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NUMERICAL APPROXIMATION OF A PARABOLIC PROBLEM WITH A NONLINEAR BOUNDARY CONDITION IN SEVERAL SPACE DIMENSIONS

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Abstract. In this paper we study the asymptotic behaviour of a semidiscrete numerical approximation for the heat equation, $u_t = \Delta u$, in a bounded smooth domain, with a nonlinear flux boundary condition at the boundary, $\frac{\partial u}{\partial \eta} = u^p$. We focus in the behaviour of blowing up solutions. First we prove that every numerical solution blows up in finite time if and only if $p > 1$ and that the numerical blow-up time converges to the continuous one as the mesh parameter goes to zero. Next, we show that the blow-up rate for the numerical scheme is different from the continuous one. Nevertheless we find that the blow-up set for the numerical approximations it is contained in a neighborhood of the blow-up set of the continuous problem, when the mesh parameter is small enough.

1. Introduction.

In this paper we study the asymptotic behaviour of a semidiscrete approximation of the following parabolic problem,

$$\left\{ \begin{array}{rcll} u_t & = & \Delta u & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} & = & u^p & \text{on } \partial\Omega \times (0,T), \\ u(x,0) & = & u_0(x) > 0 & \text{on } \Omega. \end{array} \right. \tag{1.1}$$

We assume that u_0 is regular in order to guarantee a smooth solution u.

A remarkable (and well known) fact is that solutions develop singularities in finite time, no matter how smooth u_0 is. In fact, for many differential equations or systems such as (1.1) the solution becomes unbounded in finite time, a phenomenum that is known as blow-up (see [W], [LP], [RR]). Other examples where this happens are problems involving reaction terms in the equation (see [SGKM], [P] and the references therein).

In our problem one has a reaction term at the boundary of power type and if $p > 1$ this blow-up phenomenum occurs in the sense that there exists a finite time T such that $\lim_{t\to T} \|u(\cdot,t)\|_\infty = +\infty$ for every initial data (see [W], [LP], [RR]). The blow-up set is localized at the boundary of the domain, that is for every subdomain $\Omega' \subset\subset \Omega$ there exists a constant $K = K(d(\Omega', \partial\Omega))$ such that $u(x,t) \le K$ for every $x \in \Omega'$ and for every $0 \le t < T$ (see [RR], [HY]). Also it is known that the blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{2(p-1)}}$ (see [HY]), in the sence that there exists

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positive constants c, C such that

$$c(T-t)^{-\frac{1}{2(p-1)}} \le \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(T-t)^{-\frac{1}{2(p-1)}}.$$

In this paper we are interested in numerical approximations of (1.1).

Since the solution u develops singularities in finite time, it is an interesting question what can be said about numerical approximations for this kind of problems. For previous work on numerical approximations of blowing up solutions of (1.1) in one space dimension we refer to [DER]. For other numerical approximations of blow-up problems we refer to [AB], [ALM1], [ALM2], [BK], [BHR], [C], [GR], [LR], the survey [BB] and references therein.

In [DER] the authors analyze a semidiscrete scheme (such as ours) in one space dimension. They find a necessary and sufficient condition for the appearance of blow-up ($p > 1$) and prove the convergence of the blow-up time of the discrete problem to that of the continuous one, when the mesh parameter goes to zero.

Here we extend these results to several space dimensions and prove some new results concerning the asymptotic behaviour (blow-up rate) and the localization of blow-up points (blow-up set) for semidiscretetizations in space. We will consider a general method for the space discretization with adequate assumptions on the coefficients.

More precisely, we assume that there exists a set of nodes $\{x_1, \dots, x_N\}$ and that our approximate solution $u_h(x,t)$ is a linear interpolant of $U(t) = (u_1(t), \dots, u_N(t))$ (that is $u_h(x_k,t) = u_k(t)$) where U is the solution of the following ODE

$$\begin{array}{rcl} MU'(t) & = & -AU(t) + BU^p(t), \\ U(0) & = & U_0. \end{array} \tag{1.2}$$

The precise assumptions on the matrices involved in the method are: M and B are a diagonal matrices with positive entries m_k and b_k and A is a nonnegative symmetric matrix, with nonpositive coefficients off the diagonal (that is $a_{ij} \le 0$ if $i \ne j$), $a_{ii} > 0$ and $\sum_{j=1}^N a_{ij} \le 0$. U_0 is the initial datum for the problem (1.2).

Writing this equation explicitly we obtain the following ODE system,

$$\left\{ \begin{array}{rcll} m_k u_k'(t) & = & -\sum_{j=1}^N a_{kj} u_j(t) + b_k u_k^p(t), & 1 \le k \le N, \\ u_k(0) & = & u_{0,k}, & 1 \le k \le N. \end{array} \right. \tag{1.3}$$

As an example, we can consider a linear finite element approximation of problem (1.1) on a regular acute triangulation of Ω (see [Ci]). In this case, if V_h is the subspace of piecewise linear functions in $H^1(\Omega)$. We impose that $u_h : [0, T_h) \to V_h$, verifies

$$\int_\Omega ((u_h)_t v_h)^I = -\int_\Omega \nabla u_h \nabla v + \int_{\partial\Omega} ((u_h)^p v)^I \tag{1.4}$$

for every $v \in V_h$. Here $(\cdot)^I$ stands for the linear Lagrange interpolation at the nodes of the mesh.

We denote with $U(t) = (u_1(t), ..., u_N(t))$ the values of the numerical approximation at the nodes x_k at time t. Then $U(t)$ verifies a system of the form (1.2) and all of our assumptions on the matrices M and B holds as we are using mass lumping and our assumptions on A are satisfied as we are considering an acute regular mesh. In this case M is the lumped mass matrix, A is the stiffness matrix, B correspond to the boundary condition obtained with lumping. As an initial datum, we take $U_0 = u_0^I$.

As another example if Ω is a cube, $\Omega = (0,1)^n$, we can use a semidiscrete finite differences method to approximate the solution $u(x,t)$ obtaining an ODE system of the form (1.3).

In §2 we start our analysis of (1.3) and prove that this method converges uniformly over $\Omega \times [0, T-\tau]$ under the assumption of the consistency of the method (for one space dimension see [ALM2] where the authors prove consistence under the regularity hypothesis $u \in C^{4,1}$). Under this assumption we find that

$$\|u - u_h\|_{L^\infty(\overline{\Omega}\times[0,T-\tau])} \leq C\rho(h)$$

where ρ is the modulus of consistency of the method.

In fact, we prove the following result,

Theorem 1.1. *Let u be a regular solution of* (1.1) *and u_h the numerical approximation given by* (1.3)*. If the method is consistent, that is if*

$$m_k u_t(x_k,t) = -\sum_{j=1}^{N} a_{kj} u(x_j,t) + b_k u^p(x_k,t) + \rho_k(h), \qquad t \in [0, T-\tau]$$

and there exists a positive function $\rho(h)$ such that

$$\max\{h^2, \frac{|\rho_k(h)|}{m_k}\} \leq \rho(h) \qquad \text{and} \qquad \lim_{h\to 0}\rho(h) = 0,$$

then there exists a constant C such that

$$\|u - u_h\|_{L^\infty(\Omega\times[0,T-\tau])} \leq C\rho(h).$$

if $\|u_0 - u_h(\cdot,0)\|_{L^\infty(\Omega)} \leq C\rho(h)$.

We say that a solution of (1.2) has finite blow-up time if there exists a finite time T_h such that

$$\lim_{t\to T_h}\|U(t)\|_\infty = \lim_{t\to T_h}\max_j u_j(t) = +\infty.$$

As a first step for our analysis of the behaviour of solutions of (1.3), we want to describe when the blow-up phenomenum occurs. In §3 we prove the following Theorem,

Theorem 1.2. *Positive solutions of* (1.3) *blow up in finite time if and only if* $p > 1$.

We want to remark that the blow-up condition, $p > 1$, is the same to that of the continuous problem, see [W], [LP], [RR].

The purpose of §4 is to extend the result of [DER] on the convergence of numerical blow-up times T_h to the continuous one T when the mesh parameter h goes to zero. To this end, we again assume that the method is consistent in the sense of Theorem 1.1 and hypotheses on u_0 that implies that the numerical solution u_h is increasing in time.

Theorem 1.3. *Let u_0 a compatible initial datum for* (1.1) *such that $\Delta u_0 \geq \alpha > 0$. Assume that the method* (1.3) *is consistent in the sence of Theorem* 1.1*. Let T and T_h be the blow-up times for u and u_h respectively, then*

$$\lim_{h\to 0} T_h = T.$$

In §5 and §6 we arrive at the main points of this article, the asymptotic behaviour (blow-up rate) and the localization of blow-up points (blow-up set) of u_h for a fixed h.

Concerning the blow-up rate for (1.3) in §5 we prove the following Theorem,

Theorem 1.4. *Let u_h be a solution of* (1.3). *Assume that $p > 1$ and that u_h blows up in finite time, T_h. Then*

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}},$$

in the sense that there exists two positive constants c, C such that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}.$$

We have to remark that the constants c, C that appear in Theorem 1.4 may depend on h.

Let us point out that the blow-up rate for the numerical scheme, $(T_h - t)^{-1/(p-1)}$, is different from the continuous one, $(T - t)^{-1/2(p-1)}$.

Finally, in §6, we turn our attention to the blow-up set for u_h, $B(u_h)$ that is the set of nodes x_k such that $\lim_{t \nearrow T_h} u_k(t) = +\infty$. Let F be the set of nodes x_j such that $u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$. By Theorem 1.4, $F \neq \emptyset$ and clearly, $F \subset B(u_h)$. By means of the blow-up rate given by Theorem 1.4 we observe a propagation property for blow-up points. We prove that the number of nodes adjacent to F that go to infinity is finite and determined only by p. To describe this propagation phenomena we need the following notion of distance between nodes, we say that two different nodes, x_i, x_j, are at distance one if and only if $a_{ij} \neq 0$. Let A_1 be the set of nodes that are at distance one from F. Inductively, we define A_l as the set of nodes that are at distance one from A_{l-1}. We say that a node x_k is at distance $d(k)$ from F if $x_k \in A_{d(k)} - A_{d(k)-1}$. If we are dealing with a linear finite element approximation, the distance of a node x_k to F is the usual distance measured as a graph. We prove that u_k blows up if and only if $d(k) \leq K$ where K depends only on p,

Theorem 1.5. *Let F be the set of nodes, x_j, such that*

$$u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$$

Then the blow-up propagates in the following way, let $p > 1$ and $K \in \mathbb{N}$ such that $\frac{K+2}{K+1} < p \leq \frac{K+1}{K}$, then the solution of (1.2), U, blows up exactly at K nodes near F. More precisely,

$$u_k(t) \to +\infty \qquad \Longleftrightarrow \qquad d(k) \leq K$$

where $d(k)$ is the distance of x_k to F in the sense described above. Moreover, if $d(k) \leq K$, the asymptotic behaviour of u_k is given by

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1} + d(k)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(k) = K$

$$u_k(t) \sim \ln(T_h - t).$$

We want to remark that more than one node can go to infinity, but the asymptotic behavior imposes $\frac{u_k(t)}{u_j(t)} \to 0$ $(t \to T_h)$ if $d(k) > d(j)$.

In the blow-up case ($p > 1$) the number of blow-up points outside F is finite and depends on the power p but is independent of h. This fact gives a sort of "numerical localization" of the blow-up set of u_h near the blow-up set of u when the mesh parameter h is small enough.

Theorem 1.6. *Let u and u_h be solutions of* (1.1) *and* (1.3) *respectively. Assume that the numerical method is consistent and that $\Delta u_0 \geq \alpha > 0$. Then if we call $B(u)$ and $B(u_h)$ the blow-up sets for u and u_h respectively, we have that given $\varepsilon > 0$ there exists h_0 such that for every $0 < h \leq h_0$,*

$$B(u_h) \subset B(u) + N_\varepsilon \qquad \forall h \leq h_0,$$

where $N_\varepsilon = \{x \in \mathbb{R}^n : \ |x| < \varepsilon\}$.

We want to remark that regardless the difference in the blow-up rate showed by Theorem 1.4; the blow-up sets are similar as is showed by Theorem 1.6.

In [RR] and [HY] it is proved that $B(u) \subset \partial\Omega$. Therefore, Theorem 1.6 implies that $B(u_h)$ is contained in a small neighborhood of $\partial\Omega$ for h small enough. Moreover, in [Hu] there is an example of single point blow-up for (1.1) and hence in this case $B(u_h)$ shrinks around that single point as h goes to zero.

On the one hand, Theorems 1.3, 1.2 and 1.6 shows that the numerical scheme (1.3) has asymptotic properties that are similar to the ones of the continuous problem (1.1) when the mesh parameter is small. On the other hand, a major difference appears in the blow-up rates (Theorem 1.4). Up to our knowledge, this is the first time that this phenomenum appears in the literature. This difference suggest that an adaptive method is needed in order to reproduce the same blow-up rate. We leave this question for future work.

We also want to remark that the results obtained in Theorems 1.2, 1.4 and 1.5, holds for a general ODE system of the form (1.3) regardless if it comes from a semidiscretization of (1.1).

The paper is organized as follows: in §2 we prove our convergence result (Theorem 1.1), in §3 the blow-up result (Theorem 1.2), in §4 we study the convergence of the blow-up times (Theorem 1.3), in §5 we consider the blow-up rate (Theorem 1.4) and finally in §6 we study the localization of the blow-up set for u_h (Theorem 1.5 and 1.6).

2. Convergence of the numerical scheme.

In this Section we prove a uniform convergence result for regular solutions of the numerical scheme (1.3). Throughout this section, we consider $0 < \tau < T$ fixed.

We want to show that $u_h \to u$ (when $h \to 0$) uniformly in $\Omega \times [0, T-\tau]$. This is a natural requirement since in such sets the exact solution is regular. Approximations of regular problems in one space dimension with a source in the equation have been analyzed in [ALM2] and also in one space dimension for a problem like (1.1) in [DER].

The precise assumption that we make on the scheme is: for u any solution on (1.1), it holds

$$m_k u_t(x_k,t) = -\sum_{i=1}^{N} a_{ik} u(x_i,t) + b_k u(x_k,t)^p + \rho_k(h) \tag{2.1}$$

and there exists a function $\rho : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\max\{|\rho_k(h)/m_k|, h^2\} \le \rho(h)$ and $\lim_{h\to 0} \rho(h) = 0$ (i.e. the scheme is consistent).

Let us begin with a comparison Lemma, that will be used throughout the paper,

Definition 2.1. *We say that $\overline{U}$ is a supersolution of (1.2) if*

$$M\overline{U}' \ge -A\overline{U} + B\overline{U}^p.$$

We say that $\underline{U}$ is a subsolution of (1.2) if

$$M\underline{U}' \le -A\underline{U} + B\underline{U}^p.$$

(The inequalities are understood coordenate by coordenate)

Lemma 2.1. *Let $\overline{U}$ and $\underline{U}$ be a super and a subsolution of (1.2) respectivelly such that $\underline{U}(0) \le \overline{U}(0)$. Then*

$$\underline{U}(t) \le \overline{U}(t).$$

Proof. Let $W = \overline{U} - \underline{U}$. Assume first that $W(0) > 0$. We observe that W verifies

$$MW' \ge -AW + B\left(\frac{\overline{U}^p - \underline{U}^p}{\overline{U} - \underline{U}}\right) W.$$

Now, set $\delta = \min\{w_j(0)\}$ and suppose that the statement of the Lemma is false. Thus, let t_0 be the first time that $\min\{w_j(t)\} = \delta/2$. At that time, there must be a j_0 such that $w_{j_0}(t_0) = \delta/2$. But on the one hand $w'_{j_0}(t_0) \le 0$ and, on the other hand, by our hypotheses on A,

$$m_{j_0} w'_{j_0} \ge -\sum_{i=1}^{N} a_{ij_0} w_i + b_{j_0}\left(\frac{\overline{u}_{j_0}^p - b_{j_0}\underline{u}_{j_0}^p}{\overline{u}_{j_0} - \underline{u}_{j_0}}\right) w_{j_0} > -\sum_{i=1}^{N} a_{ij_0}\frac{\delta}{2} + p\underline{u}_{j_0}^{p-1} w_{j_0} \ge 0$$

a contradiction. By an approximation argument using continuity of (1.2) with respect to initial data, the result follows. $\square$

Now we are ready to prove our convergence result.

Proof of Theorem 1.1: Let us start by defining the error functions

$$e_k(t) = u_k(t) - u(x_k,t) \tag{2.2}$$

By (2.1), these functions verify

$$m_k e'_k = -\sum_{i=1}^{N} a_{ik} e_i + b_k(u_k^p - u^p(x_k,t)) + \rho_k(h)$$

Let $t_0 = \max_{t\in[0,T-\tau]}\{\max_k |u_k(t) - u(x_k,t)| \le 1\}$. We will see that $t_0 = T - \tau$ for h small enough.

In $[0,t_0]$, $E = (e_1,...,e_N)$ is a subsolution of

$$ME' \le -AE + KBE + \rho(h)M(1,...,1)^t$$

where $K = p(\|u\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{p-1}$.

Let us now define the following function that will be used as a supersolution for (1.2). Let $a \in C^2(\overline{\Omega})$ be such that $a(x) \geq \delta > 0$ in Ω, $\partial a/\partial\eta > Ka$ on $\partial\Omega$ and let $b(t) = \exp(Lt)$ where L is to be determined.

Then, it is easy to check that $w(x,t) = Ca(x)b(t)\rho(h)$ verifies

$$\begin{array}{rcll} w_t & \geq & \Delta w & \text{in } \Omega \\ \frac{\partial w}{\partial \eta} & \geq & Kw & \text{in } \partial\Omega \end{array}$$

Now, by the consistency of the scheme, one can verify that

$$W = Cb(t)\rho(h)(a(x_1), ..., a(x_N))$$

is a supersolution of

$$MW' \geq -AW + KBW + \rho(h)M(1, ..., 1)^t$$

for L big enough depending on K but not on h.

Next, as $\rho(h) \geq h^2$, we can choose C large and independent of h, such that $E(0) \leq W(0)$. It follows by a comparison argument (Lemma 2.1) that

$$E(t) \leq W(t), \quad \forall t \in [0, t_0].$$

By a symmetric argument, it follows that

$$|E(t)| \leq Cb(T-\tau)\|a\|_{L^\infty(\Omega)}\rho(h).$$

From this fact it is easy to see that $t_0 = T - \tau$ for h small enough, and the result follows.$\square$

3. Blow-up for the numerical scheme.

In this section we prove Theorem 1.2 which states a condition for the existence of blow-up of the discrete solution.

Let us define, $T_h = \sup\{t \text{ such that } u_h(s) \text{ is defined for } s \in [0,t]\}$. If T_h is finite, then by a classical result from ODE theory we have

$$\lim_{t \nearrow T_h} \max_j u_j(t) = +\infty.$$

As we mentioned in the introduction, this means that u_h blows up at time T_h.

Let us begin with the following Lemma,

Lemma 3.1. *Let U be the solution of* (1.2)*. If $U_0 > 0$, then U is unbounded.*

Proof. Assume by contradiction that U is uniformly bounded. Then, we observe that

$$\Phi_h(U) = \frac{1}{2}\langle A^{1/2}U, A^{1/2}U\rangle - \frac{1}{p+1}\sum_{k=1}^{N} b_k u_k^{p+1}$$

is a Lyapunov functional for (1.3). In fact by direct computation we have

$$\frac{d}{dt}\Phi(U)(t) = -\langle MU', U'\rangle.$$

As (1.2) has only $U \equiv 0$ as a fixed point, it follows that (see [H]) $U \to 0$ as $t \to \infty$. As our scheme verifies the minimum principle, $\min_j u_j(t) \geq \min_j u_{0,j} > 0$, a contradiction. $\square$

To prove Theorem 1.2 we need the following result.

Lemma 3.2. *Let U be the solution of* (1.2). *Then*

$$\max_j u_j(t) = u_k(t)$$

where k is such that $b_k \neq 0$ and the $\max_j u_j(t) > \max_j u_{0,j}$.

Proof. Follows easily as our hypotheses on the matrices M, B and A implies the maximum principle. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2: By the previous lemma we have that $U(t)$ is unbounded, using that $p > 1$ and Lemma 3.2 we obtain that there exists a time t_0 and a node x_k with $b_k \neq 0$ such that $-a_{kk}u_k(t_0)+b_k u_k^p(t_0) \geq \frac{b_k}{2} u_k^p(t_0)$. Hence, by our assumptions on A,

$$m_k u_k'(t_0) = -\sum_{j=1}^{N} a_{kj} u_j(t_0) + b_k u_k^p(t_0) \geq \frac{b_k}{2} u_k^p(t_0).$$

As a consequence of this $u_k(t)$ must be increasing for $t \geq t_0$ and verifies

$$m_k u_k'(t) \geq \frac{b_k}{2} u_k^p(t).$$

Again, as $p > 1$, u_k goes to infinity in finite time, and hence $U(t)$ has finite time blow-up as we wanted to show. □

4. Convergence of the blow-up times

Now we prove the convergence of the blow-up times, Theorem 1.3. We use ideas from [DER].

We begin with the following Lemma,

Lemma 4.1. *Let U a solution of (1.2) such that $u_k'(0) \geq \delta u_k^p(0)$, $1 \leq k \leq N$. Then $u_k'(t) \geq \delta u_k^p(t)$, $1 \leq k \leq N$ for every $t < T_h$.*

Proof. Let $w_k(t) = u_k'(t) - \delta u_k(t)^p$. We want to use the minimum principle to show that $w_k(t)$ is positive. To this end, we observe that w_k verifies

$$\begin{aligned}
m_k w_k' + \sum_{j=1}^N a_{kj} w_j &= m_k(u_k'' - \delta p u_k^{p-1} u_k') + \sum_{j=1}^N a_{kj}(u_j' - \delta u_j^p) \\
&= -\delta m_k p u_k^{p-1} u_k' + b_k p u_k^{p-1} u_k' - \delta \sum_{j=1}^N a_{kj} u_j^p \\
&= -\delta p u_k^{p-1} \left(\sum_{j=1}^N a_{kj} u_j + b_k u_k^p \right) + b_k p u_k^{p-1} u_k' - \delta \sum_{j=1}^N a_{kj} u_j^p \\
&= b_k p u_k^{p-1} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^p - p u_k^{p-1} u_j) + a_{kk}(1-p) u_k^p \right) \\
&= b_k p u_k^{p-1} w_k - \delta \left(\sum_{j \neq k} a_{kj}(u_j^p - p u_k^{p-1}(u_j - u_k)) - u_k^p) + \right. \\
&\quad \left. + \sum_{j=1}^N a_{kj}(1-p) u_k^p \right).
\end{aligned}$$

As u^p is convex and by our hypotheses on the matrix A it follows that $W = (w_1, \dots, w_N)$ verifies

$$MW' \geq -AW + BpU^{p-1}W.$$

As the minimum principle holds for this equation, the result follows. □

With this Lemma we can prove Theorem 1.3.

Proof of Theorem 1.3 As the scheme is consistent one can check that the hypothesis $\Delta u_0 \geq \alpha > 0$ implies the hypothesis of the previous lemma for h small enough, with δ independent of h. So we have that $u_k'(t) \geq \delta u_k^p(t)$, integrating we obtain

$$\int_t^{T_h} \frac{u_k'(s)}{u_k^p(s)}\, ds \geq \delta(T_h - t).$$

Therefore, changing variables we get

$$\delta(T_h - t) \leq \int_{u_k(t)}^{+\infty} \frac{1}{x^p}\, dx,$$

and so

$$\delta(T_h - t) \leq \int_{\|u_h(\cdot,t)\|_{L^\infty(\Omega)}}^{+\infty} \frac{1}{x^p}\, dx. \tag{4.1}$$

Since $p > 1$ this last inequality implies that if $\|u_h(\cdot,t)\|_{L^\infty(\Omega)}$ is large enough, then t is close to T_h. Given $\varepsilon > 0$, as δ is independent of h, we can choose M (also independent of h) large enough to ensure that

$$\frac{1}{\delta}\int_M^{+\infty}\frac{1}{x^p}\,dx < \frac{\varepsilon}{2}. \tag{4.2}$$

Now, as u blows up at time T we can choose $\tau < \frac{\varepsilon}{2}$ such that

$$\|u(\cdot, T-\tau)\|_{L^\infty(\Omega)} \geq 2M.$$

Then by Theorem 1.1, if h is small enough,

$$\|u_h - u\|_{L^\infty(\Omega\times[0,T-\tau])} \leq C\rho(h) \leq M,$$

and hence

$$\|u_h(\cdot, T-\tau)\|_{L^\infty(\Omega)} \geq M.$$

By (4.1) and (4.2),

$$|T_h - (T-\tau)| \leq \frac{1}{\delta}\int_{\|u_h(\cdot,T-\tau)\|_{L^\infty(\Omega)}}^{+\infty}\frac{1}{x^p}\,dx \leq \frac{1}{\delta}\int_M^{+\infty}\frac{1}{x^p}\,dx < \frac{\varepsilon}{2}.$$

Therefore,

$$|T_h - T| \leq |T_h - (T-\tau)| + |\tau| < \varepsilon.$$

This finish the proof.$\square$

5. Blow-up rate.

In this Section we consider positive solutions of (1.3) with h fixed and we denote by C a positive constant that may depend on h and may vary from one line to another.

Proof of Theorem 1.4: Let us begin by defining

$$w(t) = \sum_{k=1}^{N} u_k(t)$$

As U blows up at time T_h, and by (1.3) we obtain that there exists t_0 such that for every $t \in [t_0, T_h)$ it holds

$$\begin{aligned} w'(t) &= -\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{a_{kj}}{m_k}u_j(t) + \sum_{k=1}^{N}\frac{b_k}{m_k}u_k^p(t) \leq C(\max_k u_k(t))^p \\ &\leq C\left(\sum_{k=1}^{N}u_k(t)\right)^p = Cw^p(t). \end{aligned}$$

For $t \in [t_0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h}\frac{w'(s)}{w^p(s)}\,ds \leq C(T_h - t).$$

Changing variables we get

$$\int_{w(t)}^{+\infty}\frac{1}{s^p}\,ds \leq C(T_h - t),$$

hence

$$w(t) \geq C(T_h - t)^{-\frac{1}{p-1}}.$$

Therefore we obtain

$$\max_j u_j(t) \geq C(T_h - t)^{-\frac{1}{p-1}}.$$

To prove the other inequality we proceed as follows: as $\max_j u_j(t) \to +\infty$ when $t \to T_h$, we have that if $u_k(t) = \max_j u_j(t)$, then $a_{kk} u_k(t) \leq \frac{b_k}{2} u_k^p(t)$ for every t close to T_h. In this case we have

$$u_k'(t) = -\sum_{j=1}^{N} \frac{a_{kj}}{m_k} u_j(t) + \frac{b_k}{m_k} u_k^p(t) \geq \frac{b_k}{2m_k} u_k^p(t).$$

Integrating again over $[t, T_h]$ we obtain

$$\int_t^{T_h} \frac{u_k'(s)}{u_k^p(s)}\, ds \geq \frac{b_k}{2m_k}(T_h - t).$$

Changing variables

$$\int_{u_k(t)}^{+\infty} \frac{1}{s^p}\, ds \geq \frac{b_k}{2m_k}(T_h - t),$$

hence

$$u_k(t) \geq C_k (T_h - t)^{-\frac{1}{p-1}}.$$

So $\max_j u_j(t)$ verifies

$$\max_j u_j(t) \sim (T_h - t)^{-\frac{1}{p-1}}$$

in the sense that

$$c(T_h - t)^{-\frac{1}{p-1}} \leq \max_j u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}.$$

The proof is finished.$\square$

6. Blow-up set.

In this Section we study the blow-up set of the solution U. We begin by distinguish the set of nodes that blows up with the same rate as the maximum from the others.

For this purpose, we make the following change of variables inspired by [GK], [HV1], [HV2],

$$\left\{ \begin{array}{l} y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t), \\ \\ (T_h - t) = e^{-s}. \end{array} \right. \tag{6.1}$$

These new variables, $Y = (y_k(s))$, verify

$$\left\{ \begin{array}{l} m_k y_k'(s) = -e^{-s} \sum_{j=1}^{N} a_{kj} y_j(s) - \frac{m_k}{p-1} y_k(s) + b_k y_k^p(s), \\ \\ y_k(-\ln(T_h)) = (T_h)^{\frac{1}{p-1}} u_0(x_k), \qquad 1 \leq k \leq N+1. \end{array} \right. \tag{6.2}$$

We observe that, as $\max u_j(t) \leq C(T_h - t)^{-\frac{1}{p-1}}$, we have that $y_j(s)$ are uniformly bounded, Let us first define the following constant that will be use troughout this section,

$$\Gamma_k = \sup_s \left| \sum_{j=1}^N a_{kj} y_j(s) \right|.$$

Lemma 6.1. *If there exists s_0 such that*

$$b_k y_k^p(s_0) - \frac{m_k}{p-1} y_k(s_0) < -\Gamma_k e^{-s_0}$$

then

$$y_k(s) \to 0 \qquad (s \to \infty).$$

Proof. ¿From (6.2 $y_k(s)$ verifies

$$m_k y_k'(s) \le \Gamma_k e^{-s} - \frac{m_k}{p-1} y_k(s) + b_k y_k^p(s).$$

Let $w_k(s)$ be a solution of

$$w_k'(s) = \Gamma_k e^{-s} - \frac{m_k}{p-1} w_k(s) + b_k w_k^p(s)$$

with $w_k(s_0) = y_k(s_0)$. We observe that,

$$w_k'(s_0) = \Gamma_k e^{-s_0} - \frac{m_k}{p-1} y_k(s_0) + b_k y_k^p(s_0) < 0.$$

We claim that $w_k'(s) < 0$ for all $s > s_0$. To prove this claim, we argue by contradiction. Assume that there exists a first time s_1 such that $w_k'(s_1) = 0$. At that time s_1 we have

$$w_k''(s_1) = -\Gamma_k e^{-s_1} - \frac{m_k}{p-1} w_k'(s_1) + p b_k w_k^{p-1}(s_1) w_k'(s_1) = -\Gamma_k e^{-s_1},$$

hence $w_k''(s_1) < 0$. Therefore w_k' is decreasing at s_1, a contradiction.

So we have proved that $w_k(s)$ is decreasing for all $s > s_0$, and $w_k(s) \ge 0$ hence there exists $l_k = \lim_{s\to\infty} w_k(s)$. As $\lim_{s\to\infty} w_k'(s) = 0$ we have that

$$-\frac{m_k}{p-1} l_k + b_k l_k^p = 0.$$

As $w_k(s_0)$ is below the only positive root of $g_k(x) = -\frac{m_k}{p-1}x + b_k x^p$ and w_k is decreasing for $s \ge s_0$, we conclude that $l_k = 0$.

By a comparison argument we have that

$$0 \le y_k(s) \le w_k(s) \to 0 \qquad (s \to \infty),$$

hence $y_k(s) \to 0$ $(s \to \infty)$. □

Lemma 6.2. *For every s, it holds*

$$b_k y_k^p(s) - \frac{m_k}{p-1} y_k(s) \le \Gamma_k e^{-s}.$$

Proof. We argue by contradiction. Suppose that there exists s_0 such that

$$b_k y_k^p(s_0) - \frac{m_k}{p-1} y_k(s_0) > \Gamma_k e^{-s_0}.$$

As before, from (6.2 $y_k(s)$ verifies

$$m_k y_k'(s) \ge -\Gamma_k e^{-s} - \frac{m_k}{p-1} y_k(s) + b_k y_k^p(s)$$

Let $w_k(s)$ be a solution of

$$w_k'(s) = -\Gamma_k e^{-s} - \frac{m_k}{p-1} w_k(s) + b_k w_k^p(s)$$

with $w_k(s_0) = y_k(s_0)$. We observe that,

$$w_k'(s_0) = -\Gamma_k e^{-s_0} - \frac{m_k}{p-1} y_k(s_0) + b_k y_k^p(s_0) > 0.$$

We claim that $w_k'(s) > 0$ for all $s > s_0$. To prove this claim, we argue by contradiction. Assume that there exists a first time s_1 such that $w_k'(s_1) = 0$, at that time s_1 we have

$$w_k''(s_1) = \Gamma_k e^{-s_1} - \frac{m_k}{p-1} w_k'(s_1) + p b_k w_k^{p-1}(s_1) w_k'(s_1) = \Gamma_k e^{-s_1}.$$

Hence $w_k''(s_1) > 0$. Therefore w_k' is increasing at s_1, a contradiction.

So we have proved that $w_k(s)$ is increasing for all $s > s_0$, hence there exists $\varepsilon > 0$ such that

$$w_k'(s) \geq \varepsilon w_k^p(s)$$

and then, using that $p > 1$, we have that w_k blows up in finite time s_2.

As before, we can use a comparison argument to get

$$y_k(s) \geq w_k(s) \to +\infty \qquad s \to \tilde{s} \leq s_2 < \infty$$

hence $y_k(s)$ blows up in finite time which contradicts the fact that it is uniformly bounded. $\square$

Lemma 6.3. *Let $y_k(s)$ be a solution of* (6.2) *then each y_k verifies*

(6.3)
$$\begin{cases} y_k(s) \to 0 \qquad (s \to +\infty), \\ or \\ y_k(s) \to l_k \quad (s \to +\infty) \text{ for some contant } l_k > 0 \text{ depending on } k \text{ and } h. \end{cases}$$

Proof. As y_k is uniformly bounded, we conclude that it is globaly defined. If $y_k(s)$ does not converge to zero, by Lemmas 6.1 and 6.2 we observe

$$\Gamma_k e^{-s} \geq b_k y_k^p(s) - \frac{m_k}{p-1} y_k(s) \geq -\Gamma_k e^{-s}.$$

Then

$$b_k y_k^p(s) - \frac{m_k}{p-1} y_k(s) \to 0 \qquad (s \to +\infty).$$

As y_k does not converges to zero, we conclude that $y_k(s) \to l_k$, where l_k is the only positive root of $g_k(y) = b_k y^p - \frac{m_k}{p-1} y$. $\square$

Now we are ready to deal with the blow-up set. We begin by the proof of the propagation result, Theorem 1.5.

Proof of Theorem 1.5: Let $F = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\}$ be the set of nodes such that

$$y_{j_i}(s) \not\to 0 \qquad (s \to \infty).$$

Let K be such that

$$\frac{K+2}{K+1} < p \leq \frac{K+1}{K}.$$

We want to see that the blow-up propagates to the K nodes adjacents to F, that is, a node x_k blows up if and only if $d(k) \le K$.

For this purpose let us begin by a considering a node x_k such that $d(k) = 1$. As $x_k \notin F$, we have that $y_k(s) \to 0$. We want to obtain the asymptotic behaviour of $y_k(s)$. To this end, first we get a bound as follows, from (6.2) $y_k(s)$ verifies

$$m_k y_k'(s) \le \Gamma_k e^{-s} - \frac{m_k}{p-1} y_k(s) + b_k y_k^p(s).$$

Using that $y_k(s) \to 0$ we have that, given $\varepsilon > 0$ there exists s_0 such that, for every $s > s_0$

$$m_k y_k'(s) \le \Gamma_k e^{-s} - \frac{m_k}{p-1} y_k(s) + b_k y_k^p(s) \le \Gamma_k e^{-s} - \left(\frac{m_k}{p-1} - \varepsilon \right) y_k(s)$$

Let $w_k(s)$ be a solution of

$$w_k'(s) = \Gamma_k e^{-s} - \left(\frac{m_k}{p-1} - \varepsilon \right) w_k(s),$$

we get

$$w(s) \le C e^{-s}$$

By a comparison argument we obtain that for every $s > s_0$,

$$y_k(s) \le w_k(s) \le C e^{-s} \tag{6.4}$$

Again, from (6.2)

$$m_k y_k'(s) + \frac{m_k}{p-1} y_k(s) = -e^{-s} \sum a_{kj} y_j(s) + b_k y_k^p(s)$$

then,

$$m_k (e^{\frac{1}{p-1}s} y_k(s))' = e^{\frac{1}{p-1}s} \left(-e^{-s} \sum a_{kj} y_j(s) + b_k y_k^p(s) \right).$$

Integrating between s_0 and s, we get

$$m_k y_k(s) = e^{-\frac{1}{p-1}s} \left(C_k + \int_{s_0}^s e^{\frac{1}{p-1}\tau} \left(-e^{-\tau} \sum a_{kj} y_j(\tau) + b_k y_k^p(\tau) \right) d\tau \right).$$

We need to find the behaviour of the last integral. With this in mind let us compute the following limit,

$$\lim_{s \to +\infty} \frac{\int_{s_0}^s e^{\frac{1}{p-1}\tau} \left(-e^{-\tau} \sum_{j=1}^N a_{kj} y_j(\tau) + b_k y_k^p(\tau) \right) d\tau}{\int_{s_0}^s e^{-\frac{p-2}{p-1}\tau} \, d\tau}.$$

If the integral diverges, we can use L'Hôspital's rule to obtain

$$\lim_{s \to +\infty} \frac{e^{(\frac{1}{p-1}-1)s} \left(-\sum_{j=1}^N a_{kj} y_j(s) + b_k e^s y_k^p(s) \right)}{e^{-\frac{p-2}{p-1}s}} =$$

$$\lim_{s \to +\infty} -\sum_{j=1}^N a_{kj} y_j(s) + b_k e^s y_k^p(s).$$

Using (6.4) we get

$$e^s y_k^p(s) \le C e^{-(p-1)s} \to 0 \qquad (s \to \infty),$$

hence we have,

$$\lim_{s\to+\infty} \frac{\int_{s_0}^{s} e^{\frac{1}{p-1}\tau}\left(-e^{-\tau}\sum_{j=1}^{N} a_{kj}y_j(\tau)+b_k y_k^p(\tau)\right)\,d\tau}{\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\tau}\,d\tau} =$$

$$\lim_{s\to+\infty} -\sum_{j=1}^{N} a_{kj}y_j(s) = \tilde{C}_k \neq 0.$$

Therefore, the integral behaves like

$$\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\tau}\,d\tau.$$

If $p\neq 2$, we have

$$y_k(s) \sim e^{-\frac{1}{p-1}s}\left(C_1 + C_2 e^{-\frac{p-2}{p-1}s}\right) = C_1 e^{-\frac{1}{p-1}s} + C_2 e^{-s}.$$

If $p = 2$ we can repeat the above calculations but in this case the integral behaves like s. Therefore

$$y_k(s) \sim \begin{cases} Ce^{-\frac{1}{p-1}s} & \text{if } p > 2,\\ Cse^{-\frac{1}{p-1}s} & \text{if } p = 2,\\ Ce^{-s} & \text{if } p < 2.\end{cases}$$

This implies that $u_k(t)$ verifies

$$u_k(t) \sim \begin{cases} C & \text{if } p > 2, \text{ and hence it is bounded},\\ -C\ln(T_h - t) & \text{if } p = 2, \text{ and hence it blows up},\\ C(T_h - t)^{\frac{p-2}{p-1}} & \text{if } p < 2, \text{ and hence it blows up.}\end{cases}$$

Now we can repeat this procedure with a node x_l that is at distance 2 from F (using the asymptotic behaviour that we have found for y_k) and so on to find that $u_l(t)$ blows up if $d(l) \le K$ and u_l is bonded if $d(l) > K$ where $K\in\mathbb{N}$ is determined by p in the following way, K verifies

$$\frac{K+2}{K+1} < p \le \frac{K+1}{K}.$$

Also we find that the asymptotic behaviour of a node x_l is given by

$$u_l(t) \sim (T_h - t)^{-\frac{1}{p-1}+d(l)}, \qquad d(l) = 1,\ldots,K,$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$

$$u_l(t) \sim \ln(T_h - t)$$

if $d(l) = K$. □

Finally we localize the blow-up set.

Proof of Theorem 1.6: We want to prove that, given $\varepsilon > 0$ there exists h_0 such that for every $0 < h \le h_0$,

$$B(u_h) \subset B(u) + N_\varepsilon. \tag{6.5}$$

We have that the blow-up set of u is contained in $\partial\Omega$ [RR], [HY]. Let us call $A = B(u) + N_{\frac{\varepsilon}{2}}$. First we claim that, for every h small enough, we have $F \subset A$ (we recall that F is the set of nodes x_j such that $y_j(s) \to C_j \neq 0$). To prove this claim we observe that there exists a constant L such that

$$|u(x,t)| \le L \qquad \forall x \in \overline{\Omega} - A,\ \forall t \in [0,T).$$

Now, Theorem 1.1 implies that

$$\|(u_h - u)(\cdot, T-\tau)\|_{L^\infty} \le C\rho(h),$$

hence given τ, for every h small enough,

$$|u_h(x, T-\tau)| \le 2L \qquad \forall x \in \overline{\Omega} \setminus A.$$

Let x_j be a node in $\overline{\Omega} \setminus A$, then it holds

$$(T_h - (T-\tau))^{\frac{1}{p-1}} u_j(T-\tau) \le 2L(T_h - (T-\tau))^{\frac{1}{p-1}}$$

and then

$$y_j(s_0) \le 2L(T_h - (T-\tau))^{\frac{1}{p-1}},$$

where $s_0 = -\ln(T_h - (T-\tau))$. By Theorem 1.3 we have that $T_h \to T$. Therefore, choosing τ and h small enough we can make $y_j(s_0)$ small and fall into the hypothesis of Lemma 6.1, proving our claim.

To finish the proof of Theorem we only have to observe that by our propagation result, Theorem 1.5, we have that, for h small enough,

$$B(u_h) \subset F + \overline{N_{Kh}} \subset A + \overline{N_{Kh}} \subset B(u) + N_\varepsilon,$$

proving (6.5). $\square$

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