SIMULTANEOUS VS. NON-SIMULTANEOUS BLOW-UP IN NUMERICAL APPROXIMATIONS OF A PARABOLIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. We study the asymptotic behavior of a semidiscrete numerical approximation for a pair of heat equations $u_t = \Delta u$, $v_t = \Delta v$ in $\Omega \times (0,T)$; fully coupled by the boundary conditions $\frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}}$, $\frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}}$ on $\partial\Omega \times (0,T)$, where Ω is a bounded smooth domain in $\mathbb{R}^d$. We focus in the existence or not of non-simultaneous blow-up for a semidiscrete approximation (U,V). We prove that if U blows up in finite time then V can fail to blow up if and only if $p_{11} > 1$ and $p_{21} < 2(p_{11}-1)$, which is the same condition as the one for non-simultaneous blow-up in the continuous problem. Moreover, we find that if the continuous problem has non-simultaneous blow-up then the same is true for the discrete one. We also prove some results about the convergence of the scheme and the convergence of the blow-up times.

1. Introduction.

In this paper we study the behavior of semidiscrete approximations of the following system. A pair of heat equations

$$u_t = \Delta u, \qquad v_t = \Delta v \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

fully coupled by the nonlinear flux boundary condition, given by

$$\frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}}, \qquad \frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}} \qquad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

and initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ in Ω. We assume that Ω is a bounded smooth domain in $\mathbb{R}^d$, $p_{ij} \geq 0$ and u_0, v_0 are positive, bounded, compatible with the boundary data and smooth enough to guarantee that u, v are regular. Solutions to this problem exist locally in time, [4]. The time T is the maximal existence time for the solution, which may be finite or infinite.

The study of reaction-diffusion systems have deserved a great deal of interest in recent years and have been used to model, for example, heat transfer, population dynamics and chemical reactions (see [14, 17] and references therein). Therefore the study of its numerical approximations become a relevant issue. Specially the study of the dynamic of such approximations, since the natural interest in this

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kind of problems is the prediction of the long-time behavior of the solutions, see [11, 13, 18], etc.

A remarkable, and well known fact, is that solutions of (1.1)-(1.2) may develop singularities in finite time, no matter how smooth the initial data are. In fact, if T is finite, the solution (u, v) blows up in the sense that

$$\limsup_{t \nearrow T} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty}) = +\infty.$$

The blow-up phenomenon for parabolic equations and systems has been widely studied in recent years, see for example [17]. For our problem, there exist solutions (u, v) that blow up in finite time, T, if and only if the exponents p_{ij} verify any of the conditions, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}$, see [16]. A priori there is no reason why both functions u and v should go to infinity simultaneously at time T. In fact, in [15] the authors prove under adequate hypotheses that there are initial data such that u blows up while v does not if and only if $p_{11} > 1$ and $p_{21} < p_{11} - 1$. They denote this phenomenon as *non-simultaneous blow-up*.

Here we prove similar results for numerical approximations of (1.1)-(1.2). For previous work on numerical approximations of blowing up solutions we refer to [1, 2, 3, 6, 7, 8, 10, 12], the survey [5] and references therein.

We will consider a general method for the space discretization with adequate assumptions on the coefficients, keeping the time variable t continuous. More precisely, we assume that for every $h > 0$ small (h is the parameter of the method), there exists a set of nodes $\{x_1, \dots, x_N\} \subset \overline{\Omega}$ $(N = N(h))$, such that the numerical approximation of (u, v) at the nodes x_k, is given by

$$U(t) = (u_1(t), \dots, u_N(t)), \qquad V(t) = (v_1(t), \dots, v_N(t)).$$

That is $(u_k(t), v_k(t))$ stands for an approximation of $(u(x_k, t), v(x_k, t))$. We assume that (U, V) is the solution of the following ODE

$$\begin{aligned} MU'(t) &= -AU(t) + B(U^{p_{11}}(t)V^{p_{12}}(t)), \\ MV'(t) &= -AV(t) + B(U^{p_{21}}(t)V^{p_{22}}(t)), \end{aligned} \tag{1.3}$$

with initial data given by $u_k(0) = u_0(x_k)$, $v_k(0) = v_0(x_k)$. In (1.3) and hereafter, all operations between vectors are understood coordinate by coordinate.

The precise assumptions on the matrices involved in the method are:

(H1) M is a diagonal matrix with positive entries m_k.
(H2) B is a diagonal matrix with nonnegative entries b_k.
(H3) A is a nonnegative symmetric matrix, with nonpositive coefficients off the diagonal (that is $a_{ij} \le 0$ if $i \ne j$) and $a_{ii} > 0$.
(H4) $\sum_{j=1}^{N} a_{ij} = 0$.

This last assumption implies, to begin with, that there is no nontrivial steady state for (1.3) (see Lemma 3.2).

The final hypothesis on the scheme is the following: if we define the graph $\mathcal{G}$ with vertices on the nodes x_k and we say that two nodes x_k, x_j are connected if and only if $a_{kj} \ne 0$, then we assume

(H5) The graph $\mathcal{G}$ is connected.

It is easy to check that hypotheses (H1)-(H5) imply the maximum principle.

We will say that a node x_k (o more generally, a node k) is a boundary node if and only if $b_k \neq 0$.

We remark that in general M, B and A depend on h.

Writing these equations explicitly we obtain the following ODE system,

$$
\begin{aligned}
m_k u_k'(t) &= -\sum_{j=1}^{N} a_{kj} u_j(t) + b_k u_k^{p_{11}}(t) v_k^{p_{12}}(t), \ 1 \le k \le N, \\
m_k v_k'(t) &= -\sum_{j=1}^{N} a_{kj} v_j(t) + b_k u_k^{p_{21}}(t) v_k^{p_{22}}(t), \ 1 \le k \le N,
\end{aligned}
\tag{1.4}
$$

with initial data $u_k(0) = u_0(x_k)$, $v_k(0) = v_0(x_k)$, for $1 \le k \le N$.

As an example, we can consider a linear finite element approximation of problem (1.1) on a regular acute triangulation of Ω (see [9]). In this case, let W_h be the subspace of piecewise linear functions in $H^1(\Omega)$.

We impose that $u_h \ , \ v_h \ : \ [0, T_h) \ \to \ W_h$, verifies

$$
\begin{aligned}
\int_\Omega ((u_h)_t w)^I &= -\int_\Omega \nabla u_h \nabla w + \int_{\partial\Omega} ((u_h)^{p_{11}} (v_h)^{p_{12}} w)^I, \\
\int_\Omega ((v_h)_t w)^I &= -\int_\Omega \nabla v_h \nabla w + \int_{\partial\Omega} ((u_h)^{p_{21}} (v_h)^{p_{22}} w)^I,
\end{aligned}
$$

for every $w \in W_h$. Here $(\cdot)^I$ stands for the linear Lagrange interpolation at the nodes of the mesh. If we call $(U(t), V(t))$ the restriction of $(u_h(\cdot, t), v_h(\cdot, t))$ to the nodes of the mesh, then (U, V) verifies a system of the form (1.3). Our assumptions on the matrices M and B hold because we are using mass lumping and our assumptions on A are satisfied as we are considering an acute regular mesh. In this example, M is called the lumped mass matrix and A the stiffness matrix. In this case, k is a boundary node, if and only if $x_k \in \partial\Omega$.

As another example if Ω is a cube, $\Omega = (0, 1)^d$, we can use a semidiscrete finite differences method to approximate the solution $u(x, t)$ obtaining an ODE system of the form (1.4), and again, k is a boundary node, if and only if $x_k \in \partial\Omega$.

We begin our analysis of (1.4) proving that this method converges uniformly over $\{x_1, \dots, x_N\} \times [0, T - \tau]$ under the assumption of consistency of the method. For a precise definition of consistency, see Definition 2.1.

In fact, we prove the following result.

Theorem 1.1. *Let (u, v) be a regular solution of (1.1)-(1.2) and (U, V) the numerical approximation given by (1.4). If the method is consistent then there exists a constant C, independent of h, such that*

$$
\max_{1 \le k \le N} \sup_{t \in [0, T-\tau]} |u(x_k, t) - u_k(t)| + |v(x_k, t) - v_k(t)| \le C\rho(h),
$$

where ρ is the modulus of consistency of the method.

As a first step for our analysis of the behavior of solutions of (1.4), we want to describe when the blow-up phenomenon occurs for the discrete problem. We say that a solution of (1.4) has finite blow-up time if there exists a finite time T_h such that

$$\lim_{t \nearrow T_h} (\|U(t)\|_\infty + \|V(t)\|_\infty) = \lim_{t \nearrow T_h} \left(\max_k u_k(t) + \max_k u_k(t) \right) = +\infty.$$

The following Theorem characterizes the existence of blowing up solutions for (1.3).

Theorem 1.2. *Every solution (U, V) of* (1.3) *blows up in finite time if and only if the exponents p_{ij} verify any of the conditions,*

$$p_{11} > 1, \qquad p_{22} > 1 \quad or \quad (p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}.$$

We want to remark that the conditions on the exponents are the same as for the continuous problem, see [16].

Moreover, we prove under further assumptions on the exponents p_{ij}, that the blow-up time for the numerical solution converges to the one of the continuous problem under adequate hypotheses on the initial data.

Theorem 1.3. *Let (u, v) be a solution of* (1.1)-(1.2) *with blow-up time T and initial datum (u_0, v_0) satisfying $\Delta u_0, \Delta v_0 \geq \kappa > 0$. Let (U, V) be the corresponding numerical solution. If the scheme is consistent and if one of the following $p_{11} > 1$, $p_{22} > 1$, or $p_{21}, p_{12} > 1$ hold, then the blow-up time T_h of the numerical solution converges to T, i.e.*

$$\lim_{h \to 0} T_h = T.$$

We observe that the hypotheses $\Delta u_0, \Delta v_0 \geq \kappa > 0$, imply monotonicity in time for the solution (u, v). That is $u_t, v_t > 0$. This monotonicity property also holds for our numerical solution (U, V) (see Lemma 4.1) and it is crucial for our arguments.

Finally we arrive to the main point of the paper. For certain choices of the parameters p_{ij} there are initial data for which one of the components of the system remains bounded while the other blows up. The next two theorems characterize the range of parameters for which non-simultaneous blow-up occurs in the discrete problem.

Theorem 1.4. *Let (U, V) a solution of* (1.4) *such that U blows up at finite time T_h and V remains bounded up to that time. Then $p_{11} > 1$ and $p_{21} < p_{11} - 1$.*

Theorem 1.5. *If $p_{11} > 1$ and $p_{21} < p_{11} - 1$, then for every initial datum V_0 for* (1.4) *there exists an initial datum U_0 such that U blows up in finite time T_h and V remains bounded up to that time.*

We want to remark that this characterization is the same as the one for the continuous problem, see [15].

We end this paper showing that the non-simultaneous blow-up is reproduced by our scheme.

Theorem 1.6. *Let u_0, v_0 be initial data for* (1.1)-(1.2) *satisfying $\Delta u_0, \Delta v_0 \geq \kappa > 0$ such that u blows up at time T and v remains bounded up to that time. Then, if the scheme is consistent, U blows up while V remains bounded, for every h small enough.*

The paper is organized as follows: in §2 we prove the convergence result (Theorem 1.1), in §3 the blow-up result (Theorem 1.2), in §4 we study the convergence of the blow-up times (Theorem 1.3), and finally in §5 we arrive at the main part of the paper, namely we prove the non-simultaneous blow-up results, Theorems 1.4, 1.5 and 1.6.

2. Convergence of the numerical scheme.

In this section we prove a uniform convergence result for the numerical scheme (1.4). Throughout this section, we consider $0 < \tau < T$ fixed.

We want to show that $(U, V) \to (u, v)$ as $h \to 0$, uniformly in $\{x_1, \dots, x_N\} \times [0, T - \tau]$. This is a natural requirement since in such time intervals the exact solution is regular. Approximations of regular solutions in one space dimension for a scalar problem with a nonlinear boundary condition have been analyzed in [10]. Also, in [3] the authors analyze the approximation in several space dimensions under similar hypotheses that we make here.

The precise assumption that we make on the scheme is the consistency of the method. We precise this concept in the following definition.

Definition 2.1. *Let w be a regular solution of*

$$w_t = \Delta w + g(x,t) \qquad \text{in } \Omega \times (0,T), \qquad \frac{\partial w}{\partial \eta} = f(x,t) \qquad \text{on } \partial\Omega \times (0,T).$$

We say that the scheme (1.3) *is consistent if for any $t \in (0, T - \tau)$ it holds*

$$m_k w_t(x_k, t) = -\sum_{j=1}^{N} a_{kj} w(x_j, t) + m_k g(x_k, t) + b_k f(x_k, t) + \rho_{k,h}(t), \tag{2.1}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max_k \frac{|\rho_{k,h}(t)|}{m_k} \le \rho(h), \qquad \text{for every } t \in (0, T - \tau),$$

with $\rho(h) \to 0$ if $h \to 0$. The function ρ is called the modulus of consistency of the method.

Let us begin with a comparison lemma that will be used throughout the paper.

Definition 2.2. *We say that $(\overline{U}, \overline{V})$ is a supersolution of* (1.3) *if*

$$M\overline{U}' \ge -A\overline{U} + B(\overline{U}^{p_{11}} \overline{V}^{p_{12}}),$$
$$M\overline{V}' \ge -A\overline{V} + B(\overline{V}^{p_{21}} \overline{V}^{p_{22}}).$$

We say that $\underline{U}$ is a subsolution of (1.3) *if*

$$M\underline{U}' \le -A\underline{U} + B(\underline{U}^{p_{11}} \underline{V}^{p_{12}}),$$
$$M\underline{V}' \le -A\underline{V} + B(\underline{V}^{p_{21}} \underline{V}^{p_{22}}).$$

The inequalities are understood coordinate by coordinate.

Lemma 2.1. *Let $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ be a super and a subsolution of* (1.3) *respectively such that $(\overline{U}, \overline{V})(0) \ge (\underline{U}, \underline{V})(0)$. Then*

$$(\overline{U}, \overline{V})(t) \ge (\underline{U}, \underline{V})(t).$$

Proof. Let $(W,Z)=(\overline{U}-\underline{U},\overline{V}-\underline{V})$. Assume first that $W(0),Z(0)>0$. We observe that W verifies

$$\begin{aligned} MW' &\geq -AW + B\left(\overline{U}^{p_{11}}\overline{V}^{p_{12}} - \underline{U}^{p_{11}}\overline{V}^{p_{12}} + \underline{U}^{p_{11}}\overline{V}^{p_{12}} - \underline{U}^{p_{11}}\underline{V}^{p_{12}}\right) \\ &= -AW + B\left[\overline{V}^{p_{12}}\left(\frac{\overline{U}^{p_{11}}-\underline{U}^{p_{11}}}{\overline{U}-\underline{U}}\right)W\right] + B\left[\underline{U}^{p_{11}}\left(\frac{\overline{V}^{p_{12}}-\underline{V}^{p_{12}}}{\overline{V}-\underline{V}}\right)Z\right]. \end{aligned}$$

And a similar inequality holds for Z. Now, set $\delta=\min\{W(0),Z(0)\}$ and suppose that the statement of the Lemma is false. Thus, let t_0 be the first time such that $\min\{W(t_0),Z(t_0)\}=\delta/2$. We can assume that W attains the minimum. At that time, there must be a node k such that $w_k(t_0)=\delta/2$. But on the one hand $w_k'(t_0)\leq 0$ and, on the other hand, by (H3) and (H4), at that time t_0,

$$\begin{aligned} m_k w_k' &\geq -\sum_{j=1}^N a_{kj}w_j + b_k\overline{v}_k^{p_{12}}\left(\frac{\overline{u}_k^{p_{11}}-\underline{u}_k^{p_{11}}}{\overline{u}_k-\underline{u}_k}\right)w_k + b_k\underline{u}_k^{p_{11}}\left(\frac{\overline{v}_k^{p_{12}}-\underline{v}_k^{p_{12}}}{\overline{v}_k-\underline{v}_k}\right)z_k \\ &> -\sum_{i=1}^N a_{kj}\frac{\delta}{2} + b_k\overline{v}_k^{p_{12}}p_{11}\xi_k^{p_{11}-1}w_k + b_k\underline{u}_k^{p_{11}}p_{12}\eta_k^{p_{12}-1}z_k > 0, \end{aligned}$$

where $\xi_k\in(\underline{u}_k,\overline{u}_k)$, $\eta_k\in(\underline{v}_k,\overline{v}_k)$, a contradiction. Using the continuity of solutions of (1.3) with respect to the initial data and an approximation argument, the result follows for general initial data. □

Now we are ready to prove our convergence result.

Proof of Theorem 1.1. Let us start by defining the error functions

$$e_{1,k}(t)=u(x_k,t)-u_k(t), \qquad e_{2,k}(t)=v(x_k,t)-v_k(t). \tag{2.2}$$

By (2.1), these functions verify

$$\begin{aligned} m_k e_{1,k}'(t) &= -\sum_{i=1}^N a_{ik}e_{1,i}(t) + b_k(u^{p_{11}}(x_k,t)v^{p_{12}}(x_k,t) - u_k^{p_{11}}(t)v_k^{p_{12}}(t)) + \rho_{k,h}^1(t), \\ m_k e_{2,k}'(t) &= -\sum_{i=1}^N a_{ik}e_{2,i}(t) + b_k(u^{p_{21}}(x_k,t)v^{p_{22}}(x_k,t) - u_k^{p_{21}}(t)v_k^{p_{22}}(t)) + \rho_{k,h}^2(t). \end{aligned}$$

Let $t_0=\max\{t:\ t<T-\tau,\ \max_k|e_{i,k}(t)|\leq 1, i=1,2\}$. We will see that $t_0=T-\tau$ for h small enough. In $[0,t_0]$ we have

$$m_k e_{1,k}'(t) \leq -\sum_{j=1}^N a_{jk}e_{1,j}(t) + Kb_k e_{1,k}(t) + Kb_k e_{2,k}(t) + \rho_{k,h}^1(t),$$

where

$$\begin{aligned} K = \max\big\{ &(\|v\|_{L^\infty(\Omega\times[0,T-\tau])}+1)^{p_{12}}p_{11}(\|u\|_{L^\infty(\Omega\times[0,T-\tau])}+1)^{p_{11}-1}, \\ &(\|u\|_{L^\infty(\Omega\times[0,T-\tau])}+1)^{p_{11}}p_{12}(\|v\|_{L^\infty(\Omega\times[0,T-\tau])}+1)^{p_{12}-1}\big\}. \end{aligned}$$

An analogous inequality holds for $e_{2,k}$. Hence, in $[0,t_0]$, $E_1=(e_{1,1};\dots;e_{1,N})$, $E_2=(e_{2,1};\dots;e_{2,N})$ is a solution of

$$\begin{aligned} ME_1' &\leq -AE_1 + KB(E_1+E_2) + \rho(h)M(1,...,1)^t, \\ ME_2' &\leq -AE_2 + KB(E_1+E_2) + \rho(h)M(1,...,1)^t. \end{aligned}$$

Let us now define the following function $(W, Z) = (w_1, \ldots, w_N, z_1, \ldots, z_N)$ that will be used as a supersolution.

Let $a \in C^2(\overline{\Omega})$ be such that $a(x) \geq \delta > 0$ in Ω, $\partial a/\partial \eta > 2Ka$ on $\partial\Omega$ and let $b(t) = \exp(Lt)$ where L is to be determined.

Then, it is easy to check that, if L is large, $w(x,t) = Ca(x)b(t)\rho(h)$ verifies

$$\begin{aligned} & w_t > \Delta w, \text{ in } \Omega \times [0, T-\tau], \\ & \frac{\partial w}{\partial \eta} > 2Kw, \text{ on } \partial\Omega \times [0, T-\tau]. \end{aligned}$$

Now, by the consistency of the scheme, one can verify that

$$W = Z = Cb(t)\rho(h)(a(x_1), ..., a(x_N))$$

is a supersolution of

$$\begin{aligned} MW' &> -AW + KB(W+Z) + \rho(h)M(1,...,1)^t, \\ MZ' &> -AZ + KB(W+Z) + \rho(h)M(1,...,1)^t, \end{aligned}$$

for C big enough depending on K but not on h.

Next as $0 = E_i(0) \leq W(0) = Z(0)$ for $i = 1, 2$, it follows by a comparison argument (Lemma 2.1) that

$$E_i(t) \leq W(t) = Z(t), \quad \forall t \in [0, t_0], \; i = 1, 2.$$

By a symmetric argument, it follows that

$$|E_i(t)| \leq Cb(T-\tau)\|a\|_{L^\infty(\Omega)}\rho(h).$$

¿From this fact, as $\rho(h) \to 0$, it is easy to see that $t_0 = T - \tau$ for h small enough, and the result follows. $\square$

3. Blow-up for the numerical scheme.

In this section we prove a blow-up result for the numerical scheme, Theorem 1.2.

First, we cite an auxiliary Lemma (whose proof can be found in [16]) about a related ordinary differential equation.

Lemma 3.1. *([16], Theorem 2.1) Let $(x(s), y(s))$ be a positive solution of*

$$x' = x^{p_{11}} y^{p_{12}}, \qquad y' = x^{p_{21}} y^{p_{22}}. \tag{3.1}$$

Then (x, y) blows up in finite time if and only if one of the following conditions holds:

$$p_{11} > 1, \; p_{22} > 1 \text{ or } (p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}.$$

Next, we prove a couple of Lemmas concerning a dynamic property of a single equation.

Lemma 3.2. *Let Z be a nonnegative solution of*

$$0 = -AZ + cBZ^\beta, \tag{3.2}$$

with $\beta, c > 0$. Then $Z \equiv 0$.

Proof. First we observe that summing up all the equations of the system (3.2) we get, by (H4),

$$0 = -\sum_{j,k=1}^{N} a_{kj} z_j + c \sum_{k=1}^{N} b_k z_k^\beta = -\sum_{j=1}^{N} z_j \left(\sum_{k=1}^{N} a_{kj} \right) + c \sum_{k=1}^{N} b_k z_k^\beta = c \sum_{k=1}^{N} b_k z_k^\beta.$$

Therefore, by (H2), $z_k = 0$ for every boundary node k. Since by (H1)-(H5) the maximum principle holds for our numerical scheme, the conclusion of the Lemma follows. □

Lemma 3.3. *Let W be a positive solution of the following equation*

$$MW' = -AW + cBW^\beta \tag{3.3}$$

with $\beta > 1$ and $c > 0$. Then W blows up in finite time.

Proof. First we observe that

$$\Phi(W) = \frac{1}{2} \langle A^{1/2} W, A^{1/2} W \rangle - \frac{c}{\beta + 1} \langle B W^\beta, W \rangle$$

is a Lyapunov functional for (3.3), therefore a solution W either converges to a stationary solution or it is unbounded. The first is impossible since, by Lemma 3.2, $Z \equiv 0$ is the unique stationary solution of (3.3) and the minimum principle holds for (3.3). Therefore W is unbounded and, as the maximum must be attained at a boundary node k, w_k satisfies

$$m_k w_k' \geq -a_{kk} w_k + c b_k w_k^\beta \geq \delta w_k^\beta.$$

Hence w_k cannot be globally defined, as we wanted to show. □

Finally, let us prove a Lemma that allows us to compare U with V in the case of strong coupling.

Lemma 3.4. *Let $p_{11} \leq 1$, $p_{22} \leq 1$ and $(p_{11} - 1)(p_{22} - 1) < p_{12} p_{21}$ and let $\alpha = (p_{12} - p_{22} + 1)/(p_{21} - p_{11} + 1)$. Observe that our assumptions on p_{ij} imply that $\alpha > 0$. If $\alpha \geq 1$ then there exists a constant $C > 0$ independent of h such that the solution of* (1.4) *satisfies*

$$C u_k(t) \geq v_k^\alpha(t), \qquad 1 \leq k \leq N \tag{3.4}$$

and if $\alpha < 1$ then there exists a constant $C > 0$ independent of h such that the solution of (1.4) *satisfies*

$$C v_k(t) \geq u_k^{1/\alpha}(t), \qquad 1 \leq k \leq N. \tag{3.5}$$

Proof. Assume first that $\alpha \geq 1$. Let $z_k(t) = C u_k(t)$ and $w_k(t) = v_k^\alpha(t)$, with C a positive constant. The function Z satisfies,

$$MZ' = -AZ + C^{1-p_{11}} B Z^{p_{11}} W^{p_{12}/\alpha}.$$

Using the convexity of the function x^α ($\alpha \geq 1$) and (H3) we have that

$$\alpha v_k^{\alpha-1} \left(-\sum_{j=1}^{N} a_{kj} v_j \right) \leq \left(-\sum_{j=1}^{N} a_{kj} v_j^\alpha \right),$$

hence,

$$MW' \le -AW + \frac{\alpha}{C^{p_{21}}} B(Z^{p_{21}} W^{\frac{p_{22}-1}{\alpha}+1}).$$

Choosing $C > 0$ large enough (but independent of h, as u_0 and v_0 are strictly positive) we can assume that

$$z_k(0) = Cu_k(0) > v_k^\alpha(0) = w_k(0). \tag{3.6}$$

We argue by contradiction. Assume, that there exists a first time t_0 and a node x_k such that

$$z_k(t_0) = w_k(t_0).$$

Using that $(C^{1-p_{11}} - \frac{\alpha}{C^{p_{21}}}) > 0$ (this can be done by choosing C large). Observing that $p_{11} + p_{12}\alpha = (p_{22} - 1)/\alpha + 1 + p_{21}$ and using (H4), at $t = t_0$ we have

$$\begin{aligned} 0 &\ge m_k(z_k - w_k)'(t_0) \\ &\ge -\sum_{j=1}^{N} a_{kj}(z_j - w_j)(t_0) + b_k \left(C^{1-p_{11}} - \frac{\alpha}{C^{p_{21}}}\right) z_k^{p_{11}+(p_{12}/\alpha)} > 0, \end{aligned}$$

a contradiction.

The case $\alpha < 1$ is analogous. $\square$

Now we prove Theorem 1.2, which states a necessary and sufficient condition for the existence of blowing up solutions of the discrete problem (1.3).

Proof of Theorem 1.2. First, let us see that if (U, V) blows up then $p_{11} > 1$, $p_{22} > 1$ or $(p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}$.

Let us call $x = \sum_{k=1}^{N} u_k,\ y = \sum_{k=1}^{N} v_k$. By (H4) we get

$$\min_k m_k x' \le \sum_{k=1}^{N} b_k u_k^{p_{11}} v_k^{p_{12}} \le N \max_k b_k x^{p_{11}} y^{p_{12}}, \quad \min_k m_k y' \le N \max_k b_k x^{p_{21}} y^{p_{22}}.$$

Hence (x, y) is a subsolution for the system (3.1). Then if (U, V) blows up the conditions of Lemma 3.1 are satisfied.

To conclude with the proof, if $p_{11} > 1$, as $v_k(t) \ge \min_j v_j(0) > \delta > 0$, U verifies

$$MU' \ge -AU + cBU^{p_{11}}$$

that is, U is a supersolution of (3.3). Therefore, by Lemma 3.3, U blows up.

The case $p_{22} > 1$ is analogous, arguying with V instead of U.

For the case $p_{11} \le 1$, $p_{22} \le 1$ and $(p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}$, we proceed as follows. Let α be as in Lemma 3.4. Then we can assume that $\alpha \ge 1$ and, using Lemma 3.4, we obtain

$$MV' \ge -AV + CBV^{p_{21}\alpha+p_{22}}.$$

Now, our assumption on the exponents p_{ij} implies that $p_{21}\alpha + p_{22} > 1$, and the result follows as in the previous cases. $\square$

4. Convergence of the blow-up times

Let us begin proving a couple of Lemmas.

Lemma 4.1. *Let (U,V) be a solution of (1.3) with $p_{11} > 1$. such that $u_k'(0) \geq \delta u_k^{p_{11}}(0)$ and $v_k'(0) \geq 0$, $1 \leq k \leq N$. Then $u_k'(t) \geq \delta u_k^{p_{11}}(t)$ and $v_k'(t) \geq 0$, $1 \leq k \leq N$ for every $t < T_h$.*

Proof. First, we claim that both $u_k'(t)$ and $v_k'(t)$ are nonnegative. In order to do that, let us define $w_k(t) = u_k'(t)$ and $z_k(t) = v_k'(t)$. Therefore, by simple computation, (W,Z) verifies

$$MW' = -AW + D_1W + D_2Z, \quad MZ' = -AZ + D_3W + D_4Z,$$

where D_i are time dependent matrices with nonnegative coefficients.

As $W(0), Z(0) \geq 0$, by the minimum principle, the claim follows.

Now, let us check that $u_k'(t) \geq \delta u_k^{p_{11}}(t)$. Let $w_k(t) = u_k'(t) - \delta u_k^{p_{11}}(t)$. We want to use the minimum principle to show that $w_k(t)$ is positive. To this end, we observe that w_k verifies

$$\begin{aligned}
& m_k w_k' + \sum_{j=1}^N a_{kj} w_j = m_k(u_k'' - \delta p_{11} u_k^{p_{11}-1} u_k') + \sum_{j=1}^N a_{kj}(u_j' - \delta u_j^{p_{11}}) \\
& = -\delta m_k p_{11} u_k^{p_{11}-1} u_k' + b_k(p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} + p_{12} v_k^{p_{12}-1} v_k' u_k^{p_{11}}) - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}} \\
& \geq -\delta m_k p_{11} u_k^{p_{11}-1} u_k' + b_k p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}} \\
& = -\delta p_{11} u_k^{p_{11}-1} \left(\sum_{j=1}^N a_{kj} u_j + b_k u_k^{p_{11}} v_k^{p_{12}} \right) + b_k p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}} \\
& = b_k p_{11} u_k^{p_{11}-1} v_k^{p_{12}} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^{p_{11}} - p_{11} u_k^{p_{11}-1} u_j) + a_{kk}(1 - p_{11}) u_k^{p_{11}} \right) \\
& = b_k p_{11} u_k^{p_{11}-1} v_k^{p_{12}} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^{p_{11}} - p_{11} u_k^{p_{11}-1}(u_j - u_k) - u_k^{p_{11}}) \right. \\
& \left. + \sum_{j=1}^N a_{kj}(1 - p_{11}) u_k^{p_{11}} \right).
\end{aligned}$$

As $f(u) = u^{p_{11}}$ is convex ($p_{11} > 1$), by our hypotheses on the matrix A it follows that $W = (w_1, \dots, w_N)$ verifies

$$MW' \geq -AW + Bp_{11}(U^{p_{11}-1} V^{p_{12}} W) \geq -AW + cBW.$$

As $W(0) > 0$ and the minimum principle holds for this equation, the result follows. $\square$

Lemma 4.2. *Let (U,V) be a solution of (1.3) with $p_{11}, p_{22} \le 1$ and $p_{12}, p_{21} > 1$ such that $u_k'(0) \ge \delta v_k^{p_{12}}(0)$ and $v_k'(0) \ge \delta u_k^{p_{21}}(0)$, $1 \le k \le N$. Then $u_k'(t) \ge \delta v_k^{p_{12}}(t)$ and $v_k'(t) \ge \delta u_k^{p_{21}}(t)$, $1 \le k \le N$ for every $t < T_h$.*

Proof. The proof is similar to the previous lemma, so we only make a sketch. Let $w_k(t) = u_k'(t) - \delta v_k^{p_{12}}(t)$ and $z_k(t) = v_k'(t) - \delta u_k^{p_{21}}(t)$. Then, arguing as in the previous Lemma, using the convexity of the function $f(s) = s^q$, $q > 1$, it follows that W, Z verifies

$$MW' \ge -AW + cBZ, \qquad MZ' \ge -AZ + cBW.$$

Finally, we use the minimum principle and our assumption on the initial data to finish the proof. $\square$

Now we prove the main result of the section, the convergence of the blow-up times.

Proof of Theorem 1.3. We begin with the case $p_{11} > 1$ and $p_{22} \le 1$. We have that u blows up at finite time T. Also the pair (U,V) blows up at finite time T_h.

As the scheme is consistent one can check that $\Delta u_0, \Delta v_0 \ge \kappa > 0$ implies the hypotheses of the previous lemma for h small enough, with δ independent of h. So we have that $u_k'(t) \ge \delta u_k^{p_{11}}(t)$ and $v_k'(t) \ge 0$. Now, applying Lemma 4.1 and integrating we obtain

$$\int_t^{T_h} \frac{u_k'(s)}{u_k^{p_{11}}}\, ds \ge \delta(T_h - t),$$

so

$$\delta(T_h - t) \le \int_{\max_k u_k(t)}^{+\infty} \frac{1}{x^{p_{11}}}\, dx. \tag{4.1}$$

Since $p_{11} > 1$, given $\varepsilon > 0$ we can choose K large and independent of h such that

$$\frac{1}{\delta}\int_K^{+\infty} \frac{1}{x^{p_{11}}}\, dx \le \frac{\varepsilon}{2}.$$

As u blows up at time T there exists $\tau < \frac{\varepsilon}{2}$ such that

$$\|u(\cdot, T-\tau)\|_\infty \ge 2K.$$

Then, by Theorem 1.1, for h small,

$$\max_k u_k(T-\tau) \ge K.$$

By (4.1)

$$|T_h - (T-\tau)| \le \frac{1}{\delta}\int_K^{+\infty} \frac{1}{x^{p_{11}}}\, dx < \frac{\varepsilon}{2},$$

therefore

$$|T_h - T| \le |T_h - (T-\tau)| + \tau < \varepsilon.$$

The case $p_{11} \le 1$ and $p_{22} > 1$ is analogous. For the case p_{11}, $p_{22} > 1$ we observe that the pair (u,v) blows up at time T and so does (U,V) at time T_h. Suppose that u blows up, as estimate (4.1) is still valid, the rest of the argument can be applied.

Finally we consider the case p_{11}, $p_{22} \leq 1$, and $1 < p_{12}, p_{21}$. Here we use Lemma 4.2 instead of Lemma 4.1 and obtain

$$u_k'(t) \geq \delta v_k^{p_{12}}(t), \qquad v_k'(t) \geq \delta u_k^{p_{21}}(t)$$

with $\delta > 0$ is independent of h. Now, let α be as in Lemma 3.4. If $\alpha \geq 1$ (the other case is analogous), by Lemma 3.4, we get

$$v_k'(t) \geq c v_k^{p_{21}\alpha},$$

as $p_{21}\alpha > 1$, the proof now follows as in the previous cases. $\square$

5. Non-simultaneous blow-up.

In this Section we consider positive solutions of (1.4) with h fixed and we denote with C a positive constant that may depend on h and may vary from one line to another.

Proof of Theorem 1.4. As V is bounded, by (1.4) we have

$$m_k u_k'(t) \leq -\sum_{j=1}^{N} a_{kj} u_j(t) + b_k u_k^{p_{11}}(t) C^{p_{12}}.$$

Let k be a boundary node that is blowing up. Then there exists a time t_0 such that for every $t \in [t_0, T_h)$ it holds

$$u_k'(t) \leq -\sum_{j=1}^{N} \frac{a_{kj}}{m_k} u_j(t) + C u_k^{p_{11}}(t) \leq C u_k^{p_{11}}(t). \tag{5.1}$$

If $p_{11} \leq 1$, u_k is bounded, a contradiction. Then p_{11} must be strictly greater than one.

For the second condition we need to get a bound from below for the blow-up rate of u_k. For $t \in [t_0, T_h)$ we can integrate (5.1) between t and T_h to obtain

$$\int_t^{T_h} \frac{u_k'(s)}{u_k^{p_{11}}(s)}\, ds \leq C(T_h - t).$$

Changing variables we get

$$\int_{u_k(t)}^{+\infty} \frac{1}{s^{p_{11}}}\, ds \leq C(T_h - t),$$

hence

$$u_k(t) \geq C(T_h - t)^{-1/(p_{11}-1)}. \tag{5.2}$$

As there exists $\delta > 0$ such that $v_k(t) > \delta$, for $1 \geq k \geq N$, $t \in [t_0, T_h)$ and using (5.2), we obtain

$$m_k v_k'(t) \geq -\sum_{j=1}^{N} a_{kj} v_j(t) + \frac{C}{(T_h - t)^{\frac{p_{21}}{p_{11}-1}}} \delta^{p_{22}}.$$

As $v_k(t)$ is bounded, we have that $p_{21} < p_{11} - 1$. This fact completes the proof. $\square$

Proof of Theorem 1.5. Let V_0 any initial data for V. There exists t_0, $\delta > 0$ such that $v_k(t) \geq \delta$ for all $0 \leq t \leq t_0$, $k = 1 \dots N$. Let k be a boundary node. For $t \in (0, t_0)$, u_k verifies

$$m_k u_k'(t) \geq -\sum_{k=1}^{N} a_{kj} u_j(t) + b_k u_k^{p_{11}}(t)\delta^{p_{22}} \geq -a_{kk} u_k(t) + C u_k^{p_{11}}(t),$$

now, if we choose $u_k(0)$ large, as $p_{11} > 1$, $u_k(t)$ cannot be globally defined. Therefore $(U(t), V(t))$ blows up at finite time $T_h < t_0$. Also, the blow-up rate is bounded by

$$\max_k u_k(t) \leq \frac{C}{(T_h - t)^{\frac{1}{p_{11}-1}}}.$$

Then V verifies

$$MV'(t) \leq -AV + BC(T_h - t)^{-\frac{p_{21}}{p_{11}-1}} V^{p_{22}}.$$

As $p_{21} < (p_{11} - 1)$, if T_h is small enough, V remains bounded until time T_h. Now we can choose $U(0)$ large such that (U, V) blows up at time T_h small enough to ensure that V is bounded hence U blows up and the result follows. □

Finally we will face the proof of Theorem 1.6.

Proof of Theorem 1.6. As u blows up and v remains bounded, by [15] we have that $p_{11} > 1$ and $p_{21} < p_{11} - 1$. Hence by Theorem 1.2 the pair (U, V) blows up in finite time T_h and by Lemma 4.1 we have that $u_k'(t) \geq \delta u_k^{p_{11}}(t)$ with δ independent of h, and therefore we obtain

$$\max_k u_k(t) \leq \frac{C}{(T_h - t)^{\frac{1}{p_{11}-1}}},$$

with C independent of h. Thus v_k verifies

$$m_k v_k' \leq -\sum_{j=1}^{N} a_{jk} v_j + \frac{K b_k v^{p_{22}}}{(T_h - t)^{\frac{p_{21}}{p_{11}-1}}}. \tag{5.3}$$

Let w be the solution of

$$\begin{aligned} &w_t = \Delta w && \text{in } \Omega \times (t_0, T_h), \\ &\frac{\partial w}{\partial \eta} = \frac{2K}{(T_h - t)^{\frac{p_{21}}{p_{11}-1}}} w^{p_{22}} && \text{on } \partial\Omega \times (t_0, T_h), \\ &w(x, t_0) = 2L && \text{on } \Omega, \end{aligned} \tag{5.4}$$

where L is a uniform bound for v. It is shown in [15] that if $T_h - t_0$ is small enough (depending only on L) then w remains bounded up to T_h. Now by our consistency assumption on the scheme it follows that

$$W(t) = (w_1(t), w_2(t), ..., w_N(t)) = (w(x_1, t), w(x_2, t), ..., w(x_N, t)),$$

is a supersolution of (5.3) and since the scheme is convergent, for h small enough, we have $V(t_0) \leq W(t_0)$. Therefore, by Lemma 2.1, $V(t) \leq W(t)$, so $V(t)$ remains bounded and hence $U(t)$ blows up. □

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