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EXISTENCE OF NON-TRIVIAL QUASI-STATIONARY DISTRIBUTIONS IN THE BIRTH–DEATH CHAIN

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Abstract

We study conditions for the existence of non-trivial quasi-stationary distributions for the birth-and-death chain with 0 as absorbing state. We reduce our problem to a continued fractions one that can be solved by using extensions of classical results of this theory. We also prove that there exist normalized quasi-stationary distributions if and only if 0 is geometrically absorbing.

MARKOV CHAINS; NORMALIZED QUASI-STATIONARY DISTRIBUTION; BIRTH AND DEATH PROCESS

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1. Introduction

We consider an irreducible discrete-time homogeneous Markov chain on a denumerable or finite state space $S = \{0\} \cup S^*$, where 0 is the only absorbing state and S is the set of transient states. A normalized quasi-stationary distribution (n-q.s.d.) for the chain is a probability measure μ on S^* with the property that the conditional distribution of the chain at time 1 given that it is not absorbed is given by μ if the initial measure is μ. It can be shown that such a μ verifies the system of equations of a left eigenvector:

$$(1.1) \qquad \mu P^* = \gamma(\mu)\mu$$

where P^* is the restriction of the stochastic matrix to S^* and $\gamma(\mu) = \mathbb{P}_\mu(X(1) \neq 0)$. We call a quasi-stationary distribution (q.s.d.) any measure satisfying (1.1).

The earliest work on this subject was done by Yaglom (1947) for the branching process. He proved the existence of a normalized quasi-stationary distribution as the limit of the conditional distribution probability of the chain given that it is not absorbed, starting with any Dirac distribution. Similar results have been established by Darroch and Seneta (1965) for finite state space, Seneta (1966) for continuous-

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time simple random walks on $\mathbb{N}$, and Seneta and Vere-Jones (1966) for discrete-time simple random walks on $\mathbb{N}$ and for the simple branching process.

For the birth and death process Good (1968) and Kijima and Seneta (1990) used results from Karlin and McGregor (1957) to study the q.s.d. which are limits of the conditional distribution probability given that it is not absorbed, starting with Dirac measures. Good proved that these limits do not depend on the particular Dirac measure and that it is null or a probability measure. Kijima and Seneta showed that the eigenvalue associated with this relevant q.s.d. is related to the exponential decay of the transition probabilities. Cavender (1978) studied the forward equations for the conditioned birth and death process and described the set of positive stationary solutions of those equations, i.e. the q.s.d. He proved that this set is a one-parameter family and is well ordered with respect to a stronger order than the usual stochastic one. It follows from his results that the mass of any q.s.d. is either $0, 1$ or ∞.

The problem of convergence to the q.s.d. when the initial distribution is different from the Dirac distribution was initially considered by Seneta and Vere-Jones (1966) when the transition probability matrix of the Markov chain is R-positive.

Our study deals mainly with the study of conditions on stochastic matrices to get non-trivial q.s.d. and normalized q.s.d. for the discrete birth and death process. Our main tools are derived from the analysis of the distribution of the absorbing time $T_0 = \inf\{n : X(n) = 0\}$ and the dynamical form that we give to the equations verified by a q.s.d.

In Section 2 we study n-q.s.d. for general Markov chains. We show that if the initial measure is an n-q.s.d. μ, then the random variable T_0 is geometrically distributed. This implies that a necessary condition for the existence of an n-q.s.d. is that the Markov chain starting from any state is geometrically absorbed at 0.

In Section 3 we begin our study of n-q.s.d. for the birth and death process. We prove that a probability measure μ is an n-q.s.d. if and only if T_0 is geometrically distributed when we start from μ, and that any n-q.s.d. has an exponential tail. Moreover, we show that if there exist non-trivial q.s.d. then all of them are normalized or all of them are of infinite mass.

In Sections 4, 5 we deal mainly with the discrete birth and death process verifying $p_x = 1 - q_x$ where

$$p_x = \mathbb{P}(X(n+1) = x+1 \mid X(n) = x), \qquad q_x = \mathbb{P}(X(n+1) = x-1 \mid X(n) = x).$$

In Proposition 4.2 we relate the non-null sequences μ satisfying $\mu P^* = \gamma(\mu)\mu$ and the sequences $(W_{\gamma,1}(x) : x \in \mathbb{N}^*)$ satisfying the dynamical equation:

$$W_{\gamma,1}(x+1) = 1 - \frac{(1-q_x)q_{x+1}}{\gamma^2 W_{\gamma,1}(x)} \quad \text{for} \quad x \in \mathbb{N}^*,$$

with initial condition $W_{\gamma,1}(1) = 1$. The variable $\gamma = \gamma(\mu)$ parametrizes the set of q.s.d.

We also show in Section 4 that the sequence of moment-generating functions of the first time of absorption $F_\gamma(x) = \mathbb{E}(\gamma^{-T_0})$ with $\gamma \in (0, 1)$ satisfies a similar dynamical equation.

In Section 5 the analysis of the dynamical equations is carried out. In particular the problem of the existence of non-trivial q.s.d. is reduced to the positivity of the approximants of a continued fraction.

In Section 6 we present our main results. By using the dynamical equations verified by q.s.d. and the moment-generating functions, we show in Theorem 6.1 that there exist n-q.s.d. if and only if the chain is geometrically absorbed. This allows us to obtain in Corollary 6.3 that the condition $\liminf_{x\to\infty} q_x > \frac{1}{2}$ implies the existence of n-q.s.d.

In Theorem 6.4 we show that the condition

$$\limsup_{x\to\infty} (1 - q_{x-1})q_x < \tfrac{1}{4}$$

implies that there exist non-trivial q.s.d. Our proof uses a result of Scott and Wall (1940), Wall (1967) on continued fractions. The method was used earlier by Callaert and Keilson (1973) to study a similar continued fraction that appears in the study of continuous-time birth and death processes with natural boundaries. We remark that there exist q.s.d. which are not normalized, in fact our analysis made after Proposition 5.4 shows that the random walk with $0 < q < \frac{1}{2}$ possesses non-trivial q.s.d. which are not normalized.

2. General definitions and preliminary results

Let $X(n)$ be a homogeneous Markov chain on the denumerable or finite state space $S = \{0\} \cup S^*$. We assume its transition probabilities $p(y, x) = \mathbb{P}(X(x+1) = x \mid X(x) = y)$ satisfy the following hypotheses:

(2.1) $$p(0, 0) = 1, \text{ i.e. 0 is an absorbing state,}$$

(2.2) $$P^* = (p(x, y) : y, x \in S^*) \text{ is an irreducible matrix,}$$

(2.3) $$\forall x \in S \text{ the set } \{y \in S : p(y, x) > 0\} \text{ is finite and non-empty.}$$

Let ν be an initial distribution concentrated in S^*. We denote by

$$\nu^{(n)}(x) = \frac{\mathbb{P}_\nu(X(n) = x)}{\mathbb{P}_\nu(X(n) \neq 0)}$$

the conditional probability of being at time n in site $x \in S^*$ given that the process has not been absorbed at 0, starting from ν. We remark that assumption (2.2) implies that $\mathbb{P}_\nu(X(n) \neq 0) \neq 0$ for any $n \in S^*$.

Definition 2.1. If $\lim_{n\to\infty} \nu^{(n)}(x) = \mu(x)$ exists for any $x \in S^*$ and $\sum_{x\in S^*} \mu(x) = 1$, then the probability measure μ is called the Yaglom limit of ν.

Let us remark that from the forward equations

$$\mathbb{P}_\nu(X(n)=x)=\sum_{y\in S^*}\mathbb{P}_\nu(X(n-1)=y)p(y,x),$$

$$\mathbb{P}_\nu(X(n)\neq 0)=\mathbb{P}_\nu(X(n-1)\neq 0)-\sum_{y\in S^*}\mathbb{P}_\nu(X(n-1)=y)p(y,0)$$

we easily deduce the following relation:

$$\nu^{(n)}(x)=\sum_{y\in S^*}\nu^{(n-1)}(y)(p(y,x)+p(y,0)\nu^{(n)}(x)). \tag{2.4}$$

From condition (2.3) the sum in the right-hand side of (2.4) is over a finite number of terms, therefore the limit $n\to\infty$ and the sum can be interchanged. Hence, if μ is the Yaglom limit of some probability measure it must satisfy the relations

$$\forall x\in S^*:\mu(x)=\sum_{y\in S^*}\mu(y)(p(y,x)+p(y,0)\mu(x)), \tag{2.5}$$

$$\forall x\in S^*:\mu(x)\geqq 0, \tag{2.6}$$

$$\sum_{x\in S^*}\mu(x)=1. \tag{2.7}$$

Irreducibility condition (2.2) implies that any Yaglom limit μ verifies $\mu(x)>0$ for any $x\in S^*$. In fact if $\mu(x)=0$, Equation (2.5) implies $\mu(y)=0$ for any $y\in S^*$ such that there exists a sequence $x_0=y, x_1, \cdots, x_n, x_{n+1}=x$ in S^* satisfying $p(x_i, x_{i+1})>0$ for every $i=0,\cdots,n$.

Recall that if μ verifies Equations (2.5), (2.6) and (2.7) above then $\mu^{(n)}=\mu$ for all $n\geqq 1$. Hence the class of Yaglom limits coincides with the set of μ's verifying (2.5), (2.6) and (2.7). An element of this set is called a normalized quasi-stationary distribution (n-q.s.d.); the set of all of them is denoted by $\mathcal{Q}_{\mathcal{N}}$. We also adopt the terminology of Cavender (1978), who uses the term quasi-stationary distribution (q.s.d.) for those μ's satisfying Equations (2.5) and (2.6). The trivial measure $\mu\equiv 0$ is the trivial q.s.d.; any other q.s.d. μ satisfies $\mu(x)>0$ for any $x\in S^*$. The set of all q.s.d. is denoted by $\mathcal{Q}$.

The first time of absorption is denoted by $T_0=\inf\{n:X(n)=0\}$, and for any probability measure ν on S^* we set

$$\gamma(\nu)=1-\sum_{x\in S^*}\nu(x)p(x,0)=\mathbb{P}_\nu(T_0>1)=\mathbb{P}_\nu(X(1)\neq 0) \tag{2.8}$$

for the probability of not being absorbed at 0 in one step. Obviously $\gamma(\nu)\in(0,1)$. With this notation we can write Equation (2.5) as

$$\mu P^*=\gamma(\mu)\mu. \tag{2.9}$$

Let us prove that a necessary condition for the existence of n-q.s.d. is that the chain is geometrically absorbed at 0. This last means that $\exists\beta<1$ such that $\forall x\in S^*$,

$(1-p_{x,0}^{(n)}) \leqq \beta^n$. This property is equivalent to

$$\exists \lambda > 1 \quad \text{such that} \quad \forall x \in S^* : \mathbb{E}_x(\lambda^{T_0}) < \infty. \tag{2.10}$$

We have the following result.

Lemma 2.2. If μ is an n-q.s.d. then T_0 is geometrically distributed with parameter $\gamma(\mu)$ when the initial distribution is μ. In particular

$$\forall \lambda < \frac{1}{\gamma(\mu)} : \mathbb{E}_\mu(\lambda^{T_0}) = \frac{\lambda(1-\gamma(\mu))}{1-\lambda\gamma(\mu)}. \tag{2.11}$$

Proof. Let μ be an n-q.s.d. Call $f_\mu(n) = \mathbb{P}_\mu(X(n) \neq 0)$. We have

$$f_\mu(n+m) = \sum_{x \in S^*} \mathbb{P}_\mu(X(n)=x)\mathbb{P}_x(X(m)\neq 0) = f_\mu(n) \sum_{x \in S^*} \mu^{(n)}(x)\mathbb{P}_x(X(m)\neq 0).$$

Since $\mu^{(n)} = \mu$ we deduce that $\sum_{x \in S^*} \mu^{(n)}(x)\mathbb{P}_x(X(m)\neq 0) = f_\mu(m)$. Hence $f_\mu(n+m) = f_\mu(n)f_\mu(m)$, so $f_\mu(n) = (f_\mu(1))^n$. Since $f_\mu(1) = \gamma(\mu)$, we conclude that for any n-q.s.d. μ, $\forall n \in \mathbb{N}$, $\mathbb{P}_\mu(X(n)\neq 0) = \gamma(\mu)^n$.

Since $\mathbb{P}_\mu(X(n)\neq 0) = \mathbb{P}_\mu(T_0 > n)$, the last result means T_0 is geometrically distributed with parameter $\gamma(\mu)$.

Hence we get the following.

Corollary 2.3. A necessary condition for the existence of n-q.s.d. is that the chain is geometrically absorbed at 0, i.e. for some $\lambda > 1$ we have $\forall x \in S^* : \mathbb{E}_x(\lambda^{T_0}) < \infty$.

Proof. Since any n-q.s.d. μ is strictly positive we have that

$$\forall \lambda < \frac{1}{\gamma(\mu)}, \ \forall x \in S^* : \mathbb{E}_x(\lambda^{T_0}) \leqq \frac{1}{\mu(x)} \mathbb{E}_\mu(\lambda^{T_0}) < \infty. \tag{2.12}$$

Remark. It can be proved from the last result that the domain of attraction of any n-q.s.d. μ with $\gamma(\mu)$ non-minimal is not reduced to a singleton. In fact, if μ, μ' are n-q.s.d. with $\gamma(\mu) > \gamma(\mu')$ and ν is a probability measure on S^* satisfying one of the following conditions:

$$\exists C > 0 \quad \text{such that} \quad \forall x \in S^* : |\nu(x) - \mu(x)| \leqq C\mu'(x),$$

$$\exists 0 < \eta < 1 \quad \text{such that} \quad \forall x \in S^* : \nu(x) = \eta\mu(x) + (1-\eta)\mu'(x),$$

then $\nu^{(n)} \xrightarrow[n\to\infty]{} \mu$ geometrically fast (see Ferrari et al. (1991)).

3. Birth and death chains

We consider a discrete-time birth and death chain $X(n)$ with absorbing state 0. The Markov chain takes values on $\mathbb{N} = \{0\} \cup \mathbb{N}^*$ and its transition probabilities are given by

$$(3.1)\qquad p(0,0)=1,$$

$$(3.2)\qquad \begin{gathered}\forall y\in\mathbb{N}^*: p(y,y-1)=q_y,\qquad p(y,y+1)=p_y,\qquad p(y,y)=1-(p_y+q_y),\\ \text{with}\quad q_y,p_y>0\quad\text{and}\quad p_y+q_y\leqq 1.\end{gathered}$$

This transition matrix $(p(y,x))$ satisfies conditions (2.1), (2.2) and (2.3).

For the birth and death process, starting with an initial measure μ concentrated on $\mathbb{N}^*$, the probability of not being absorbed in one step is $\gamma(\mu)=1-\mu(1)q_1$. In this case the system of equations (2.5) takes the form

$$(3.3)\qquad \forall y\in\mathbb{N}^*: (p_y+q_y)\mu(y)=q_{y+1}\mu(y+1)+p_{y-1}\mu(y-1)+q_1\mu(1)\mu(y).$$

Based upon Lemma 2.2 we can establish the following characterization of n-q.s.d.

Proposition 3.1. A probability measure μ is an n-q.s.d. for the birth and death chain if and only if the absorbing time T_0 has a geometric distribution when the initial measure is μ.

Proof. From Lemma 2.2 the condition is necessary; let us show that it is also sufficient. From the proof of Lemma 2.2 we get that any probability measure μ verifies

$$\mathbb{P}_\mu(X(n+m)\neq 0)=\mathbb{P}_\mu(X(n)\neq 0)\sum_{x\in\mathbb{N}^*}\mu^{(n)}(x)\mathbb{P}_x(X(m)\neq 0).$$

The hypothesis implies $\mathbb{P}_\mu(X(n)\neq 0)=(1-\mu(1)q_1)^n$. So

$$\forall n\in\mathbb{N}^*: (1-\mu(1)q_1)=\sum_{x\in\mathbb{N}^*}\mu^{(n)}(x)\mathbb{P}_x(X(1)\neq 0).$$

This is

$$1-\mu(1)q_1=\mu^{(n)}(1)(1-q_1)+\sum_{x>1}\mu^{(n)}(x)=\mu^{(n)}(1)(1-q_1)+(1-\mu^{(n)}(1)).$$

Then it follows that $\forall n\in\mathbb{N}^*$, $\mu^{(n)}(1)=\mu(1)$.

Let $y\geqq 2$. Assume we have shown that $\forall n\in\mathbb{N}^*$, $\mu^{(n)}(x)=\mu(x)$ holds for any $x<y$. For $n\in\mathbb{N}^*$ write (with the convention $p_0=0$):

$$\begin{aligned}\mu^{(n)}(y-1)=q_y\mu^{(n-1)}(y)+(1-p_{y-1}-q_{y-1})\mu^{(n-1)}(y-1)+p_{y-2}\mu^{(n-1)}(y-2)\\ +q_1\mu^{(n-1)}(1)\mu^{(n)}(y-1).\end{aligned}$$

Then by the induction hypothesis we deduce

$$\mu(y-1)=q_y\mu^{(n-1)}(y)+(1-p_{y-1}-q_{y-1})\mu(y-1)+p_{y-2}\mu(y-2)+q_1\mu(1)\mu(y-1).$$

This implies that $\forall n\in\mathbb{N}^*$, $\mu^{(n)}(y)=\mu(y)$.

From Lemma 2.2 we get that n-q.s.d. have at least exponential decay.

Proposition 3.2. If μ is an n-q.s.d., then $\forall x\in\mathbb{N}^*$, $\mu(x)\leqq\gamma(\mu)^{(x-1)}$. In particular, the moment-generating function of μ, $\sum_{n\in\mathbb{N}^*}\lambda^n\mu(n)$, is finite if $\lambda<1/\gamma(\mu)$.

Proof. We have $\mathbb{P}_x(X(n) \neq 0) \leqq (1/\mu(x))\mathbb{P}_\mu(X(n) \neq 0) = (1/\mu(x))\gamma(\mu)^n$. Since $\mathbb{P}_x(X(x-1) \neq 0) = 1$ we get the result.

Now, if μ is a non-trivial q.s.d. we have $\mu(1) > 0$ and we deduce from (3.3) that

$$\sum_{y=1}^{x} \mu(y) = 1 - \frac{1}{\mu(1)q_1}(q_{x+1}\mu(x+1) - p_x\mu(x)). \tag{3.4}$$

Therefore if μ is a non-trivial q.s.d. such that $\mu(x) \xrightarrow[x\to\infty]{} 0$, then μ is a probability measure, that is, an n-q.s.d. So, as mentioned by Cavender (1978), any q.s.d. μ satisfies either $\sum_{y\in\mathbb{N}^*} \mu(y) = 0$, or 1, or ∞.

We have the following necessary conditions.

Proposition 3.3. If there exist n-q.s.d. then $\exists\lambda > 1$ such that $\mathbb{E}_1(\lambda^{T_0}) < \infty$; and if there exists q.s.d. of infinite mass then $\mathbb{E}_1(T_0) = \infty$.

Proof. The first part is Corollary 2.3. Now assume there exists a q.s.d. μ of infinite mass. Let x_0 be such that $\sum_{y=1}^{x_0} \mu(y) > 1$. From equality (3.4) we deduce that for any $x \geqq x_0$ we have $q_{x+1}\mu(x+1) - p_x\mu(x) < 0$, so

$$\mu(x+1) \leqq \frac{p_x}{q_{x+1}}\mu(x) \leqq \left(\prod_{y=x_0}^{x} \frac{p_y}{q_{y+1}}\right)\mu(x_0).$$

Then

$$\sum_{x\geqq x_0} \mu(x+1) \leqq \sum_{x\geqq x_0}\left(\prod_{y=x_0}^{x} \frac{p_y}{q_{y+1}}\right)\mu(x_0).$$

Since $\sum_{x\geqq x_0} \mu(x+1) = \infty$ we deduce that

$$\sum_{x\geqq 1}\left(\prod_{y=1}^{x} \frac{p_y}{q_{y+1}}\right) = \infty.$$

Now if $\mathbb{P}_1(T_0 < \infty) < 1$, there exists a positive mass of T_0 at ∞, so, $\mathbb{E}_1(T_0) = \infty$. On the other hand if $\mathbb{P}_1(T_0 < \infty) = 1$ the necessary and sufficient condition in order that $\mathbb{E}_1(T_0) = \infty$ is

$$\sum_{x\geqq 1}\left(\prod_{y=1}^{x} \frac{p_y}{q_{y+1}}\right) = \infty$$

(Karlin and Taylor (1975), Theorem 7.1). Hence the result.

Corollary 3.4. If a birth and death chain has non-trivial q.s.d. then all of them are probability measures or all of them are of infinite mass.

4. Birth and death chains: dynamical systems associated to q.s.d. and to moment generating functions

The birth and death chain we consider is assumed to satisfy conditions (3.1) and (3.2).

Lemma 4.1. A sequence $\mu = (\mu(x) : x \in \mathbb{N}^*)$ of non-null terms satisfies the system of equations (3.3) if and only if for some $\gamma \neq p_1 + q_1$ it is equal to the sequence $\mu_\gamma = (\mu_\gamma(x) : x \in \mathbb{N}^*)$ constructed as follows:

$$\mu_\gamma(1) = \frac{1}{q_1}(p_1 + q_1 - Z_\gamma(1)) \tag{4.1}$$

$$\forall x \geqq 2 : \mu_\gamma(x) = \mu_\gamma(1) \prod_{y=1}^{x-1} \left(\frac{Z_\gamma(y)}{q_{y+1}} \right) \tag{4.2}$$

where the sequence $(Z_\gamma(y) : y \in \mathbb{N}^*)$ is constituted of non-null terms that satisfy the conditions

$$Z_\gamma(1) = \gamma, \tag{4.3}$$

$$\forall y \geqq 2 : Z_\gamma(y) = f_{\gamma,y}(Z_\gamma(y-1)), \tag{4.4}$$

where

$$f_{\gamma,y}(z) = \gamma + p_y + q_y - p_1 - q_1 - \frac{p_{y-1} q_y}{z}. \tag{4.5}$$

Proof. Assume μ is a sequence of non-null terms and satisfies (3.3). Call $\gamma = p_1 + q_1 - q_1 \mu(1)$. Now any $\mu(x)$ is a function of γ and the vector $(p_1, \cdots, p_{x-1}, q_1, \cdots, q_x)$. We set

$$Z_\gamma(y) = \frac{\mu(y+1) q_{y+1}}{\mu(y)}. \tag{4.6}$$

The equation (3.3) for $y = 1$ is equivalent to $Z_\gamma(1) = \gamma$, and for $y \geqq 2$ it can be written

$$\mu(1) q_1 \frac{\mu(y)}{\mu(y-1)} q_y = \left(\frac{\mu(y)}{\mu(y-1)} q_y - p_{y-1} \right) q_y - \left(\frac{\mu(y+1)}{\mu(y)} q_{y+1} - p_y \right) \frac{\mu(y)}{\mu(y-1)} q_y,$$

or equivalently $\mu(1) q_1 Z_\gamma(y-1) = (Z_\gamma(y-1) - p_{y-1}) q_y - (Z_\gamma(y) - p_y) Z_\gamma(y-1)$. Hence $Z_\gamma(y) = f_{\gamma,y}(Z_\gamma(y-1))$, with $f_{\gamma,y}(z)$ satisfying (4.5). From (4.6) we deduce property (4.2).

Reciprocally assume $(Z_\gamma(x) : x \in \mathbb{N}^*)$ is a sequence of non-null terms satisfying conditions (4.3), (4.4) and (4.5) and $\mu = \mu_\gamma$ is defined by (4.1) and (4.2). Then any term of μ is non-null, and we shall prove that it satisfies the system of equations (3.3): from (4.1) and (4.2) for $x = 2$ we get the equality:

$$Z_\gamma(1) = \frac{\mu(2)}{\mu(1)} q_2 = -\mu(1) q_1 + p_1 + q_1,$$

which is equivalent to Equation (3.3) for $y = 1$. Analogously (4.4) and (4.5) imply that Equation (3.3) is also satisfied for $y \geqq 2$.

According to Lemma 4.1, in order to prove the existence of a non-trivial q.s.d. it suffices to exhibit strictly positive solutions for the dynamical system given by (4.4) and (4.5) with initial condition (4.3). Let us assume for simplicity that

$$\forall y \in \mathbb{N}^* : p_y + q_y = 1. \tag{4.7}$$

Then the dynamics (4.4) and (4.5) can be written as

$$\forall y \geqq 2 : Z_\gamma(y) = f_{\gamma,y}(Z_\gamma(y-1)) \quad \text{with} \quad f_{\gamma,y}(z) = \gamma - \frac{(1-q_{y-1})q_y}{z}$$

with initial condition $Z_\gamma(1) = \gamma$, for $\gamma \neq 1$. Remark that parameter γ defined in (4.1), (4.3) corresponds to $\gamma(\mu)$ defined in (2.8).

From the condition

$$\mu(2) = \mu(1) \frac{Z_\gamma(1)}{q_2} \neq 0$$

we deduce that $\gamma \neq 0$, 1. Now let us make the following change of variables:

$$\forall y \in \mathbb{N}^* : W_{\gamma,1}(y) = \frac{Z_\gamma(y)}{\gamma} \quad \text{assuming} \quad \gamma \neq 0. \tag{4.8}$$

Then we can write our last result in the following form.

Proposition 4.2. Assume $\forall y \in \mathbb{N}^* : p_y + q_y = 1$. Then a non-null vector $\mu = (\mu(x) : x \in \mathbb{N}^*)$ satisfies the system of equations (3.3) if and only if for some $\gamma \neq 0, 1$ it is equal to the vector $\mu_\gamma = (\mu_\gamma(x) : x \in \mathbb{N}^*)$ satisfying:

$$\mu_\gamma(1) = \frac{1}{q_1}(1-\gamma) \tag{4.9}$$

$$\forall x \geqq 2 : \mu_\gamma(x) = \mu_\gamma(1)\gamma^{x-1} \prod_{y=1}^{x-1} \frac{W_{\gamma,1}(y)}{q_{y+1}} \tag{4.10}$$

where $(W_{\gamma,1}(y) : y \in \mathbb{N}^*)$ is a non-null sequence satisfying

$$W_{\gamma,1}(1) = 1 \tag{4.11}$$

$$\forall y \geqq 2 : W_{\gamma,1}(y) = g_{\gamma,y}(W_{\gamma,1}(y-1)) \tag{4.12}$$

with

$$g_{\gamma,y}(w) = 1 - \frac{(1-q_{y-1})q_y}{\gamma^2 w}. \tag{4.13}$$

Proof. It is direct from (4.8) and Lemma 4.1.

When $\gamma = 1$ we set $\mu_1 \equiv 0$. This follows from (4.9). But notice that the evolution given by (4.11), (4.12), (4.13) is non-trivial for $\gamma = 1$.

We are interested to find conditions for existence of non-trivial q.s.d. Since we must necessarily have $\mu_\gamma(1) > 0$, $\mu_\gamma(2) > 0$, from Equations (4.9) and (4.10), this last one evaluated at $x = 2$, we deduce that $\gamma < 1$ and $\mu_\gamma(1)\gamma W_{\gamma,1}(1) = \mu_\gamma(1)\gamma > 0$. Therefore a necessary condition is

$$0 < \gamma < 1. \tag{4.14}$$

Hence there exists a non-trivial q.s.d. if and only if for some $0 < \gamma < 1$ we have that the sequence $(W_{\gamma,1}(y) : y \in \mathbb{N}^*)$ determined by (4.11), (4.12) and (4.13) is strictly positive.

We shall study the class of such sequences and find conditions on vectors $q = (q_x : x \in \mathbb{N}^*)$ implying that they are strictly positive.

Definition 4.3. Let $\gamma \in (0, 1]$, $r \in (0, 1]$. We denote by $W_{\gamma,r} = (W_{\gamma,r}(x) : x \in \mathbb{N}^*)$ the vector given by the following equalities (recall $1/\infty = 0$, $1/0 = \infty$):

$$W_{\gamma,r}(1) = r, \tag{4.15}$$

$$\forall x \geqq 2 : W_{\gamma,r}(x) = g_{\gamma,x}(W_{\gamma,r}(x-1)) \tag{4.16}$$

with $g_{\gamma,x}(w)$ the transformation introduced in (4.13).

Now we shall prove that the sequence of moment-generating functions of the first time of absorption, starting from $x \in \mathbb{N}^*$, belongs to the class of sequences introduced in the last definition.

Recall that the birth and death chain is geometrically absorbed at 0 if

$$\exists \lambda > 1 \quad \text{such that} \quad \forall x \in \mathbb{N}^* : \mathbb{E}_x(\lambda^{T_0}) < \infty. \tag{4.17}$$

Remark that the last condition holds if for some $\lambda > 1 : \mathbb{E}_1(\lambda^{T_0}) < \infty$.

We denote by $\tilde{\lambda}_1$ the supremum of the λ satisfying (4.17). We set $\gamma = 1/\lambda$ and $\tilde{\gamma} = 1/\tilde{\lambda}_1$. Write $F_\gamma(x) = \mathbb{E}_x(\gamma^{-T_0})$. Therefore, for any $\gamma > \tilde{\gamma}$, $x \in \mathbb{N}$, we get $F_\gamma(x) < \infty$.

We have $F_1(x) = 1$ for any $n \in \mathbb{N}$, and $F_\gamma(0) = 1$ for any $\gamma > \tilde{\gamma}$. Also $F_\gamma(x) = \sum_{n=x}^{\infty} \gamma^{-n} \mathbb{P}_x(T_0 = n)$.

Consider the forward equations:

$$\begin{aligned}\forall x \in \mathbb{N}^* : \mathbb{P}_x(T_0 = n) = (1 - p_x - q_x)\mathbb{P}_x(T_0 = n-1) + p_x \mathbb{P}_{x+1}(T_0 = n-1) \\ + q_x \mathbb{P}_{x-1}(T_0 = n-1).\end{aligned}$$

Multiplying the above equality by $\gamma^{-n} = (1/\gamma)\gamma^{-(n-1)}$ and summing over all $n \geqq x$ we obtain

$$F_\gamma(x) = \frac{1}{\gamma}((1 - p_x - q_x)F_\gamma(x) + p_x F_\gamma(x+1) + q_x F_\gamma(x-1)). \tag{4.18}$$

These equations imply that if $F_\gamma(1) < \infty$ then $F_\gamma(x) < \infty$ for any $x \in \mathbb{N}^*$.

Now assume condition (4.7) holds: $\forall x \in \mathbb{N}^*$, $p_x = 1 - q_x$. Then

$$\forall x \in \mathbb{N}^* : \gamma F_\gamma(x) = (1 - q_x)F_\gamma(x+1) + q_x F_\gamma(x-1). \tag{4.19}$$

Define

$$\forall \gamma \in (\tilde{\gamma}, 1],\ x \in \mathbb{N}^*: \mathcal{U}_\gamma(x) = \frac{1}{\gamma}\frac{F_\gamma(x+1)}{F_\gamma(x)}(1-q_x). \tag{4.20}$$

Hence Equation (4.19) can be written as

$$\forall \gamma \in (\tilde{\gamma}, 1],\ x \geqq 2: \mathcal{U}_\gamma(x) = 1 - \frac{(1-q_{x-1})q_x}{\gamma^2 \mathcal{U}_\gamma(x-1)} \tag{4.21}$$

i.e. $\mathcal{U}_\gamma(x) = g_{\gamma,x}(\mathcal{U}_\gamma(x-1))$, where $g_{\gamma,x}(x)$ is the same function that we have introduced in (4.13). We have

$$\forall \gamma \in (\tilde{\gamma}, 1]: \mathcal{U}_\gamma(1) = 1 - \frac{q_1}{\gamma F_\gamma(1)}. \tag{4.22}$$

Since $F_\gamma(1) > (1/\gamma)q_1$, the initial condition satisfies

$$\forall \gamma \in (\tilde{\gamma}, 1]: r_\gamma = \mathcal{U}_\gamma(1) \in (0, 1). \tag{4.23}$$

Hence with the notation of (4.15), (4.16) and (4.13) in Definition 4.3, we have

$$\forall \gamma \in (\tilde{\gamma}, 1]: \mathcal{U}_\gamma(x) = W_{\gamma, r_\gamma}(x). \tag{4.24}$$

Proposition 4.4. Assume $\tilde{\gamma} < 1$, where $\tilde{\gamma}$ is the infimum of the values $\gamma < 1$ such that $F_\gamma(1) = \mathbb{E}_1(\gamma^{-T_0}) < \infty$. Then the sequence of moment-generating functions $F_\gamma(x) = \mathbb{E}_x(\gamma^{-T_0}) < \infty$ satisfies

$$\forall \gamma \in (\tilde{\gamma}, 1],\ x \geqq 2: F_\gamma(x) = F_\gamma(1)\gamma^{x-1} \prod_{y=1}^{x-1} \frac{W_{\gamma, r_\gamma}(y)}{(1-q_y)}. \tag{4.25}$$

Proof. From the analysis made and equality (4.20).

Proposition 4.5. If $\mu_\gamma(x)$ and $F_\gamma(x)$ exist for any $x \in \mathbb{N}^*$ then

$$\forall x \in \mathbb{N}^*: \mu_\gamma(x) F_\gamma(x) = \frac{(1-\gamma)}{\gamma}(W_{\gamma,1}(x) - W_{\gamma, r_\gamma}(x))^{-1}.$$

Proof. For $x \geqq 2$ we have

$$\begin{aligned} W_{\gamma,1}(x) - W_{\gamma,r_\gamma}(x) &= g_{\gamma,x}(W_{\gamma,1}(x-1)) - g_{\gamma,x}(W_{\gamma,r_\gamma}(x-1)) \\ &= \frac{(1-q_{x-1})q_x}{\gamma^2}\left(\frac{W_{\gamma,1}(x-1) - W_{\gamma,r_\gamma}(x-1)}{W_{\gamma,1}(x-1)W_{\gamma,r_\gamma}(x-1)}\right). \end{aligned}$$

By induction we deduce

$$W_{\gamma,1}(x) - W_{\gamma,r_\gamma}(x) = \frac{1}{(\gamma^2)^{x-1}}\left(\prod_{y=1}^{x-1} \frac{(1-q_y)q_{y+1}}{W_{\gamma,1}(y)W_{\gamma,r_\gamma}(y)}\right)(W_{\gamma,1}(1) - W_{\gamma,r_\gamma}(1)).$$

From expressions (4.10) and (4.25) we find:

$$W_{\gamma,1}(x) - W_{\gamma,r_\gamma}(x) = \frac{\mu_\gamma(1)F_\gamma(1)}{\mu_\gamma(x)F_\gamma(x)}(W_{\gamma,1}(1) - W_{\gamma,r_\gamma}(1)).$$

Now

$$\mu_\gamma(1)F_\gamma(1)(W_{\gamma,1}(1)-W_{\gamma,r_\gamma}(1))=\frac{1}{q_1}(1-\gamma)F_\gamma(1)\Big(1-\Big(1-\frac{q_1}{\gamma F_\gamma(1)}\Big)\Big)=\frac{(1-\gamma)}{\gamma}.$$

The result follows.

5. Analysis of the dynamical equations

We are interested in obtaining conditions, in terms of γ, r, q, which allow us to assert that the sequence $W_{\gamma,r}$ introduced in Definition 4.3 is a strictly positive real sequence. To simplify notation we do not make explicit the dependence of $W_{\gamma,r}$ and $g_{\gamma,x}$ on the vector q.

Remark that for all applications concerning q.s.d. we must take $\gamma\in(0, 1)$ (see (4.14)). But the analysis of $\gamma=1$ will help us to understand the problem of existence of positive solutions for $\gamma\in(0, 1)$.

The function $g_{\gamma,x}:(0, \infty]\to(-\infty, 1]$ introduced in (4.13) is onto, one-to-one and strictly increasing. So its inverse $g_{\gamma,x}^{-1}:(-\infty, 1]\to(0, \infty]$, which is given by

$$g_{\gamma,x}^{-1}(w)=\frac{(1-q_{x-1})q_x}{\gamma^2(1-w)} \tag{5.1}$$

satisfies the same properties.

Consider the following quantities

$$\forall y\geqq x\geqq 2: h_\gamma(x, y)=g_{\gamma,x}^{-1}\circ\cdots\circ g_{\gamma,y-1}^{-1}\circ g_{\gamma,y}^{-1}(0). \tag{5.2}$$

Lemma 5.1. Let $\gamma\in(0, 1]$, $r\in(0, 1]$. Then $W_{\gamma,r}(y)>0$ for any $y\in\mathbb{N}^*$ if and only if the following conditions are satisfied:

$$\forall y\geqq x\geqq 2: h_\gamma(x, y)<1, \tag{5.3}$$

$$\forall y\geqq 2: h_\gamma(2, y)<r. \tag{5.4}$$

Proof. Let us assume $W_{\gamma,r}(y)>0$ for any $y\in\mathbb{N}^*$. From the definition of $g_{\gamma,x}$ in (4.13) we must necessarily have:

$$\text{for}\quad y\geqq 2: 0<W_{\gamma,r}(y)<1. \tag{5.5}$$

Since $g_{\gamma,y}^{-1}$ is increasing in $(-\infty, 1)$ we deduce $g_{\gamma,y}^{-1}(0)<g_{\gamma,y}^{-1}(W_{\gamma,r}(y))=W_{\gamma,r}(y-1)$ for $y\geqq 3$. Therefore, by induction we get

$$h_\gamma(x, y)=g_{\gamma,x}^{-1}\circ\cdots\circ g_{\gamma,y}^{-1}(0)<W_{\gamma,r}(x-1)$$

for $y\geqq x\geqq 3$. For $x=2$ we find

$$h_\gamma(2, y)=g_{\gamma,2}^{-1}\circ\cdots\circ g_{\gamma,y}^{-1}(0)<W_{\gamma,r}(1)=r$$

for $y\geqq 2$.

Now let us assume that the conditions (5.3) and (5.4) are satisfied. For any $y = x \geqq 2$ we have $h_\gamma(y, y) > 0$. Assume that for some pair $y > x \geqq 2$ we have $h_\gamma(x, y) = g_{\gamma,x}^{-1}(h_\gamma(x+1, y)) < 0$. Then by definition of $g_{\gamma,x}^{-1}$ on (5.1) we would necessarily have $h_\gamma(x+1, y) > 1$ which contradicts (5.3). Thus (5.3), (5.4) imply that

$$(5.6) \qquad 0 < h_\gamma(x, y) \quad \text{for any} \quad y \geqq x \geqq 2.$$

Therefore $0 < g_{\gamma,2}^{-1} \circ \cdots \circ g_{\gamma,y}^{-1}(0) < r$. Since $g_{\gamma,2}$ is increasing on $(0, \infty)$ we get $0 < h_\gamma(3, y) = g_{\gamma,3}^{-1} \circ \cdots \circ g_{\gamma,y}^{-1}(0) < g_{\gamma,2}(r) = W_{\gamma,r}(2)$. Applying $g_{\gamma,3}$ to the above inequality implies $0 < h_\gamma(4, y) < g_{\gamma,3}(W_{\gamma,r}(2)) = W_{\gamma,r}(3)$. Therefore by induction we have $0 < h_\gamma(x+1, y) < W_{\gamma,r}(x)$ for any $y \geqq x+1 > 2$.

In particular the last lemma shows that if $\gamma \in (0, 1]$, $r \in (0, 1]$ and $W_{\gamma,r}$ is a strictly positive vector, then $W_{\gamma,s}$ is also strictly positive for any $s \in [r, 1]$.

Lemma 5.2. Let $r \in (0, 1]$. Then the set

$$(5.7) \qquad \Gamma_r = \{\gamma \in (0, 1] : W_{\gamma,r}(y) > 0 \text{ for any } y \in \mathbb{N}^*\}$$

is either empty or a closed interval $[\gamma_r, 1]$ for some $\gamma_r \in (0, 1]$.

Proof. Let us suppose that there exists $\bar\gamma \in \Gamma_r$. Take $\gamma \in [\bar\gamma, 1]$. Since $g_{\gamma,x}^{-1}(w)$ is decreasing in the set $\gamma > 0$, for some fixed $x \geqq 2$ and $w < 1$, we find that $h_\gamma(y, y) = g_{\gamma,y}^{-1}(0) \leqq g_{\bar\gamma,y}^{-1}(0) = h_{\bar\gamma}(y, y)$ for any $y \geqq 2$. Since $h_\gamma(y, y) < 1$ the inequality is preserved when we apply $g_{\gamma,y-1}^{-1}$ to it. So, in general, we get $h_\gamma(x, y) \leqq h_{\bar\gamma}(x, y)$ for any $y \geqq x \geqq 2$. From Lemma 5.1 we deduce that $\gamma \in \Gamma_r$, so Γ_r is an interval $[\gamma_r, 1]$ where $\gamma_r = \inf \Gamma_r$. We cannot have $\gamma_r = 0$ because for γ positive small enough we would find $h_\gamma(2, 2) > 1$, so $\gamma_r > 0$.

Now let us prove that the interval Γ_r is closed. Remark that

$$(5.8) \qquad \text{for all} \quad y \geqq x > 2 \quad \text{we have} \quad H(x, y) = \sup_{\gamma \in \Gamma_r} h_\gamma(x, y) < 1.$$

In fact if for some $y \geqq x > 2$ we would have $H(x, y) = 1$, then for any $\varepsilon > 0$ there would exist $\gamma \in \Gamma_r$ such that $h_\gamma(x, y) > 1 - \varepsilon$. Since $g_{\gamma,x-1}^{-1}(w)$ increases to ∞ when w increases to 1, for some $\gamma \in \Gamma_r$ we would have $h_\gamma(x-1, y) = g_{\gamma,x-1}^{-1}(h_\gamma(x, y))$, contradicting the fact that $\gamma \in \Gamma_r$.

Then, by continuity of $g_{\gamma,x}^{-1}$ on $\gamma > 0$ and since $h_\gamma(x, y) < H(x, y) < 1$ for any $\gamma > \gamma_r$ we deduce $h_{\gamma_r}(x, y) \leqq H(x, y) < 1$ for any $y \geqq x > 2$. Now by continuity we deduce also that $h_{\gamma_r}(2, y) \leqq r$ for any $y \geqq 2$. Assume for some $y \geqq 2$ that we have $h_{\gamma_r}(2, y) = r$. We have $0 < g_{\gamma_r,y+1}^{-1}(0) < 1$ so if $y \geqq 3$ we get $g_{\gamma_r,y}^{-1}(0) < g_{\gamma_r,y}^{-1}(0) \circ g_{\gamma_r,y+1}^{-1}(0)$. By induction $h_{\gamma_r}(3, y) < h_{\gamma_r}(3, y+1) < 1$. Then $h_{\gamma_r}(2, y) < h_{\gamma_r}(2, y+1)$ so $h_{\gamma_r}(2, y+1) > r$, which is a contradiction. So $\gamma_r \in \Gamma_r$.

Lemma 5.3. For fixed $r \in (0, 1]$, $x \in \mathbb{N}^*$, the function $W_{\gamma,r}(x)$ is increasing in $\gamma \in \Gamma_r$.

Proof. We prove by induction that

$$\forall x \in \mathbb{N}^* \quad \text{we have:} \quad \gamma \leqq \gamma' \quad \text{in } \Gamma_r \text{ implies} \quad W_{\gamma,r}(x) \leqq W_{\gamma',r}(x).$$

First we have $W_{\gamma,r}(1)=W_{\gamma',r}(1)$. Now assume $W_{\gamma,r}(x')\leqq W_{\gamma',r}(x')$ is satisfied for any $x'<x$. Since $\gamma,\gamma'\in\Gamma_r$, these quantities are >0. Let us show that the inequality holds for $x'=x$. Using the definition of $g_{\gamma,x}$ it is easy to check that

$$g_{\gamma,x}(w)\leqq g_{\gamma',x}(w)\leqq g_{\gamma',x}(w') \quad \text{for any} \quad 0<\gamma\leqq\gamma'\leqq 1, \qquad w'\geqq w\geqq 0.$$

Then

$$W_{\gamma,r}(x)=g_{\gamma,x}(W_{\gamma,r}(x-1))\leqq g_{\gamma',x}(W_{\gamma,r}(x-1))\leqq g_{\gamma',x}(W_{\gamma,r}(x-1))=W_{\gamma',r}(x).$$

Hence the result follows.

Now let us show that $h_\gamma(x,y)$ is decreasing for $\gamma\in\Gamma_r$. We have that $g_{\gamma,x}^{-1}(w)$ is increasing for $w<1$, when $x\geqq 2$ and $\gamma\in\Gamma_r$ are fixed, and decreasing for $\gamma\in\Gamma_r$, when $x\geqq 2$ and $w<1$ are fixed. So $h_\gamma(y,y)=g_{\gamma,y}^{-1}(0)$ is decreasing for $\gamma\in\Gamma_r$. By induction, if $h_{\gamma'}(x,y)\leqq h_\gamma(x,y)$ for $\gamma<\gamma'\in\Gamma_r$, then $h_{\gamma'}(x-1,y)=g_{\gamma',x-1}^{-1}(h_{\gamma'}(x,y))\leqq g_{\gamma',x-1}^{-1}(h_\gamma(x,y))\leqq g_{\gamma,x-1}^{-1}(h_\gamma(x,y))=h_\gamma(x-1,y)$.

Assume $\gamma\in\Gamma_r$. Since $0<g_{\gamma,y+1}^{-1}<1$ we deduce that $h_\gamma(x,y)\leqq h_\gamma(x,y+1)$, so $h_\gamma(x,y)$ increases in y. Denote

$$h_\gamma(x,\infty)=\lim_{y\to\infty}h_\gamma(x,y). \tag{5.9}$$

Now let us write $h_\gamma(x,y)$ as an approximant of a continued fraction:

$$h_\gamma(x,y)=\cfrac{a_\gamma(x)}{1-\cfrac{a_\gamma(x+1)}{1-\cfrac{a_\gamma(x+2)}{\cfrac{\cdots}{1-a_\gamma(y)}}}} \tag{5.10}$$

where $a_\gamma(x)=p_{x-1}q_x/\gamma^2$.

Now by induction on $y-x$ it can be shown that (also see Wall (1967), Theorem 11.1, pp. 45–48):

$$h_1(x,y)=(1-q_{x-1})\left(1-\left(1+\sum_{l=x}^{y}\prod_{m=x}^{l}\frac{q_m}{p_m}\right)^{-1}\right). \tag{5.11}$$

Hence Γ_1 is non-empty:

$$\forall y\geqq x\geqq 2: h_1(x,y)<1, \quad \text{i.e.} \quad 1\in\Gamma_1 \tag{5.12}$$

and

$$h_1(x,\infty)=(1-q_{x-1})(1-(1+L(x))^{-1}) \quad \text{where} \quad L(x)=\sum_{y\geqq x}\prod_{m=x}^{y}\frac{q_m}{p_m}. \tag{5.13}$$

Recall that a birth and death chain has certain absorption if and only if $L(1)=\infty$ (Karlin and Taylor (1975)), Theorem 7.1, pp. 148–149) or equivalently $L(x)=\infty$ for any $x\geqq 2$. We get the following result.

Proposition 5.4. For any $r \in (0, 1]$, a necessary condition in order that there exists $\gamma \in (0, 1)$ such that $W_{\gamma,r}(y) > 0$ for any $y \in \mathbb{N}^*$ is $\sup_{x\in\mathbb{N}^*} (1 - q_x)(1 - (1 + L(x+1))^{-1}) < 1$. In particular, if the birth and death chain has certain absorption this necessary condition is $\inf q_x > 0$, $x \in \mathbb{N}^*$.

Proof. We know that $h_\gamma(x, y)$ decreases with $\gamma \in \Gamma_r$. Therefore for any $\gamma \in \Gamma_r$ we get

$$\begin{aligned} p_x(1-(1+L(x+1))^{-1})\gamma^{-2} = h_1(x-1, \infty)\gamma^{-2} &= \frac{p_{x-1}q_x}{\gamma^2(1-h_1(x+2,\infty))} \\ &\leqq \frac{p_{x-1}q_x}{\gamma^2(1-h_\gamma(x+1,\infty))} \\ &= h_\gamma(x+2,\infty) \leqq 1. \end{aligned}$$

Therefore $\sup_{x\in\mathbb{N}^*}(1 - q_x)(1 - (1 + L(x + 1))^{-1}) = 1$ would imply that for any $\gamma \in (0, 1)$ there exists x such that $p_x(1 - (1 + L(x + 1))^{-1})\gamma^{-2} > 1$. So $\Gamma_r \cap (0, 1) = \varnothing$.

Now, it is easily shown that the non-symmetric random walks, i.e. $q_x = \bar{q} \in (0, 1)$ constant for any $x \in \mathbb{N}^*$ such that $\bar{q} \neq \frac{1}{2}$, with 0 an absorbing state, have non-trivial q.s.d.s (Cavender (1978)). If $1 > \bar{q} > \frac{1}{2}$ they are normalized, if $0 < \bar{q} < \frac{1}{2}$ they are of infinite mass. In the symmetric case, $\bar{q} = \frac{1}{2}$, it is easily deduced from (Wall (1967)), Theorem 8.2, that there do not exist non-trivial q.s.d.

Denote by $\bar{g}_{\gamma,x}(w) = 1 - ((1 - \bar{q})\bar{q}/\gamma^2 w)$ the transformation (4.13) for the non-symmetric random walk, by $\bar{g}_{\gamma,x}^{-1}$ its inverse function and by $\bar{h}_\gamma(x, y) = \bar{g}_{\gamma,x}^{-1} \circ \cdots \circ \bar{g}_{\gamma,y}^{-1}(0)$. By direct computation it is easily shown (also see Wall (1967), Theorem 8.2, p. 39) that

(5.14) if $\bar{q} \neq \frac{1}{2}$ then for any $\gamma \in [2\sqrt{\bar{q}(1-\bar{q})}, 1]$ we have $\forall y \geqq x \geqq 2 : h_\gamma(x, y) < 1$.

Proposition 5.5. If $\limsup_{x\in\mathbb{N}^*} (1 - q_x)q_{x+1} < \frac{1}{4}$ then there exists $\gamma \in (0, 1)$ such that $W_{\gamma,1}(y) > 0$ for any $y \in \mathbb{N}^*$.

Proof. Let $k \geqq 2$ be such that $M_k = \sup_{y\geqq k} (1 - q_{y-1})q_y < \frac{1}{4}$. Take $\bar{q} \in (\frac{1}{2}, 1)$ satisfying $M_k < \bar{q}(1 - \bar{q})$ and consider the random walk of parameter $\bar{q}$ with 0 an absorbing state. Take $\gamma \in D_k = [2\sqrt{\bar{q}(1-\bar{q})}, 1]$.

For any $y \geqq k$ we have

$$g_{\gamma,y}^{-1}(w) = \frac{p_{y-1}q_y}{\gamma^2(1-w)} < \frac{(1-\bar{q})\bar{q}}{\gamma^2(1-w)} = \bar{g}_{\gamma,y}^{-1}(w) \quad \text{for any} \quad w < 1.$$

So $h_\gamma(y, y) < \bar{h}_\gamma(y, y) < 1$. Now take $y - 1 \geqq k$. Since $g_{\gamma,y-1}^{-1}(w)$ is increasing in $w<1$ we deduce $h_\gamma(y - 1, y) < g_{\gamma,y-1}^{-1}(\bar{h}_\gamma(y, y)) < \bar{g}_{\gamma,y-1}^{-1}(\bar{h}_\gamma(y, y))$. Then $h_\gamma(y - 1, y) < \bar{h}_\gamma(y - 1, y) < 1$. By induction we obtain

(5.15) for any $\gamma \in D_k$, $\quad \forall y \geqq x \geqq k : h_\gamma(x, y) < \bar{h}_\gamma(x, y) < 1$.

From Lemma 5.1 and (5.15) to end the proof it is enough to prove that

$$\text{(5.16)}\qquad \begin{array}{l}\text{for some}\quad \gamma \in (2\sqrt{\bar{q}(1-\bar{q})},\, 1) \quad \text{and for}\quad x = 2, \cdots, k-1 \\ \text{we have}\quad \forall y \geqq x, \qquad h_\gamma(x, y) < 1.\end{array}$$

From (5.15) we deduce that for any $\gamma \in D_k$ and $x \geqq k$, the continued fraction $h_\gamma(x, \infty)$ exists and satisfies $h_\gamma(x, \infty) \leqq 1$. Moreover we have

$$\sup_{\gamma \in D_k} \left(\sup_{y \geqq k} \frac{p_{y-1} q_y}{\gamma^2} \right) < \tfrac{1}{4}.$$

Hence, Worpityky's theorem (see Wall (1967), Theorem 10.1, pp. 42–43, also Theorem 18.1, pp. 78–79) implies that $h_\gamma(k, \infty)$ is continuous when γ varies in D_k. In particular $h_\gamma(k, \infty)$ is continuous at $\gamma = 1$.

From (5.14) we have $h_1(k, \infty) = (1 - q_{k-1})(1 - (1 + L(k))^{-1}) \leqq (1 - q_{k-1})$. Then for any $\varepsilon_k > 0$ small enough we can find $\gamma_k \in (2\sqrt{\bar{q}(1-\bar{q})}, 1)$ such that for any $\gamma \in [\gamma_k, 1]$ we have $h_\gamma(k, \infty) \leqq (1 - q_{k-1}) + \varepsilon_k$. Remark that $h_\gamma(k, y) \leqq h_\gamma(k, \infty) \leqq (1 - q_{k-1}) + \varepsilon_k$ for any $y \geqq k$.

Now, we shall prove (5.16) by induction in a decreasing way. Let $x = k - 1$. Evidently we have $h_\gamma(k-1, k-1) < 1$. Take $\gamma \in [\gamma_k, 1]$. For $y \geqq k$ we have $h_\gamma(k-1, y) = g_{\gamma,k-1}^{-1}(h_\gamma(k, y))$, and $h_\gamma(k, y) \leqq (1 - q_{k-1}) + \varepsilon_k < 1$. So

$$h_\gamma(k-1, y) \leqq g_{\gamma,k-1}^{-1}((1 - q_{k-1}) + \varepsilon_k) = \frac{(1 - q_{k-2}) q_{k-1}}{\gamma^2(1 - ((1 - q_{k-1}) + \varepsilon_k))} \leqq \frac{(1 - q_{k-2}) q_{k-1}}{\gamma_k^2 (q_{k-1} - \varepsilon_k)}.$$

Take $0 < \varepsilon_{k-1} < q_{k-2}$. Let γ_k be sufficiently near to 1 so that ε_k is small enough in order that

$$\frac{(1 - q_{k-2}) q_{k-1}}{\gamma_k^2 (q_{k-1} - \varepsilon_k)} < (1 - q_{k-2}) + \varepsilon_{k-1} < 1.$$

Hence (5.16) is shown for $x = k - 1$. Since $\varepsilon_{k-1} \to 0$ when $\gamma_k \to 1$, $\varepsilon_k \to 0$, we can apply the same argument to prove by induction that (5.16) holds, and we get the result.

6. Main results on the existence of q.s.d.

In this section we obtain the main results concerning existence of q.s.d.s and n-q.s.d. for the birth and death chain. This process is defined by a vector $q = (q_x : x \in \mathbb{N}^*)$ with $q_x \in (0, 1)$. We assume equality $p_x = 1 - q_x$ $\forall x \in \mathbb{N}^*$ in (3.2) and condition (3.1).

All the notation introduced in previous sections will be used. Using Lemma 5.2 and the discussion after it we get that the set of q.s.d.s is

$$\mathcal{Q} = \{\mu_\gamma : \gamma \in (0, 1] \text{ such that } W_{\gamma,1}(y) > 0 \text{ for any } y \in \mathbb{N}^*\}$$

where μ_γ is given by (4.9) and (4.10). The set Γ_1 is of the form $[\gamma_1, 1]$ for some $\gamma_1 \in (0, 1]$. If $\gamma_1 = 1$ the unique q.s.d.s is the trivial one $\mu_1 \equiv 0$, and $\gamma_1 < 1$ is equivalent to the existence of a non-trivial q.s.d. In this way we have shown that the set of q.s.d. is parametrized by $\gamma \in [\gamma_1, 1)$ with $\gamma = 1 - \mu(1)q_1$, a result of Cavender (1978). Moreover from Corollary 3.4 when there exist non-trivial q.s.d. all of them are normalized or all of them are of infinite mass.

The first result concerning existence of n-q.s.d. is the following.

Theorem 6.1. In the birth and death chain there exists a normalized q.s.d. if and only if the chain is geometrically absorbed at 0.

Proof. From Corollary 2.3 we get that the condition of geometric absorption is necessary. Let us show it is also sufficient. Assume it holds; this means that

$$\exists \lambda > 1 \quad \text{such that} \quad \forall x \in \mathbb{N}^* : \mathbb{E}_x(\lambda^{T_0}) < \infty$$

or with the notation of Section 4:

$$\exists 0 < \gamma < 1 \quad \text{such that} \quad \forall x \in \mathbb{N}^* : F_\gamma(x) = \mathbb{E}_x(\gamma^{-T_0}) < \infty.$$

Now $F_\gamma(1) = r_\gamma \in (0, 1)$. From (4.25) we deduce $W_{\gamma, r_\gamma}(y) > 0$ for any $y \in \mathbb{N}^*$. Then, Lemma 5.1 gives

$$\forall y \geqq x \geqq 2, \qquad h_\gamma(x, y) < 1, \qquad \forall y \geqq 2, \qquad h_\gamma(2, y) < r_\gamma < 1.$$

So, $\gamma \in \Gamma_1$. Hence there exists a non-trivial q.s.d. On the other hand, Proposition 3.3 allows us to deduce that all the non-trivial q.s.d. are normalized.

We obtain the following estimation for $\gamma_1 = \inf \Gamma_1$.

Corollary 6.2. If $\mathbb{E}_1(T_0) < \infty$ then $\gamma_1 = \tilde{\gamma}$, the infimum of the values γ satisfying $\mathbb{E}_1(\gamma^{-T_0}) < \infty$.

Proof. If $\gamma_1 = 1$, then there does not exist non-trivial q.s.d., so $\tilde{\gamma} = 1$. Suppose $\gamma_1 < 1$. Since $\mathbb{E}(T_0) < \infty$ we deduce the result from Proposition 3.3 and the last theorem.

Also from Theorem 6.1 we get the following.

Corollary 6.3. If $\liminf_{x \to \infty} q_x > \frac{1}{2}$ then there exist n-q.s.d.

Proof. The hypothesis implies the finiteness of $\mathbb{E}_1(\gamma^{-T_0})$, so the last theorem gives us the result.

An explicit condition for the existence of non-trivial q.s.d. is the following.

Theorem 6.4. If $\limsup_{x \to \infty} (1 - q_{x-1})q_x < \frac{1}{4}$ then there exists a non-trivial q.s.d.

Proof. In fact a necessary and sufficient condition for the existence of a non-trivial q.s.d. is the existence of $0 < \gamma < 1$ belonging to Γ_1. Then Proposition 5.5 implies the result.

Remark. The constant $\frac{1}{4}$ in Theorem 6.1 cannot be sharpened. In fact, the symmetric random walk, $q_x = \frac{1}{2}$, for any $x \in \mathbb{N}^*$, does not have a non-trivial q.s.d.

Also remark that the conditions $\limsup_{x\to\infty}(1-q_{x-1})q_x<\frac{1}{4}$ and $\limsup_{x\to\infty}q_x<\frac{1}{2}$ imply that there exists a q.s.d. of infinite mass.

From Proposition 5.4 we get the following necessary condition.

Proposition 6.5. If the birth and death chain has certain absorption then a necessary condition for the existence of a non-trivial q.s.d. is $\inf_{x\in\mathbb{N}^*}q_x>0$.

Proposition 6.6. If $\gamma_1\leqq\gamma<\gamma'$ then the associated q.s.d. is exponentially divergent:

$$\frac{\mu_{\gamma'}(x)}{\mu_\gamma(x)}\geqq\frac{(1-\gamma')}{(1-\gamma)}\left(\frac{\gamma'}{\gamma}\right)^{x-1}\xrightarrow[x\to\infty]{}\infty. \tag{6.1}$$

Proof. Consider the expressions (4.9), (4.10), i.e. $\mu_\gamma(1)=(1/q_1)(1-\gamma)$ and $\forall x\geqq 2$:

$$\mu_\gamma(x)=\mu_\gamma(1)\gamma^{x-1}\prod_{y=1}^{x-1}W_{\gamma,1}(y)\prod_{y=1}^{x-1}\frac{1}{q_{y+1}}.$$

So,

$$\frac{\mu_{\gamma'}(x)}{\mu_\gamma(x)}=\frac{(1-\gamma')}{(1-\gamma)}\left(\frac{\gamma'}{\gamma}\right)^{x-1}\prod_{y=1}^{x-1}\frac{W_{\gamma',1}(y)}{W_{\gamma,1}(y)}.$$

From Lemma 5.3 we deduce $W_{\gamma',1}(y)\geqq W_{\gamma,1}(y)$, then the result.

This gives further insight on Corollary 3 proved by Cavender (1978), which asserts that if $\gamma_1\leqq\gamma<\gamma'$ then $\mu_{\gamma'}(x)/\mu_\gamma(x)\to\infty$.

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