Aproximaciones numéricas para problemas con blow-up.

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What is blow-up?

$$u_t = \mathcal{F}(u)$$

The operator $\mathcal{F}$ is defined in certain functional space E.

Blow-up occurs when the solution $u = u(\cdot, t)$ grows up to infinity as t approaches some finite time T (the blow-up time).

Examples:

1. The ODE

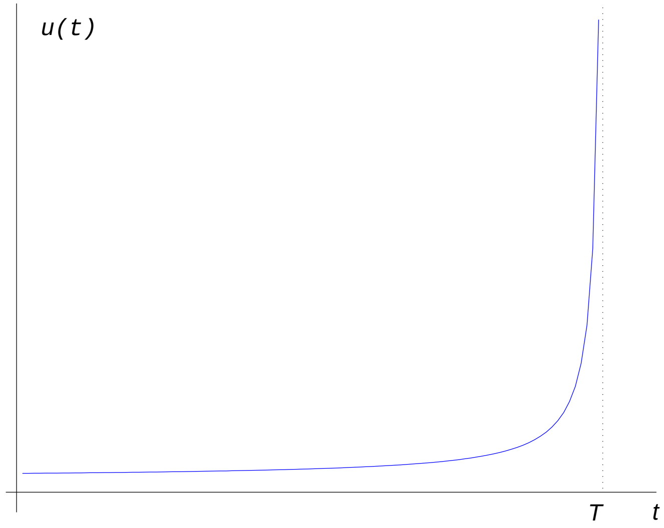
$$\dot{u}(t) = u^p(t), u(0) = u_0 \qquad p > 1$$

The solution

$$u(t) = C_p(T - t)^{-1/(p-1)},$$

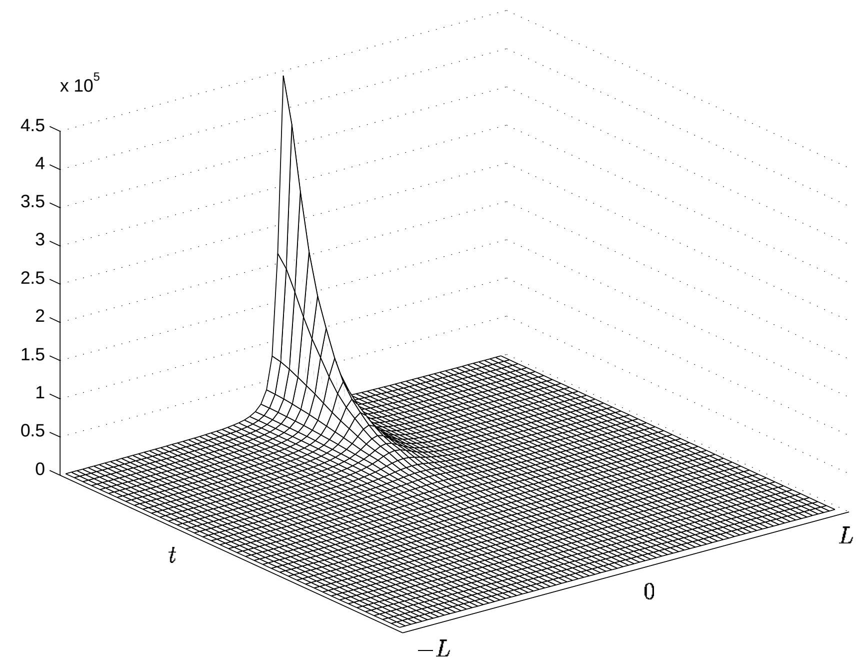
$$T = \frac{1}{u_0^{p-1}(p-1)}, \qquad C_p = (p-1)^{-1/(p-1)}.$$

blows up at time T



2. The PDE

$$
\begin{array}{ll}
u_t = \Delta u + u^p, & \Omega \times (0,T), \\
u = 0, & \partial\Omega \times (0,T), \\
u(x,0) = u_0(x), & \Omega.
\end{array}
$$



If u_0 is smooth and large enough the solution u is regular for every $0 \le t < T$ but

$$\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

Kaplan, 63

What do we study when we study blow-up?

1. Does blow-up occur?

2. When?

3. Where?

4. How?

5. What happens when perturbing the problem?

6. **How to compute it numerically?**

1. Does blow-up occur?

For a specific problem,

The solution blows up or is globally defined?

Every solution blows up or just the ones in a certain class?

Is it possible to characterize this class?

2. When?

What can we say about the maximal existence time T where a solution blows up?

Is it posible to estimate T in terms of the parameters, the initial data or the evolution of the solution as time goes forward?

3. Where?

If a solution u blows up at time T, we define the blow-up set

$$B(u) = \{x \in \overline{\Omega} / \exists (x_n, t_n), x_n \to x, t_n \nearrow T, u(x_n, t_n) \to \infty\}.$$

Any information about this set is welcome: dimension, number of points, location, measure, etc.

4. How?

Which is the behavior of the solution near the blow-up time T? (blow-up rate)

For example, solutions with blow-up to the problem

$$\begin{array}{ll} u_t = \Delta u + u^p, & \Omega \times (0,T), \\ u = 0, & \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \Omega. \end{array}$$

behave like $C_p(T-t)^{-\frac{1}{p-1}}$

i.e. if x is a blow-up point

$$\frac{u(x,t)}{C_p(T-t)^{-\frac{1}{p-1}}} \to 1.$$

Giga-Kohn, 85, 87, 89

6. How to compute it numerically?

(This thesis)

How a numerical method should be in order to get similar answers to the previous questions for both the continuous problem and the numerical approximations?

Some references:

Blow-up in parabolic PDEs	Numerical blow-up
- Kaplan 63, Fujita 66, 68	- Ushijima, Nakagawa 75,76,77
- Giga-Kohn 85, 87,89, etc.	- Chen 86
- Bandle-Brunner (survery) 98	- Berger-Kohn 88
- Galaktionov-Vazquez (survey) 99	- Budd et. al. 96
- Smarskii et. al. (book), 95	- Durán-Etcheverry-Rossi 98

Why blow-up is not just a singularity?

$$\dot{u}(t) = u^p(t), \qquad p > 1 \qquad u(t) = C_p(T-t)^{-1/(p-1)}$$

$$C_p = (p-1)^{-1/(p-1)}$$

$$\begin{array}{ll} u(0) = u_0 & u(0) = u_0 + \varepsilon \\ T = \frac{1}{u_0^{p-1}(p-1)} & T_\varepsilon = \frac{1}{(u_0+\varepsilon)^{p-1}(p-1)} < T \end{array}$$

The error function $e(t) = u_\varepsilon(t) - u(t)$ blows up at time $T_\varepsilon < T$, where u is regular.

$$\Rrightarrow$$

Hence

- Standard convergence results does not hold in this case.

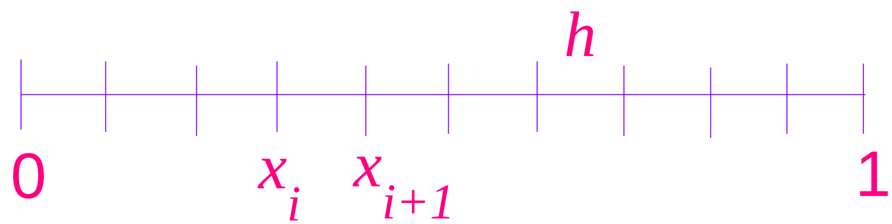
- We can not (a priori) expect the numerical approximations of blow-up problems to reproduce every property of the continuous solution.

- Usual techniques for regular problems or even those for problems with fixed singularities do not apply for these problems.

- New methods have to be developed in order to get the asymptotic properties of the solution.

A standard numerical scheme: the method of lines.

$$
\begin{aligned}
u_t &= u_{xx} + u^p && \text{in } (0,1)\times[0,T),\\
u(1,t) &= u(0,t) = 0 && \text{on } [0,T),\\
u(x,0) &= u_0(x) \geq 0 && \text{on } [0,1].
\end{aligned}
$$



$$
\begin{cases}
u_1(t) = 0,\\
u_i'(t) = \frac{1}{h^2}(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)) + u_i^p(t),\\
u_{N+1}(t) = 0,\\
u_i(0) = u_0(x_i), \qquad 1 \leq i \leq N+1.
\end{cases}
$$

Coincidences and differences.

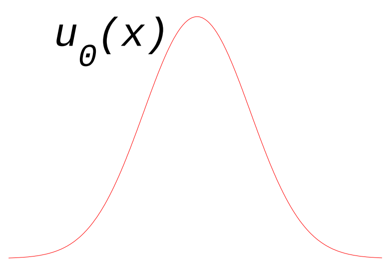
Continuous solutions $\leftrightarrow$ Numerical approximations

$$
\begin{array}{ccc}
u & \qquad\qquad\qquad & u_h \\
T & & T_h \\
B(u) & & B(u_h)
\end{array}
$$

1. Heat equation with a source.

$$
\begin{array}{ll}
u_t = \Delta u + u^p, & \Omega \times (0,T), \\
u = 0, & \partial\Omega \times (0,T), \\
u(x,0) = u_0(x), & \Omega.
\end{array}
$$

-Convergence of the method
-Similar conditions to get blow-up
-Same blow-up rate
-Convergence of the numerical blow-up times
$|T_h - T| \le C h^{\gamma}, \qquad \gamma > 0$
-The blow-up propagates in numerical approximations



$$\Rightarrow B(u) = \{0\}, \qquad B(u_h) = [-Kh, Kh], \qquad K = \left[\frac{1}{p-1}\right]$$

If $p \approx 1$ $B(u_h)$ is much bigger than $B(u)$. However $B(u_h) \xrightarrow{(h \to 0)} B(u)$.

2. Porous medium equation a source. $\Omega = (-L, L)$

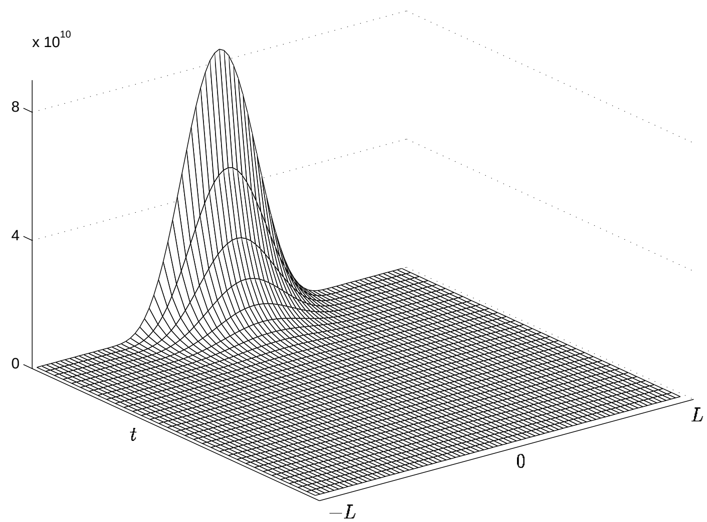
$$\begin{array}{ll} u_t = (u^m)_{xx} + u^m, & \Omega \times (0,T), \\ u = 1, & \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \Omega. \end{array}$$

-Similar conditions to get blow-up

-Same blow-up rate

-Convergence of the numerical blow-up times to the continuous one

-Big differences in the blow-up sets



Regional blow-up
in the continuous solution

Global blow-up
in the numerical scheme

The numerical scheme

$$
\begin{aligned}
&u_{-N}(t) = 1, \\
&u_k'(t) = \frac{1}{h^2}(u_{k+1}^m(t) - 2u_k^m(t) + u_{k-1}^m(t)) + u_k^m(t), \\
&u_N(t) = 1, \\
&u_k(0) = \varphi(x_k), \qquad\qquad -N+1 \le k \le N-1.
\end{aligned}
$$

The numerical scheme

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&u_{-N}(t) = 1,\\
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&u_N(t) = 1,\\
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\end{aligned}$$

Every node behaves like

$$u_k(t) \sim w_k(h)(T_h - t)^{-\frac{1}{m-1}}$$

But...

The numerical scheme

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But...

$w_k(h) \overset{(h \to 0)}{\longrightarrow} 0$ if $u_k(t)$ should not blow-up.

The numerical scheme

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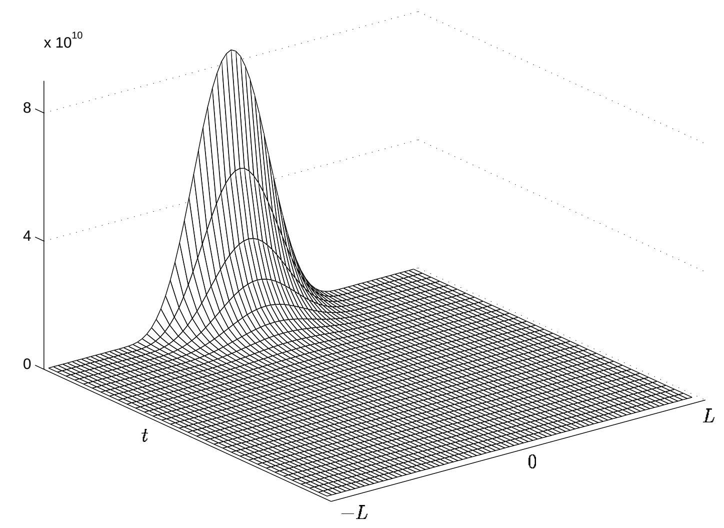
&u_N(t) = 1, \\
&u_k(0) = \varphi(x_k), \qquad\qquad -N+1 \le k \le N-1.
\end{aligned}
$$

Every node behaves like

$$u_k(t) \sim w_k(h)(T_h - t)^{-\frac{1}{m-1}}$$



But...

$w_k(h) \overset{(h \to 0)}{\longrightarrow} 0$ if $u_k(t)$ should not blow-up.

3. Heat equation with nonlinear boundary conditions.

$u_t = u_{xx}$	$(0,1) \times [0,T)$,	-Every solution blows up [DER]
$u_x(0,t) = 0$	$[0,T)$,	- $T_h \to T$ [DER]
$u_x(1,t) = u^p(1,t)$,	$p > 1$ $[0,T)$,	-Blow-up propagates
$u(x,0) = u_0(x) > 0$	$(0,1)$.	**-Different blow-up rates!**

Continuous solutions

$$\|u(\cdot,t)\|_{L^\infty(0,1)} \sim \frac{C}{(T-t)^{1/2(p-1)}}$$

Numerical solutions

$$\|u_h(\cdot,t)\|_{L^\infty(0,1)} \sim \frac{C}{(T-t)^{1/(p-1)}}$$

3. Heat equation with nonlinear boundary conditions.

$$\begin{array}{lrl}
u_t = u_{xx} & (0,1)\times[0,T), & \text{-Every solution blows up [DER]} \\
u_x(0,t) = 0 & [0,T), & \text{- } T_h \to T \text{ [DER]} \\
u_x(1,t) = u^p(1,t), & p>1\ [0,T), & \text{-Blow-up propagates} \\
u(x,0) = u_0(x) > 0 & (0,1). & \textbf{-Different blow-up rates!}
\end{array}$$

Continuous solutions

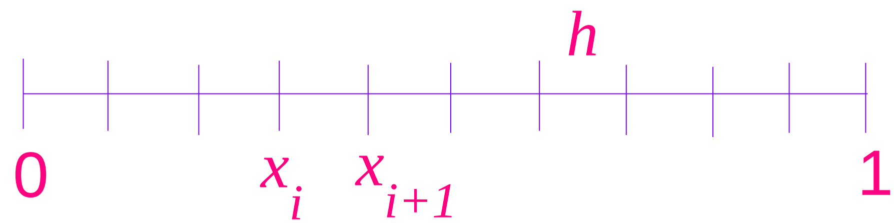
$$\|u(\cdot,t)\|_{L^\infty(0,1)} \sim \frac{C}{(T-t)^{1/2(p-1)}}$$

Numerical solutions

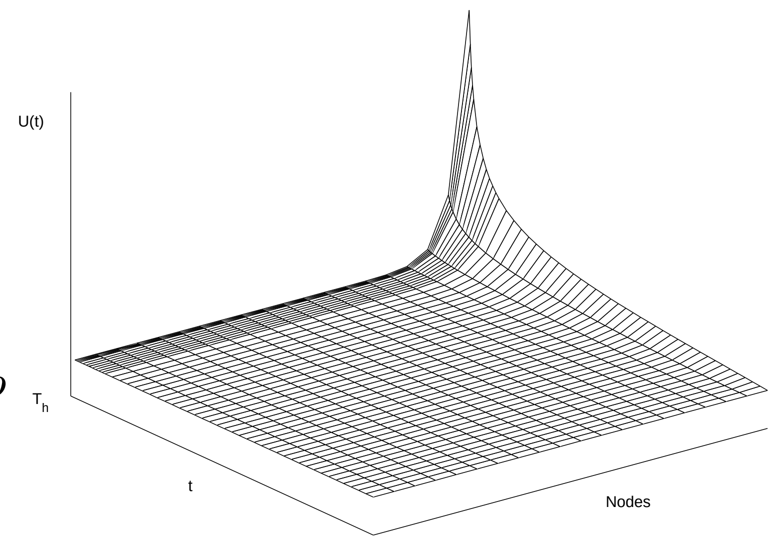
$$\|u_h(\cdot,t)\|_{L^\infty(0,1)} \sim \frac{C}{(T-t)^{1/(p-1)}}$$

A mesh adaptive algorithm is requiered

The fixed mesh method



$$u_0'(t) = \frac{2}{h_0^2}(u_1(t) - u_0(t)),$$
$$u_i'(t) = \frac{1}{h_i^2}(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)),$$
$$u_N'(t) = \frac{2}{h_N^2}(u_{N-1}(t) - u_N(t)) + \frac{2}{h_N}(u_N(t))^p$$



$$\max_{1\le i\le N} u_i(t) = u_N(t), \qquad u_N'(t) \sim \frac{2}{h_N}(u_N(t))^p.$$

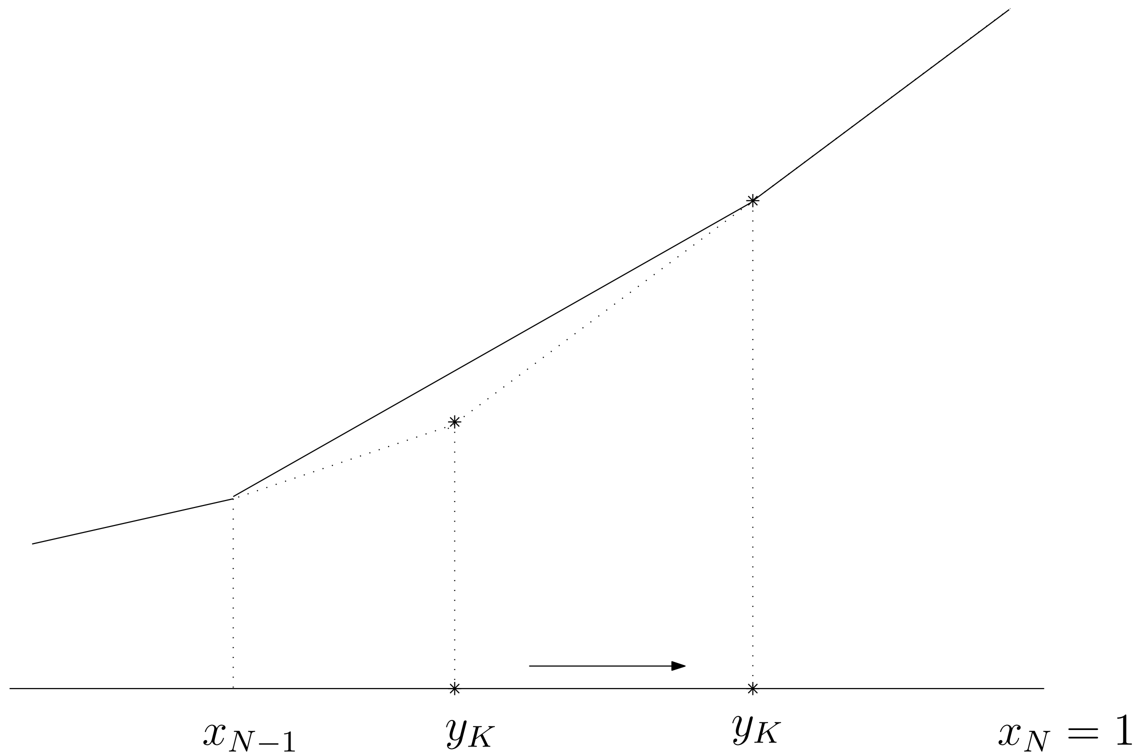
The adaptive in space method: If we want to get the correct rate $u_N(t) \sim C(T_h - t)^{-1/2(p-1)}$ we need

$$u'_N(t) \sim (u_N(t))^q, \qquad q \text{ such that } \frac{1}{q-1} = \frac{1}{2(p-1)}, \qquad q = 2p-1.$$

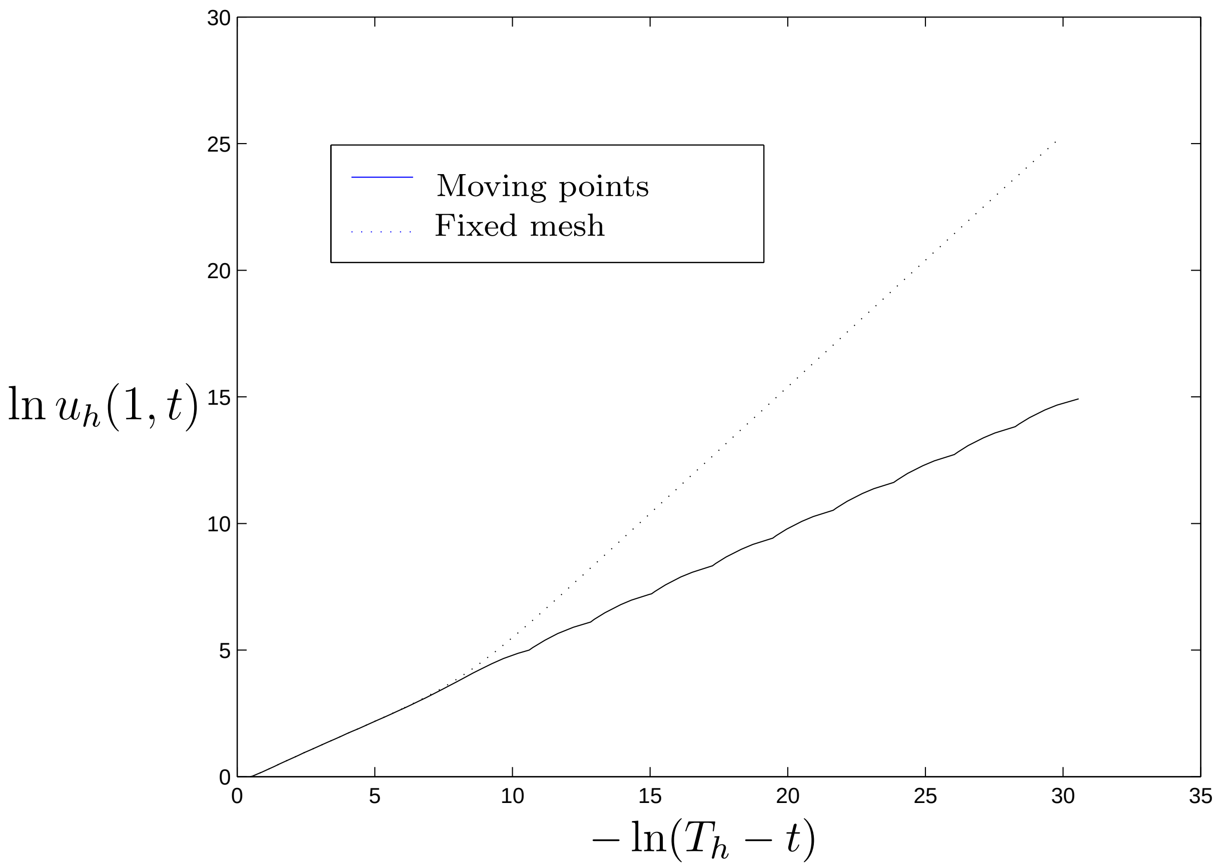
We impose

$$c_1 \leq \frac{u'_N(t)}{u_N^{2p-1}(t)} = \frac{\frac{2}{h_N^2}(u_{N-1}(t) - u_N(t)) + \frac{2}{h_N}(u_N(t))^p}{u_N^{2p-1}(t)} \leq c_2$$

Moving points method



Blow-up rates, $p = 2$



Order of convergence and regularity.

The problem:

$$\begin{array}{lll} u_t = \Delta u + u^p, & \Omega \times (0,T), & \\ u = 0, & \partial\Omega \times (0,T), & p \text{ is subcritical} \\ u(x,0) = u_0(x) > 0, & \Omega. & \end{array}$$

u_h the solution of the same problem replacing $u_0(x)$ by $u_0(x) + h(x)$.

$$|T - T_h| \leq C \|h\|^{\gamma}_{L^\infty(\Omega)}, \qquad \gamma > 0.$$

Using that u and u_h have the same blow-up rate (independent of h) this techniques can be applied to bound $|T - T_h| \leq Ch^\gamma$ in

- Numerical approximations for blow-up problems

- Perturbations of the continuous problem (initial datum, reaction power, diffusion coefficient, etc.)

In case of perturbations of the initial datum the bound can be improved

$$|T(u_0 + h) - T(u_0)| \leq C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^\theta$$

The map $u_0 \mapsto T(u_0)$ is "almost Lipschitz".

THE END