DEPENDENCE OF THE BLOW-UP TIME WITH RESPECT TO PARAMETERS AND NUMERICAL APPROXIMATIONS FOR A PARABOLIC PROBLEM

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ABSTRACT. We find a bound for the modulus of continuity of the blow-up time for the problem $u_t = \lambda\Delta u + u^p$, with initial datum $u(x,0) = \phi(x) + hf(x)$ respect to the parameters λ, p and h. We also find an estimate for the rate of convergence of the blow-up times for a semi-discrete numerical scheme.

INTRODUCTION

We study the behavior of the blow-up time for the following semilinear parabolic problem,

$$
\begin{array}{lll}
 & u_t = \lambda\Delta u + u^p, & \text{in } \Omega \times (0,T), \\
(1.1) & u = 0, & \text{on } \partial\Omega \times (0,T), \\
 & u(x,0) = u_0(x) = \phi(x) + hf(x), & \text{for all } x \in \Omega,
\end{array}
$$

where $p > 1$, $\lambda > 0$ and $h \in \mathbb{R}$ are parameters, Ω is a bounded smooth domain in $\mathbb{R}^n$ and u_0 is regular in order to guarantee existence, uniqueness and regularity of the solution. We also assume that p is subcritical, that is $p < (n+2)/(n-2)$.

A remarkable fact is that the solution of (1.1) may develop singularities in finite time, no matter how smooth u_0 is. For many differential equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow-up). Typical examples where this happens are problems involving reaction terms in the equation like (1.1), see [SGKM, P] and the references therein.

Key words and phrases. Blow-up, semilinear parabolic equations, semidiscretization in space.

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In our problem one has a reaction term in the equation of power type and if $p > 1$ this blow-up phenomenon occurs in the sense that there exists a finite time T, that we will call the *blow-up time*, such that $\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty$ for initial data large enough, see [SGKM]. The study of the blow-up problem for (1.1) has attracted a considerable attention in recent years, see for example, [B, BC, GK1, GK2, GV, HV1, HV2, M, Z].

We are interested in the study of the dependence of the blow-up time, T, on the parameters that appears in the equation, p, λ and h.

It is known that the blow-up time is continuous with respect to the initial data u_0, see [BC, GV, M, Q].

In this paper we want to improve these results showing that there exists a modulus of continuity for T not only with respect to the initial condition but also with respect to the diffusion coefficient λ and the reaction power p. In fact, we will find bounds of the form

$$|T - T_h| \le C|h|^\gamma. \tag{1.2}$$

Our main idea for the proof rely on estimates on the first time where two different solutions spread. These estimates are based on the rate at which solutions blow up.

The blow-up rate for solutions of (1.1) with subcritical nonlinearity, $1 < p < (n+2)/(n-2)$, is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-1/(p-1)}$, see [GK1], [HV1], [HV2]. In particular in [FKZ, MS] it is proved that for p, u_0 and λ near p_0, ϕ and λ_0 there exists a uniform constant C such that

$$u(x,t) \le C(T-t)^{-\frac{1}{p-1}}. \tag{1.3}$$

The fact that the constant C is uniform is crucial for our arguments.

We want to remark that the restriction $1 < p < (N+2)/(N-2)$ is not technical, since if this is not the case the blow-up time is not continuous as a function of the initial data, see [GV].

The ideas developed to prove estimates like (1.2) can be applied when one deals with numerical methods to approximate solutions of (1.1). In fact we find an estimate for the rate of convergence of the numerical blow-up time in a semi-discrete approximation of problem (1.1).

We split the paper in two parts: in the first part we find an estimate for the modulus of continuity of the blow-up time in problem (1.1) respect to a parameter. In the second part we find the estimate for the rate of convergence of the blow-up time in a numerical semidiscrete approximation of the problem.

Along the rest of the paper we will denote by C a constant that do not depends on the relevant parameters involved but may change from one line to another.

PART I: DEPENDENCE OF THE BLOW-UP TIME WITH RESPECT TO PARAMETERS

In this part we will prove the continuity of the blow-up time respect to parameters and to the initial datum in problem (1.1). To begin with we show that the set of parameters $\{h, p, \lambda\}$ that produce blow-up is open. This can be seen introducing the following energy functional

$$\Phi(u)(t) \equiv \frac{\lambda}{2}\int_\Omega |\nabla u(s,t)|^2\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds.$$

This functional it is useful to characterize the solutions that blow up. In [GK2, CPE] the authors prove that the fact that u blows up in finite time is equivalent to the existence of a time t_0 such that $\Phi(u)(t_0) < 0$. Now let u be a blowing up solution of (1.1). Hence there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. As u in $\Omega \times [0, t_0]$ is continuous with respect to the parameters, we get that $\Phi(\tilde{u})(t_0) < 0$ for every $(\tilde{h}, \tilde{p}, \tilde{\lambda})$ near (h, p, λ). This shows that the set $\{(h, p, \lambda) \,:\, u \text{ blows up}\}$ is open.

To clarify the exposition we will deal with perturbations of h, p, λ separately, however the main lines of the arguments are similar.

1. Perturbations in the nonlinear source.

Let us consider the problem

$$\begin{array}{lll} u_t = \Delta u + u^\alpha, & (x,t) \in \Omega \times (0, T_\alpha), & \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, T_\alpha), & . \\ u(x,0) = u_0(x), & x \in \Omega. & \end{array} \tag{1.4}$$

Our first result shows a modulus of continuity of T_α respect to α.

Theorem 1.1. *Let u and u_h be solutions of problem (1.4) with $p > 1$, $\alpha = p + h$ (h small), respectively, and let T and T_h their blow-up times. Then there exist positive constants C and γ independent of h such that for every h small enough,*

$$|T - T_h| \le C|h|^\gamma.$$

Proof: We observe that we can assume $h > 0$. Let us introduce the error function $e(x,t) = u_h(x,t) - u(x,t)$, we have

$$e_t = \Delta e + u_h^{p+h} - u^p = \Delta e + \frac{u_h^p - u^p}{u_h - u} e + u_h^{p+h} - u_h^p =$$
$$\Delta e + p\xi(x,t)^{p-1} e + u_h^p (u_h^h - 1) \leq \Delta e + p\xi^{p-1} e + u_h^p h u_h^{h+1}.$$

Now fix $a > 0$ and let t_0 such that $e(t) \leq a$ in $[0, t_0]$, in that interval we have $\xi \leq u + a$ and as there exist a constant C such that $u(x,t) \leq C(T-t)^{-1/(p-1)}$, for $t \in [0, t_0]$ we have

$$u_h(x,t) \leq u(x,t) + a \leq C(T-t)^{-1/(p-1)}.$$

Therefore, we have

$$e_t \leq \Delta e + \frac{C}{T-t} e + \frac{Ch}{(T-t)^{\frac{p+h+1}{p-1}}}.$$

Let $\overline{e}$ be the solution of

$$\overline{e}' = \frac{C}{T-t}\overline{e} + \frac{Ch}{(T-t)^{\frac{p+h+1}{p-1}}}, \qquad \overline{e}(0) = 0.$$

Integrating we obtain

$$\overline{e}(t) = Ch(T-t)^{-C} + C_2 h (T-t)^{-\frac{2+h}{p-1}},$$

and by a comparison argument we have

$$e(t) \leq \overline{e}(t) \leq Ch(T-t)^{-C}.$$

This is an important point in the proof, since we obtain a bound on the asymptotic behavior of $e(t)$. In the following perturbations we will see that once we obtain this kind of bounds, the rest of the proof is as follows: we choose $t_1 = t_1(h) \leq t_0$ the first time such that

$$Ch(T-t_1)^{-C} = a,$$

so we have

$$T - t_1 = \left(\frac{Ch}{a}\right)^{1/C}.$$

This time t_1 also verifies

$$|T_h - t_1| \leq \left(\frac{C}{\|u_h(t_1,\cdot)\|_{L^\infty(\Omega)}}\right)^{p+h-1} \leq \left(\frac{C}{\|u(t_1,\cdot)\|_{L^\infty(\Omega)} - a}\right)^{p+h-1} \leq$$
$$C\left(C(T-t_1)^{-\frac{1}{p-1}} - a\right)^{-(p+h-1)} \leq C(T-t_1)^{\frac{p+h-1}{p-1}} \leq C(T-t_1)^{\frac{1}{2}}.$$

Hence we obtain

$$|T - T_h| \leq |T - t_1| + |T_h - t_1| \leq$$

$$|T-t_1| + C|T-t_1|^{\frac{1}{2}} \le Ch^{\frac{1}{2C}} = Ch^{\gamma}.$$

This completes the proof. □

Remark: Let us observe that the exponent γ and the constant C that appears in Theorem 1.1 are not uniform, they depend on the solution u that we are considering. This can not be improved even for the ODE $u' = u^p$ where solutions are explicit.

2. Perturbations in the initial datum.

Now we turn our attention to the dependence of the blow-up time with respect to the initial data. We consider the problem

$$(1.5) \qquad \begin{array}{ll} (u_h)_t = \Delta u_h + u_h^p, & (x,t) \in \Omega \times (0,T_h), \\ u_h(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T_h), \ . \\ u_h(x,0) = u_0(x) + hf(x), & x \in \Omega. \end{array}$$

We denote by u the solution for $h = 0$ and T its blow-up time.

Theorem 1.2. *Let u_h, u solutions of problem (1.5) with $h > 0$ and $h = 0$ respectively, and let T_h and T their blow-up times. Then there exist positive constants C and γ such that for every h small enough,*

$$|T - T_h| \le C|h|^{\gamma}.$$

Proof: As in Theorem 1.1 we call $e(x,t) = u_h(x,t) - u(x,t)$. Then e verifies,

$$\begin{aligned} e_t = & \ \Delta e + \frac{u_h^p - u^p}{u_h - u} e \\ = & \ \Delta e + p\xi(x,t)^{p-1} e \le \Delta e + p\xi^{p-1} e. \end{aligned}$$

Let again t_0 be such that $e(t) \le a$ in $[0,t_0]$ for fixed a. Using that $u(x,t) \le C(T-t)^{-\frac{1}{p-1}}$, in that interval $[0,t_0]$, we have

$$e_t \le \Delta e + \frac{C}{T-t} e.$$

Consider $\overline{e}$, the solution of

$$\overline{e}' = \frac{C}{T-t}\overline{e}, \qquad \overline{e}(0) = |h| \|f\|_{L^\infty},$$

that is

$$\overline{e}(t) = C|h|(T-t)^{-C}.$$

Arguing by comparison we obtain

$$e(t) \le \overline{e}(t) \le C|h|(T-t)^{-C}.$$

Now we proceed as in Theorem 1.1 to complete the proof. First, we choose $t_1 = t_1(h) \le t_0$ with the property that is the first time such that

$$C|h|(T-t_1)^{-C} = a,$$

that is

$$T - t_1 = \left(\frac{C|h|}{a}\right)^{1/C}.$$

This time verifies

$$|T_h - t_1| \le \left(\frac{C}{\|u_h(t_1,\cdot)\|_{L^\infty(\Omega)}}\right)^{p-1} \le \left(\frac{C}{\|u(t_1,\cdot)\|_{L^\infty(\Omega)} - a}\right)^{p-1} \le$$

$$C\left(C(T-t_1)^{-\frac{1}{p-1}} - a\right)^{-(p-1)} \le C(T-t_1).$$

Hence

$$|T - T_h| \le |T - t_1| + |T_h - t_1| \le$$

$$C|T - t_1| \le C|h|^{\frac{1}{C}} = C|h|^\gamma,$$

as we wanted to prove. □

3. Perturbations in the diffusion coefficient.

Next we obtain a similar result for perturbations in the diffusion parameter, λ. We consider the problem

$$\begin{array}{lll} & (u_\lambda)_t = \lambda \Delta u_\lambda + u_\lambda^p, & (x,t) \in \Omega \times (0, T_\lambda), \\ (1.6) & u_\lambda(x,t) = 0, & (x,t) \in \partial\Omega \times (0, T_\lambda), \ . \\ & u_\lambda(x,0) = u_0(x), & x \in \Omega, \end{array}$$

Theorem 1.3. *Let u and u_λ be solutions of problem (1.6) with $\lambda_0 > 0$ and $\lambda > 0$ (close to λ_0), respectively, and let T and T_λ their blow-up times. Then there exist positive constants C and γ such that for every λ close enough from λ_0,*

$$|T - T_\lambda| \le C|\lambda - \lambda_0|^\gamma$$

Proof: Once again we call $e(x,t) = u_\lambda(x,t) - u(x,t)$, then

$$\begin{aligned} e_t = \; & \lambda_0 \Delta e + \frac{u_\lambda^p - u^p}{u_\lambda - u} e + (\lambda - \lambda_0)\Delta u_\lambda \\ = \; & \lambda_0 \Delta e + p\xi(x,t)^{p-1} e + (\lambda - \lambda_0)\Delta u_\lambda \qquad (1.7) \\ \le \; & \lambda_0 \Delta e + p\xi^{p-1} e + (\lambda - \lambda_0)\Delta u_\lambda. \end{aligned}$$

Let again t_0 such that $e(t) \leq a$ in $[0, t_0]$ for fixed a, in that interval we have

$$e_t \leq \lambda_0 \Delta e + \frac{C}{T-t} e + |\lambda - \lambda_0| \frac{C}{(T-t)^\alpha}. \tag{1.8}$$

Now, let $\overline{e}$ be the solution of

$$\overline{e}' = \frac{C}{T-t}\overline{e} + |\lambda - \lambda_0| \frac{C}{(T-t)^\alpha}, \qquad \overline{e}(0) = 0.$$

Integrating we obtain

$$\overline{e}(t) \leq C|\lambda - \lambda_0|(T-t)^{-C},$$

and by a comparison argument we have

$$e(t) \leq \overline{e}(t) \leq C|\lambda - \lambda_0|(T-t)^{-C}.$$

Now we proceed as in the proof of the previous Theorems, Theorem 1.1 and Theorem 1.2, to conclude. □

PART II: RATE OF CONVERGENCE OF THE BLOW-UP TIME FOR A NUMERICAL SCHEME

In this section we study the behavior of the blow-up time for numerical approximations of problem (1.1), that is,

$$\begin{array}{ll} u_t = \Delta u + u^p, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array} . \tag{2.1}$$

Since the solutions of this problem may develop singularities one can asks about the behavior of the numerical approximations: do the approximations reproduce the blow-up phenomena? and the blow-up rate? which is the behavior of the blow-up times? and many other questions. For previous works on numerical approximations of this kind of problems and answers to the above questions we refer to [ALM1, ALM2, BB2, BK, BHR, C, GR, GQR, LR] the survey [BB] and references therein.

Here we will focus just in the behavior of the numerical blow-up time as the mesh parameter goes to zero. In [GR] it is proved that the numerical blow-up time converges to the continuous one as the mesh parameter goes to zero. Using the same ideas of Part I, we will improve this convergence showing that there exists a bound of the form

$$|T - T_h| \leq Ch^\gamma.$$

We will consider a general semidiscrete scheme (we keep t as a continuous variable) with adequate assumptions on the coefficients. More precisely, we assume that there exists a set of nodes $\{x_1, \dots, x_N\}$ and that the approximate solution $u_h(x,t)$ is a linear interpolant of $U(t) = (u_1(t), \dots, u_N(t))$ (that is $u_h(x_k,t) = u_k(t)$) where U is the solution of the following system of ODEs,

$$
\begin{array}{rcl}
MU'(t) & = & -AU(t) + MU^p(t), \\
 & & \\
U(0) & = & U_0.
\end{array} \qquad (2.2)
$$

Here U_0 is the initial datum for the problem (2.2) and M and A are $N \times N$ matrices.

The precise assumptions on the matrices involved in the method are:

(H1) M is a diagonal matrix with positive entries m_k.

(H2) A is a nonnegative symmetric matrix, with non-positive coefficients off the diagonal (that is $a_{ij} \le 0$ if $i \ne j$) and $a_{ii} > 0$.

(H3) $\sum_{j=1}^N a_{ij} = 0$.

The last hypothesis guarantees the maximum principle for the numerical scheme.

Writing this equation explicitly we obtain the following ODE system,

$$
\begin{array}{rclr}
m_k u_k'(t) & = & -\sum_{j=1}^N a_{kj} u_j(t) + m_k u_k^p(t), & 1 \le k \le N, \\
 & & & \\
u_k(0) & = & u_{0,k}, & 1 \le k \le N.
\end{array} \qquad (2.3)
$$

As an example, we can consider a linear finite element approximation of problem (2.4) on a regular acute triangulation of Ω (see [Ci]). In this case, if V_h is the subspace of piecewise linear functions in $H_0^1(\Omega)$. We impose that $u_h : [0, T_h) \to V_h$, verifies

$$
\int_\Omega ((u_h)_t v)^I = -\int_\Omega \nabla u_h \nabla v + \int_\Omega ((u_h)^p v)^I
$$

for every $v \in V_h$. Here $(\cdot)^I$ stands for the linear Lagrange interpolation at the nodes of the mesh.

We denote with $U(t) = (u_1(t), ..., u_N(t))$ the values of the numerical approximation at the nodes x_k at time t. Then $U(t)$ verifies a system of the form (2.2) and all of the assumptions on the matrix M hold as we are using mass lumping and our assumptions on A are satisfied as we are considering an acute regular mesh. In this case M is the lumped mass matrix and A is the stiffness matrix. As initial datum, we take $U_0 = u_0^I$.

As another example if Ω is a cube, $\Omega = (0,1)^n$, we can use a semidiscrete finite differences method to approximate the solution $u(x,t)$ obtaining an ODE system of the form (2.3).

We require to the general scheme that we introduce here to be consistent. A precise definition of consistency is given below

Definition 2.1. *We say that the scheme* (2.3) *is consistent if for any solution u of* (2.1) *holds*

$$m_k u_t(x_k,t) = -\sum_{j=1}^{N} a_{kj} u(x_j,t) + m_k u^p(x_k,t) + \rho_k(h,t), \tag{2.4}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on h and a universal constant θ such that

$$\max_k \left| \frac{\rho_{k,h}(t)}{m_k} \right| \le \frac{\rho(h)}{(T-t)^\theta}, \qquad \text{for every } t \in (0,T_h),$$

with $\rho(h) \to 0$ if $h \to 0$.

We want to remark that this consistency hypothesis is valid in the two examples cited above with $\rho(h) = Ch^2$. The power $C(T-t)^{-\theta}$ is a bound for the spatial derivatives of u.

Using ideas from [GR] and [GQR] it can be proved that the method converges uniformly in sets of the form $\Omega \times [0, T-\tau]$. Moreover, using the energy functional

$$\Phi_h(U(t_0)) \equiv \frac{1}{2}\langle A^{1/2}U(t_0); A^{1/2}U(t_0)\rangle - \sum_{i=1}^{N+1} m_{ii} \frac{((U(t_0))_i)^{p+1}}{p+1},$$

one can check that, given any initial data u_0 such that the continuous solution u blows up, then the numerical approximation u_h also blows up in finite time, T_h, for every h small enough. Since our interest here is to analyze the convergence rate of $|T - T_h|$ we refer to those papers ([GR] and [GQR]) for the details.

Before involving us in the main result of this part we will prove some lemmas. We will need the following definition,

Definition 2.2. *We say that $\overline{U}$ is a supersolution of* (2.2) *if*

$$M\overline{U}' \ge -A\overline{U} + M\overline{U}^p.$$

We say that $\underline{U}$ is a subsolution of (2.2) *if*

$$M\underline{U}' \le -A\underline{U} + M\underline{U}^p.$$

The inequalities are understood coordinate by coordinate.

Lemma 2.1. *(Comparison Lemma) Let* $\overline{U}$ *and* $\underline{U}$ *be a super and a subsolution of* (2.2) *respectively, such that the initial data verify*

$$\overline{U}(0) \geq \underline{U}(0).$$

Then

$$\overline{U}(t) \geq \underline{U}(t).$$

Proof: Let $W = \overline{U} - \underline{U}$. We can assume, by an approximation argument, that we have strict inequalities in Definition 2.2 and that $W(0) > \delta > 0$. We observe that W verifies

$$\begin{aligned} MW' &> -AW + M\left(\overline{U}^p - \underline{U}^p\right) \\ &= -AW + M\left(\frac{\overline{U}^p - \underline{U}^p}{\overline{U} - \underline{U}}\right)W. \end{aligned}$$

Now, suppose that the conclusion of the Lemma is false. Thus, let t_0 be the first time such that $\min W(t) = \frac{\delta}{2}$. At that time, there must be a node j such that $w_j(t_0) = \frac{\delta}{2}$. As w_j has a minimum there we must have $w_j'(t_0) \leq 0$, but, on the other hand, by our hypotheses on A,

$$\begin{aligned} m_j w_j' &> -\sum_{i=1}^{N} a_{ij} w_i + m_j \frac{\overline{u}_j^p - \underline{u}_j^p}{\overline{u}_j - \underline{u}_j} w_j \\ &\geq -\sum_{i=1}^{N} a_{ij} \frac{\delta}{2} + m_j p \underline{u}_j^{p-1} w_j \geq 0, \end{aligned}$$

a contradiction that completes the proof. $\square$

Next we prove a lemma that ensures that the blow-up rate of the numerical solutions has the same upper bound than the continuous ones. And that this bound does not depend on h, the mesh size. We need to assume a technical hypothesis on the initial data.

Lemma 2.2. *Assume* $\Delta u_0 + u_0^p > 0$ *in* Ω, *then there exists a constant* C, *independent of* h *such that*

$$u_h(x,t) \leq \frac{C}{(T_h - t)^{\frac{1}{p-1}}}$$

Proof: First we observe that the hypothesis $\Delta u_0 + u_0^p > 0$ implies that there exist $\delta > 0$, independent of h, such that $u_k'(0) \geq \delta u_k^p(0),\ 1 \leq k \leq N$. Next we show that this inequality holds for every $0 \leq t < T_h$.

Let $w_k(t) = u_k'(t) - \delta u_k^p(t)$. We want to use the minimum principle to show that $w_k(t)$ is positive. To this end, we observe that w_k verifies

$$
\begin{aligned}
m_k w_k' + \sum_{j=1}^{N} a_{kj} w_j &= m_k(u_k'' - \delta p u_k^{p-1} u_k') + \sum_{j=1}^{N} a_{kj}(u_j' - \delta u_j^p) \\
&= -\delta m_k p u_k^{p-1} u_k' + m_k p u_k^{p-1} u_k' - \delta \sum_{j=1}^{N} a_{kj} u_j^p \\
&= -\delta p u_k^{p-1} \left(\sum_{j=1}^{N} a_{kj} u_j + m_k u_k^p \right) + m_k p u_k^{p-1} u_k' - \delta \sum_{j=1}^{N} a_{kj} u_j^p \\
&= m_k p u_k^{p-1} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^p - p u_k^{p-1} u_j) + a_{kk}(1-p) u_k^p \right) \\
&= m_k p u_k^{p-1} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^p - p u_k^{p-1}(u_j - u_k) - u_k^p) \right. \\
&\quad \left. + \sum_{j=1}^{N} a_{kj}(1-p) u_k^p \right).
\end{aligned}
$$

As $f(u) = u^p$ is convex ($p > 1$) and from our hypotheses on the matrix A it follows that $W = (w_1, \ldots, w_N)$ verifies

$$MW' \geq -AW + MpU^{p-1}W.$$

Since $W(0) > 0$ and the minimum principle holds for this equation, we obtain that every node verify

$$u_k'(t) \leq \delta u_k(t)^p.$$

If x_k is a node that blows up we can integrate the above inequality between t and T_h to get

$$\int_t^{T_h} \frac{u_k'}{u_k^p}(s)\, ds \geq \delta(T_h - t),$$

so that

$$u_k(t) \leq C(T_h - t)^{-\frac{1}{p-1}},$$

where C depends only on p and δ (but not on h). This proves the result. $\square$

We are ready to prove the main result related to the numerical approximations, the rate of convergence for the numerical blow-up time.

Theorem 2.4. *Let u be a solution of problem* (2.1) *that verifies the hypothesis of the above lemma and blows up at time T, let u_h the numerical solution given by* (2.3)*. Assume also that the scheme is consistent (according to Definition 2.1) with $\rho(h) \le Ch^{\alpha}$, for some $\alpha > 0$. Then there exist positive constants C and γ such that for h small enough,*

$$|T - T_h| \le Ch^{\gamma}.$$

Proof: Let us start by defining the error functions

$$e_k(t) = u(x_k, t) - u_k(t).$$

By the consistency assumption (2.4), these functions verify

$$m_k e_k' \;=\; -\sum_{j=1}^{N} a_{kj} e_i + m_k(u^p(x_k,t) - u_k^p) + \rho_k(h).$$

Let $t_0 = \max\{t : \; t < T - \tau, \; \max_k |e_k(t)| \le a\}$, in $[0, t_0]$ we have

$$m_k e_k' \le -\sum_{j=1}^{N} a_{kj} e_i + m_k p \xi_k(t)^{p-1} e_k(t) + \rho_k(h),$$

As $\xi_k(t), u_k(t) \le u(x_k,t) + a \le \frac{C}{(T-t)^{\frac{1}{p-1}}}$, in $[0, t_0]$, $E = (e_1, ..., e_N)$ satisfies

$$ME' \le -AE + \frac{C}{T-t} ME + \frac{\rho(h)}{(T-t)^{\theta}} M(1, ..., 1)^t. \tag{2.5}$$

Let $\overline{E}(t)$ be the solution of

$$\overline{E}' = \frac{C}{T-t}\overline{E} + \frac{\rho(h)}{(T-t)^{\theta}}, \qquad \overline{E}(0) = E(0) = 0.$$

Integrating, we obtain

$$\overline{E}(t) = C_1 \rho(h)(T-t)^{-\theta+1} + C_2 \rho(h)(T-t)^{-C}.$$

As $\overline{E}(t)$ is a supersolution for (2.5) and the comparison Lemma 2.1 holds we get

$$E(t) \le \overline{E}(t) \le C\rho(h)(T-t)^{-C}.$$

Now we proceed as in the Theorems of the previous sections, let $t_1 \le t_0$ be the first time such that

$$C\rho(h)(T - t_1)^{-C} = a,$$

that is

$$T - t_1 = \left(\frac{C\rho(h)}{a}\right)^{1/C},$$

this time t_1 also verifies (by the previous Lemma)

$$|T_h - t_1| \leq \left(\frac{C}{u_k(t_1)}\right)^{p-1} \leq \left(\frac{C}{\|u(\cdot,t_1)\|_{L^\infty(\Omega)} - a}\right)^{p-1}$$

$$C\left(C(T-t_1)^{-\frac{1}{p-1}} - a\right)^{-(p-1)} \leq C(T-t_1),$$

and so we obtain

$$|T - T_h| \leq |T - t_1| + |T_h - t_1| \leq$$

$$|T - t_1| + C|T - t_1| \leq (1 + C)\left(\frac{C\rho(h)}{a}\right)^{1/C} = Ch^\gamma.$$

This completes the proof. □

We observe that this proof shows, in particular, the convergence of the numerical scheme in sets of the form $\Omega \times [0, T - \tau]$. However, using a different approach it can be obtained a better estimate for the rate of convergence of the solutions of the numerical scheme in sets of the form $\Omega \times [0, T - \tau]$ for a fixed τ. In fact this rate of convergence coincides with the modulus of consistency (see [GQR]).

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