Practica 7. Procesos de Poisson (FG).
Procesos Estocásticos 2015.

1. Exercises

Exercise 1 Let X and Y be independent exponential random variables with parameters λ_1 and λ_2 respectively,

1. Compute the law of $Z = \min(X, Y)$.
2. Compute the conditional law of Z given $X = x$?

Exercise 2 Show that the sum of n independent random variables with Poisson distribution with means $\lambda_1, \ldots, \lambda_n$, respectively has Poisson law with mean $\lambda_1 + \cdots + \lambda_n$.

Exercise 3 In one urn we put N_1 balls type 1, N_2 balls type 2 and N_3 balls type 3, where N_i, $i = 1, 2, 3$ are independent Poisson random variables with expectations $\mathbb{E}N_i = \lambda_i$.
(a) Choose a ball at random from the urn. Show that given the event $\{N_1+N_2+N_3 \geq 1\}$, the probability to have chosen a ball type i is $\lambda_i/(\lambda_1 + \lambda_2 + \lambda_3)$.
(b) Show that the result of (a) is the same if we condition to the event $\{N_1 + N_2 + N_3 = n\}$ for a fixed arbitrary $n \geq 1$.
(c) Show that, given the event $\{N_1 + N_2 + N_3 = n \geq 1\}$, the law of the type of the balls in the urn is a trinomial with parameters n and $p_i = \lambda_i/(\lambda_1 + \lambda_2 + \lambda_3)$.
(d) Generalize item (c) for $k \geq 1$ different types of balls:

$$\mathbb{P}(N_i = n_i | N = n) = \frac{n!}{n_1! \ldots n_k!} \lambda_1^{n_1} \ldots \lambda_k^{n_k} (\lambda_1 + \cdots + \lambda_k)^{-n},$$

for $n_1 + \cdots + n_k = n \geq 1$.

Exercise 4 Let $\mathbf{N}(t)$ be a Poisson process with parameter λ. For $s < r < t$ compute

$$P(\mathbf{N}(s) = k, \mathbf{N}(r) - \mathbf{N}(s) = j | \mathbf{N}(t) = n).$$

Exercise 5 Show the Theorem: Let $\mathbf{M}(\cdot)$ be a bi-dimensional Poisson process with rate 1. Let $S_1, S_2, \ldots$ be the ordered times of occurrence of events in the strip $[0, \lambda]$. Let $W_1, W_2, \ldots$ be the second coordinates of those event times. Then, $(S_{i+1} - S_i)_{i\geq 1}$ are iid random variables exponentially distributed with rate λ and $(W_i)_{i\geq 1}$ are iid random variables with uniform distribution in $[0, \lambda]$. Furthermore $(S_{i+1} - S_i)_{i\geq 1}$ and $(W_i)_{i\geq 1}$ are independent.

Exercise 6 (i) Let N be a random variable with Poisson law of mean λ. Let $B = X_1 + \cdots + X_N$, where X_i is a Bernoulli random variable with parameter p independent of N. Prove that B is Poisson with parameter λp. Hint: construct a process $\mathbf{M}(\cdot)$ in $[0, 1] \times [0, \lambda]$ and mark the points accordingly to the fact that they are above or below the line $y = \lambda p$.
(ii) Prove that if T_n are the successive instants of a Poisson process with parameter λ, then the process defined by the times

$$S_n = \inf\{T_l > S_{n-1} : X_l = 1\},$$

with $S_1 > 0$, is a Poisson process with parameter λp. Hint: construct simultaneously (S_n) and (T_n).

Exercise 7 Assume that a random variable X satisfies the following property. For all $t \geq 0$

$$\mathbb{P}(X \in (t, t+h) | X > t) = \lambda h + o(h).$$

Show that X is exponentially distributed with rate λ.

Exercise 8 For a Poisson process of rate λ,
(a) Compute the law of T_2, the instant of the second event.
(b) Compute the law of T_2 given $\mathbf{N}(0, t) = 4$.

Exercise 9 Let $\mathbf{N}(0, t)$ be a Poisson process with parameter λ. Compute
(a) $\mathbb{E}\mathbf{N}(0, 2)$, $\mathbb{E}\mathbf{N}(4, 7)$,
(b) $\mathbb{P}(\mathbf{N}(4, 7) > 2 \,|\, \mathbf{N}(2, 4) \geq 1)$,
(c) $\mathbb{P}(\mathbf{N}(0, 1) = 2 \,|\, \mathbf{N}(0, 3) = 6)$,
(d) $\mathbb{P}(\mathbf{N}(0, t) = \text{odd})$,
(e) $\mathbb{E}(\mathbf{N}(0, t)\mathbf{N}(0, t+s))$.

Exercise 10 For a Poisson process $\mathbf{N}(0, t)$ compute the joint distribution of S_1, S_2, S_3, the arrival instants of the first three events.

Exercise 11 Compute the law of the age $A(t)$ and residual time $R(t)$ of a Poisson process with parameter λ, where

$$A(t) \;:=\; t - S_{\mathbf{N}(t)} \qquad R(t) \;:=\; S_{\mathbf{N}(t)+1} - t \tag{12}$$

Exercise 13 Let $\mathbf{N}(t)$ be a stationary Poisson process. Show that $\mathbf{N}(t)/t$ converges to $1/\lambda$, as $t \to \infty$.

Exercise 14 Women arrive to a bank according to a Poisson process with parameter 3 per minute and men do so according to a Poisson process with parameter 4 per minute. Compute the following probabilities:
(a) The first person to arrive is a man.
(b) 3 men arrive before the fifth woman.
(c) 3 clients arrive in the first 3 minutes.
(d) 3 men and no woman arrive in the first 3 minutes.
(e) Exactly three women arrive in the first 2 minutes, given that in the first 3 minutes 7 clients arrive.

Exercise 15 Assume that only two types of clients arrive to a supermarket counter. Those paying with credit card and those paying cash. Credit card holders arrive according to a Poisson process $\{\mathbf{N}_1(t), t \geq 0\}$ with parameter 6 per minute, while those paying cash arrive according to a Poisson process $\{\mathbf{N}_2(t), t \geq 0\}$ with rate 8 per minute.
(a) Show that the process $\mathbf{N}(t) = \{\mathbf{N}_1(t) + \mathbf{N}_2(t), t \geq 0\}$ of arrivals of clients to the supermarket is a Poisson process with rate 14.
(b) Compute the probability that the first client arriving to the supermarket pays cash.
(c) Compute the probability that the first 3 clients pay with credit card, given that in the first 10 minutes no client arrived.

Exercise 16 Vehicles arrive to a toll according to a Poisson process with parameter 3 per minute (180 per hour). The probability each vehicle to be a car depends on the time of the day and it is given by the function $p(s)$:

$$p(s) = \begin{cases} s/12 & \text{if } 0 < s < 6 \\ 1/2 & \text{if } 6 \le s < 12 \\ 1/4 & \text{if } 12 \le s < 18 \\ (1/4) - s/24 & \text{if } 18 \le s < 24 \end{cases}.$$

The probability each vehicle to be a truck is $1 - p(s)$.
(a) Compute the probability that at least a truck arrives between 5:00 and 10:00 in the morning.
(b) Given that 30 vehicles arrived between 0:00 and 1:00, compute the law of the arrival time of the first vehicle of the day.

Exercise 17 Construct a non homogeneous Poisson process with rate $\lambda(t) = e^{-t}$. Compute the mean number of events in the interval $[0, \infty]$.

Exercise 18 Show that for a non homogeneous Poisson process with rate $\lambda(t)$, the law of $\{S_1, \dots, S_n\}$ in $[0, t]$ given $\mathbf{N}(0, t) = n$ is the same as the law of n independent random variables with density

$$\frac{\lambda(r)}{\int_0^t \lambda(s)ds}; \quad r \in [0, t].$$