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Growing processes on a strip

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Summary. We review some growing processes in the strip $\mathbb{Z} \times \{-N, \ldots, N\}$. The Richardson model, the asymmetric voter model, the Branching exclusion process and the solid on solid model are considered. The common characteristic of the models is the existence of a microscopic interface. This is a random position evolving in time with the property that the process presents different densities asymptotically to the right and left of the interface. Law of large numbers an central limit theorems hold for the position of the interface. Some of the processes present hydrodynamical behavior.

1. Introduction.

Interacting particle systems have been considered as models for the microscopic evolution of physical systems. Phase transition, ergodicity, shock waves, metastability and large deviations are some of the problems studied. In this paper we are interested in particle systems in the strip $\mathbb{Z} \times \{-N, \ldots, N\}$ that present microscopic interfaces. Roughly, a microscopic interface is a random point evolving in time with the property that, uniformly in time, the asymptotic density of particles to the right and left of that point are different. In one dimensional systems, this phenomenon appears in different models. We distinguish between two type of models: dissipative and conservative. Dissipative are those systems for which the density of particles changes in time. Generally these systems have at most two extremal invariant measures and approach equilibrium in a fast way. We consider here the voter model and the contact process. Extremal invariant measures means that other invariant measures are convex combinations of these two. For the contact process one of the equilibrium measures concentrates mass on the empty configuration. For the voter model (and the Richardson model that is a particular case of the voter model) the extremal invariant measures that concentrate mass on the full and empty configurations respectively. Conservative systems have constant density of particles. We consider here the simple exclusion process. This process has a one parameter family of extremal invariant measures. Even if the exclusion process is not rigorously a growing process, it shares with these the existence of a microscopic interface. When a microscopic interface exists, it describes the way that two invariant measures coexist at a microscopic level. Or, in a more physical language, two phases of the system coexist. Asymptotically from the interface one expects to see the two measures. The examples we study here suggest that the following is true. 1. Dissipative systems. The semi infinite system has a microscopic interface if and only if the infinite system presents phase transition –*i.e.* if there exists more than one invariant measure for the infinite system. For some of these systems there exists phase transition if some parameter is bigger than a critical value. Law of large numbers hold for the position of the interface at time t divided by t. Central limit theorems only hold for values strictly bigger than the critical value. 2. Conservative systems. The system has a microscopic interface if the one particle motion is not symmetric. In equilibrium the interface satisfies law of large numbers and central limit theorems. The systems can also evolve in a random environment. This means that the dynamics may depend on some local parameter that is chosen in a random fashion at the beggining and fixed for ever. This is called a random environment. Then the process behaves according to the environment. For some of these models there exist more than one critical parameter. There is a first critical value such that for values of the parameter bigger than the critical value the system survives, when starting with a single particle. There is a second critical value such that when the parameter is bigger than this second value, the system grows linearly. Most results are proven only for the one dimensional case. We are interested however in the behaviour of the systems in the strip $\{-N, \dots, N\} \times \mathbb{Z}$. The jump from one line to $2N+1$ lines is far from trivial. But we expect that the study of the systems in the strip will help to understand the existence of interfaces in two dimensions. This usually hard problem is solved for the solid on solid model.

The paper is hybrid. We show some results for the Richardson grow model that is a special case of the voter model and for a combination of this model with the symmetric exclusion process. For the other models we review the results, give a sketch of some of the proofs and state open problems. First we describe the state space and give some general preliminary notation and results. Then we consider the simplest among the dissipatives systems: the Richardson model. In the last section we treat briefly the exclusion process.

2. Definitions and preliminary results.

Our object of study is the evolution of a process $\eta_t \in \mathbf{X} = \{0,1\}^{\Lambda_N}$, where $\Lambda_N = \{x = (x_1, x_2) \in \mathbb{Z}^2 : |x_2| \leq N\}$. To fix ideas we consider periodic boundary conditions, *i.e.* $(x_1, N+1) = (x_1, -N)$ in the formulas below. Elements of $\mathbf{X}$ are called configurations, and elements of Λ_N are called sites. A typical configuration is a function from Λ_N to $\{0,1\}$. We say that for the configuration η, the site x is occupied if $\eta(x) = 1$, otherwise x is empty. We identify η with the subset of Λ_N of occuppied sites: $\eta = \{x \in \Lambda_N : \eta(x) = 1\}$.

The state space $\mathbf{X}$ has a non countable number of configurations. We consider a countable subspace $\mathbf{X}_0 = \{\eta \in \mathbf{X} : \sum_{x_1<0}(1 - \eta(x_1, x_2)) < \infty$ and $\sum_{x_1>0} \eta(x_1, x_2) < \infty\}$. In words, $\mathbf{X}_0$ is the set of configurations that has a finite number of occupied sites to the right of the origin and a finite number of empty sites to its left.

On $\mathbf{X}_0$ we consider probability measures. Since the space is countable we can give a positive mass to each configuration of $\mathbf{X}_0$ and a probability measure or simply a measure is a function $\mu : \mathbf{X}_0 \to [0,1]$ such that $\mu(\eta) \geq 0$ for all $\eta \in \mathbf{X}_0$ and $\sum_{\eta\in\mathbf{X}_0} \mu(\eta) = 1$.

The definition of measures on $\mathbf{X}$ is a bit more complicated. Since the space is non countable, in general single configurations have zero measure, and one has to define the measure on a subset $\mathcal{F}$ of the family of parts of $\mathbf{X}$. We do not want to enter in details here, but $\mathcal{F}$ is called a sigma algebra and it suffices to define the measure on a countable subset of $\mathcal{F}$, the algebra of the cylindric sets: $\{\{\eta \in \mathbf{X} : \eta(x) = 1$, for $x \in A\}$, $A \subset \Lambda_N$ finite$\}$.

We say that a sequence of configurations η_n converges to η, as $n \to \infty$, if for any finite set $\Lambda \subset \Lambda_N$ there exists $n(\Lambda)$ such that for any $n \geq n(\Lambda)$, it holds $\eta_n(x) = \eta(x)$ for $x \in \Lambda$. A sequence of measures μ_n converges to μ if for any cylindric f, $\mu_n f \to \mu f$. If η_n is a sequence of random variables with distribution $Ef(\eta_n) = \mu_n f$ for a sequence of measures

μ_n, then μ_n converges to μ if η_n converges weakly (or in distribution) to μ. A process is called Feller if

$$\eta_n \to \eta \text{ imply that } \eta_t^{\eta_n} \to \eta_t^{\eta}$$

in distribution. This is equivalent to

$$\mu_n \to \mu \text{ imply that } \mu_n S(t) \to \mu S(t)$$

where $\nu S(t) f = \int \nu(d\eta) E_\eta f(\eta_t)$.

After the construction, the first thing that one would like to study is the existence or not of equilibrium states. We say that a measure μ on $\mathbf{X}$ is invariant if for all $t \geq 0$, $\mu S(t) = \mu$. In our case, the condition $\mu L f = 0$ for all cylinder f is necessary and sufficient for the invariance of μ. Define also

$$\frac{d}{dt} S(t) f(\eta)\big|_{t=0} = Lf(\eta); \quad \frac{d}{dt} S(t) f(\eta)\big|_{t=s} = S(s) L f(\eta) = L S(s) f(\eta) \tag{3.1}$$

The operator L is defined on $C(\mathbf{X})$, the set of continuous functions on $\mathbf{X}$ and is called the generator of the process. It plays the role that probability transition functions do in discrete time Markov processes. It describes the intuitive instantaneous behavior of the process.

The following result is useful to establish the existence of invariant measures.

Theorem. *Feller processes on compact spaces admit at least one invariant measure.*

Proof. The set of probability measures on a compact space is compact (Skorohod). Take μ on $\mathbf{X}$. Let $\mu_t = \mu S(t)$. Let $\bar{\mu}_t$ be the Cesaro limit

$$\bar{\mu}_t = \frac{1}{t} \int_0^t \mu_s ds$$

Since $\{\bar{\mu}_t\}$ is a compact set, there exists at least a convergent subsequence $\{\bar{\mu}_{t(n)}\}$. Call $\bar{\mu}$ the limit of the subsequence. Since the process is Feller, one can interchange limit with

$S(u)$ and get

$$
\begin{aligned}
\bar{\mu}S(u) &= (\lim_{n\to\infty} \bar{\mu}_{t(n)})S(u) \\
&= \lim_{n\to\infty} \bar{\mu}_{t(n)}S(u) \\
&= \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \mu S(s)S(u)ds \\
&= \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \mu S(s+u)ds \\
&= \lim_{n\to\infty} \frac{1}{t_n}\int_u^{t_n+u} \mu S(s)ds \\
&= \lim_{n\to\infty} \frac{1}{t_n}\left(\int_0^{t_n} \mu S(s)ds + \int_0^{u} \mu S(s)ds - \int_{t_n}^{t_n+u} \mu S(s)ds\right) \\
&= \bar{\mu}
\end{aligned}
$$

This proof is taken from Liggett (1985) to which we refer for details.

3. The Richardson model.

3.1. Construction. The Richardson model is a purely growing model. The motion is governed by the following rules. Occupied sites remain occupied. Empty sites become occupied at rate equal to the number of occupied nearest neighbor sites. For this model it has been proven that there exists an asymptotic shape for the system starting with only one occupied site in $\mathbb{Z}^d$.

We give the construction of a version of the process. This is called graphical construction. The important property of this construction is that it allow to realize simultaneously two or more versions of the process with different initial configurations (coupling). At each bond (x, y), $x, y \in \Lambda_N$ such that $|x - y| = 1$ associate a Poisson point process (Ppp) with rate 1. Each of these processes is a sequence of times, that we call $\omega(x, y, n)$, $n \in \mathbb{N}$ with the property that $\{\omega(x, y, n) - \omega(x, n-1)\}_{n,x}$ is a family of mutually independent random variables with exponential distribution with parameter 1. That is, with marginals $P(\omega(x, y, n) - \omega(x, y, n-1) > t) = e^{-t}$. Neglect the set of probability zero where at least two of these times coincide, *i.e.* the set $\{\omega :$ there exist $(x, y, n), (x', y', n')$ such that $\omega(x, y, n) = \omega(x', y', n')\}$. We say that an infection mark between x and y is present at the times $\omega(x, y, n)$, $n \in \mathbb{N}$. Call ω a configuration of marks and $(\Omega,\mathcal{F},P)$ the probability space induced by the Ppp described above.

Fix now a time $\bar{t}$. The set $\{\omega : \text{for all } x_1 > 0 \text{ there exists } x_2 \text{ such that } \omega((x_1, x_2), (x_1 + 1, x_2), 1) < \bar{t}\}$ has probability zero, as well as the event defined in the same way but with $x_1 < 0$. This means that for almost all ω there exist a coordinate $x_1 > 0$ such that there are no infection marks in $(0, \bar{t})$ connecting (x_1, x_2) with $(x_1 + 1, x_2)$ for all x_2 in $(0, \bar{t})$. Repeating the same argument, we can say that with probability one there is a sequence of coordinates x_1^i, $i \in \mathbb{Z}$ such that there are no connections between the sites with first coordinates x_1^i and those with first coordinate $x_1^i + 1$ in the time interval $(0, \bar{t})$. We consider only the ω belonging to this set of probability one. For each ω, we construct the process separately in the boxes $\{(x_1, x_2) : x_1 \in [x_1^i + 1, x_1^{i+1}] \cap \Lambda_N\}$. Of course x_1^i is a function of ω and $\bar{t}$ for each i. Put an initial configuration η at time zero, at the top of our graph –the time is flowing down– and construct η_t, $0 \le t \le \bar{t}$, as a function of η and ω. Since there are no infection marks connecting different boxes, there is no interaction among them. Since the boxes have finite lenght, we are able to label the marks inside each box by order of appearance. If the first mark appears for the bond (x, y) and before the mark either x or y are occupied, then after the mark both x and y are occupied. Otherwise the two sites remain empty after the mark. Then we go to the second mark and repeat the procedure up to the last mark in the box and go to the next box constructing in this way $\eta^\eta_{\omega,t}$, $0 \le t \le \bar{t}$, with initial configuration η. For times greater than $\bar{t}$, use $\eta_{\bar{t}}$ as initial configuration and repeat the procedure to construct the process between $\bar{t}$ and $2\bar{t}$, and so on. The generator of the process is

$$Lf(\eta) = \sum_{x,y \in \Lambda_N} \eta(y)(1 - \eta(x))[f(\eta^x) - f(\eta)] \tag{3.2}$$

where the sum runs over x, y such that $|x - y| = 1$ and $\eta^x(z)$ equals $1 - \eta(x)$ for $z = x$ and $\eta(z)$ for $z \neq x$.

3.2. The shape theorem.

Exceptionally, and only in this subsection we consider the process defined in the whole lattice $\mathbb{Z}^2$ in order to describe the most important result for this model: the shape theorem. Let the process evolve according to the same rules as in the previous section in the full plane $\mathbb{Z}^2$. Call ξ_t the set of occupied sites by time t. Then the theorem says that ξ_t has an asymptotic shape:

Theorem. *(Richardson shape theorem.) There exists a norm $||.||$ on $\mathbb{R}^2$ such that, if $B(r) = \{y \in \mathbb{R} : ||y|| \le r\}$ is the ball of radious r, then*

$$P(B(t-\varepsilon) \subset \xi_t \subset B(t+\varepsilon) \text{ for all } t \text{ sufficiently large }) = 1$$

Nothing is known about the norm. It should reflect the differences of the square lattice when looked at differents angles. The theorem was first proven by Richardson (1973) in a weaker form and then by Schürger (1979). See also Durrett (1988).

3.3. The growing process in the strip.

We come back to the strip. It is clear that if one starts with a configuration that has at least one particle, then all the sites eventually become occupied. We consider initial configurations in $\mathbf{X}_0$, *i.e.* with only a finite number of particles to the right of the origin and a finite numer of empty sites to its left. For such configurations there is always one (or more) rightmost occupied site. We refer to the first coordinate of the rightmost occupied site by rightmost particle. We shall prove that the process as seen from the rigthmost particle has an unique invariant measure μ. Furthermore this measure μ concentrates mass on configurations with only a finite number of empty sites to the left of the rightmost particle.

Let $\eta \in \mathbf{X}_0$ be a configuration with a rightmost particle. Define $Z(\eta) := \max\{x_1 : \eta(x_1, x_2) = 1\}$ as the first coordinate of the rightmost particle(s) of η. For $z \in \mathbb{Z}$ let $\tau_z\eta(x_1, x_2) := \eta(x_1 + z, x_2)$, be the horizontal translation of η by z. We denote by η_t the configuration at time t and $Z_t = Z(\eta_t)$. We consider $\xi_t := \tau_{Z_t}\eta_t$, the process as seen from the rightmost particle.

Lemma 3.1. *The process ξ_t is Feller.*

Proof. Let f be a cylinder function depending on the finite set $A \subset \Lambda_N$. We can assume that this set depends on sites whose first coordinate is bigger than $-M$ and smaller than

0. Fix t and ω and let $x_1^{-i} < -M$ be the first site to the left of $-N$ with no marks from (x_1^{-i}, x_2) to $(x_1^{-i}+1, x_2)$, $x_2 \in (-N, N)$. Then $f(\xi_t)$ depends on the values of the configuration η_t in the set $A_t = \tau_{X(t)} A \subset \Lambda_N^i = ([x_1^{-i}, x_1^1] \times [-N, N]) \cap \Lambda_N$. Take $n(\Lambda_N^i)$ such that for all $n \geq n(\Lambda_N^i)$, $\eta^{(n)}(x) = \eta(x)$ for all $x \in \Lambda_N^i$. Since there is no interaction in $[0, t]$ between what happens in Λ_N^i and its complementar, we have that for $n \geq n(\Lambda_N^i)$, $\eta_t^{\eta^{(n)}}(x) = \eta_t^{\eta}(x)$ for all $x \in \Lambda_N^i$. Hence, for $n > n(\Lambda_N^i)$, $\xi_t^{\eta^{(n)}}(x) = \xi_t^{\eta}(x)$ for all $x \in \Lambda_N$.♣

Lemma 3.1 together with Theorem 2.1 imply the following.

Corollary. *The process ξ_t admits at least one invariant measure μ.*

This proves the existence of the invariant measure but does not imply the unicity. In order to show unicity we study the properties that an invariant measure must satisfy. For each configuration η let $H(\eta) = \sum_{x_2 \leq Z(\eta)} (1 - \eta(x_1, x_2))$ be the number of empty sites to the left of the rightmost particle of the η.

Proposition 3.3. *If μ is invariant for ξ_t, then μ must concentrate on configurations with a finite number of empty sites to the left of the rightmost particle. Moreover*

$$\mu H = \int \mu(d\xi) H(\xi) \leq N^3$$

Proof. If μ is invariant, then $\mu L' f = 0$ for all cilinder f, where L' is the generator of the process ξ_t. We want to conclude that $\mu L' H = 0$. However H is not a continuous function. But it can be approached by cylinder functions: for $K > 0$ let $H_K(\xi) = \max\{H(\xi), K\}$. Clearly $H_K(\xi)$ is non decreasing in K and for all ξ we have

$$\lim_{K \to \infty} H_K(\xi) = H(\xi)$$

Now, since H_K is cilinder, we have $\mu L' H_K = 0$ for all K and by the monotone convergence theorem, $\mu L' H = 0$. Now, a simple computation gives

$$0 = \mu L' H(\xi) = \mu((N-1)V(\xi) = 0) - \mu \sum_{x,y} \xi(x)(1 - \xi(y)).$$

where the sum is taken over the set $\{(x, y) : |x - y| = 1, y_2 \leq 0, x_2 \leq 0\}$ and $V(\xi) = \#\{x_2 : \xi(0, x_2) = 1\}$. The formula above just states that in equilibrium the number of holes to the left of the rightmost particle must be constant in average. The second term is the rate at which a hole dissapear and the first term is the rate $(N-1)$ holes are created times $(N-1)$. Since $\#\{x_2 : \xi(0,x_2)=0\} \leq N$, we have

$$\mu\left(\sum_{x,y:|x-y|=1} \xi(x)(1-\xi(y))\right) \leq 2N^2 \tag{3.4}$$

On the other hand, $\mu(\xi : |\xi| = \infty) = 1$ because particles are created and all finite states are transient. This implies that

$$\mu\left(\xi : \sum_{x:x_1<0} (1-\xi(x)) < \infty\right) = 1$$

Since all invariant measures μ concentrate on $\mathbf{X}_0$, which is a denumerable space we have that, starting with the invariant measure μ, our process is a denumerable Markov chain which has at most one invariant measure. (The state space is irreductible.) To prove $\mu H \leq N^3$, notice that for each $x_2 < 0$ there exists at least one x_1 such that $\eta(x_1,x_2) = 1$.♣

We have proven the following

Corollary. *The process ξ_t has a unique invariant measure μ which concentrates on $\mathbf{X}_0$. Furthermore μ satisfies (3.4).*

Let $v = \mu V = \mu(\xi : \#x_2 : \xi(0,x_2) = 1)$ be the instantaneous velocity of the process in equilibrium. Clearly $v \leq N$ We prove now the following law of large numbers:

$$P(\lim_{t\to\infty}(Z_t/t) = v) = 1$$

The proof is a consequence of the fact that the process is a positive recurrent continuous time Markov chain. Let $S_i = \inf\{t > S_{i-1} : \xi_t = \bar\eta, \xi_{(t-)} \neq \bar\eta\}$, where $\bar\eta$, the Heaviside configuration, has all sites with the first coordinate less or equal to zero occupied and the other sites empty. The random times S_i are renewal times, *i.e.* $T_i = S_{i+1} - S_i$ are independent identically distributed random variables. We say that a renewal occurs at each time S_i. Let $N(t)$ be the number of renewals occurred in the interval $[0,t]$. Let $Y_i = Z_{S_{i+1}} - Z_{S_i}$ and $X_n = \sum_{i=1}^n Y_i$. Then we can write

$$Z_t = \sum_{i=1}^{N(t)} Y_i + (Z_t - X_{N(t)})$$

Hence as S_i is a renewal process

$$\frac{Z_t}{t} = \frac{\sum_{i=1}^{N(t)} Y_i + (Z_t - X_{N(t)})}{\sum_{i=1}^{N(t)} T_i + (t - S_{N(t)})}$$

Dividing numerator and denominator by $N(t)$ and taking t to infinity, we get $(Z_t - X_{N(t)})/N(t) \to 0$ and $(t - S_{N(t)})/N(t) \to 0$. Hence

$$\lim_{t\to\infty} \frac{Z_t}{t} = \frac{EY_1}{ET_1}$$

From where $v = EY_1/ET_1$.

To prove a Central Limit Theorem one proves that $ET_1^2 < \infty$ and $EY_1^2 < \infty$. This is not so hard to prove for this model as the number of empty sites to the left of the rightmost particle can be dominated by a process with all moments. The reader may ask at this point, why we did not use this domination to prove the positive recurrence of the process. The reason is that the domination argument does not work for the next model, while the general argument given here does work.

4. The branching exclusion process.

We modify the Richardson model by letting the particles to move. To each pair (x, y) of nearest neighbor sites of the strip we associate another Poisson point process of rate $\gamma/2$. We call stirring marks, the ocurrence times of these Ppp. When a stirring mark is present, involving two nearest neighboring sites, after the mark the contents of the two sites is exchanged. The resulting process is called branching exclusion process. Using the same argument as for the Richardson model in a strip one can prove that this process admits a unique invariant measure concentrating on $\mathbf{X}_0$ and that under this measure the velocity v is bounded by N. This was done by Bramson et al. (1986).

The law of large numbers is easy to obtain for this model, but to prove the central limit theorem as we did for the Richardson model requires the computation of the second moment of T, the time necessary to coming back to the initial configuration. This is difficult because we do not have any information on the invariant measure and due to the exclusion interaction, the domination argument does not work. Nevertheless the central limit can be proven using a somehow more subtle argument. This has been done for the unidimensional case ($N = 1$) by Cammarota and Ferrari (1991) following an idea of Kukzek (1989) who proved the result for the contact process. It is not clear how to do it in the strip.

We see now the hydrodynamical limit. For notational convenience we consider $N = 1$, but a slightly more complicated result holds for any N. The hydrodynamical limit arises by considering a family of processes η_t^γ and rescaling the space as $\sqrt{\gamma}$.

Theorem 4.1. *(The hydrodynamical limit.)*

$$\lim_{\gamma \to \infty} E\tau_{\sqrt{\gamma}r} f(\eta_t^\gamma) = \nu_{u(r,t)} f$$

where $u(r,t)$ satisfies the Kolmogorov, Petrovsky, Piscunof (KPP) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + u(1-u)$$

with initial condition $u(r,0) = 1$ for $r < 0$ and $u(r,0) = 0$ for $r > 0$.

The KPP equation admits travelling wave solutions. For any $v \geq \sqrt{2}$ there exists a function q_v such that $u(r,t) = q_v(r - vt)$ is solution of the equation for all $t \geq 0$.

Sketch of the proof. Consider first the function $f(\eta) = \eta(0)$. Then $\tau_x f(\eta) = \eta(x)$. Now, if we want to compute the time derivative of $E\eta_t^\gamma(\sqrt{\gamma}r)$ we get as in the Kolmogorov forward equation (we omit the superlabel γ to simplify notation)

$$\tag{4.2} \begin{aligned} \frac{\partial E\eta_t(\sqrt{\gamma}r)}{\partial t} &= \gamma E\big[(-2\eta_t(\sqrt{\gamma}r) + \eta_t(\sqrt{\gamma}r + 1) + \eta_t(\sqrt{\gamma}r - 1))\big] \\ &\quad + \eta_t(\sqrt{\gamma}r)((1 - \eta_t(\sqrt{\gamma}r + 1) + (1 - \eta_t(\sqrt{\gamma}r - 1)) \end{aligned}$$

If one is able to prove that, as $\gamma \to \infty$, $\sup_{x,y} |E(\eta_t(x)\eta_t(y)) - E\eta_t(x)E\eta_t(y)| = 0$, *i.e.* , the distribution of the particles become independent random variables, then the equation (4.2) approaches the KPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + u(1-u)$$

This independence can be proven. The proof uses duality. The site x at time t will be occupied if and only if at least one of the sites of A_t^x is occupied at time 0, where A_t^x is defined as follows. Assume that for $A \subset \Lambda_N$, $A_0^A = A$. Then A_t^A is the branching exclusion process with initial configuration with all sites in A occupied and the other sites empty. But use the marks of the Ppp backwards in time from t to 0. Call $A_t^x = A_t^{\{x\}}$. Then it is easy to see the a.s. duality formula

$$\eta_t(x) = 1 \text{ if and only if } \eta_0(z) = 1 \text{ for some } z \in A_t^x$$

and the additive property

$$\eta_t(x)\eta_t(y) = 1 \text{ if and only if } \eta_0(z) = 1 \text{ for some } z \in A_t^x \text{ and } \eta_0(z) = 1 \text{ for some } z \in A_t^y$$

So that, if one can prove that A_t^x and A_t^y behave like independent, as $\gamma \to \infty$, then the independence of $\eta_t(x)$ and $\eta_t(y)$ will follow. The independence of A_t^x and A_t^y was proved by De Masi, Ferrari and Lebowitz (1985). The idea is that in a finite time interval only a finite number of branchings for the dual process will happen. On the other hand, a very large number of exchanges will occur. A coupling is done between (A_t^x, A_t^y) and a pair of independent processes $(\bar{A}_t^x, \bar{A}_t^y)$. Under this coupling, the probability that $\bar{A}_t^x \neq A_t^x$ or $\bar{A}_t^y \neq A_t^y$ goes to zero as $1/\sqrt{\gamma}$. ♣

For fixed γ, the process in equilibrium has a constant average velocity v_γ. An interesting feature of this approximation is that $v_\gamma/\sqrt{\gamma}$ converges, as $\gamma \to \infty$ to $\sqrt{2}$. This is indeed the critical velocity for the existence of travelling wave solutions of the KPP equation (Bramson et al. (1986)).

Consider now the system as seen from $W_t(0) = \max\{x_1 : \eta_t(x_1, 0) = 1\}$, the position of the rightmost particle in the x axis. It follows from the results described above that for each N there exists an invariant measure for the process $\tau_{W_t}\eta_t$. Is there a limiting distribution for this process when $N \to \infty$?

5. The biased voter model.

For this process to each bond we associate two Poisson point processes, one with intensity δ and the other with intensity λ. When a δ mark appears involving two sites, one of which is empty, then after the mark the two sites become empty. When a λ mark appears involving two sites with at least one occupied, then after the mark the two sites become occupied. We can think that at each site there is a voter with two possible opinions: 0 or 1. The rate of changing opinion from 0 to 1 is λ times the number of neighbors with opinion 1, while the rate of changing opinion from 1 to 0 is δ times the number of nearest neighbors with opinion 0. This process has two invariant measures: those that concentrate mass in the full and empty configuration respectively.

The model is trivial for $N = 1$: when starting from the configuration full of particles to the left of the origin and no particles to its right, this configuration remains unchanged in time unless for a random translation Z_t, where Z_t is an asymmetric random walk jumping with rate λ to the right and with rate δ to the left.

To prove that there exists an invariant measure as seen from the rightmost particle for $N > 1$ we use an argument similar to the one we used for the Richardson model. The first difficulty with this argument is that the voter model as seen from the rightmost particle is not Feller in $\mathbf{X}_0$. To see that consider the measure giving mass one to the configuration $\eta^n = \{0\} \cup \{-\infty, \dots, -n\}$. Then η^n converges to $\{0\}$. On the other hand η_t^n converges to a non trival convex combination of $\bar{\eta}_t$ and η_t^0, which is different of η_t^0 that may be not well defined. This counter example shows that the voter model as seen from the rightmost particle is not Feller in $\mathbf{X}_0$. But if we consider the subset of $\mathbf{X}_0$ containing configurations with infinitely many particles to the left of the rightmost one, it is not difficult to prove that in this set the process is Feller, by an argument as the one in the Richardson model. The problem is that this subset is not compact, so the Cesaro limits may not belong to the subset. For the asymmetric voter model it is not hard to prove that all weak limits of ξ_t concentrate on the subset, concluding the proof of the existence of at least one invariant measure. Then one gets (3.10) with the left hand side multiplied by $(\lambda - \delta)$ and the right hand side multiplied by λ. In this way the invariant measure must be unique and concentrating on $\mathbf{X}_0$ for $\lambda \neq \delta$. The existence of an invariant measure for ξ_t in $\mathbf{X}_0$ imply the law of large numbers.

In the symmetric case, that is, when $\lambda = \delta$, the argument does not work. We conjecture that in this case there is no invariant measure for the process as seen from the rightmost particle. The weak limits as $t \to \infty$ of the distribution of ξ_t would concentrate on configurations with a finite number of particles that can not be invariant.

We propose the following problem. Is there a "weakly biased" case with a non trivial hydrodynamical limit? By weakly biased we mean to consider $\delta = 1$ and $\lambda = 1 + (1/\sqrt{\gamma})$. Then, one would look at the process at a (macroscopic) time γt at a (macroscopic) position $f(\gamma)r$.

6. The biased voter model in a random environment.

The voter model can be considered in the so called "random environment". The results that we are going to see were proven only for dimension $d = 1$ by Ferreira (1988) and reviewed by Bramson, Durrett and Schonmann (1991). A random variable δ_b is attached at each bond $b = (x, y)$, $|x - y| = 1$. These variables are mutually independent. Consider a realization of the variable δ_b as the rate of a Poisson point processes associated with the bond b. Now realize the process in the following manner: When the λ clock associated with the bond $b = (x, y)$ rings, if either x or y is occupied, then after the mark the two sites are occupied. Otherwise nothing happens. When the δ_b clock rings, if either x or y is empty, then after the mark the two sites are empty.

In one dimension, when one starts with the Heaviside configuration, at latter times one gets the same configuration shifted by a random variable Z_t. This variable makes a random walk in the environment described above: if $Z_t = z$, it jumps to the right at rate λ and to the left at rate δ_z. The results come from the study of this random walk in random environment due to Solomon (1975):

If $E \log(\lambda/\delta_b) < 0$ then $P_\delta(Z_t = 0 \; i.o.) = 0$ for a.e. environment δ.

If $E \log(\lambda/\delta_b) > 0$ then $P_\delta(Z_t = 0 \; i.o.) = 1$ for a.e. environment δ.

This induces the definition of a critical value

$$\lambda_c = \inf\{\lambda > 0 : P_{\delta,\lambda}(Z_t = 0 \; i.o.) > 0 \text{ for a.e. } \delta\} = \exp(E \log \delta_b)$$

On the other hand, it is true that Z_t/t converges almost surely to $\alpha(\lambda)$ as $t \to \infty$, where

$\alpha(\lambda)$ is the constant satysfying

$$\alpha(\lambda) = \begin{cases} > 0 & \text{if } \lambda \in (E\delta_b, \infty), \\ = 0 & \text{if } \lambda \in (1/E\delta_b, E\delta_b), \\ < 0 & \text{if } \lambda \in (-\infty, 1/E\delta_b), \end{cases}$$

So that, under the random environment δ, we can define a critical value λ_α, where

$$\lambda_\alpha = \inf\{\lambda > 0 : \alpha(\lambda) > 0\} = E\delta_b$$

So that the biased voter model in random environment has two critical values, one for recurrence of the extreme and the other for linear growth. The recurrence condition is equivalent to the non survival of the process when starting from a single particle.

The following open problems arise. Does the existence of two different critical parameters appear in the strip for any N? We conjecture that the answer is yes. If our conjecture is true, this puts another problem. Bramson, Durrett and Schonmann conjecture that the existence of two critical values is not true in two dimensions. So the difference between these two critical paramenters should somehow go to zero as $N \to \infty$. Is there an invariant measure for the process as seen from W_t, the position of the rightmost occupied site at the x axis?

7. The contact process.

The contact process is one of the first particle systems studied. In spite of its simplicity, it presents a phase transition even in one dimension. We describe it in our box Λ_N: It consists in the superposition of two processes: one is the infection process used to define the Richardson model. Particles appear at empty sites at a rate λ times the number of nearest neighbor occupied sites. The other is just a death process: particles dissapear at rate one, independent of the rest of the configuration.

In the one dimensional case, when $N = 1$ is has been proven that there exists a critical parameter λ_c such that starting with the initial Heaviside configuration, and calling Z_t the position of the rightmost particle, the following hold

(1) As $t \to \infty$, the normalized position Z_t/t converges almost surely to $\alpha = \alpha(\lambda)$,

where

$$\alpha = \begin{cases} > 0 & \text{if } \lambda > \lambda_c \\ = 0 & \text{if } \lambda = \lambda_c \\ = -\infty & \text{if } \lambda < \lambda_c \end{cases}$$

Furthermore α is a non decreasing function of λ and is continuous to the right in $\lambda = \lambda_c$.

(2) If $\lambda \geq \lambda_c$, the process $\xi_t = \tau_{Z_t}\eta_t$, *i.e.* the process as seen from the rightmost particle, has a unique invariant measure. The process starting from the Heaviside configuration converges to this invariant measure.

(3) If $\lambda > \lambda_c$, then the normalized process $(Z_t - \alpha t)/\sqrt{t}$ converges to a Gaussian random variable of 0 mean and positive variance σ^2.

(4) $1 \leq \lambda_c \leq 2$

The proof of (1) is a consequence of the subadditive ergodic theorem and the attractive properties of the process. The proof can be found in Durrett (1984). The existence of an invariant measure in (2) can be shown in a constructive way as in Galves and Presutti (1987a, 1987b), or using an abstract argument as the one we used for the voter model, as in Durrett (1984). The contact process as seen from the rightmost particle is not Feller for the same reason as the voter model. The constructive way works for $\lambda > \lambda_c$ and the other works also for λ_c. In order to show that the limiting measure concentrates on the subset of $\mathbf{X}$ with infinitely many particles to the left of the rightmost particle one uses that the rightmost particle has a non negative a.s. limit. The convergence of the process starting from any configuration with infinitely many particles to the left of the rightmost particle to the invariant measure in the supercritical case was proven by Galves and Presutti (1987), as well as the central limit theorem of (3). An alternative proof of the central limit theorem is found in Kuzcek (1989) . For $\lambda = \lambda_c$ one expects that the fluctuations are superdiffussive.

The lower bound in (4) is easy to obtain by comparing the process with a branching process. The upperbound was obtained by Holley and Liggett (1978). The expected value is $\lambda_c = 1.64$.

These results are closely related to the problem of survival of the contact process. It is said that the contact process η_t survives if, starting with the full configuration, the process converges to a measure different from the mass concentrated on the empty configuration.

Otherwise we say that the process dies out. It turns out that for $\lambda \geq \lambda_c$, the process survives and for $\lambda < \lambda_c$ the process dies out. It is proven that for any dimension d there exists λ_c^d such that the process survives if $\lambda \geq \lambda_c^d$ and dies if $\lambda < \lambda_c^d$. The fact that the contact process dies at $\lambda = \lambda_c$ is deep and was proven only recently by Bezuindenhout and Grimett (1990).

Here the following open problems arise.

1. Is it possible to improve the bounds for λ_c?. Computer simulations propose $\lambda_c = 1.64$.

2. Is it possible to extend results (1) ,(2) and (3) to the strip λ_N for any N?

3. Assuming that the conjecture 2 holds. Call $\lambda_c(N)$ the critical value for the strip of lenght N. Then, Bezuindenhout and G. Grimmett (1990) proved that, as $N \to \infty$, $\lambda_c(N)$ converges to λ_c^2 (the critical value of λ for $d = 2$). Is there a limiting measure for the process as seen from the rightmost particle on the x axis?

4. Does ξ_t converge in distribution, as $t \to \infty$ when $\lambda < \lambda_c$? For $N = 1$ Schonmann (198?) proved for oriented percolation that in this case does not exist an invariant measure. This was extended by Andjel (1988) for the contact process. Nevertheless it may be convergence in distribution. It is conjectured that the answer to this question is yes and that the result must be a quasi-stationary distribution. This is a measure ν concentrating on configurations with a positive and finite number of particles satisfying the property: "starting with ν and conditioning that at time t the process is not empty, then the process has distribution ν". One of the difficulties here is that in general there is not just one quasi stationary measure, but infinitely many (when any). So the problem is to determine to which of those the process converges.

As well as for the voter model one can consider the contact process in a random environment. The same results as for the biased voter model were obtained by Bramson, Durrett and Schonmann (1991). Also Liggett (1991, 1992) and Klein (199?) worked in these kind of models.

8. The Solid on Solid model.

To understand the name of this model one should consider a vertical strip and consider the particles as falling from the sky, so that in the state space one never accepts that an empty site can be under a particle. But to avoid confusions we describe the process in our horizontal strip. Three Poisson point processes are associated to each vertical bond, with rates β_0, β_1 and β_2 respectively. Consider that $\eta(x) = 1$, $\eta(x + e_1) = 0$ and the β_i clock attached to the bond $(x, x + e_1)$ rings, where $e_1 = (1, 0)$, $e_2 = (1, 0)$. If $i = \eta(x+e_1+e_2)+\eta(x+e_1-e_2)$, then after the mark, $\eta(x+e_1)$ changes its value from 0 to 1. In words, for each $k \in \{-N, \dots, N\}$ there is a value $Y(k)$ such that $\eta(x_1, k) = 1\{x_1 \leq Y(k)\}$ and $Y(k)$ increases at rate β_i if i of $\{Y(k-1), Y(k+1)\}$ are bigger that $Y(k)$.

This model is an approximation of the Ising model when the interactions in the vertical direction are considered as for zero temperature.

Mauro (1992), using a technique proposed by Gates and Wescott (1990) proved that for any values of β_i such that $\beta_0 < \beta_1 < \beta_2$ there exists an invariant measure for the process $\xi_t = \tau_{Y(0)}\eta_t$ on the space $\mathbf{X}_0$. Compare with $\beta_0 = \beta_1 = \beta_2$ under which $Y_t(k)$ are independent and the process converges to a measure that gives mass zero to $\mathbf{X}_0$. Under the invariant measure $Y_t(0)/t$ converges almost surely to a limiting velocity v. If one requires the extra condition

$$(*) \qquad \beta_1 = (\beta_0 + \beta_2)/2$$

much more can be said. The invariant measure can be described explicitely:

$$\mu(Y) = (1/Z)\exp(-\beta \sum_{k=-N}^{N-1} |Y(k) - Y(k+1)|)$$

where $\beta = \log(\beta_0/\beta_2)^2$ and Z is the normalizing factor such that $\sum_Y \mu(Y) = 1$. The central limit theorem holds, *i.e.* $(Y_t(0) - vt)/\sqrt{t}$ converges weakly to a Gaussian random variable. The condition $\beta_1 = (\beta_0 + \beta_2)/2$ is technical. This process is a Markov chain in a denumerable state space. Hence, the process is positive recurrent. To prove the central limit theorem one shows that the expected value of the square of the return time for a given state, is finite. This is equivalent to say that the expected hitting time of a given configuration, starting with the invariant measure is finite. The explicit form of the invariant measure is used to prove the finiteness of the second moment of the absorbing time. We do not expect a different behaviour for other values of β_i: nothing induces us to think that if there exists an invariant measure, then the central limit theorem does not hold, even if the second moment of the return time is not finite. We do not know how the variance of the limiting Gaussian variable behaves with N.

The symmetric case. Consider that $Y(k)$ can also decrease and do that at rate β_i, where i is the number of its nearest neighbors that are strictly smaller that $Y(k)$. Under condition (*) one proves the law of large numbers with asymptotyc velocity 0 (by symmetry) and also the central limit theorem. In this case the limiting variance is show to be proportional to $1/N$. The process in the plane ($N = \infty$) is well defined and Ferrari (1987) showed that the limiting variance in the plane is zero.

The measure μ described above is the Gibbs measure for this mode and is the one introduced in statistical mechanics for studying this model. Une property of this measure is that considering the x axis as time, $Y(k)$ is a Markov chain. This gives us that the spatial fluctuations of the measure are of the order of $\sqrt{k}$. The average velocity behaves as the number of maximuns of the Markov chain.

When the strip is two dimensional (or more), that is in the state space $\{-N, \dots, N\}^2 \times \mathbb{Z}$ even the existence of the invariant measure is difficult to show. Only in very few cases there are rigorous proofs, even if the heuristics are clear. See detais in Gates and Wescott (1990) and Mauro (1992).

9. The exclusion process.

To describe the process we consider that a Ppp is associated at each bond of the strip. The vertical bonds work as in the stirring process described above: when a stirring mark appears, the content of the two sites involved with the mark is interchanged. The horizontal bonds work as the stirring marks in the symmetric case, but in the asymmetric case we let the content of the sites to be exchanged only if the site to the left is occupied and the site to the right is empty. In the two cases the exclusion constraint is present, but the one particle motion has a drift in the asymmetric case.

The product measures are invariant for both the symmetric and the asymmetric process. Assume that one starts with a product measure with density λ to the left of the origin and density $\rho \in [0,1]$ to the right of the origin. Assume also that $\lambda + \rho = 1$. Then the behaviour is very different for the two processes. For the symmetric case the process converges to $\nu_{(\lambda+\rho)/2}$ and for the asymmetric case the process converves to $(\nu_\rho + \nu_\lambda)/2$. The first result follows from the duality of the process (see Liggett (1985)) while the second is harder to prove (Andjel, Bramson and Liggett (1988). The reason for this behaviour is that for the asymmetric case it is possible to prove the existence of a microscopic interface as defined in the introduction. This was proven by Ferrari, Kipnis and Saada (1991) and Ferrari (1992).

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