z,$$

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- The time step h can not be constant. It must be adapted as the solution increases.
- We propose $h_i = \frac{h}{b(X_i)}$. i.e. $t_{i+1} - t_i = \frac{h}{b(X_i)}$

  $$X_{i+1} = X_i + h_i b(X_i) + \sigma(X_i)\Delta W_i = X_i + h + \sigma(X_i)\Delta W_i$$

The numerical solution

$$X(t_i) = X_i,$$

$$X(t) = X_i + (t - t_i)b(X_i) + \sigma(X_i)(W(t) - W(t_i)), \quad \text{for } t \in [t_i, t_{i+1}).$$

is a well defined process up to time

$$T_h = \sum_{i=1}^{\infty} h_i = \sum_{i=1}^{\infty} \frac{h}{b(X_i)}.$$

We say that the numerical solution explode in finite time T_h if

$$T_h < \infty$$

Mean Square Convergence (while solutions are bounded)

Theorem 1: Fix a time $S > 0$ and a constant $M > 0$. Consider the stopping times given by

$$R^M := \inf\{t : x(t) = M\} \quad R_h^M := \inf\{t : X(t) = M\}$$

$$\tau_h = R^M \wedge R_h^M \wedge S$$

Then

$$\lim_{h \to 0} \mathbb{E}\left[\sup_{0 \le t \le \tau_h} |x(t) - X(t)|^2\right] = 0.$$

Explosions in the Numerical Scheme

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For example, if $b(s) \sim s^p$ $(p > 1)$, the explosion rate is

$$X(t_i)(T_h - t_i)^{1/(p-1)} \to \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \text{ as } t_i \nearrow T_h.$$

Proof:

$$X_{i+1} = X_i + h_i b(X_i) + \sigma(X_i)\Delta W_i = X_i + h + \sigma(X_i)\Delta W_i$$

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$$X_{i+1} = z + ih + \sum_{j=1}^{i} \sigma(X_j)\Delta W_j$$

Since "$\Delta W_j \sim N(0, h_j)$",

$$\frac{\sum_{j=1}^{i} \sigma(X_j)\Delta W_j}{i} \sim \frac{\sum_{j=1}^{i} \sqrt{\frac{h}{b(X_j)}}\sigma(X_j) Z_j}{i} \to 0, \text{ then}$$

$\frac{X_i}{ih} \to 1$ a.s., and hence

$$\sum_{i=i_0}^{\infty} \tau_i = \sum_{i=i_0}^{\infty} \frac{h}{b(X_i)} \sim \int_{X_{i_0-1}}^{\infty} \frac{1}{b(u)}\, du < +\infty, \quad \text{a.s.}$$

Convergence of the Numerical Explosion Times

Theorem 3: For every $\varepsilon > 0$

$$\lim_{M\to\infty} \lim_{h\to 0} \mathbb{P}(|R_h^M - T| > \varepsilon) = 0.$$

$$R_h^M := \inf\{t : X(t) = M\}$$

Numerical Experiments

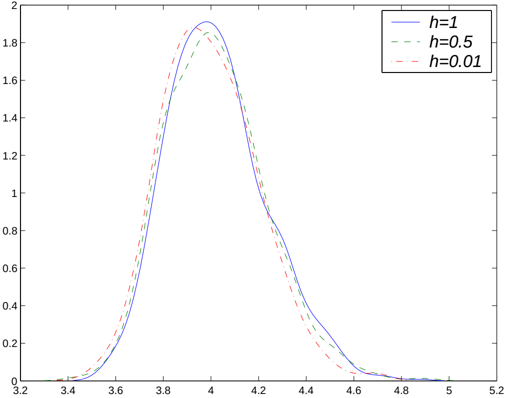
$$b(s) = |s|^{1.1} + 0.1$$

$$\sigma(s) = \sqrt{|s|^{1.1} + 0.1}$$

$$z = 10$$

$$M = 10^5$$

1000 sample paths.



Kernel density estimator of R_h^M.

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Asymptotic behavior of the numerical solution.

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Some sample paths