Continuity of the blow-up time and numerical approximations for $u_t = \lambda \Delta u + u^p$.

Joint work with

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The problem:

$$
\begin{array}{ll}
u_t = \lambda \Delta u + u^p, & \Omega \times (0,T), \\
u = 0, & \partial\Omega \times (0,T), \\
u(x,0) = u_0(x), & \Omega.
\end{array}
$$

The domain $\Omega \subset \mathbb{R}^n$ is bounded and smooth, and p is superlineal and subcritical, i.e.,

$$1 < p < p_s = (n+2)/(n-2).$$

The initial datum u_0 is smooth, nonnegative and nontrivial ($u_0 \not\equiv 0$).

This model is used e.g. to describe heat propagation with constant thermal conductivity in a medium with a nonlinear source due, for example, to chemical reaction.

1. Existence, uniqueness and regularity for small times.

2. There is a maximal time of existence, T. If $T < \infty$

$$\lim_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty.$$

In this case we say that the solution *blows up* at time $T = T(\lambda, p, u_0)$.

Several authors proved that $u_0 \mapsto T$ is continuos under different assumptions and using different techniques, e.g.

Baras, P.; Cohen, L. *J. Funct. Anal.* (1987)

Merle, F. *Comm. Pure Appl. Math.* (1992)

Quittner, *P. Houton J. Math.* To appear.

MAIN RESULTS

1. We extend this results finding a modulus of continuity for $\eta = (\lambda, p, u_0) \mapsto T$ which has the form

$$|T(\eta) - T(\eta_0)| \leq C(\eta_0)\|\eta - \eta_0\|^{\gamma}, \qquad \gamma > 0.$$

2. We improve this result for perturbations on the initial data proving

$$|T(u_0 + h) - T(u_0)| \leq C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^{\theta}$$

3. In the second part of the talk we analyze the relation between this modulus of continuity and the rate of convergence for the blow-up time in numerical approximations for this problem.

Some facts about solutions of this equation:

1. The energy functional

$$\Phi(u)(t) = \frac{\lambda}{2}\int_{\Omega} |\nabla u|^2 \, ds - \int_{\Omega} \frac{u^{p+1}}{p+1} \, ds,$$

characterize the solutions with blow-up in the sense that

$$\Phi(u)(t_0) < 0 \text{ for some } t_0 \iff T < \infty.$$

(This fact proved that the set composed of solutions with blow-up si open)

Giga, Y.; Kohn, R.V. *Indiana Univ. Math. J.* (1987).

Cortazar; Del Pino; Elgueta. Comm. Partial Differential Equations (1999).

2. If u blows up at time T

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \sim (T-t)^{-\frac{1}{p-1}}$$

in the sense that there exist $\kappa, \tilde{\kappa} = \tilde{\kappa}(\lambda, p, u_0)$ such that

$$\kappa(T-t)^{-\frac{1}{p-1}} \leq \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \tilde{\kappa}(T-t)^{-\frac{1}{p-1}}$$

$\kappa = (p-1)^{-\frac{1}{p-1}}$, $\tilde{\kappa}$ can be taken locally independent of λ, p, u_0.

This is the key for our arguments!!

Giga; Kohn. (1987)

F. Kammerer, C.; Zaag, H. *Nonlinearity.* (2000)

3. Maximum Principle.

Idea of the proof: perturbations in the initial datum.

The perturbed problem

$$
\begin{cases}
(u_h)_t = \Delta u_h + u_h^p, & \Omega \times (0, T_h), \\
u_h = 0, & \partial\Omega \times (0, T_h), \\
u_h(x,0) = u_0(x) + h(x), & \Omega.
\end{cases}
$$

When $h = 0$ we denote u, T the solution and the blow-up time of this problem. So we define the error function

$$
e(x,t) = u_h(x,t) - u(x,t),
$$

which verifies

$$
\begin{cases}
e_t = \Delta e + u_h^p - u^p, & \Omega \times (0, \tilde{T}), \\
e = 0, & \partial\Omega \times (0, \tilde{T}), \\
e(x,0) = h(x), & \Omega.
\end{cases}
$$

Let t_0 the first time such that $\|e(\cdot, t_0)\|_\infty = 1$. In $[0, t_0]$ e verifies

$$\begin{aligned} e_t &= \Delta e + \frac{u_h^p - u^p}{u_h - u} e \\ &\leq \Delta e + C(T-t)^{-1} e \end{aligned}$$

$$e(x, 0) \leq h(x).$$

By comparison arguments

$$e(x,t) \leq C\|h\|_{L^\infty}(T-t)^{-C}.$$

The error remains small until times very close to the blow-up time if $\|h\|_{L^\infty}$ is small enough.

From this bound we can obtain

$$|T - T_h| \leq C \left(\|h\|_{L^\infty}\right)^{1/C}$$

$\square$

The exponent γ in

$$|T - T_h| \leq C\|h\|_{L^\infty}^{\gamma}$$

depends on the uniform constant that bounds the blow-up rate

$$u(x,t) \leq \frac{\tilde{\kappa}}{(T-t)^{\frac{1}{p-1}}}$$

To obtain a sharper estimate for the modulus of continuity it is necessary to have a better knowledge of this constant.

Merle and Zaag (2000) found the best constant $\kappa = (p-1)^{-1/(p-1)}$ and a bound for a second term of lower order

$$u_h(x,t) \leq \kappa(T_h - t)^{-\frac{1}{p-1}} + \left(\frac{n\kappa}{2p} + \varepsilon\right) \frac{(T_h - t)^{-\frac{1}{p-1}}}{|\ln(T_h - t)|}.$$

This allows us to obtain

$$|T - T_h| \leq C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^{\frac{n+2}{2}+\varepsilon}$$

$\square$

Conjecture:

$$|T - T_h| \leq C\|h\|_{L^\infty}$$

i.e $u_0 \mapsto T$ is Lipschitz.

Numerical Approximations

$$\boxed{\text{Order of convergence}} \leftrightarrow \boxed{\text{Regularity}}$$

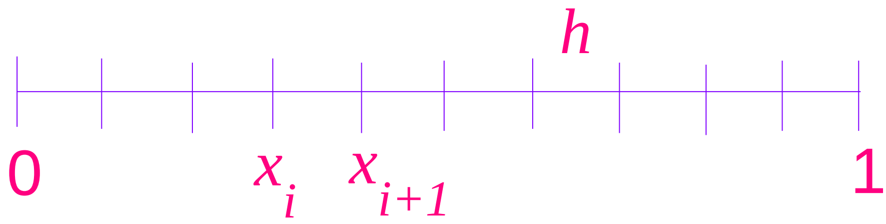
A numerical semi-discrete approximation of this problem is a vector $U(t) = (u_1(t), \dots, u_N(t))$ that approximates the solution $u(x,t)$ at some fixed nodes $\{x_1, \dots, x_N\} \subset \overline{\Omega}$.

This vector $U(t)$ must verify a system like

$$\begin{array}{lcl} MU'(t) & = & -AU(t) + MU(t)^p \\ u_i(0) & = & u_0(x_i),\ 1 \le i \le N. \end{array}$$

M is the mass matrix obtained with lumping and A is the stiffness matrix.

Example: the one dimensional case.



$$\begin{cases} u_1(t) = 0, \\ u_i'(t) = \frac{1}{h^2}(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)) + u_i^p(t), \\ u_{N+1}(t) = 0, \\ u_i(0) = u_0(x_i), \qquad 1 \leq i \leq N+1. \end{cases}$$

It can be proved that continuous solutions with blow-up produce numerical approximations that also blow-up (and with the same blow-up rate) if the parameter of the method, h, is small enough.

$$\|U(t)\|_\infty \leq C(T_h - t)^{-\frac{1}{p-1}}.$$

As before we can define the error function $E(t) = (e_1(t), \ldots, e_N(t))$.

$$e_i(t) = u_i(t) - u(x_i, t).$$

Under adequate assumptions on the matrices A and M similar bounds for this error function can be obtained.

$$\begin{aligned} E'(t) &\leq \frac{C}{T-t} E(t) + Ch^{\alpha}(T-t)^{-\theta} \\ E(0) &\leq \|E(0)\|_{\infty}. \end{aligned}$$

And hence,

$$E(t) \leq Ch(T-t)^{-C}.$$

Arguing as before we get

$$|T_h - T| \leq Ch^{\gamma}.$$

$\square$