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Point processes

Lecture notes 2021, draft

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$\{x \in \mathbb{Z}^d : Y_x = 1,\, Y_x \text{ iid in } \{0,1\}\}$ Binomial or Bernoulli process.

Point processes

From the book *Poisson processes* by J.F.C. Kingman. Oxford Studies in Probability. Claredon Press. 1993, Reprinted 2002. [43]

We consider a space R and a subset of $\mathcal{P}(R)$ of *measurable* sets, satisfying

1) empty set is measurable.

2) complement of measurable set is measurable

3) countable union of measurable sets is measurable

This is a σ-field or σ-algebra.

We want that for a point process S and for all measurable $A \subset R$,

$$N_S(A) := \sum_{s \in S} \mathbf{1}\{s \in A\} \tag{0.1}$$

is a random variable and want that the points in R are measurable. We ask that the *diagonal* $\{(x, y) \in R \times R : x = y\}$ is measurable. This implies $\{x\}$ is measurable for all $x \in R$.

We can think N_S as a random counting measure on R.

Usually $R = \mathbb{R}^d$ and the σ-algebra as the Borel sets. The diagonal in $\mathbb{R}^d \times \mathbb{R}^d$ is closed, so measurable. Later we consider Poisson processes of straight lines and of trajectories of random walks.

A *point process* is a random countable subset S of R.

1 Poisson process

Want a point process S on R such that for $N = N_S$:

(1) for disjoint $A_1, A_2, \dots, A_n$ we have $N(A_i)$ are independent.

(2) $N(A) \sim \text{Poisson}(\mu(A))$ where $\mu : A \mapsto \mu(A)$ satisfies $0 \leq \mu(A) \leq \infty$.

If N satisfies (1) and (2) we have

$$EN(A) = \mu(A) \tag{1.1}$$

and if A_i is a partition of A with $\mu(A_i) < \infty$, then

$$\sum_i N(A_i) = N(A) \tag{1.2}$$

and

$$\sum_i EN(A_i) = EN(A), \text{ that is, } \sum_i \mu(A_i) = \mu(A) \tag{1.3}$$

so that μ is a measure on R (sigma-additive and $\mu(\emptyset) = 0$).

Let us compute the joint distribution of $(N(A_j) : j = 1, \dots, n)$: Consider the family

$$\{B = A_1^* \cap \cdots \cap A_i^* : A_i^* \in \{A, A^c\}\} \tag{1.4}$$

Sets in this family are disjoint. Denote $B_1, \ldots, B_{2^n}$ the elements of that set. Then,

$$A_j = \cup_{i \in \gamma_j} B_i \tag{1.5}$$

and

$$N(A_j) = \sum_{i \in \gamma_j} N(B_i) \tag{1.6}$$

By (2) $N(B_i)$ are Poisson($\mu(B_i)$) and since B_i are disjoint, by (1), B_i are independent. So we can write the joint distribution of A_i:

$$(N(A_1), \ldots, N(A_n)) = (\sum_{i \in \gamma_1} N(B_1), \ldots, \sum_{i \in \gamma_n} N(B_n)). \tag{1.7}$$

For instance, for $n = 2$, we want the joint distribution of (A_1, A_2):

$$B_1 = A_1 \cap A_2, \quad B_2 = A_1 \cap A_2^c, \quad B_3 = A_1^c \cap A_2, \quad B_4 = A_1^c \cap A_2^c$$

and

$$A_1 = B_1 \cup B_2, \quad A_2 = B_1 \cup B_3. \tag{1.8}$$

so that, calling $X_i := N(B_i)$ we have that X_i are independent and

$$\begin{aligned} E(N(A_1)\, N(A_2)) &= E((X_1 + X_2)(X_1 + X_3)) \\ &= EX_1^2 + E(X_1X_3) + E(X_2X_1) + E(X_2X_3) \\ &= EX_1^2 + EX_1\, EX_3 + EX_2\, EX_1 + EX_2\, EX_3 \end{aligned} \tag{1.9}$$

On the other hand, the product of the expectations gives

$$EN(A_1)\, EN(A_2) = (EX_1)^2 + EX_1\, EX_3 + EX_2\, EX_1 + EX_2\, EX_3, \tag{1.10}$$

so that

$$\text{Cov}(N(A_1)\, N(A_2)) = \text{Var}\, X_1 = \text{Var}(N(A_1 \cap A_2)) = E(N(A_1 \cap A_2)).$$

because the mean equals the variance for Poisson random variables.

Exercise, use (1.7) to compute

$$E(N(A_1)\, N(A_2)\, N(A_3)) \tag{1.11}$$

Intensity If μ is absolutely continuous with respect to Lebesgue,

$$\mu(A) = \int_A \lambda(x)\, \mathrm{d}x \tag{1.12}$$

where $\mathrm{d}x = \mathrm{d}x_1 \dots \mathrm{d}x_n$, the function $\lambda : \mathbb{R}^d \to \mathbb{R}_+$ is called the *intensity measure.*

If λ is continuous at x and $A \ni x$,

$$\mu(A) \approx \lambda(x)|A|, \qquad \text{es decir} \qquad \lim_{\varepsilon\to 0} \frac{\mu(A_\varepsilon(x))}{|A_\varepsilon(x)|} = \lambda(x) \tag{1.13}$$

where $A_\varepsilon(x)$ is a decreasing sequence of sets containing x with volume going to 0 with ε.

When $\lambda(x) \equiv \lambda \in \mathbb{R}_+$, we say that the Poisson process is *homogeneous.*

Superposition Theorem We want to show that the superposition of Poisson processes is Poisson. For that we need to show that independent Poisson processes have empty intersection.

Lemma 1.1 (Disjointness Lemma)**.** *Let S_1 and S_2 be independent Poisson processes on R, and let A be a measurable set with $\mu_1(A)$ and $\mu_2(A)$ finite. Then S_1 and S_2 are disjoint with probability 1 on A:*

$$P(S_1 \cap S_2 \cap A = \emptyset) = 1. \tag{1.14}$$

Proof. See Kingman pag 15. □

Exercise. Prove the disjointness Lemma when the processes in $\mathbb{R}^d$ have continuous intensities λ_1 and λ_2, repectively.

Take $A = [0, 1]$ and $\mu_i(dx) = dx$, $i = 1, 2$ the Lebesgue measure. Let $\mathcal{B}_n =$ partition of $[0, 1]$ with $|B| = 2^{-n}$ for all $B \in \mathcal{B}_n$. Observe that $\#\mathcal{B}_n = 2^n$.

$$\{S_1 \cap S_2 \cap A \neq \emptyset\} \subset C_n, \tag{1.15}$$

where

$$C_n := \cup_{B\in\mathcal{B}_n} \{S_1 \cap B \neq \emptyset, S_2 \cap B \neq \emptyset\}$$

for all $n \geq 1$

$$\begin{aligned} P(C_n) &= \sum_{B\in\mathcal{B}_n} P(S_1 \cap B \neq \emptyset, S_2 \cap B \neq \emptyset\} \\ &\leq \sum_{B\in\mathcal{B}_n} EN_{S_1\cap B} EN_{S_2\cap B} \\ &= \sum_{B\in\mathcal{B}_n} 2^{-2n} \ = \ 2^n 2^{-2n} = 2^{-n} \end{aligned} \tag{1.16}$$

Borel Cantelli:

$$P(C_n \text{ i.o.}) \le \sum_n P(C_n) < \infty. \tag{1.17}$$

This means that for almost all S_1, S_2 there is an $n(S_1, S_2)$ such $S_1 \cap B = \emptyset$ or $S_2 \cap B = \emptyset$ for all $B \in B_n$.

Theorem 1.2 (Superposition of Poisson processes)**.** *Let $S_1, S_2, \dots$ be independent Poisson processes with mean measures $\mu_1, \mu_2, \dots$ respectively. Then $S := \cup_n S_n$ is a Poisson process with mean measure $\mu := \sum_n \mu_n$.*

Proof. Let $N_n(A) :=$ number of points in A for the Poisson process S_n. Then, by the countably additivity theorem,

$$N(A) = \sum_n N_n(A) \tag{1.18}$$

has distribution Poisson($\sum_n \mu_n$). Here we used the disjointness Lemma to be sure that there are no superposition of points.

If $\sum_n \mu_n(A) = \infty$, then $\sum_n N_n(A) = N(A) = \infty$.

To show independence of $N(A_i)$ for disjoint A_i, it suffices to observe that de double array of variables $N_n(A_i)$ are independent. □

Theorem 1.3 (Restriction Theorem)**.** *Let S be a Poisson process on the space R with mean measure μ and $B \subset R$. Then $S_B := S \cap B$ is a Poisson Process with mean measure $\mu_B(A) = \mu(A \cap B)$.*

Proof. Let N and N_B be the counting measures of S and S_B, respectively. We need to show that $N_B(A)$ is Poisson for all A, and the independence property. This follows from the definitions. □

Def: a measure ν has an atom at y if $\nu(\{y\}) > 0$. In particular, μ^* has an atom in $y \in T$ if $f^{-1}(y)$ is a set of R with positive μ measure.

Theorem 1.4 (Mapping)**.** *Let S be a Poisson process on the space R with mean measure μ. Let $f : R \to T$ be a measurable function such that*

$$\mu^*(B) := \mu(f^{-1}(B)) \tag{1.19}$$

has no atoms. Then $S^* := f(S)$ is a Poisson process on T with mean measure μ^*.

Proof. This is a bit technical in general spaces. It is an exercise if $R = \mathbb{R}^d$ and μ has intensity measure $\lambda(x) > 0$ for all x. □

Example. Let S be an homogeneous Poisson process with mean measure $\mu(A) = |A|$ and $\lambda : \mathbb{R} \to \mathbb{R}_+$ a positive intensity $\lambda(x) > 0$ for all x. Define $M : \mathbb{R} \to \mathbb{R}$ by $M(0) = 0$ and $M(x) = \int_0^x \lambda(y)dy$. Then

$$S^* = M(S) \quad \text{is a Poisson process with intensity } \lambda. \tag{1.20}$$

If $\lambda(x)$ is 0 for some subset of $\mathbb{R}$ with positive Lebesgue measure, we are in trouble.

Projections Let μ be a mean measure on $\mathbb{R}^d$ with intensity $\lambda(x)$ and $\hat{S}$ be a homogeneous Poisson process of intensity 1 on $\mathbb{R}^d \times \mathbb{R}_+$. Define $f(x_1, \ldots, x_n) = x_1$

Then $f(S)$ is a Poisson process on $\mathbb{R}^1$ with intensity

$\lambda^*(x) := \int \lambda(x, \ldots, x_n) dx_2 \ldots dx_n$.

Polar coordinates Let S be a PP with constant intensity λ on $\mathbb{R}^2$. Let

$$f(x, y) = ((x^2 + y^2)^{1/2}, \tan^{-1}(y/x)), \tag{1.21}$$

and

$$\mu^*(B) = \int \int_{f^{-1}(B)} \lambda \, \mathrm{d}x \, \mathrm{d}u = \int \int_B \lambda \, r \, \mathrm{d}r \, \mathrm{d}\theta. \tag{1.22}$$

Then (r, θ) form a Poisson process in the strip $\{(r, \theta) : r > 0, \, 0 \le \theta < 2\pi\}$ with rate function $\lambda^*(r, \theta) = \lambda r$. The r-projection gives a Poisson process S^1 in $\mathbb{R}$ with intensity

$$\lambda^1(r) = \int_0^{2\pi} \lambda^*(r, \theta) \mathrm{d}\theta = 2\pi\lambda r. \tag{1.23}$$

1.1 The Bernoulli process

Let S be a Poisson process with measure μ on R, $\mu(R) < \infty$. Condition S to have n points. What is the distribution of those points? Let $A_1, \ldots, A_n$ be disjoint subsets of R, then

$$P(N_S(A_1) = n_1, \ldots, N_S(A_k) = n_k \,|\, N_S(R) = n)$$

$$= \frac{\prod_{j=1}^k e^{-\mu(A_j)} (\mu(A_j))^{n_j} (n_j!)^{-1}}{e^{-\mu(R)} (\mu(R))^n (n!)^{-1}}$$

$$= \frac{n!}{n_1! \dots n_k!} \Big(\frac{\mu(A_0)}{\mu(R)}\Big)^{n_0} \cdots \Big(\frac{\mu(A_k)}{\mu(R)}\Big)^{n_k}$$

where $n_0 = n - (n_1 + \dots + n_k)$ and $A_0 = R \setminus (A_1 \cup \dots \cup A_k)$. That is, the point numbers in A_i's have multinomial distribution with parameters $p_i = \mu(A_i)/\mu(R)$.

Definition 1.5. *A random process S on R with n points and such that for any partition $A_0, \dots, A_k$ of R and $n_k \le n$,*

$$P(N_S(A_1) = n_1, \dots, N_S(A_k) = n_k) = \frac{n!}{n_1! \dots n_k!} \Big(\tfrac{\mu(A_0)}{\mu(R)}\Big)^{n_0} \cdots \Big(\tfrac{\mu(A_k)}{\mu(R)}\Big)^{n_k},$$

where $n_0 = n - (n_1 + \dots + n_k)$, is called Bernoulli process *on A with parameters n and $p(A) = \frac{\mu(A)}{\mu(R)}$.*

Notice that p is a probability on R.

Construction of a Bernoulli process. Take $X_1, \dots, X_n$ iid random variables with values on R and distribution p:

$$P(X_i \in A) = p(A). \tag{1.24}$$

Then $S := \{X_1, \dots, X_n\}$ (as a set) is a Bernoulli process with counting measure

$$N_S(A) = \sum_{i=1}^{n} \mathbf{1}\{X_i \in A\} = \#(S \cap A) \tag{1.25}$$

Existence theorem Let μ be a non-atomic measure on R that can be decomposed as

$$\mu = \sum_n \mu_n \tag{1.26}$$

such that $\mu_n(R) < \infty$ for all n. Define N_n and $X_{n,j}$, $j \ge 1$ independent random variables with

$$N_n \sim \text{Poisson}(\mu_n(R)) \tag{1.27}$$

and $X_{n,j}$ is a random variable with distribution

$$P(X_{n,j} \in A) = p_n(A) := \frac{\mu_n(A)}{\mu_n(R)}, \qquad \text{for all } j. \tag{1.28}$$

Theorem 1.6 (Existence Theorem)**.** *The process*

$$S := \cup_n \{X_{n,1}, \dots, X_{n,N_n}\}$$

is a Poisson process on R with mean measure μ.

Proof. Denote $S_n := \{X_{n,1}, \dots, X_{n,N_n}\}$. This process consists of a Poisson random number of independent points on R, each with distribution p_n.

First we prove that the process S_n is a Poisson process.

Observe that, given $N_n = m$, the process is Bernoulli:

$$\begin{aligned} &P(N_n(A_1) = m_1, \dots, N_n(A_k) = m_k \mid N_n = m) \\ &= \frac{m!}{m_0! \dots m_k!} \left(\tfrac{\mu_n(A_0)}{\mu_n(R)}\right)^{m_0} \dots \left(\tfrac{\mu_n(A_k)}{\mu_n(R)}\right)^{m_k}. \end{aligned} \tag{1.29}$$

where $m_0 = m - (m_1 + \cdots + m_k)$ and $A_0 = R \setminus (A_1 \cup \cdots \cup A_k)$.

By the total probability lemma ($P(C) = \sum_m P(B_m)\, P(C|B_m)$, if (B_m) is a partition of C), we have

$$\begin{aligned} &P(N_n(A_1) = m_1, \dots, N_n(A_k) = m_k) \\ &= \sum_{m \geq m_1 + \cdots + m_k} \tfrac{e^{-\mu_n(R)}(\mu_n(R))^m}{m!} \tfrac{m!}{m_0! \dots m_k!} \left(\tfrac{\mu_n(A_0)}{\mu_n(R)}\right)^{m_0} \dots \left(\tfrac{\mu_n(A_k)}{\mu_n(R)}\right)^{m_k} \\ &= \sum_{m \geq m_1 + \cdots + m_k} \tfrac{e^{-\mu_n(A_0)}(\mu_n(A_0))^{m_0}}{m_0!} \tfrac{e^{-\mu_n(A_1)}(\mu_n(A_1))^{m_1}}{m_1!} \dots \tfrac{e^{-\mu_n(A_k)}(\mu_n(A_k))^{m_k}}{m_k!} \\ &= \tfrac{e^{-\mu_n(A_1)}(\mu_n(A_1))^{m_1}}{m_1!} \dots \tfrac{e^{-\mu_n(A_k)}(\mu_n(A_k))^{m_k}}{m_k!}. \end{aligned}$$

Hence, the process $S_n := \{X_{n,1}, \dots, X_{n,N_n}\}$ is a Poisson process. Since S_n are independent, the superposition Theorem implies that S is a Poisson process with mean measure μ. □

1.2 Campbell Theorem

Want to study variables of a point process S of the form

$$\Sigma := \sum_{s \in S} f(s) \tag{1.30}$$

Examples. (a) Radioactivity. Suppose each point of $S \subset \mathbb{R}$ produces an effect that decays exponentially. The cumulated effect for a site $x \in \mathbb{R}$ can be computed with Σ with the function $f_x(s) = \exp(x - s)\mathbf{1}\{s < x\}$.

(b) The gravitational field in $\mathbb{R}^3$: assuming all stars $s \in S$ have the same mass, $f_x(s) = \frac{1}{\|x-s\|}$.

Theorem 1.7 (Campbell Theorem)**.** *Let S be a Poisson process on R with mean measure μ. Let $f : R \to \mathbb{R}$ be measurable. Then, the sum*

$$\Sigma := \sum_{s \in S} f(s) \tag{1.31}$$

is absolutely convergent with probability one if and only if

$$\int_R \min(|f(x)|, 1)\mu(\mathrm{d}x) < \infty. \tag{1.32}$$

Under this condition,

$$E(e^{\theta\Sigma}) = \exp\Big\{\int_R \big(e^{\theta f(x)} - 1\big)\mu(\mathrm{d}x)\Big\}. \tag{1.33}$$

for any θ complex when the integral on the right converges or for purely imaginary θ. Moreover,

$$E\Sigma = \int_R f(x)\mu(\mathrm{d}x) \tag{1.34}$$

meaning that the expectation exists if and only if the integral converges, in which case they are equal. If (1.34) *converges, then*

$$V\Sigma = \int_R (f(x))^2 \mu(\mathrm{d}x). \tag{1.35}$$

where V is variance.

Proof. We prove first for *simple functions*, that is, a function that takes only a finite number of values and vanishes outside a set of finite μ measure. Let $A_1, \dots, A_k$ be disjoint measurable subsets of R with $m_j := \mu(A_j) < \infty$ and let $f(x) = a_j$ for $x \in A_j$, with $f(x) = 0$ if $x \notin \cup_j A_j$. We have then that $N_j := N_S(A_j)$ are independent with law Poisson(m_j) and

$$\Sigma = \sum_{x \in S} f(x) = \sum_j a_j N_j. \tag{1.36}$$

Since we know the characteristic function of the Poisson random variable, for real or complex θ we have

$$\begin{aligned} Ee^{\theta\Sigma} &= Ee^{\theta \sum_j a_j N_j} = \prod_j Ee^{\theta a_j N_j} \quad \text{(independence of PP)} \\ &= \prod_j \exp\big(e^{(\theta a_j - 1)} m_j\big) \quad \text{(characteristic of Poisson rv)} \end{aligned}$$

$$= \exp\Big(\sum_j \int_{A_j} (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)\Big)$$

$$= \exp\Big(\int_R (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)\Big) \tag{1.37}$$

If $f(x) \geq 0$, it is better to consider $\theta = -u$ for real $u \geq 0$. In this case,

$$Ee^{-u\Sigma} = \exp\Big(\int_R (e^{-uf(x)} - 1)\mu(\mathrm{d}x)\Big)$$

We know that any non-negative function f can be expressed as the limit $f = \lim_j f_j$ where f_j is an increasing family of simple functions, that is, $f_i \leq f_{i+1}$. Taking $\theta = -u$ for real $u \geq 0$, and setting $\Sigma_j = \sum_{s \in S} f_j(s)$, we have

$$\begin{aligned} E(e^{-u\Sigma}) &= \lim_j E(e^{-u\Sigma_j}) \\ &= \lim_j \exp\Big(\int_R (e^{-uf_j(x)} - 1)\mu(\mathrm{d}x)\Big)) \\ &= \exp\Big(\int_R (e^{-uf(x)} - 1)\mu(\mathrm{d}x)\Big). \end{aligned} \tag{1.38}$$

by monotone convergence theorem. Under condition (1.32) the integral above converges and goes to zero as $u \to 0$. This implies Σ is a random variable; otherwise, if $\Sigma = \infty$ with positive probability, $E(e^{-u\Sigma}) \equiv 1$.

Expanding (1.37) and denoting $\mu f := \int f(x)\mu(\mathrm{d}x)$, we get

$$\int_R (e^{\theta f(x)} - 1)\mu(\mathrm{d}x) = \theta \mu f + \frac{\theta^2}{2!}\mu f^2 + \dots$$

taking the exponential of both sides and expanding again,

$$\exp(\int_R (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)) = 1 + \theta\mu f + \frac{\theta^2}{2!}((\mu f)^2 + \mu f^2) + \dots \tag{1.39}$$

On the other hand,

$$Ee^{\theta\Sigma} = 1 + \theta E\Sigma + \frac{\theta^2}{2!}E\Sigma^2 + \dots \tag{1.40}$$

We conclude then

$$E\Sigma = \mu f, \qquad V\Sigma = \mu f^2. \tag{1.41}$$

This proves the formula for positive f. We leave as an exercise to prove for all f. □

1.3 The characteristic functional

Taking Campbell formula for $f \geq 0$ and $\theta = -1$ we get

$$Ee^{-\Sigma_f} = \exp\Big\{-\int_R (1 - e^{-f(x)})\mu(\mathrm{d}x)\Big\} \tag{1.42}$$

where $\Sigma_f := \sum_{s \in S} f(s)$.

Proposition 1.8 (Characterizing functional characterizes process)**.** *If S satisfies* (1.42) *for f non-negative and simple, then S is a Poisson process.*

Proof. Let $A_1, \dots, A_k$ disjoint with $m_i := \mu(A_i) < \infty$. Let $f(s) = \sum_i a_i \mathbf{1}\{s \in A_i\}$. Then $\Sigma_f = \sum_j a_j N(A_j)$ and

$$Ee^{-\Sigma_f} = \exp\Big\{-\sum_{j=1}^{k}(1 - e^{-a_j})m_j\Big\} \tag{1.43}$$

Hence, taking $z_j := e^{-a_j}$ we have

$$E(z_1^{N(A_1)} \dots z_k^{N(A_k)}) = \prod_j e^{m_j(z_j - 1)}. \tag{1.44}$$

but this is the product of the moment generation functions of Poisson(m_j) random variables calculated at arbitray points $z_j \in (0, 1)$. This implies that $N(A_j)$ are independent Poisson(m_j) random variables, which in turn implies S is a PP(μ). □

1.4 Avoidance functions characterize point processes

Given a point process S define the *avoidance function* $\alpha : \mathcal{F} \to [0, 1]$ by

$$\alpha(A) := P(S \cap A = \emptyset) \tag{1.45}$$

If S is a PP(μ), we have $\alpha(A) = e^{-\mu(A)}$. Avoidance function are sometimes called *void probabilities.*

For any $a \leq b \in \mathbb{R}^d$, define the *cube* $(a, b] \subset \mathbb{R}^d$ by

$$(a, b] := \{(x_1, \dots, x_d) \in \mathbb{R}^d : a_i < x_i \leq b_i, \text{ for all } i\}$$

Theorem 1.9 (Rényi Theorem)**.** *Let μ be a non-atomic measure on $\mathbb{R}^d$ with $\mu(A) < \infty$ if A is bounded. Let S be a point process on $\mathbb{R}^d$ such that for any finite union of cubes A,*

$$P(S \cap A = \emptyset) = e^{-\mu(A)} \tag{1.46}$$

Then S is a Poisson process with mean measure μ.

Two ways of using the theorem. (1) if A is a finite union of cubes and $N_S(A)$ is Poisson($\mu(A)$), then (1.46) and the theorem implies S is PP(μ). Notice that here we have not assumed independence.

(2) if $\alpha(A) > 0$ for bounded A and the sets $\{S \cap A = \emptyset\}$ and $\{S \cap B = \emptyset\}$ are independent for disjoint A, B, then

$$\alpha(A \cup B) = P(\{S \cap A = \emptyset\} \cap \{S \cap B = \emptyset\}) = \alpha(A)\,\alpha(B) \tag{1.47}$$

so that $\mu(A) := -\log \alpha(A)$ is finitely additive and non-atomic and the Theorem implies that S is a Poisson process with mean measure μ. Here we have not assumed Poisson distribution for $N(A)$.

Sketch proof. Let a k-cube be a cube $(a, b] \subset \mathbb{R}^d$ with $a = (a_1, \dots, a_d)$ and $a_i = z_i 2^{-k}$, $b_i = (z_i + 1)2^{-k}$ for integers $z_i \in \mathbb{Z}$. For each k, k-cubes are a partition of $\mathbb{R}^d$.

For k-cubes $C_1, \dots, C_n$ we have

$$\begin{aligned} P(\cap_r \{S \cap C_r = \emptyset\}) &= P(S \cap \cup_r C_r = \emptyset) \\ &= e^{-\mu(\cup_r C_r)} \quad \text{by hipothesis} \\ &= e^{-\sum_r \mu(C_r)} \quad (C_r \text{ disjoint and } \mu \text{ measure}) \\ &= \prod_r e^{-\mu(C_r)}. \end{aligned} \tag{1.48}$$

Let G be open bounded set and

$$\begin{aligned} N_k(G) &:= \#\{k\text{-cubes } C \text{ contained in } G \text{ with } S \cap C_r \neq \emptyset\} \\ &= \sum_{C \subset G} (1 - \mathbf{1}\{S \cap C = \emptyset\}) \end{aligned} \tag{1.49}$$

where the sum is over k-cubes. We have

$$N(G) = \lim_k N_k(G) \tag{1.50}$$

By (1.48), the events $\{S \cap C_r = \emptyset\}$ are independent, hence the generating function of the variable $N_k(G)$ is product of generating functions of Bernoulli:

$$E(z^{N_k(G)}) = \prod_{C \subset G} (e^{-\mu(C)} + (1 - e^{-\mu(C)})\, z)$$

in $|z| < 1$, where the product is over k-cubes. Last expression equals

$$= \prod_{C \subset G} (z + (1 - z)e^{-\mu(C)}) \approx \prod_{C \subset G} (z + (1 - z)(1 - \mu(C)))$$

$$\approx \prod_{C\subset G} e^{-(1-z)\mu(C)} = e^{-(1-z)\sum_C \mu(C)} = e^{-(1-z)\mu(G)} \tag{1.51}$$

where the $\approx$ is a bit technical, see Kingman. It is motivated by the expansion of the exponentials:

$$\begin{aligned} z + (1-z)e^{-\mu} &= z + (1-z)(1-\mu+\mu^2/2+\dots) \\ &= 1-(1-z)\mu+(1-z)\mu^2/2+\dots \\ e^{-(1-z)\mu} &= 1-(1-z)\mu+((1-z)\mu)^2/2+\dots \end{aligned} \tag{1.52}$$

Identity (1.51) shows that $N(G)$ is a Poisson($\mu(G)$) random variable.

The same argument works to show independence of $N(G_1), \dots, N(G_k)$ for mutually disjoint $G_1, \dots, G_k$. This shows that S is Poisson process with mean measure μ. □

Exercise: Prove that under the conditions of the theorem, $N(G_1), \dots, N(G_k)$ are independent for mutually disjoint $G_1, \dots, G_k$.

Coupling Assume P_i is the distribution of a random element X_i, assumming values in a space R_i. A *coupling* between X_1 and X_2 is a random vector $(\hat{X}_1, \hat{X}_2) \in R_1 \times R_2$ with distribution called $\tilde{P}$, satisfying

$$\tilde{P}(\hat{X}_1 \times R_2 \in A) = P_1(X_1 \in A), \tag{1.53}$$

$$\tilde{P}(R_1 \times \hat{X}_2 \in A) = P_2(X_2 \in A) \tag{1.54}$$

that is, the marginal variables $\hat{X}_i \sim X_i$, where $\sim$ means "with the same distribution".

Alternative proof via Coupling An alternative proof can be obtained via coupling. See Section 5 of Chapter 1 of Thorisson.

Coupling Bernoulli and Poisson. Let $U \sim$ Uniform$[0,1]$ and $\alpha > 0$, a parameter. Define

$$X := \mathbf{1}\{U \geq e^{-\alpha}\} \tag{1.55}$$

$$Y := \sum_{j\geq 0} j\mathbf{1}\{U \in I_j\}, \quad \text{where } I_j \text{ has volume } |I_j| = e^{-\alpha}\alpha^j/j! \tag{1.56}$$

and form a partition of $[0,1]$ are disjoint sets.

This is a *coupling* between those variables. We have defined a random vector (X, Y) in the space where U is defined, such that its marginals are Bernoulli and Poisson, respectively:

$X \sim$ Bernoulli$(1-e^{-\alpha})$ and $Y \sim$ Poisson(α).

We have

$$\begin{aligned} P(X=0) &= P(Y=0) = P(U \le e^{-\alpha}) \\ P(X=1) &= P(U \in I_1 \cup I_2 \cup I_3 \cup \dots) \\ P(Y=1) &= P(U \in I_1) \end{aligned} \tag{1.57}$$

Hence

$$P(X \neq Y) = P(U \in I_2 \cup I_3 \cup \dots) \le \frac{\alpha^2}{2}. \tag{1.58}$$

Proof of $N_k(G)$ converges to Poisson($\mu(G)$) via coupling.

Fix k and let $\mathcal{C}_k(G)$ be the k-cubes intersecting the open set G. Then, the variables

$$X_C := \mathbf{1}\{N_S(C) \neq 0\}, \qquad C \in \mathcal{C}_k(G) \tag{1.59}$$

are independent, by (1.46) (exercise).

For each k-cube $C \subset G$ define a coupling (X_C, Y_C), such that:

$X_C := \mathbf{1}\{N_S(C) \neq 0\}$ and $Y_C \sim$Poisson$(\mu(C))$.

$$X_G := \sum_C X_C, \quad Y_G := \sum_C Y_C \tag{1.60}$$

where the sum is over k-cubes. By the coupling,

$$P(X_G \neq Y_G) \le \sum_{C \subset G} P(X_C \neq Y_C) \le \frac{1}{2} \sum_C \mu(C)^2 \tag{1.61}$$

If $\mu(C) = \int_{C \subset G} \lambda(x) dx$ y $\lambda(x) \le M$, using $|C| = 2^{-dk}$

$$\sum_{C \subset G} \mu(C)^2 \le M^2 \sum_{C \subset G} |C|^2 = M^2 \, |G| \, 2^{-dk}, \tag{1.62}$$

where the sum is over k-cubes. This implies $P(X \neq Y, \text{ i.v.}) = 0$. □

Exercise: extend the proof to any non atomic measure μ.

1.5 Slivnyak-Mecke Theorem

Proposition 1.10. *Let S be a Poisson process in $\mathbb{R}^d$ with measure $\mu(dx) = \lambda(x)dx$, B a Borel set with $\mu(B) < \infty$, and G a measurable set of point configurations. We have*

$$P(X \cap B \in G) = \sum_{n\geq 0} \frac{\exp(-\mu(B))}{n!} \int_B \cdots \int_B \mathbf{1}\{\{x_1, \ldots, x_n\} \in G\} \\ \times \lambda(x_1) \ldots \lambda(x_n)\, dx_1 \ldots dx_n. \quad (1.63)$$

Let $h : S \mapsto h(S) \geq 0$ be a nonnegative measurable function. We have

$$Eh(X \cap B) = \sum_{n\geq 0} \frac{\exp(-\mu(B))}{n!} \int_B \cdots \int_B h(\{x_1, \ldots, x_n\}) \\ \times \lambda(x_1) \ldots \lambda(x_n)\, dx_1 \ldots dx_n. \quad (1.64)$$

Proof. Exercise. It follows from the construction of the Poisson process. □

Theorem 1.11 (Slivnyak-Mecke)**.** *Let S be a Poisson process with measure $\mu(dx) = \lambda(x)dx$ on $R = \mathbb{R}^d$. Let $h : (x; X) \mapsto h(x; X) \in \mathbb{R}$, where X is a denumerable set of R. Then*

$$E\Big(\sum_{s\in S} h(s; S \setminus \{s\})\Big) = \int_R Eh(x; S)\lambda(x)\, \mathrm{d}x.$$

Proof when $\mu(S) < \infty$.

$$E\Big(\sum_{s\in S} h(s; S \setminus \{s\})\Big)$$

$$= \sum_{n\geq 0} \frac{\exp(-\mu(B))}{n!} \sum_{i=1}^n \int_B \cdots \int_B h(x_i; \{x_1, \ldots, x_n\} \setminus \{x_i\}) \\ \times \lambda(x_1) \ldots \lambda(x_n)\, dx_1 \ldots dx_n.$$

$$= \sum_{n\geq 0} \frac{\exp(-\mu(B))}{(n-1)!} \int_B \cdots \int_B h(x_n; \{x_1, \ldots, x_{n-1}\}) \\ \times \lambda(x_1) \ldots \lambda(x_n)\, dx_1 \ldots dx_n.$$

$$= \int_R Eh(x; S)\lambda(x)\, \mathrm{d}x.$$

□

Theorem 1.12 (Extended Slivnyak-Mecke)**.** *Let S be a Poisson process with intensity $\lambda(\cdot)$ on $R = \mathbb{R}^d$. Let $h : (s_1, \ldots, s_n; S) \mapsto h(s_1, \ldots, s_n; S) \in \mathbb{R}$, where S is a denumerable set of R. Then*

$$E\Big(\sum_{s_1,\ldots,s_n\in S} h(s_1, \ldots, s_n; S \setminus \{s_1, \ldots, s_n\})\Big) \quad (1.65)$$

$$= \int_R \cdots \int_R Eh(x_1, \ldots, x_n; S)\lambda(x_1) \ldots \lambda(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

where the sum is over distinct $s_1, \ldots, s_n$*.*

Proof. We have solved it for $n = 1$. For $n \geq 2$ define

$$\tilde{h}(s, S) = \sum_{s_2, \ldots, s_n \in S} h(s, s_2, \ldots, s_n; S \setminus \{s_2, \ldots, s_n\}) \tag{1.66}$$

We have that (1.65) is

$$E \sum_{s \in S} \tilde{h}(s, S \setminus \{s\}) = \int_R \tilde{h}(s, S)\lambda(s)\mathrm{d}s$$
$$= \int_R \cdots \int_R Eh(s, s_2, \ldots, s_n; S)\lambda(s_2) \ldots \lambda(s_n)\, \mathrm{d}s_2 \ldots \mathrm{d}s_n\, \lambda(s)\mathrm{d}s. \qquad \square$$

See Møller-Waagpetersen [39], Theorem 3.3. for a proof.

Moment measures Define the n-th moment measure of a point process S by

$$\mu_n(C) = E\Big(\sum_{s_1, \ldots, s_n \in S} \mathbf{1}\{(s_1, \ldots, s_n) \in C\}\Big), \qquad C \subset R^n. \tag{1.67}$$

and the reduced measure by

$$\alpha_n(C) = E\Big(\sum_{s_1, \ldots, s_n \in S}^{\neq} \mathbf{1}\{(s_1, \ldots, s_n) \in C\}\Big), \qquad C \subset R^n. \tag{1.68}$$

The measure μ_n determines the joint moments of the count measure $N(\cdot)$: For $B_1, \ldots, B_n \subset R$ we have

$$\mu_n(B_1 \times \cdots \times B_n) = E \prod_{i=1}^n N(B_i). \tag{1.69}$$

and if the B_i are disjoint, and S is a Poisson process:

$$\alpha_n(B_1 \times \cdots \times B_n) = E \prod_{i=1}^n N(B_i) = \prod_{i=1}^n \mu(B_i). \tag{1.70}$$

1.6 One dimensional Poisson processes

Let S be a Poisson process on $\mathbb{R}$ with mean measure $\mu(A) = \lambda|A|$. In the one dimensional case, the intensity is called *rate* λ. We enumerate the points of S as follows: $S = \{X_i : i \in \mathbb{Z}\}$ with

$$\cdots < X_{-2} < X_{-1} < X_0 < 0 < X_1 < X_2 < \dots \tag{1.71}$$

X_i are random variables, depending on N. For instance, if we think X_i as the time of the i-th arrival, we have that $N(0,x]$ is the number of arrivals between time 0 and x. It is clear that the n-th arrival occurs after time x if and only if there are less than n arrivals between 0 and x:

$$\{X_n > x\} = \{N(0,x] < n\}. \tag{1.72}$$

Notice that $(X_1, X_2, \dots)$ has the same law as $(-X_0, -X_{-1}, \dots)$.

Theorem 1.13 (Interval theorem)**.** *Let $S = \{X_i : i \in \mathbb{Z}\}$ be a PP(λ). The random variables $Y_1 = X_1$, $Y_n = X_n - X_{n-1}$ are independent with exponential(λ) distribution.*

Proof. Consider the process

$$S' := \{X_2 - X_1, X_3 - X_1, X_4 - X_1, \dots\} \tag{1.73}$$

We want to show that X_1 is independent of S' and that S' is PP(λ). It suffices to show that for given f, the characteristic functionals

$$\Sigma = \sum_{n\geq 1} f(X_n), \qquad \Sigma' = \sum_{n\geq 2} f(X_n - X_1) \tag{1.74}$$

have the same distribution.

Denote $\xi_k = 2^{-k}\lceil 2^k X_1\rceil$ the least integer multiple of 2^{-k} greater than X_1. Then ξ_k is a random variable converging to X_1: $\xi_k \searrow_k X_1$. We have

$$\Sigma' = \lim_k \Sigma^k, \qquad \Sigma^k := \sum_{n\geq 2} f(X_n - \xi_k), \tag{1.75}$$

Now for any z, x,

$$P(\Sigma^k \leq z,\, X_1 \leq x) = \sum_\ell P(\Sigma^k \leq z,\, X_1 \leq x,\, \xi_k = l2^{-k}). \tag{1.76}$$

When $\xi_k = l2^{-k}$, the points X_n in (le^{-k}, ∞) form a Poisson process and are independent of $\{X_1 < l2^{-k}, \xi_k = l2^{-k}\}$. Hence

$$P(\Sigma^k \leq z,\, X_1 \leq x,\, \xi_k = l2^{-k}) = P(\Sigma \leq z)\, P(X_1 \leq x,\, \xi_k = l2^{-k}) \tag{1.77}$$

Substituting in (1.76),

$$P(\Sigma^k \leq z,\, X_1 \leq x) = P(\Sigma \leq z)\, P(X_1 \leq x) = P(\Sigma \leq z)\, P(N(0,x] \geq 1) \tag{1.78}$$

Letting $k \to \infty$, Σ^k converges almost surely to Σ' and we are done.

Applying induction, we can prove that $Y_1, \dots, Y_m$ are independent and independent of $S^m = \{X_{m+1} - X_m, X_{m+2} - X_m, \dots\}$. □

Waiting time paradox We saw that $X_{i+1} - X_i$, $i \geq 1$ are iid exponential (λ). However $X_1 - X_0$ has distribution Gamma$(2, \lambda)$. The density function of $X_1 - X_0$ is $g_2(x) = \lambda x g(x)$, where g is the density of the "typical" interval. This is a general result for stationary point processes.

1.6.1 Law of large numbers

From the interval theorem, the distribution of X_n is Gamma(n, λ); indeed it is the sum of n exponential(λ):

$$X_n = Y_1 + \cdots + Y_n. \tag{1.79}$$

Y_j iid exponential(λ).

Theorem 1.14 (law of large numbers)**.**

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\lambda}. \tag{1.80}$$

Proof. $EY_n = \frac{1}{\lambda}$ and $VY_n = \frac{1}{\lambda^2}$, hence by the lln for iid with finite mean and variance, we get (1.80). □

As a corollary we get a lln for $N(0,t]$. Since

$$N(0,t] \to_t \infty, \tag{1.81}$$

it follows that

$$\lim_t \frac{X_{N(0,t]}}{N(0,t]} = \frac{1}{\lambda}. \tag{1.82}$$

and since

$$X_{N(0,t]} \leq t < X_{N(0,t]+1} \tag{1.83}$$

we get

$$\lim_t \frac{t}{N(0,t]} = \frac{1}{\lambda}. \tag{1.84}$$

We conclude

$$\lim_t \frac{N(0,t]}{t} = \lambda. \qquad \text{a.s.} \tag{1.85}$$

We give another proof of (1.85) without using the interval theorem.

Theorem 1.15. *Let S be a PP(λ) on $(0,\infty)$. Then*

$$\lim_t \frac{N(0,t]}{t} = \lambda. \qquad a.s. \tag{1.86}$$

Proof. First proof. Using Chevichev:

$$P(|\frac{1}{t}N(0,t] - \lambda t| > \varepsilon) \leq \frac{\lambda}{\varepsilon^2 t} \tag{1.87}$$

taking $t_k = k^2$:

$$\sum_k P(|\frac{1}{k^2}N(0,k^2] - \lambda t| > \varepsilon) \leq \sum_k \frac{\lambda}{\varepsilon^2 k^2} < \infty \tag{1.88}$$

Hence, by Borel Cantelli,

$$\lim_k \frac{1}{k^2} N(0,k^2] = \lambda, \quad \text{a.s.} \tag{1.89}$$

Taking $k = k(t) = \lfloor \sqrt{t} \rfloor$,

$$N(0,k^2] \leq N(0,t] \leq N(0,(k+1)^2] \tag{1.90}$$

which dividing by $(k+1)^2 > t \geq k^2$ implies

$$\frac{N(0,k^2]}{(k+1)^2} \leq \frac{N(0,t]}{t} \leq \frac{N(0,(k+1)^2]}{k^2} \tag{1.91}$$

Since $k^2/(k+1)^2$ converges to 1, we get the result.

Second proof. It suffices to see that $N_k := N(k,k+1]$ are iid and that

$$N(0,t] = \sum_{i=0}^{\lfloor t \rfloor} N_k + N(\lfloor t \rfloor, t - \lfloor t \rfloor] \tag{1.92}$$

and use the lln for iid. □

This extends to Poisson processes in general spaces. For instance, for homogeneous Poisson processes in $\mathbb{R}^d$, let $B(0, a_k)$ is the ball of center at the origin and radious a_k, where a_k is the radious of the ball of volume k. Then we have $a_k \to \infty$ and

$$\lim_{k\to\infty} \frac{N(B(0,a_k))}{|B(0,a_k)|} = \lambda. \tag{1.93}$$

To prove it, let $Y_k := N(B(0, a_k))\backslash N(B(0, a_{k-1}))$. Then Y_k are iid Poisson(λ),

$$N(B(0,a_k) = X_1 + \cdots + Y_k. \tag{1.94}$$

and we can use the same proof.

1.6.2 Non homogeneous processes in $\mathbb{R}$

Non homogeneous processes in one dimension are monotone transformations of homogeneous processes.

Let μ be a positive locally integrable measure on $\mathbb{R}$. Let S be a PP(μ) in $\mathbb{R}$ and define $M(0) = 0$ and for $r < t$,

$$M(t) - M(r) = \mu(r, t]. \tag{1.95}$$

μ is non-atomic if and only if M is continuous. We have seen that

$$\tilde{S} := \{M(s) : s \in S\} \tag{1.96}$$

is a homogeneous PP(1) on the interval $(M(-\infty), M(\infty))$.

Notice that the inter-point distances $X_{k+1} - X_k$ are not independent. To obtain properties of S, we study $\tilde{S}$.

Example. The distances to the origin of the points of a homogeneous Poisson process. Let S_2 be a homogeneous PP(λ) in $\mathbb{R}^2$ and let $S = \{\|s\| : s \in S_2\} \subset \mathbb{R}_+$, the set of distances to the origin of the points in S_2. Then S is a Poisson process of density $\lambda(x) = 2\pi\lambda x$, so that

$$M(x) = \pi\lambda x^2. \tag{1.97}$$

and $\tilde{S} = M(S)$ is a Poisson process of rate 1 on $\mathbb{R}_+$. So that if $S = \{X_1, X_2, \dots\}$ with $X_i < X_{i+1}$, then $\tilde{S} = \{\tilde{X}_1, \tilde{X}_2, \dots\}$ with $\tilde{X}_i = \pi\lambda X_i^2$ is a PP(1).

Export law of large numbers Let $S = \{X_i : i \in Z\}$ be a non-homogeneous process with cumulated density $M(x)$. We know that $\tilde{S} = \{M(X_i) : i \in Z\}$ is a homogeneous Poisson Process in $(-M(-\infty), M(\infty))$ and that if $M(\infty) = \infty$, then

$$\lim_n \frac{\tilde{X}_n}{n} = 1; \quad \lim_t \frac{\tilde{N}(0,t]}{t} = 1 \tag{1.98}$$

Using

$$\tilde{N}(0, M(t)] = N(0,t] \tag{1.99}$$

we get

$$1 = \lim_t \frac{\tilde{N}(0, M(t)]}{M(t)} = \lim_t \frac{N(0,t]}{M(t)} \tag{1.100}$$

Law of large numbers for empirical processes Let μ be a measure and $\varepsilon > 0$ be a parameter.

Consider a family of Poisson processes S^ε with mean measure $\varepsilon^{-1}\mu$ and denote N^ε its counting measure. $N^\varepsilon(A)$ is a Poisson($\varepsilon^{-1}\mu(A)$) random variable. The empirical measure π^ε is defined by

$$\pi^\varepsilon(A) := \varepsilon N^\varepsilon(A). \tag{1.101}$$

Note that the measure π^ε is random and its expectation and variance are given by

$$E\pi^\varepsilon(A) = \varepsilon E N^\varepsilon(A) = \varepsilon\varepsilon^{-1}\mu(A) = \mu(A).$$
$$V\pi^\varepsilon(A) = \varepsilon^2 V N^\varepsilon(A) = \varepsilon^2\varepsilon^{-1}\mu(A) = \varepsilon\mu(A).$$

Theorem 1.16 (Law of large numbers for the empirical process)**.** *We have*

$$\lim_{\varepsilon\to 0} \pi^\varepsilon(A) = \mu(A). \quad a.s. \tag{1.102}$$

Proof. To prove it, take $\varepsilon = \frac{1}{n}$ to get

$$P(|\pi^{1/n}(A) - \mu(A)| > \delta) \le \frac{V\pi^{1/n}(A)}{\delta^2} = \frac{\mu(A)}{\delta^2\, n} \xrightarrow[n\to\infty]{} 0. \tag{1.103}$$

This shows convergence in probability. The argument of Theorem 1.15 can be applied to show strong convergence (1.102). □

1.7 Marked Poisson processes

Theorem 1.17 (Coloring Theorem)**.** *Let S be a PP(μ) and $(p_1, \dots, p_k)$ be a probability. To each point of $s \in S$ assign color i with probability p_i, independently of S and the colors of the other points. Let S_i be the set of points with color i. Then $S_1, \dots, S_k$ are independent Poisson processes with mean measure $\mu_i := p_i\mu$, respectively.*

Proof. Exercise using the superposition theorem. S is the superposition of independent Poisson processes $S_1, \dots, S_k$, with mean measures $p_1\mu, \dots, p_k\mu$, respectively. □

Marked processes In general. Let S be a PP(μ) in R and for some space M, let $(m_s, s \in S)$ be a family of independent random variables in M satisfying that m_s (marks) may depend on s but it is independent of $S \setminus \{s\}$. Consider the random set

$$S^* = \{(s, m_s) : s \in S\} \tag{1.104}$$

Consider a family of mark distributions $(p(x, \cdot), x \in R)$, where $p(x, A)$ for a mesurable $A \subset M$ is the probability that the mark $m_x \in A$. To construct S^*, start with S and then, for each $s \in S$ choose m_s independently of $S \setminus s$ with law $p(s, \cdot)$.

Theorem 1.18 (Marking theorem)**.** *S^* is a Poisson process on $S \times M$ with mean measure*

$$\mu^*(C) = \int\int_C \mu(\mathrm{d}x)\, p(x, \mathrm{d}m), \qquad C \subset S \times M. \tag{1.105}$$

Proof. We use the Characteristic functional. For $f : S \times M \to \mathbb{R}_+$ denote $\Sigma^* := \sum_{(s,m_s)\in S^*} f(s, m_s)$ and compute

$$\begin{aligned} E(e^{-\Sigma^*}|S) &= \prod_{s\in S} E(e^{-f(s,m_s)}|S) \\ &= \prod_{s\in S} \int_M e^{-f(s,m)}\, p(s, dm) \\ &= \prod_{s\in S} e^{-f^*(s)}, \end{aligned} \tag{1.106}$$

where $f^* := -\log \int_M e^{-f(s,m)}\, p(s, dm)$. Hence,

$$E(e^{-\Sigma^*}) = E(E(e^{-\Sigma^*}|S)) = E \prod_{s\in S} e^{-f^*(s)}$$

Use Campbell formula to get

$$
\begin{aligned}
E \prod_{s\in S} e^{-f^*(s)} &= \exp\Big(-\int_R (1-e^{-f^*})\mathrm{d}\mu\Big) \\
&= \exp\Big(-\int_R\int_M (1-e^{-f(x,m)})\mu(\mathrm{d}x)\, p(x,\mathrm{d}m)\Big) \\
&= \exp\Big(-\int_{R\times M} (1-e^{-f})\mathrm{d}\mu^*\Big). \qquad \square
\end{aligned}
$$

Corollaries 1. If $\mu(R) < \infty$, the points $\{m_s : s \in S\}$ form a Poisson process on M with mean measure μ_m given by

$$\mu_m(B) := \int_R \int_B \mu(\mathrm{d}x)\, p(x,\mathrm{d}m), \qquad B \subset M. \tag{1.107}$$

2. If m takes countable values denoted $a_1, a_2, \dots$, the point process S_i with i-marks is $\mathrm{PP}(\mu_i)$, where

$$\mu_i(A) = \int_A \mu(\mathrm{d}x)\, p(x,\{a_i\}) \tag{1.108}$$

and $S_1, S_2, \dots$ are independent. Notice that the color distribution may depend on the position of the point x.

Rain It rains over $R = \mathbb{R}^2$. Drops are in the space-time $R \times M$, where $M = \mathbb{R}_+$ corresponds to the time (= distance) a given drop will touch the pavement R. The point process S^* is a Poisson process with mean measure $\lambda\, \mathrm{d}x\, \mathrm{d}t$. Fix a time T (duration of the rain) and observe that the projected set

$$S := (s : (s,t) \in S^*,\, t \le T)$$

is a Poisson process in R with mean measure $\lambda T\, \mathrm{d}x$.

Each fallen drop s has a random radious $r_s \ge 0$. We consider then the marked Poisson process $\{(s, r_s) : s \in S\}$. Call *wet set* the random set

$$\text{Wet Set} := \cup_{s\in S} B(s, r_s), \tag{1.109}$$

where $B(x,r) \subset R$ is the ball of center x and radious r; let $b := E|B(x,r)|$. Each connected component of the Wet Set is called *cluster*. We say that there is *wetting* if the probability $\theta(\lambda, T, \rho)$ that the origin belongs to a cluster with infinite area is positive.

The classical result in *percolation* is that $\theta > 0$ if $\lambda T b$ is sufficiently large and that $\theta < 0$ if that product is sufficiently small.

Theorem 1.19 (Boolean percolation; Penrose [41], Peter Hall [26], Georgii [21], Meester-Roy [38]). *For fixed ρ, T, there are $0 < a_1 \le a_2 < \infty$ such that*

$$\begin{aligned} \theta(\lambda, T, \rho) > 0 \quad & \textit{if} \quad \lambda T b > a_2, \\ \theta(\lambda, T, \rho) = 0 \quad & \textit{if} \quad \lambda T b < a_1. \end{aligned} \tag{1.110}$$

Galaxies Galaxies are points of a Poisson process S on $R = \mathbb{R}^3$. Each galaxy $s \in S$ has a random mass $m_s > 0$ with continuous distribution with density $\rho(s, m)$. The process $S^* = \{(s, m_s)\}$ is Poisson with intensity

$$\lambda^*(x, m) = \lambda(x)\rho(x, m), \tag{1.111}$$

that is, with mean measure $\mu^*(\mathrm{d}(x, m)) = \lambda(x)\,\mathrm{d}x\,\rho(x, m)\,\mathrm{d}m$.

The gravitatory field at the origin $0 \in \mathbb{R}^3$ is the vector (F_1, F_2, F_3) given by

$$F_j = \sum_{s \in S} \frac{G m_s s_j}{(s_1^2 + s_2^2 + s_3^2)^{3/2}} \tag{1.112}$$

where G is the gravitational constant. The characteristic functional of S^* applied to $t_1 F_1 + t_2 F_2 + t_3 F_3$ gives

$$E(e^{\mathrm{i}t_1 F_1 + \mathrm{i}t_2 F_2 + \mathrm{i}t_3 F_3}) \tag{1.113}$$

$$= \exp\Big(\int_{\mathbb{R}^3} \int_0^\infty (e^{\mathrm{i}Gmt\cdot\psi(x)} - 1)\lambda(x)\rho(x, m)\,\mathrm{d}x, \mathrm{d}m\Big), \tag{1.114}$$

where

$$\psi_j(x) = (x_1^2 + x_2^2 + x_3^2)^{-3/2} x_j \tag{1.115}$$

and $t \cdot \psi$ is the scalar product. This holds if (by Campbell Theorem)

$$\int_{\mathbb{R}^3} m|\psi(x)|\lambda(x)\,\rho(x, m)\,\mathrm{d}x, \mathrm{d}m < \infty \tag{1.116}$$

If we denote the expected mass of a galaxy at x by

$$\bar{m}(x) := \int m\rho(x, m)\mathrm{d}m \tag{1.117}$$

(1.116) is equivalent to

$$\int_{\mathbb{R}^3} \frac{\bar{m}|\psi(x)|\lambda(x)}{x_1^2 + x_2^2 + x_3^2}\,\mathrm{d}x < \infty \tag{1.118}$$

Under (1.117) we would have the joint distribution of F_1, F_2, F_3.

The problem is that in an uniform universe, where $\bar{m}(x)$ and $\lambda(x)$ are constant, then (1.118) does not hold.

The ideal gas An element $(q, v) \in R := \mathbb{R}^2$ is a *particle* with position $q \in \mathbb{R}$ and velocity $v \in \mathbb{R}$. Assume S is a Poisson process with mean measure F, with intensity $f : \mathbb{R}^2 \to \mathbb{R}_+$.

The *ideal gas* dynamics let each particle $x = (q, v) \in S$ move at speed v, conserving speed, with no interaction with other particles. The operator T_t is defined by

$$T_t S := \{(q, v) \in \mathbb{R}^2 : (q - vt, v) \in S\} \tag{1.119}$$

$T_t S$ is the configuration of particles at time t,

Notice that T_t is a bijection of $\mathbb{R}^2$, with inverse T_{-t}.

By the Mapping Theorem, $T_t S$ is a Poisson process with mean measure F_t defined by

$$dF_t(A) = F(T_{-t}A). \tag{1.120}$$

Exercise. Show that if $f(q, v) = \lambda\, f_2(v)$ such that $\int |v|^2 f_2(v) dv < \infty$, then

$$S \text{ is a PP}(F) \text{ if and only if } T_t S \text{ is a PP}(F). \tag{1.121}$$

This is a process with homogeneous spatial intensity λ and with independent speeds, satisfying a second moment condition.

1.8 Cox Processes

A Cox process is a "Poisson process with a random mean measure μ".

For instance, let S_i be a Poisson process with intensity λ_i and K be a random variable in $\mathbb{N}$ with probability $P(K = i) = p_i$, independent of $(S_i)_{i \in \mathbb{N}}$

Then, the process $S = S_K$ is a Cox process. The random mean measure μ in this case has distribution

$$P(\mu(\cdot) = \int_{\cdot} \lambda_i(x) dx) = p_i. \tag{1.122}$$

Given a random variable Λ in the space of intensities, with some distribution F, a Cox process is a process that, conditioned to $\Lambda = \lambda$, is a Poisson process with intensity λ.

$$P(S \cap A = \emptyset \,|\, \Lambda = \lambda) = e^{-\int_A \lambda(x) \mathrm{d}x}. \tag{1.123}$$

so that,

$$P(S \cap A = \emptyset) = E\big[P(S \cap A = \emptyset \,|\, \Lambda)\big] \tag{1.124}$$

$$= \int e^{-\int_A \lambda(x)\mathrm{d}x} F(d\lambda). \tag{1.125}$$

In this case, μ can be written as

$$\mu(\cdot) = \int_{\cdot} \Lambda(x) dx. \tag{1.126}$$

so that

$$EN(A) = E[E(N(A) \mid \mu]] = \int \int_{\cdot} \Lambda(x) dx F(d\lambda). \tag{1.127}$$

Compute the second moment:

$$\begin{aligned} E(N(A))^2 &= E[E(N(A)^2 | \mu)] \\ &= E(\mu(A) + \mu(A)^2) \\ &= E\mu(A) + (E\mu(A))^2 + \mathrm{Var}(\mu(A)) \end{aligned} \tag{1.128}$$

so that, for a Cox process:

$$\mathrm{Var}(N(A)) \geq EN(A)$$

The identity only holds if $\mu(A)$ concentrates on a point (degenerate). Cox processes are "over dispersed". $N(A)$ has greater variance than a Poisson random variable with the same mean.

Void probabilities, characteristic functional, correlations The void probabilities of a Cox process are given by

$$\begin{aligned} \alpha(A) &= P(S \cap A = \emptyset) \\ &= E[P(S \cap A = \emptyset | \Lambda)] \\ &= E \exp\Big(- \int_A \Lambda(x) \mathrm{d}x\Big) \end{aligned} \tag{1.129}$$

this is the expectation of a function of the random density Λ.

The generating functional is given by

$$E^{-\Sigma_f} = E \exp\Big(- \int_R (1 - e^{-f(x)}) \Lambda(x) \mathrm{d}x\Big) \tag{1.130}$$

The intensity function is given by

$$\rho(x) = E\Lambda(x). \tag{1.131}$$

and the n-point correlation function for distinct $x_1, \ldots, x_n$:

$$\psi(x_1, \ldots, x_n) = E(\Lambda(x_1) \ldots \Lambda(x_n)) \tag{1.132}$$

so that, for mutually disjoint $A_1, \ldots, A_n$

$$E(N(A_1) \ldots N(A_n)) = E\Big(\int_{A_1} \cdots \int_{A_n} \Lambda(x_1) \ldots \Lambda(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n\Big). \tag{1.133}$$

Ecology The members of a population form a non-homogeneous PP($\lambda(\cdot)$) denoted S.

For each spatial position x, let $\phi(x)$ be the mean number of daughters of a plant at x and $g(x, \cdot)$, the density of the continuous distribution of the position of each daughter. Let the daughters of a plant at x be a Poisson process S_x with intensity

$$\lambda_x(y) = \phi(x) g(x, y - x).$$

Given S, let the daughter processes $(S_s)_{s \in S}$ be independent Poisson processes of intensity λ_s and define the cumulative daughter process by

$$S' = \cup_{s \in S} S_s. \tag{1.134}$$

By the superposition theorem, the set of daughters S' is a Poisson process of (random) intensity

$$\Lambda'(y) = \sum_{s \in S} \phi(s)\, g(s, y - s).$$

$$E\Lambda'(y) = E[\sum_{s \in S} \phi(s) \sum_{s \in S} g(s, y - s)] \tag{1.135}$$

$$= \int_{\mathbb{R}^2} \phi(x) g(x, y - x) \lambda(x) dx. \tag{1.136}$$

In particular, if

$$\lambda(x) \equiv \lambda,\ \phi(x) \equiv \phi,\ g(x, y - x) = g(y - x), \tag{1.137}$$

that is, the functions do not depend on x, we have

$$E\Lambda'(y) = \int_{\mathbb{R}^2} \phi\, g(y - x)\, \lambda\, dx = \lambda\phi. \tag{1.138}$$

And, if $\phi = 1$ we have the same density in each generation.

Evolution of the generations. The random intensity of the granddaughter generation under (1.137) is

$$\Lambda''(x) = \phi \sum_{s' \in S'} g(x - s') \tag{1.139}$$

where $s' \in S'$ are the positions of the daughters. Since, given Λ'' the granddaughter process S'' is Poisson, one could use Campbell to compute its distribution. We compute the first and second order moments. As before,

$$E(\Lambda'(x) \,|\, \Lambda') = \int_{\mathbb{R}^2} \phi g(x - y) \Lambda'(y) dy \tag{1.140}$$

and

$$E(\Lambda'(x)) = \phi \int_{\mathbb{R}^2} g(x - y) E\Lambda'(y) dy \tag{1.141}$$

Hence, if $E(\Lambda'(y)) = \lambda$, then $E(\Lambda''(y)) = \phi\lambda$. In particular, if $\phi = 1$ we have the same expected intensity λ in each generation.

$$E(\Lambda''(x)\Lambda''(y) \,|\, \Lambda') = \int_{\mathbb{R}^2} g(x - z)\Lambda'(z) dz \int_{\mathbb{R}^2} g(y - \tilde{z})\Lambda'(\tilde{z}) d\tilde{z} \tag{1.142}$$

$$+ \int_{\mathbb{R}^2} g(x - z) g(y - z) \Lambda'(z) dz \tag{1.143}$$

So that

$$E(\Lambda''(x)\Lambda''(y)) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x - z) g(y - \tilde{z})\, E(\Lambda'(z)\Lambda'(\tilde{z}))\, dz d\tilde{z} \tag{1.144}$$

$$+ \lambda \int_{\mathbb{R}^2} g(x - z) g(y - z) dz \tag{1.145}$$

In particular, if Λ' is second order stationary, with covariance

$$E(\Lambda'(x)\Lambda'(y)) = \lambda^2 + \sigma'(x - y). \tag{1.146}$$

for some σ' (depends only on the difference). Then Λ'' is also second order stationary with covariance

$$\lambda^2 + \sigma''(x - y) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x - z) g(y - \tilde{z})\, (\lambda^2 + \sigma'(x - y))\, dz d\tilde{z} \tag{1.147}$$

$$+ \int_{\mathbb{R}^2} g(x - z) g(y - z) dz \tag{1.148}$$

and writing

$$\gamma(x) = \int_{\mathbb{R}^2} g(x-y)g(y)dy \tag{1.149}$$

we conclude

$$\sigma''(x) = \int_{\mathbb{R}^2} \sigma(x-y)\gamma(y)dy + \lambda\gamma(x). \tag{1.150}$$

A useful recursive formula.

Exercise (optional). Let S_n the process of the n-th generation of plants when $\phi = 1$. We know that the density is $\phi = 1$ for all n. Compute the asymptotic distribution of S_n. That is, for instance,

$$\lim_{n\to\infty} P(S_n \cap \Lambda = \emptyset)$$

Neyman-Scott processes These are special cases of the ecology models of previous paragraph.

Let C be a stationary Poisson Process on $\mathbb{R}^d$ with intensity $\kappa > 0$. Conditioned on C, let X_c be independent Poisson processes on $\mathbb{R}^d$ with intensity function

$$\rho_c(x) = \alpha k(x-c)$$

where the *kernel* $k(\cdot)$ is a density function; that is, $k(x) \geq 0$ and $\int k = 1$.

Then $X = \cup_{c\in C} X_c$ is a *Neyman-Scott* process with cluster centres C and clusters X_c. More general Neyman-Scott processes do not ask C to be Poisson.

X is a Cox process with (random) intensity

$$\Lambda(x) = \sum_{c\in C} \alpha k(x-c). \tag{1.151}$$

2 Cadenas de Markov a tiempo continuo

Sea $\mathbb{X}$ un conjunto finito o numerable, llamado espacio de estados. El proceso $(X_t)_{t\in\mathbb{R}}$, $X_t \in \mathbb{X}$ es un proceso Markoviano de saltos si

$$P(X_{t+h} = y \,|\, X_t = x,\, X_s = x_s,\, 0 \leq s < t) = h\, q(x,y) + o(h). \tag{2.1}$$

$q(x,y)$ es la *tasa de salto* del estado x al estado $y \neq x$. Note que la tasa de salto en el instante t depende sólo del estado X_{t-}.

Denotamos

$$p_t(x,y) = P(X_t = y \mid X_0 = x) \tag{2.2}$$

Por ejemplo, la medida de conteo de el proceso de Poisson $S \subset \mathbb{R}$ de intensidad λ induce un proceso Markoviano de salto. Defina $N(t) := \#(S \cap [0,t])$.

$$\begin{aligned} P(N(t+h) = x+1)|N(t) = x) &= P(N(t+h] - N(t) \\ &= \lambda\, h\, e^{-\lambda h} + P(\text{otras cosas})\,. \end{aligned}$$

El evento “otras cosas” está contenido en el evento “hay 2 o más puntos del proceso de Poisson en el intervalo de tiempo $[t, t+h)$” y la probabilidad de ese evento es $o(h)$. Esto implica que $q(x, x+1) = \lambda$. El resto de las tasas es 0, $q(x,y) = 0$ si $y \neq x+1$.

Construcción usando procesos de Poisson bi-dimensionales Esta construcción está basada en [14] y [11]. Queremos construir un proceso X_t con tasas $q(x,y)$, es decir, que satisfaga (2.1). La tasa de salida del estado x se denota

$$\lambda_x := \sum_y q(x,y).$$

Asumimos $\sup_x \lambda_x < \infty$. Consideremos un proceso de Poisson bi-dimensional S de intensidad 1, con medida de conteo $M(A) := \#(S \cap A)$, A Boreliano en $\mathbb{R}^2$. Para cada estado x consideramos una partición del intervalo $I_x = [0, \lambda_x]$ en Borelianos $B(x,y)$ de medida $q(x,y)$, $y \neq x$:

$$\begin{aligned} &I_x = \dot\cup_{y \in \mathbb{X}\setminus\{x\}} B(x,y); \quad B(x,y) \cap B(x,y') = \emptyset, \text{ for } y \neq y'. \\ &|B(x,y)| = q(x,y), \quad x \neq y. \end{aligned} \tag{2.3}$$

Fijemos $Y_0 = x_0$, un estado arbitrario y $T_0 = 0$. Sea T_1 la primera coordenada del primer evento del proceso $M(\cdot)$ en la banda $[0,\infty) \times I_{x_0}$:

$$T_1 := \inf\{t > 0 : M([0,t] \times I_{Y_0}) > 0\}.$$

Defina $(T_1, U_1) \in S$, el único punto que realiza el ínfimo, con $U_1 \in I_{Y_0}$. Como $(B(Y_0, y) : y \in \mathbb{X})$ es una partición de I_{Y_0}, hay un único $Y_1 \in \mathbb{X}$ tal que $U_1 \in B(Y_0, Y_1)$.

Iterativamente, asumimos que T_{n-1}, Y_{n-1} están determinados y definimos

$$T_n := \inf\{t > T_{n-1} : M((I_{Y_{n-1}} \times T_{n-1}, t] > 0\}.$$

Defina $(T_n, U_n) \in S$, el único punto que realiza el ínfimo, con $U_n \in I_{Y_{n-1}}$. Como $(B(Y_{n-1}, y) : y \in \mathbb{X})$ es una partición de $I_{Y_{n-1}}$, hay un único $Y_n \in \mathbb{X}$ tal que $U_n \in B(Y_{n-1}, Y_n)$.

Defina

$$X_t := Y_n, \text{ si } t \in [T_n, T_{n+1}), \quad \text{para } t \in [0, \infty) \tag{2.4}$$

Así, para cada realización del proceso de Poisson bidimensional M, construimos una realización del proceso $(X_t : t \in [0, \infty))$. T_n es el n-ésimo instante de salto; Y_n es el n-ésimo estado visitado por el proceso.

El proceso $(X_t)_{t \geq 0}$ es una función, denotada ϕ, del estado inicial x_0 y del proceso de Poisson bidimensional M:

$$(X_t)_{t \geq 0} = \phi(x_0, M). \tag{2.5}$$

Proposition 2.1. *El proceso* $(X_t : t \in [0, T_\infty))$ *definido en* (2.4) *satisface* (2.1).

Proof. Por definición,

$$\mathbb{P}(X_{t+h} = y \,|\, X_t = x) \tag{2.6}$$

$$= \mathbb{P}\big\{M((t, t+h] \times B(x, y)) = 1\big\} + \mathbb{P}(\text{otras cosas}), \tag{2.7}$$

donde el evento $\{\text{otras cosas}\}$ está contenido en el evento

$$\{M((t, t+h] \times [0, \lambda_x]) \geq 2\},$$

(M contiene dos o más puntos en el rectángulo $[0, \lambda_x] \times (t, t+h]$). Por definición de $M(\cdot)$, tenemos

$$\mathbb{P}(M((t, t+h] \times [0, \lambda_x]) \geq 2) = o(h) \quad \text{y} \tag{2.8}$$

$$\mathbb{P}(M((t, t+h] \times B(x, y) = 1) = hq(x, y) + o(h). \tag{2.9}$$

Esto demuestra la proposición. □

Nacimiento puro. X_t en $\{0, 1, 2, \dots\}$ con tasas

$$q(x, x+1) = \lambda x \tag{2.10}$$

$$q(x, y) = 0, \text{ en los otros casos.} \tag{2.11}$$

La tasa de llegadas en el instante t es proporcional al número de llegadas hasta ese instante. Los intervalos son

$$B(x, x+1) = [0, \lambda x) \tag{2.12}$$

Fila M/M/1 Consideramos el proceso X_t en $\{0,1,2,\dots\}$ con tasas $\lambda,\mu\in\mathbb{R}_{\geq 0}$ dadas por

$$q(x,x+1)=\lambda, \qquad x\geq 0, \tag{2.13}$$
$$q(x+1,x)=\mu, \qquad x\geq 0. \tag{2.14}$$

La tasa de llegadas es constante igual a λ. La tasa de servicios es μ. Los intervalos son

$$B(x,x+1)=[0,\lambda) \tag{2.15}$$
$$B(x+1,x)=[0,\mu) \tag{2.16}$$

X_t cuenta el número de clientes en el sistema en el instante t.

Este proceso se puede realizar también con dos procesos de Poisson homogeneos independientes A y S, de intensidades λ y μ, respectivamente. (Ejercicio).

2.1 Kolmogorov equations

It is useful to use the following matrix notation. Let Q be the matrix with entries

$$q(x,y) \qquad \text{se } x\neq y \tag{2.17}$$
$$q(x,x)=-\lambda_x \;=\; -\sum_{y\neq x} q(x,y). \tag{2.18}$$

and P_t be the matrix with entries

$$p_t(x,y) \;=\; \mathbb{P}(X_t=y\,|\,X_0=x).$$

Con esta notación, las ecuaciones de Chapman-Kolmogorov dicen

$$P_{t+s} \;=\; P_tP_s. \tag{2.19}$$

for all $s,t\geq 0$. To see it compute

$$\begin{aligned} p_{t+s}(x,y) &= \mathbb{P}(X_{t+s}=y\,|\,X_0=x)\\ &= \sum_z \mathbb{P}(X_s=z\,|\,X_0=x)\mathbb{P}(X_{t+s}=y\,|\,X_s=z)\\ &= \sum_z p_s(x,z)p_t(z,y). \end{aligned} \tag{2.20}$$

This is the (x,y) entry of P_tP_s.

Proposition 2.2 (Kolmogorov equations)**.** *The following identities hold*

$$P_t' = QP_t \quad \textit{(Kolmogorov Backward equations)}$$

$$P_t' = P_tQ \quad \textit{(Kolmogorov Forward equations)}$$

for all $t \geq 0$, where P_t' is the matrix having as entries $p_t'(x,y)$ the derivatives of the entries of the matrix P_t.

Proof. Backward equations. Using Chapman-Kolmogorov,

$$\begin{aligned} p_{t+h}(x,y) - p_t(x,y) &= \sum_z p_h(x,z)p_t(z,y) - p_t(x,y) \\ &= (p_h(x,x) - 1)p_t(x,y)) + \sum_{z \neq x} p_h(x,z)p_t(z,y). \end{aligned}$$

Dividing by h and taking h to zero we obtain $p_t'(x,y)$ in the left hand side. To compute the right hand side, observe that

$$p_h(x,x) = 1 - \lambda_x h + o(h).$$

Hence

$$\lim_h \frac{p_h(x,x) - 1}{h} = -\lambda_x = q(x,x).$$

Analogously, for $x \neq y$

$$p_h(x,y) = q(x,y)h + o(h)$$

and

$$\lim_h \frac{p_h(x,y)}{h} = q(x,y).$$

This shows the Kolmogorov Backward equations. The forward equations are proven analogously. To start, use Chapman-Komogorov to write

$$p_{t+h}(x,y) = \sum_z p_t(x,z)p_h(z,y). \qquad \square$$

Las ecuaciones backward dicen $P_t' = QP_t$. Si P_t fuera un número, tendríamos

$$P_t = e^{Qt}$$

Formalmente podemos definir la matriz

$$e^{Qt} = \sum_{n\geq 0} \frac{(Qt)^n}{n!} = \sum_{n\geq 0} Q^n \frac{t^n}{n!}$$

y diferenciar para obtener

$$\frac{d}{dt} e^{Qt} = \sum_{n\geq 1} Q^n \frac{t^{n-1}}{(n-1)!} = Q \sum_{n\geq 1} Q^{n-1} \frac{t^{n-1}}{(n-1)!} = QP_t$$

A pesar que en general el producto no es conmutativo, tenemos que $QP_t = P_t Q$. Para verlo de otra manera,

$$QP_t = \sum_{n\geq 0} QQ^{n-1} \frac{t^{n-1}}{(n-1)!} = \sum_{n\geq 0} Q^{n-1} \frac{t^{n-1}}{(n-1)!} Q = P_t Q.$$

2.2 Invariant measures

Definition 2.3. We say that π is an *stationary distribution* or *invariant measure* for the transition matrix P_t, with rate matrix Q if

$$\sum_x \pi(x) p_t(x, y) = \pi(y) \tag{2.21}$$

$$\sum_x \pi(x) = 1 \tag{2.22}$$

that is, if the distribution of the initial state is given by π, then the distribution of the process at time t is also given by π for any $t \geq 0$. In matrix form, an invariant measure satisfies

$$\pi = \pi P_t, \quad t \geq 0. \tag{2.23}$$

Theorem 2.4. *A distribution π is stationary for a process with rates $q(x, y)$ if and only if $\pi Q = 0$, that is,*

$$\sum_x \pi(x) q(x, y) = \pi(y) \sum_z q(y, z). \tag{2.24}$$

Condition (2.24) can be interpreted as a *flux condition*: the entrance rate under π to state y is the same as the exit rate from y. For this reason the equations (2.24) are called *balance equations.*

Proof. Differentiating (2.21) we get

$$
\begin{aligned}
0 &= \sum_x \pi(x) p_t'(x,y) = \sum_x \pi(x) \sum_z p_t(x,z) Q(z,y) \\
&= \sum_z \sum_x \pi(x) p_t(x,z) Q(z,y) = \sum_z \pi(z) Q(z,y)
\end{aligned}
$$

where we have applied the forward equations. This proves (2.24).

Reciprocally, applying Kolmogorov backwards equations we get

$$(\pi P_t)' = \pi P_t' = \pi Q P_t = 0;$$

This implies that πP_t is constant. Since $P_0 = I$ (the identity matrix), we get $\pi P_t = \pi P_0 = \pi$. ☐

Clima. Hay 3 estados: sol, nublado, lluvia. El tiempo que está en sol es exponencial 1/3 de donde pasa a nublado, queda un tiempo exponencial 1/4 cuando empieza a llover y llueve un tiempo exponencial 1, cuando vuelve a sol. La matriz de tasas es

$$
\begin{pmatrix}
-1/3 & 1/3 & 0 \\
0 & -1/4 & 1/4 \\
1 & 0 & -1
\end{pmatrix}
$$

$\pi Q = \pi$ equivale a las ecuaciones

$$
\begin{array}{llll}
-\frac{1}{3}\pi(1) & & +\pi(3) & = 0 \\
\frac{1}{3}\pi(1) & -\frac{1}{4}\pi(2) & & = 0 \\
& \frac{1}{4}\pi(2) & -\pi(3) & = 0
\end{array}
$$

La solución es $\pi = (\frac{3}{8}, \frac{4}{8}, \frac{1}{8})$.

Generadores La matriz de tasas Q es llamada *generador*. Aplicada a una función $f : \mathbb{X} \to \mathbb{R}$, tenemos

$$
\begin{aligned}
Qf(x) &= \sum_{y \in \mathbb{X}} q(x,y) f(y) && (2.25) \\
&= \sum_{y \in \mathbb{X} \setminus \{x\}} q(x,y) \, [f(y) - f(x)]. && (2.26)
\end{aligned}
$$

Esta forma es la que se usa en espacios no numerables. Se interpreta como que la tasa de salto de x a y es $q(x,y)$.

Esa notación es cómoda también para caracterizar P_t:

$$E[f(X_t)|X_0 = x] = P_t f(x) \tag{2.27}$$

y sus derivadas:

$$\frac{d}{dt} P_t f(x) = Q P_t f(x) = P_t Q f(x) \tag{2.28}$$

por las ecuaciones de Kolmogorov.

2.3 Recurrencia de Harris y convergencia

En el siguiente teorema probamos simultaneamente la existencia y la convergencia a velocidad exponencial bajo una condición que es conocida como *recurrencia de Harris*. El enfoque está basado en [11]. Defina

$$\gamma(z) := \min_x Q(x,z), \qquad \gamma(Q) := \sum_z \gamma(z) \tag{2.29}$$

Theorem 2.5. *Sea X_t un proceso de Markov con tasas Q. Si $\gamma > 0$, entonces (X_t) tiene una única distribución estacionaria π. Además, el proceso converge a π en variación total, a velocidad exponencial con coeficiente γ:*

$$\sup_x \frac{1}{2} \sum_z |\pi(z) - P_t(x,z)| < e^{-\gamma t}.$$

Proof. Primero vamos a demostrar que si hay una distribución estacionaria π, entonces vale la convergencia en variación total. Después demostraremos la existencia y unicidad de π.

Sin pérdida de generalidad asumimos que el espacio de estados es $\{1, \dots, K\}$, para algún $K > 0$.

Acoplamiento. La construcción (2.5) del proceso X_t en función de x_0 y M sirve para realizar un acoplamiento de procesos con condiciones iniciales distintas pero con el mismo M:

$$(X_t, X'_t)_{t\geq 0} = (\phi(x_0, M), \phi(x'_0, M)). \tag{2.30}$$

Ahora vamos a diseñar las particiones $B(x,y)$ en forma conveniente.

Sea $(J(z))_{z\in\mathbb{X}}$ una partición del intervalo $[0,\gamma)$ en intervalos de tamaño

$$|J(z)| = \gamma(z).$$

Para cada estado x construya intervalos disjuntos sucesivos $(I(x,z))_{z\in\mathbb{X}\setminus\{x\}}$ de tamaño

$$|I(x,z)| = q(x,z) - \gamma(z);$$

localizados a la derecha de los intervalos $J(z)$, es decir $I(x,z) \subset [\gamma,\infty)$.

Defina

$$B(x,z) := J(z) \cup I(x,z)$$

Como $B(x,z)$ tiene medida $q(x,z)$, la construcción de X_t usando el proceso de Poisson bidimensional M se puede hacer con estas particiones.

Considere $(X_t, X'_t)_{t\geq 0}$ con estado inicial $(X_0, X'_0) = (x_0, x'_0)$, definido en (2.30). Cada marginal es governada por los puntos de M, y la partición B recién construída; cada marginal tiene su estado inicial.

Si el proceso en el instante $t-$ se encuentra en el estado (x,x') y M contiene al punto (t,u), con

$$u \in \cup_z J(z) = [0,\gamma),$$

entonces hay tres casos:

(a) $u \in J(z)$ para $z \notin \{x,x'\}$, en ese caso ambas marginales saltan a z;

(b) $u \in J(x)$, en ese caso la segunda marginal salta a x y la primera marginal se queda en x;

(c) $u \in J(x')$, en ese caso la primera marginal salta a x' y la segunda marginal se queda en x'.

En síntesis, si $u \in [0,\gamma)$ entonces $X_t = X'_t$. Los procesos *coalescen.*

Esto es así porque por debajo de γ la partición $B(x,z)$ no depende de x:

$$B(x,z) \cap [0,\gamma) = B(x',z) \cap [0,\gamma), \quad \text{para todo } z, x, x'.$$

El instante de coalescencia se denota τ:

$$\tau = \inf\{t > 0 : M([0,t] \times [0,\gamma)) > 0\}.$$

Por lo tanto,

$$t \geq \tau \quad \text{implica} \quad X_t = X'_t, \tag{2.31}$$

además

$$\tau \sim \text{ Exponencial}(\gamma): \quad P(\tau > t) = e^{-\gamma t}. \tag{2.32}$$

Para concluir, escribimos

$$\begin{aligned}
\sum_z |P_t(y,z) - P_t(x,z)| &= \sum_z |P(X_t = z) - P(X'_t = z)| \\
&= \sum_z |E(\mathbf{1}\{X_t = z\} - \mathbf{1}\{X'_t = z\})| \\
&\le E\Big(\sum_z |\mathbf{1}\{X_t = z\} - \mathbf{1}\{X'_t = z\}|\Big) \\
&= 2E\mathbf{1}\{X_t \neq X'_t\} = 2P(X_t \neq X'_t) \\
&\le 2P(\tau > t) = 2e^{-\gamma t}, \text{ usando (2.31) y (2.32).}
\end{aligned}$$

Si el estado inicial X'_0 es aleatorio con distribución estacionaria π,

$$\begin{aligned}
\sum_z |\pi(z) - P_t(x,z)| &= \sum_z \Big|\sum_y \pi(y)P_t(y,z) - \sum_y \pi(y)P_t(x,z)\Big| \\
&\le \sum_y \pi(y) \sum_z |P_t(y,z) - P_t(x,z)| \\
&\le \sum_y \pi(y) 2e^{-\gamma t} = 2e^{-\gamma t}
\end{aligned}$$

Esto demuestra la convergencia a velocidad exponencial a la distribución estacionaria, cuando $t \to \infty$.

2.4 Existencia y simulación perfecta de π

Denote $(X^x_{[s,t]})_{t\ge s}$ el proceso que tiene una condición inicial $X^x_{[s,s]} = x$, en el instante s y utiliza los puntos de M en la banda $[s,\infty) \times [0,\infty)$. Esa construcción es invariante por traslaciones:

$$(X^x_{[s,s+t]})_{t\ge 0} \text{ tiene la misma distribución que } (X^x_{[0,t]})_{t\ge 0}.$$

Observe que $(X^x_{[s,t]})_{t\ge s}$ está definido para todo s en función del mismo M. Sea

$$\tau(t) := \sup\{s \le t : M([s,t] \times [0,\gamma)) > 0\}$$

En el instante $\tau(t)$ hay un punto $u(t) \in [0,\gamma)$ tal que

$$(u(t), \tau(t)) \in M.$$

Defina

$$z(t) := z, \quad \text{si} \quad u(t) \in J(z). \tag{2.33}$$

Defina $(Z_t)_{t\in\mathbb{R}}$ por

$$Z_t := X^{z(t)}_{[\tau(t),t]}, \quad t \in \mathbb{R}. \tag{2.34}$$

Z_t es Markov con tasas Q y es estacionaria: $P(Z_t = z)$ no depende de t. Por lo tanto, llamando π a la distribución de Z_t, vale que π es invariante para Q. □

Ejemplo. Estados $\{1, 2, 3\}$.

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 4 & -5 & 1 \\ 2 & 3 & -5 \end{pmatrix} \tag{2.35}$$

$$(\gamma(z)) = (2, 1, 1) \qquad \gamma = 4. \tag{2.36}$$

$$(J(z)) = ([0, 2), [2, 3), [3, 4))$$

$$(Q(i, z) - \gamma(z)) = \begin{pmatrix} * & 0 & 1 \\ 2 & * & 0 \\ 0 & 2 & * \end{pmatrix}$$

$$I(i, j) = \begin{pmatrix} * & \emptyset & [4, 5) \\ [4, 6) & * & \emptyset \\ \emptyset & [4, 6) & * \end{pmatrix}$$

2.5 Convergencia al equilibrio

Theorem 2.6 (Teorema de convergencia)**.** *Sea X_t un proceso de Markov en un espacio finito o numerable $\mathbb{X}$ con tasas q. Si $p_t(x, y) > 0$ para todo $x, y \in \mathbb{X}$, y π es una medida invariante para q, entonces π es la única medida invariante y*

$$\lim_{t\to\infty} p_t(x, y) = \pi(y), \quad x \in \mathbb{X}. \tag{2.37}$$

Proof. Ejercicio en el caso de recurrencia de Harris. Omitida en el caso general □

2.6 Reversibilidad

Sea Q una matriz de tasas con una distribución estacionaria π. Sea $(X_t)_{t\geq 0}$ una realización de la cadena de Markov con distribución inicial

$$P(X_0 = x) = \pi(x).$$

Considere el proceso $(X_t^*)_{t\in[0,s]}$ dado por $X_t^* = X_{s-t}$.

Lemma 2.7. *El proceso X_t^* es Markov con tasas*

$$q^*(x,y) = \frac{\pi(y)q(y,x)}{\pi(x)}. \tag{2.38}$$

Además π es una distribución estacionaria para Q^*.

Proof. Por definición,

$$\begin{aligned}
p_t^*(x,y) &= P(X_t^* = y | X_0^* = x) \\
&= P(X_{s-t} = y | X_s = x) \\
&= \frac{P(X_{s-t} = y, X_s = x)}{P(X_s = x)} \\
&= \frac{P(X_0 = y, X_t = x)}{P(X_s = x)} \\
&= \frac{P(X_t = x | X_0 = y)\, P(X_0 = y)}{P(X_0 = x)} \\
&= \frac{\pi(y)p_t(y,x)}{\pi(x)}
\end{aligned}$$

Las tasas del proceso X_t^* se obtienen derivando p_t^* y usando las ecuaciones de Kolmogorov en $t = 0$:

$$q^*(x,y) = \frac{d}{dt}\frac{\pi(y)p_t(y,x)}{\pi(x)}\Big|_{t=0} = \frac{\pi(y)q(y,x)}{\pi(x)}. \tag{2.39}$$

Note que la tasa de salida de x es la misma para el proceso directo y reverso:

$$\lambda_x^* := \sum_y q^*(x,y) = \sum_y \frac{\pi(y)}{\pi(x)} q(y,x) = \sum_y \frac{\pi(x)}{\pi(x)} q(x,y) = \lambda_x,$$

usando en la tercera identidad que π es estacionaria para Q. Finalmente,

$$\sum_x \pi(x) p_t^*(x,y) = \sum_x \pi(x)\frac{\pi(y)}{\pi(x)} p_t(y,x) = \pi(y),$$

lo que demuestra que π es estacionaria para p_t^*. □

Corollary 2.8. *Supongamos que la matriz de tasas Q tiene una medida invariante π. Sea $(\hat{X}_t)_{t\geq 0}$ un proceso de Markov con tasas q y condición inicial $P(\hat{X}_0 = x) = \pi(x)$. Sea $(\hat{X}^*_t)_{t\geq 0}$ un proceso de Markov con tasas q^* y condición inicial $\hat{X}^*_0 = \hat{X}_0$.*

El proceso $(X_t)_{t\in\mathbb{R}}$ definido por

$$X_t = \begin{cases} \hat{X}_t, & t \geq 0 \\ \hat{X}^*_{-t}, & t \leq 0, \end{cases} \tag{2.40}$$

es un proceso de Markov estacionario con tasas q y el proceso $X^*_t = X_{-t}$ es tambien estaciorio con tasas q^*. Ambos procesos tienen medida invariante π.

Procesos reversibles Cuando π satisface las ecuaciones de balance detallado:

$$\pi(x)q(x,y) = \pi(y)q(y,x) \tag{2.41}$$

tenemos

$$q^*(x,y) = \frac{\pi(y)}{\pi(x)}q(y,x) = q(x,y).$$

La película yendo para atrás o para adelante tiene la misma distribución. Es decir $(X_t)_{t\in\mathbb{R}}$ y $(X^*_t)_{t\in\mathbb{R}}$ tienen la misma distribución.

2.7 Simulación. Metrópolis y Baño caliente

El objetivo es generar muestras de una distribución π dada. Para eso vamos a construir cadenas de Markov cuya distribución estacionaria es π.

Empezamos con una matriz de saltos arbitraria $p(x,y)$ que será usada para *proponer* un salto. Tipicamente $p(x,y) > 0$ cuando $\pi(x)/\pi(y)$ es fácil de calcular y hay "pocos" estados y con $p(x,y) > 0$.

Metropolis. En el algoritmo Metrópolis, el salto será aceptado con probabilidad

$$r(x,y) = \min\Big\{\frac{\pi(y)p(y,x)}{\pi(x)p(x,y)}, 1\Big\}$$

(por ejemplo, se elige p para saltos a los vecinos, o teniendo en cuenta el cálculo de $\pi(y)/\pi(x)$). La matriz de tasas Metrópolis se define por

$$q(x,y) = p(x,y)\, r(x,y), \quad x \neq y$$

Lemma 2.9. *La distribución π es reversible para el proceso con tasas q.*

Proof. Supongamos que $\pi(y)p(y,x) > \pi(x)p(x,y)$. En este caso,

$$\pi(x)q(x,y) = \pi(x)p(x,y)\,1$$
$$\pi(y)q(y,x) = \pi(y)p(y,x)\frac{\pi(x)p(x,y)}{\pi(y)p(y,x)} = \pi(x)p(x,y).$$

Es decir que se satisfacen las ecuaciones de balance detallado. □

Baño caliente. La matriz de *baño caliente* se define por

$$q(x,y) = p(x,y)\frac{\pi(y)}{\pi(y)+\pi(x)} \tag{2.42}$$

donde $p(x,y)$ es la matriz que define los saltos. Proponemos y y lo aceptamos con probabilidad proporcional a $\pi(y)$.

La medida π es reversible para q. Ejercicio.

Processes restricted to a subset of $\mathbb{X}$. Let X_t be a process with rate matrix Q and π is reversible for Q, that is,

$$\pi(x)Q(x,y) = \pi(y)Q(y,x). \tag{2.43}$$

Let $A \subset \mathbb{X}$ and define a matrix Q_A such that the jumps outside A vanish, that is,

$$Q_A(x,y) := Q(x,y)\mathbf{1}\{y \notin A\}, \tag{2.44}$$

then the measure $\pi(\cdot|A)$ is reversible for Q_A.

Hence, to simulate a random variable with distribution $\pi(\cdot|A)$ one can use the Markov process with rate matrix Q_A.

Ejemplo. Estacionamiento discreto. Considere el espacio $\mathbb{X} = \{0,1\}^{\{1,\dots,K\}}$, para un natural K. Son vectores η de K coordenadas con entradas $\eta(x) = 0$ o $\eta(x) = 1$, $x \in \{1,\dots,K\}$. Considere la medida $\pi(\eta) = \frac{1}{Z}\mathbf{1}\{\eta(x)\eta(x{+}1) = 0\}$, donde $K{+}1 = 1$ por convención (condiciones de contorno periódicas), y Z es la normalización. Es la medida condicionada a que no haya dos coordenadas vecinas con valor 1. Defina $\eta^y(x) = \eta(x)\mathbf{1}\{x \neq y\} + (1-\eta(x))\mathbf{1}\{x = y\})$ (cambiar la coordenada en y). Decimos que η^y es *vecino* de η. Sea $p(\eta,\eta^y) = 1$, $y = 1\dots,K$ y $p(\eta,\eta') = 0$ para η' no vecino de η. Describa la matriz de tasas del proceso baño caliente en este ejemplo.

Simulación Usando los teoremas de convergencia se pueden obtener muestras aproximadas de π, o las esperanzas en relación a π de funciones objetivo. Por ejemplo:

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to_n \sum_x f(x)\pi(x).$$

3 Branching processes

Proceso de ramificación Z_n = tamaño de una población en el instante n. Cada individuo tiene un número aleatorio de hijos distribuidos como una variable aleatoria $\xi \geq 0$ con media $E\xi = \mu < \infty$ y distribución $P(\xi = j) = p_j$. Sean $\xi_{n,k}, n, k \geq 1$ iid con la misma distribución de ξ. Aquí $\xi_{n,k}$ es el número de hijos que tiene el k-ésimo individuo vivo en el instante n. Definimos $Z_0 = 1$ y para $n \geq 1$,

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k}$$

Condicionando a Z_{n-1},

$$EZ_n = E(E(Z_n|Z_{n-1})) = \mu EZ_{n-1} = \mu^2 EZ_{n-2} = \mu^n.$$

Theorem 3.1 (subcrítico). *Si* $\mu < 1$ *entonces*

$$P(Z_n \geq 1, \ \textit{para todo } n \geq 0) = 0$$

Proof. Por la desigualdad de Markov

$$P(Z_n \geq 1) \leq EZ_n = \mu^n \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{si } \mu < 1.$$

Como $\{Z_n \geq 1\} \nearrow \{Z_n > 0, \ \text{para todo } n \geq 0\}$, podemos concluir. □

Theorem 3.2 (crítico). *Si* $\mu = 1$, $P(\xi = 0) > 0$ *y* $P(\xi = 1) < 1$, *entonces*

$$P(Z_n \geq 1, \ \textit{para todo } n \geq 0) = 0$$

Theorem 3.3 (supercrítico). *Si* $\mu > 1$, *entonces*

$$P(Z_n \geq 1, \ \textit{para todo } n \geq 0) > 0$$

Función generadora de momentos Defina la función $\phi : [0,1] \to [0,1]$ por $\phi(0) = p_0$ y para $s \in (0,1]$,

$$\phi(s) = \sum_{j\geq 0} s^j p_j$$

ϕ es continua en $[0,1]$. Para $s \in (0,1)$ tenemos

$$\phi'(s) = \sum_{j\geq 1} j s^{j-1} p_j > 0$$
$$\phi''(s) = \sum_{j\geq 2} j(j-1) s^{j-2} p_j > 0$$

Las desigualdades se deben a que $p_0 + p_1 < 1$ (porque si no, $\mu = p_1 \leq 1$, contradiciendo $\mu > 1$).

Las desigualdades implican que ϕ es estrictamente creciente y estrictamente convexa en el intervalo $(0,1)$.

Además

$$\lim_{s \nearrow 1} \phi'(s) = \mu.$$

Defina

$$\theta_n = P(Z_n = 0 | Z_0 = 1).$$

Como $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$, tenemos $\theta_n \leq \theta_{n+1}$. Además $\theta_0 = 0$, $\theta_n \leq 1$. Por lo tanto $\theta_n \nearrow \theta_\infty \leq 1$.

Condicionando a la primera generación,

$$\begin{aligned}
\theta_n &= \sum_{j\geq 0} P(Z_n = 0 | Z_1 = j) P(Z_1 = j | Z_0 = 1) \\
&= \sum_{j\geq 0} \left(P(Z_{n-1} = 0 | Z_0 = 1)\right)^j p_j \\
&= \phi(\theta_{n-1}).
\end{aligned}$$

La segunda igualdad se explica así: la probabilidad que el proceso se extinga en el instante n dado que hay j individuos en el instante 1 es igual a la probabilidad que cada una de las familias de los j individuos vivos en el instante 1 se haya extinguido en el instante n. Para concluir observe que las j familias evolucionan independientemente, y hay $n-1$ generaciones entre el instante 1 y el n.

Sacando límites, vemos que $\theta_\infty = \phi(\theta_\infty)$. Es decir, θ_∞ es un punto fijo de ϕ.

Si ρ es un punto fijo de ϕ, es decir $\rho = \phi(\rho)$, tenemos

$$\rho = \sum_{k\geq 0} \rho^k p_k \geq \rho^0 p_0 = P(Z_1 = 0) = \theta_1.$$

Como ϕ creciente, la desigualdad implica $\phi(\theta_1) \leq \phi(\rho) = \rho$.

Iterando, obtenemos

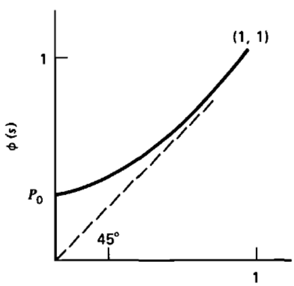
$$\theta_n = \phi(\theta_{n-1}) \leq \phi(\rho) = \rho \quad \text{para todo } n.$$

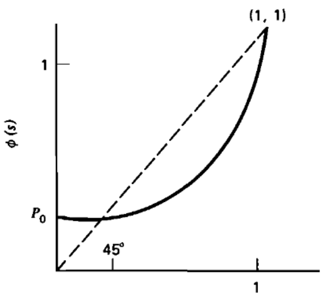
Es decir que θ_n converge al menor de los puntos fijos.

Demostración del teorema crítico. Si $\phi'(1) = \mu = 1$ y $p_1 < 1$, como ϕ es estrictamente convexa $\phi(s) > s$ para $s \in (0,1)$ y ϕ tiene 1 como único punto fijo. Por lo tanto $\theta_n \to 1$. □

Demostración del teorema supercrítico. Si $\phi'(1) = \mu > 1$, entonces hay un único $\rho < 1$ tal que $\phi(\rho) = \rho$. Para ver esto, observe que $\phi(0) = p_0 \geq 0$, $\phi(1) = 1$ y $\phi'(1) = \mu > 0$, lo que implica que hay un único punto fijo ρ menor que 1. Unicidad es consecuencia de la estricta convexidad de ϕ. Por lo tanto $\theta_n \nearrow \rho < 1$. □

Distinguimos dos casos:





A la izquierda $\phi(s) > s$ para todo $s \in (0,1)$ y a la derecha $\phi(s) = s$ para algún $s \in (0,1)$. En la figura de la izquierda $\phi'(1) \leq 1$ y en la de la derecha $\phi'(1) > 1$.

Ejemplo. Considere que la distribución del número de hijos es Poisson con parámetro λ. Es decir

$$p_j = \frac{e^{-\lambda}\lambda^j}{j!}$$

La función generadora de momentos es

$$\phi(s) = \exp(\lambda(s-1))$$

Por lo que la ecuación para el punto fijo es

$$\rho = \exp(\lambda(\rho-1))$$

4 Gibbsian point processes

This section is based on the paper [20] by Georgii.

4.1 Discrete Gibbsian models

The set of spin configurations is $\xi = (\xi_i)_{i\in\mathbb{Z}^d}$, where ξ_i belongs to a finite set S. $\Omega = S^{\mathbb{Z}^d}$ is the *configuration space*, provided of the product topology and the Borel sigma-algebra $\mathcal{F}$. We are interested in probability measures on the space $(\Omega, \mathcal{F})$. *Lattice systems.* For $\xi \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ denote $\xi_\Lambda = (\xi_i)_{i\in\Lambda}$; same notation as the projection of Ω on S^Λ.

Specifications Prescribe the probability of a finite set of spins when the other spins are fixed. That is, we look for probability measures P on $(\Omega, \mathcal{F})$ with conditional probabilities

$$G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) \tag{4.1}$$

for finite Λ.

Examples: 1) Markovian case. (4.1) depends only on the spins in the boundary of Λ defined by $\partial\Lambda := \{i \notin \Lambda : |i-j| = 1 \text{ for some } j \in \Lambda\}$, that is,

$$G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) = G_\Lambda(\xi_\Lambda|\xi_{\partial\Lambda}) \tag{4.2}$$

2) Gibbsian case. Hamiltonian H_Λ and Boltzmann-Gibbs formula

$$G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) = Z^{-1}_{\Lambda|\xi_{\Lambda^c}} \exp[-H_\Lambda(\xi)], \tag{4.3}$$

where $Z^{-1}_{\Lambda|\xi_{\Lambda^c}} = \sum_{\xi':\xi'_\Lambda=\xi_\Lambda} \exp[-H_\Lambda(\xi')]$. (4.3) are the DLR equations.

See the book of Georgii [18] for a exhaustive account of this matter

Definition 4.1 (Gibbs measures)**.** *A probability measure P on $(\Omega, \mathcal{F})$ is a* Gibbs measure *for $\mathbf{G} = (G_\Lambda)_{\Lambda finite}$ if*

$$P(\xi_\Lambda \text{ occurs in } \Lambda \,|\, \xi_{\Lambda^c} \text{ occurs off } \Lambda) = G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) \tag{4.4}$$

for P-almost all ξ and all finite $\Lambda \subset \mathbb{Z}^d$.

Gibbs measures do not exist automatically. However, in the present case of a finite state space S, Gibbs measures do exist whenever G is Markovian in the sense of (2), or almost Markovian in the sense that the conditional probabilities (1) are continuous functions of the outer configuration ξ_{Λ^c}. In this case one can show that any weak limit of $G_\Lambda(\cdot|\xi_{\Lambda^c})$ for fixed ξ $\Lambda \nearrow \mathbb{Z}^d$ is a Gibbs measure.

A basic observation is that the Gibbs measures for a given consistent family G of conditional probabilities form a convex set $\mathcal{G}$. Therefore one is interested in its extremal points, which are characterized in the next theorem.

Let $\mathcal{F}_\Lambda$ be the sigma algebra generated by $\big\{\{\xi_i = k_i\} : i \in \Lambda, k_i \in S\big\}$ and $\mathcal{T} := \cap_{\Lambda:|\Lambda|<\infty}\mathcal{F}_{\Lambda^c}$, the *tail* sigma algebra –generated by sets not depending on the values of any finite set of spins.

Theorem 4.2 (Extremal Gibbs measures)**.** *The following statements hold:*

(a) A Gibbs measure $P \in \mathcal{G}$ is extremal in $\mathcal{G}$ if and only if P is trivial on $\mathcal{T}$, i.e., if and only if any tail measurable real function is P-almost surely constant.

(b) Distinct extremal Gibbs measures are mutually singular on $\mathcal{T}$.

(c) Any non-extremal Gibbs measure is the barycenter of a unique probability weight on the set of extremal Gibbs measures. (Convex combination of extremal measures).

A proof can be found in Georgii [18], Theorems (7.7) and (7.26).

(a) means that the extremal Gibbs measures are macroscopically deterministic: on the macroscopic level all randomness disappears, and an experimenter will get non-fluctuating measurements of macroscopic quantities like magnetization or energy per lattice site.

(b) asserts that distinct extremal Gibbs measures show different macroscopic behavior. So, they can be distinguished by looking at typical realizations of the spin configuration through macroscopic glasses.

(c) implies that any realization which is typical for a non-extremal Gibbs measure is in fact typical for a suitable extremal Gibbs measure. In physical terms: any configuration which can be seen in nature is governed by an extremal Gibbs measure, and the non-extremal Gibbs measures can only be interpreted in a Bayesian way as measures describing the uncertainty of the experimenter. These observations lead us to the following definition.

Definition 4.3 (Phase transition)**.** *Any extremal Gibbs measure is called a* phase *of the corresponding physical system. If distinct phases exist, one says that a* phase transition *occurs.*

So, in terms of this definition the existence of phase transition is equivalent to the nonuniqueness of the Gibbs measure.

One mechanism related to phase transition is the formation of infinite clusters in suitably defined random graphs. Such infinite clusters serve as a link between the local and global behavior of spins, and make visible how the individual spins unite to form a specific collective behavior.

4.1.1 Holley inequality

Suppose the state space S is a subset of $\mathbb{R}$ and thus linearly ordered. Then the configuration space Ω has a natural partial order given by $\xi \leq \xi'$ if and only if $\xi_i \leq \xi'_i$ for all i, and we can speak of increasing real functions $f : \Omega \to \mathbb{R}$.

Let P, P' be two probability measures on Ω. We say that P is stochastically smaller than P', denoted $P \leq P'$, if $\int f dP \leq \int f dP'$ for all measurable bounded increasing f on Ω. This is equivalent to have a coupling $\hat{P}$ on Ω^2 with marginals P and P' such that $\hat{P}(\xi \leq \xi') = 1$.

A sufficient condition for stochastic monotonicity is given in the proposition below. Although this condition refers to the case of finite products (for which stochastic monotonicity is similarly defined), it is also useful in the case of infinite product spaces. This is because (by the very definition) the relation $\leq$ is preserved under weak limits.

Proposition 4.4 (Holley inequality). *Let Λ be a finite index set, and μ, μ' two probability measures on $\{0,1\}^\Lambda$ giving positive weight to each element of $\{0,1\}^\Lambda$. Suppose the single-site conditional probabilities at any $i \in \Lambda$ satisfy*

$$\mu(\xi_i = 1 | \xi_{\Lambda \setminus \{i\}}) \leq \mu'(\xi'_i = 1 | \xi'_{\Lambda \setminus \{i\}}) \text{ for } \xi \leq \xi',$$

then $\mu \leq \mu'$.

Proof. Let

$$p(i, \xi_{\Lambda \setminus \{i\}}) = \mu(\xi(i) = 1 | \xi_{\Lambda \setminus \{i\}} \text{ occurs off } i) \tag{4.5}$$

the conditional probability that the site i takes value 1 given the configuration $\xi_{\Lambda \setminus \{i\}}$ in the other sites. Then when $\xi_i = 1$ we have

$$\mu(\xi) = p(i, \xi_{\Lambda \setminus \{i\}}) \mu(\xi_{\Lambda \setminus \{i\}}) \tag{4.6}$$

$$\mu(\xi^i) = (1 - p(i, \xi_{\Lambda \setminus \{i\}})) \mu(\xi_{\Lambda \setminus \{i\}}) \tag{4.7}$$

where $(\xi^i)_j = \xi_j$ if $j \neq i$; $(\xi^i)_i = 1 - \xi_i$, a configuration differing from ξ only at site i. In the above case $(\xi^i)_i = 0$.

Define analogously $p'(i, \xi_{\Lambda\setminus\{i\}})$. By hypothesis, we have

$$p(i, \xi_{\Lambda\setminus\{i\}}) \leq p(i, \xi'_{\Lambda\setminus\{i\}}) \text{ whenever } \xi \leq \xi'. \tag{4.8}$$

Consider a pure jump Markov process $\xi(t)$ on $\{0,1\}^\Lambda$ with the following evolution. Each site i, at rate 1, is updated with a Bernoulli distribution with parameter $p(i, \xi_{\Lambda\setminus\{i\}}(t-))$.

The positive entries of the matrix are

$$Q(\xi, \xi^i) = \begin{cases} p(i, \xi_{\Lambda\setminus\{i\}}) & \text{if } \xi_i = 0 \\ 1 - p(i, \xi_{\Lambda\setminus\{i\}}) & \text{if } \xi_i = 1 \end{cases} \tag{4.9}$$

The measure μ is reversible for the process $\xi(t)$. Indeed, (4.6)-(4.7) imply

$$\mu(\xi)\, Q(\xi, \xi^i) = \mu(\xi^i)\, Q(\xi^i, \xi). \tag{4.10}$$

Let $P_t(\xi, \zeta) = P(\xi(t) = \zeta | \xi(0) = \xi)$ the semigroup associated to Q. Then, the ergodic theorem for finite jump Markov processes says

$$\lim_{t\to\infty} P_t(\xi, \zeta) = \mu(\zeta). \tag{4.11}$$

Let $N = (N_i : i \in \Lambda)$ be a collection of independent marked Poisson processes in $\mathbb{R} \times [0,1]$ with intensity $dt\, du$.

We construct the process as a function of N, $\xi(t) = (\xi(t))_{t\geq 0}[N]$, as follows. Fix an initial configuration $\xi(0)$ and assume we know $\xi(s)$ up to time $t-$ and $(t, u) \in N_i$. Then at time t update ξ_i as follows:

$$\xi_i(t) = \mathbf{1}\{u \leq p(i, \xi_{\Lambda\setminus\{i\}}(t-))\} \tag{4.12}$$

The updating does not depend on the value of $\xi_i(t-)$. The process so defined has transition matrix Q. Indeed if $\xi_i = 1$, we have

$$\begin{aligned} &P(\xi(t+\delta) = \xi^i | \xi(t) = \xi) \\ &= P\big(N_i \cap ([t, t+\delta) \times [p(i, \xi_{\Lambda\setminus\{i\}}(t)), 1\big) = 1\big) + P(\text{other things}) \\ &= \delta(1 - p(i, \xi_{\Lambda\setminus\{i\}})) + o(\delta). \end{aligned} \tag{4.13}$$

and analogously when $\xi_i = 0$,

$$P(\xi(t+\delta) = \xi^i | \xi(t) = \xi) = \delta p(i, \xi_{\Lambda\setminus\{i\}}) + o(\delta). \tag{4.14}$$

Define Q' and P'_t analogously with μ'.

Take $\xi(0) \leq \xi'(0)$ and consider the coupling

$$\big((\xi(t))_{t\geq 0}[N], (\xi'(t))_{t\geq 0}[N]\big) \tag{4.15}$$

Both marginals use the same marked Poisson processes N.

If $(t, u) \in N_i$ and the configurations are ordered at time $t-$, we have

$$\xi_i(t) = \mathbf{1}\{u \leq p(i, \xi_{\Lambda\setminus\{i\}}(t-))\} \leq \mathbf{1}\{u \leq p(i, \xi'_{\Lambda\setminus\{i\}}(t-))\} = \xi'_i(t)$$

where the inequality comes from (4.8). Hence, $\xi \leq \xi'$ implies $(\xi(t))_{t\geq 0}[N] \leq (\xi'(t))_{t\geq 0}[N]$ for all t. In turn, this implies that their distributions satisfy $P_t(\xi, \cdot) \leq P'_t(\xi', \cdot)$. Use (4.11) to conclude that $\mu \leq \mu'$. □

4.2 Bernoulli percolation

Consider $\mathbb{Z}^d$, $d \geq 2$ as a graph with vertex set $\mathbb{Z}^d$ and edge set $E(\mathbb{Z}^d) = \{e = \{i, j\} \subset \mathbb{Z}^d : |i - j| = 1\}$.

Parameters $0 \leq p_s, p_b \leq 1$, site and bond probabilities.

Random subgraph $\Gamma = (X, E)$ of $(\mathbb{Z}^d, E(\mathbb{Z}^d))$, where $X = \{i \in \mathbb{Z}^d : \xi_i = 1\}$, $E = \{e \in E(X) : \eta_e = 1\}$, where $E(X) = \{e \in E(\mathbb{Z}^d) : e \subset X\}$ is the set of edges between the sites of X, and $(\xi_i : i \in \mathbb{Z}^d)$, $(\eta_e : e \in E(\mathbb{Z}^d))$ are independent Bernoulli variables with $P(\xi_i = 1) = p_s$, $P(\eta_e = 1) = p_b$.

This is *Bernoulli mixed site-bond percolation.* Setting $p_b = 1$ we obtain pure site percolation, and $p_s = 1$ corresponds to pure bond percolation.

Let $\{0 \leftrightarrow \infty\}$ denote the event that Γ contains an infinite path starting from 0, and $\theta(p_s, p_b; \mathbb{Z}^d) = P(0 \leftrightarrow \infty)$ be its probability.

By Kolmogorov's zero-one law, we have $\theta(p_s, p_b; \mathbb{Z}^d) > 0$ if and only if Γ contains an infinite connected component –called *cluster*– with probability 1. In this case one says that *percolation* occurs.

The following proposition asserts that this happens in a non-trivial region of the parameter square, which is separated by the so-called critical line from the region where all clusters of Γ are almost surely finite. The change of behavior at the critical line is the simplest example of a critical phenomenon.

Proposition 4.5 (Bernoulli percolation)**.** *The function $\theta(p_s, p_b; \mathbb{Z}^d)$ is increasing in p_s, p_b and d. Moreover,*

(1) $\theta(p_s, p_b; \mathbb{Z}^d) = 0$ when $p_s p_b$ is small enough;

(2) $\theta(p_s, p_b; \mathbb{Z}^d) > 0$ when $d \geq 2$ and $p_s p_b$ is sufficiently close to 1.

Proof. The monotonicity in p_s and p_b follows from coupling by defining the $\eta_e(U_e) = \mathbf{1}\{U_e \leq p_b\}$ and $\eta_x(U_x) = \mathbf{1}\{U_x \leq p_s\}$, where U_e U_x are iid Uniform$[0,1]$ or using Holley inequality; the monotonicity in d follows by noticing that an infinite percolation cluster in $\mathbb{Z}^2$ is contained in an infinite percolation cluster in $\mathbb{Z}^d$ for $d \geq 3$.

The following arguments are taken from Grimmett [25].

(1) A *self avoiding walk* (SAW) is a path that visit no vertex more than once. Let σ_n be the number of SAW starting at the origin with length n. Let N_n be the set of those walks having all edges and sites open (open SAW). Then

$$\theta = P(N_n \geq 1 \text{ for all } n \geq 1).$$

This is because the cluster of the origing has infinitely many points if and only if there are open SAW starting at the origin of all lenghts. The above expresion equals

$$\lim_{n\to\infty} P(N_n \geq 1)$$

Now

$$P(N_n \geq 1) \leq EN_n = (p_s p_b)^n \sigma_n$$

An upperbound for σ_n is

$$\sigma_n \leq (2d)(2d-1)^{n-1},\ n \geq 1$$

because the first step of a SAW can be performed in $2d$ different ways and each subsequent step can be performed at most in $2d-1$ different ways (since the walk is self avoiding, it cannot come back).

Hence

$$\theta \leq \lim_{n\to\infty} (2d)(2d-1)^{n-1}(p_s p_b)^n$$

But this is 0 if $p_s p_b < (2d-1)^{-1}$. This gives

$$p_s p_b < \frac{1}{2d-1} \quad \text{implies} \quad \theta = 0.$$

(2) *Planar duality.* Given the graph $\mathbb{Z}^2$ with edges $E(\mathbb{Z}^2)$, construct a *dual* graph $\Gamma^* = (V^*, E(V^*))$ with vertices $V^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ and edges $E(V^*) = \{\{x,y\} \subset V^* : |x-y| = 1\}$. This is the graph $\Gamma = (\mathbb{Z}^2, E(\mathbb{Z}^2))$ translated by $(\frac{1}{2}, \frac{1}{2})$. See Fig. 1

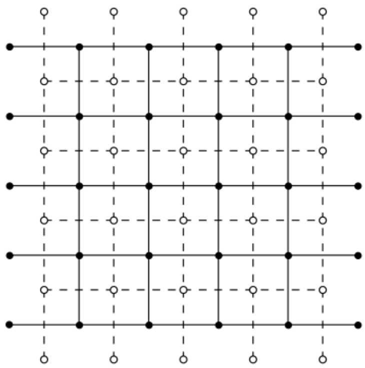


Figure 1: The dual graph.

Each edge $e = \{x, y\}$ of the graph Γ is crossed by an edge e^* of the dual graph Γ^*. Bond Percolation model in Γ^* by declaring

$$e^* \text{ open if and only if } e = \{x, y\} \text{ open and } x \text{ open.}$$

Circuit: A selfavoiding path $x_1, \ldots, x_n$ such that $\|x_i - x_{i+1}\| = 1$ and $\|x_n - x_1\| = 1$, with edges $\{x_i, x_{i+1}\}$ and $\{x_n, x_1\}$.

Peierls argument. If there exists a closed circuit in the dual graph containing the origin, there cannot be an infinite open path containing the origin, that is,

$$\{0 \not\leftrightarrow \infty\} \subset \{\text{there is a closed circuit of the dual graph } \Gamma^* \tag{4.16}$$
$$\text{containing the origin}\} \tag{4.17}$$

Let M_n be the number of closed circuits of the dual graph of lenght n sorrounding the origin. Then

$$1 - \theta = P(0 \not\leftrightarrow \infty) = P\Big(\sum_{n\geq 4} M_n \geq 1)\Big) \leq E\Big(\sum_{n\geq 4} M_n\Big),$$

where we used that the shorter circuit sorrounding the origin has lenght 4. The above expression equals

$$\sum_{n\geq 4} EM_n \leq \sum_{n\geq 4} (n4^n)(1 - p_s p_b)^n$$

In fact, the number of circuits of lenght n sorrounding the origin in the dual graph must cross in at least one point the semiline $[0, \infty)$. The leftmost point the circuit intersects this semiline can take at most n values: $\{\frac{1}{2}, \frac{3}{2}, \dots, n + \frac{1}{2}\}$. Starting the circuit from this point, at each step it can take at most 4 different values. Hence, the number of circuits of lenght n sorrounding the origin in the dual graph is dominated by $n4^n$. This circuit is closed if all bonds are closed, hence the probability that it is closed is dominated by $(1 - p_s p_b)^n$. It is possible to take $\varepsilon > 0$ such that

$$\sum_{n \geq 4} n\varepsilon^n < 1$$

Then $p_s p_b \geq 1 - \varepsilon$ implies $\theta > 0$. □

4.3 Continuum percolation

This section is based on Georgii [19]. The model consists on Poisson points connected by Bernoulli edges.

Let $\mathcal{X}$ = countable subsets X of $\mathbb{R}^d$.

σ-algebra generated by the counting variables $N(B) := \#(X \cap B)$ for bounded Borel sets $B \subset \mathbb{R}^d$; $(X_\Lambda := X \cap \Lambda)$.

$\mathcal{E}$ = locally finite subsets of $E(\mathbb{R}^d) = \{\{x, y\} \subset \mathbb{R}^d : x \neq y\}$. $\mathcal{E}$ = possible edge configurations with an analogous σ-algebra.

For $X \in \mathcal{X}$ let $E(X) = \{e \in E(\mathbb{R}^d) : e \subset X\}$ = possible edges between the points of X, and $\mathcal{E}(X) = \{E \in \mathcal{E} : E \subset E(X)\}$ = edge configurations between the points of X.

Random graph $\Gamma = (X, E)$ **in** $\mathbb{R}^d$ as follows.

• Pick a random point configuration $X \in \mathcal{X}$ according to the Poisson point process π^z on $\mathbb{R}^d$ with intensity $z > 0$.

• For given $X \in \mathcal{X}$, pick a random edge configuration $E \in \mathcal{E}(X)$ according to the Bernoulli measure μ_X^p on EX for which the events $\{E \ni e\}$, $e = \{x, y\} \in E(X)$, are independent with probability $\mu_X^p(E \ni e) = p(x - y)$; here $p : \mathbb{R}^d \to [0, 1]$ is a given even measurable function.

The distribution of the random graph Γ on $X \times E$ is

$$P^{z,p}(dX, dE) = \pi^z(dX)\mu_X^p(dE) \tag{4.18}$$

It is called the *Poisson random-edge model*, or *Poisson random-connection model*, Penrose [41] Meester and Roy [38].

Boolean model: $p(x - y) = \mathbf{1}\{|x - y| \leq 2r\}$ for some $r > 0$.

x and y are connected if and only if $B(x, r)$ and $B(y, r)$ overlap. Same as random set $\Xi = \cup_{x \in X} B(x, r)$, for random X with distribution π^z.

Percolation probability of a typical point: As before, $\{x \leftrightarrow \infty\}$ is the set of configurations satisfying that x belongs to an infinite cluster of $\Gamma = (X, E)$. The density of points belonging to an infinite cluster is a function of z, p denoted θ:

$$\theta(z, p; \mathbb{R}^d) := \int \frac{\#\{x \in X_\Lambda : x \leftrightarrow \infty\}}{|\Lambda|} P^{z,p}(dX, dE)$$

for an arbitrary bounded box Λ with volume $|\Lambda|$. By translation invariance, $\theta(z, p; \mathbb{R}^d)$ does not depend on Λ.

Palm measure In terms of the Palm measure, $\theta(z, p; \mathbb{R}^d) = \hat{P}^{z,p}(0 \leftrightarrow \infty)$.

The Palm measure of $P^{z,p}$, denoted $\hat{P}^{z,p}$ is the distribution of $P^{z,p}$ conditioned to have a point at the origin. In the case of Poisson processes, this means just to add a point at the origin and divide by the rate:

$$\hat{E}^{z,p}(f(X, E)) := \frac{1}{z} E^{z,p}(f(X \cup \{0\}, E)) \tag{4.19}$$

$$= \frac{1}{z} \int f(X \cup \{0\}, E)\, \pi^z(dX)\, \mu^p_{X \cup \{0\}}(dE). \tag{4.20}$$

Theorem 4.6 (Percolation phase transition. Penrose, [41])**.** *$\theta(z, p; \mathbb{R}^d)$ is increasing as function of z and $p(\cdot)$. Moreover, $\theta(z, p; \mathbb{R}^d) = 0$ when $z \int p(x)dx$ is sufficiently small, while $\theta(z, p; \mathbb{R}^d) > 0$ when $z \int p(x)dx$ is large enough.*

Sketch proof. Monotonicity follows from a stochastic comparison argument. Exercise.

Since $z \int p(x)dx$ is the expected number of edges emanating from each point, a branching argument shows that $\theta(z, p; \mathbb{R}^d) = 0$ when $z \int p(x)dx < 1$. Exercise: complete the details.

It remains to show that $\theta(z, p; \mathbb{R}^d) > 0$ when $z \int p(x)dx$ is large enough. For simplicity assume $\int p(x)dx = 1$ and $p(x - y) \geq \delta > 0$ whenever $|x - y| \leq 2r$. The following is taken from Georgii and Haggstrom [22].

Divide $\mathbb{R}^d$ into cubic cells $\Delta(i), i \in Z^d$, with diameter at most r an pick a sufficiently large number n.

Call a cell $\Delta(i)$ *good* if it contains a connected component of the graph Γ with at least n points. This does not depend on the other cells and

$$P(\Delta(i) \text{ is good}) \geq \pi^z(N_i \geq n)[1 - (n-1)(1 - \delta^2)^{n-2}] =: p_s \tag{4.21}$$

where N_i is the number of points in $\Delta(i)$, and the second term is an estimate for the probability that one of the n points is not connected to the first point by a sequence of two edges. We say that $x \to_2 z$ if there is an y such that $x \to y \to z$. For each x there are $(n-1)$ z's, and $x \not\to_2 z$ if the $n-2$ possible connections fail (there are $(n-2)$ possible y's to connect; connections occur independently with probability at least δ^2.

p_s is arbitrarily close to 1 when n and z are large enough.

Call two good adjacent cells $\Delta(i), \Delta(j)$ *linked* if there exists an edge from some point in the connected component of $\Delta(i)$ to some point in the connected component of $\Delta(j)$. Conditionally on the event that $\Delta(i)$ and $\Delta(j)$ are good, this has probability at least $1-(1-\delta)^{n^2} =: p_b$, which is also close to 1 when n is large enough.

We have constructed a site-bond percolation model in $\mathbb{Z}^d$ with p_s and p_b as a function of our original continuum percolation model with parameters z and p.

$$\begin{aligned}&\{\text{there exists an infinite cluster of linked good cells}\}\\ &\subset \{\text{there exists an infinite cluster in the Poisson random-edge model}\}.\end{aligned}$$

Hence,

$$\theta(z,p;\mathbb{R}^d) \geq \frac{n}{|\Delta(0)|}\theta(p_s,p_b;\mathbb{Z}^d), \tag{4.22}$$

where $\theta(p_s,p_b;\mathbb{Z}^d)$ is the probability that the origin is connected to infinity in the discrete model. The right hand side dominates from below the average number of points of X in $\Delta(0)$ connected to infinity divided by the volume of $\Delta(0)$.

Hence $\theta(z,p;\mathbb{R}^d) > 0$ when z is large enough. □

4.3.1 Stochastic domination

Assume the random elements X, Y assume values in a space R, with a partial order denoted $\leq$. We say that

$$X \leq Y, \qquad \text{stochastically} \tag{4.23}$$

if there exists a coupling $(\hat{X},\hat{Y}) \in R^2$, that is, a random vector in R^2 with distribution $\tilde{P}$, with marginals $\hat{X} \sim X$ and $\hat{Y} \sim Y$ ($\sim$ means "with the same distribution") such that

$$\tilde{P}(X \leq Y) = 1. \tag{4.24}$$

Proposition 4.7. *X is stochastically dominated by Y if and only if for all nondecreasing $f : R \to \mathbb{R}$ we have*

$$Ef(X) \leq Ef(Y). \tag{4.25}$$

Proof. Exercise when $R = \mathbb{R}$, see §3.2 in Thorisson [48]. For general R the result "(4.25) implies $X \leq Y$ stochastically" is known as Strassen Theorem. □

Examples: (1) $X_i \sim$ Bernoulli(p_i), with $p_1 \leq p_2$. Define $\hat{X}_i = \mathbf{1}\{U \leq p_i\}$, to show that $X_1 \leq X_2$, stochastically.

(2) Poisson processes X_i with intensities z_i, $z_1 \leq z_2$. Construct the coupling as an exercise.

Stochastic domination of point processes

A simple point process P on a bounded Borel subset Λ of $\mathbb{R}^d$ (a probability measure on $\mathcal{X}_\Lambda = \{X \in \mathcal{X} : X \subset \Lambda\}$) has **Papangelou (conditional) intensity** $\gamma : \Lambda \times \mathcal{X}_\Lambda \to [0, \infty[$ if P satisfies

$$\int P(dX) \sum_{x \in X} f(x, X \setminus \{x\}) = \int dx \int P(dX) \gamma(x|X) f(x, X) \tag{4.26}$$

for any measurable function $f : \Lambda \times \mathcal{X}_\Lambda \to [0, \infty[$.

$\gamma(x|X)dx$ is the "conditional intensity" for the existence of a particle in dx when the remaining configuration is X.

The Poisson process π^z_Λ on Λ has Papangelou intensity $\gamma(x|X) = z$.

Holley Preston inequality Consider the partial order induced by the inclusion relation on $\mathcal{X}_\Lambda$.

Proposition 4.8 (Holley-Preston inequality, Georgii and Kuneth [23]). *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Borel set and μ, μ' probability measures on $\mathcal{X}_\Lambda$ with Papangelou intensities γ resp. γ'. Suppose $\gamma(x|X) \leq \gamma'(x|X')$ whenever $X \subset X'$ and $x \notin X' \setminus X$. Then $\mu \leq \mu'$.*

This inequality is useful to compare systems with dependencies with Poisson-Bernoulli percolation models. Proof postponed to the next section.

4.4 The continuum Ising model

Point particles in $\mathbb{R}^d$ of two different types, plus and minus.

A configuration is a pair $\xi = (X^+, X^-)$.

Configuration space: $\Omega = \mathcal{X}^2$.

Repulsive interspecies interaction pair potential of finite range, given by an even function $J : \mathbb{R}^d \to [0, \infty]$ with bounded support.

The Hamiltonian in a bounded Borel set $\Lambda \subset \mathbb{R}^d$ of a configuration $\xi = (X^+, X^-)$ is given by

$$H_\Lambda(\xi) := \sum_{x\in X^+, y\in X^- : \{x,y\}\cap\Lambda\neq\emptyset} J(x-y). \tag{4.27}$$

Example: classical Widom-Rowlinson model (1970) with a hard-core interspecies repulsion: $J(x-y) = \infty$ when $|x-y| \le 2r$ and $J(x-y) = 0$ otherwise.

Assumption:

$$\text{There exist } \delta, r > 0 \text{ such that } J(x-y) \ge \delta \text{ if } |x-y| \le 2r. \tag{4.28}$$

Definition. The *Gibbs distribution in* Λ *with activity* $z > 0$ *and boundary condition* $\xi_{\Lambda^c} = (X^+_{\Lambda^c}, X^-_{\Lambda^c}) \in \mathcal{X}^2_{\Lambda^c}$ is

$$G_\Lambda(d\xi_\Lambda|\xi_{\Lambda^c}) = Z^{-1}_{\Lambda|\xi_{\Lambda^c}} \prod_{\substack{x\in X^+\\ y\in X^-\\ \{x,y\}\not\subset\Lambda^c}} \mathbf{1}\{\|x-y\| > r\}\, \pi^z_\Lambda(dX^+_\Lambda)\, \pi^z_\Lambda(dX^-_\Lambda) \tag{4.29}$$

where the normalization factor $Z_{\Lambda|\xi_{\Lambda^c}}$, also called *partition function* is given by

$$Z^{-1}_{\Lambda|\xi_{\Lambda^c}} = \int_{\mathcal{X}\times\mathcal{X}} \prod_{\substack{x\in X^+\\ y\in X^-\\ \{x,y\}\not\subset\Lambda^c}} \mathbf{1}\{\|x-y\| > r\}\, \pi^z_\Lambda(dX^+_\Lambda)\, \pi^z_\Lambda(dX^-_\Lambda) \tag{4.30}$$

so that, $G(\cdot|\xi_{\Lambda^c})$ is a probability distribution in $\mathcal{X}\times\mathcal{X}$.

Definition. The set of *Gibbs measures* $\mathcal{G} = \mathcal{G}(z)$ are the measures having G_Λ as conditional probabilities, for all Λ bounded Borel set in $\mathbb{R}^d$.

That is, G defined on the set of infinite point configurations $\mathcal{X}\times\mathcal{X}$ is Gibbs for the specifications $(G_\Lambda(\cdot|\xi) : \Lambda$ bounded, $\xi \in \mathcal{X}\times\mathcal{X})$ if

$$G(f(\xi)|\xi_{\Lambda^c}) = G_\Lambda(f(\xi)|\xi_{\Lambda^c}) \tag{4.31}$$

The family G_Λ must be *consistent*, meaning that for bounded sets $\Lambda \subset \Delta$, the following must be satisfied

$$G_\Delta(f(\xi)|\xi_{\Delta^c}) = G_\Delta(\, G_\Lambda(f(\xi)\,|\,\xi_{\Lambda^c})\,|\,\xi_{\Delta^c}) \tag{4.32}$$

Existence of Gibbs measures The Papangelou intensity of the measure $G_\Lambda(\cdot|\xi_{\Lambda^c})$ is given by

$$\gamma(x|X^+,X^-) = \left\{ \begin{array}{ll} z\prod_{y\in X^-} \mathbf{1}\{\|x-y\|>r\} & \text{if } x\in\Lambda^+ \\ z\prod_{y\in X^+} \mathbf{1}\{\|x-y\|>r\} & \text{if } x\in\Lambda^- \end{array} \right\} \le z.$$

where we think Λ^+,Λ^- as copies of Λ and $X^+\subset\Lambda^+$, $X^-\subset\Lambda^-$.

Holley-Preston implies that $G_\Lambda(\cdot|\xi_{\Lambda^c}) \le \pi^z_\Lambda \times \pi^z_\Lambda$. Compactness theorems for point processes show that for each $\xi\in\mathcal{X}^2$, $G_\Lambda(\cdot|\xi_{\Lambda^c})$ has an accumulation point P as $\Lambda \nearrow \mathbb{R}^d$.

Exercise. Use (4.32) to show that any limit $P\in\mathcal{G}$.

Uniqueness and phase transition We will show that the Gibbs measure is unique when z is small, whereas a phase transition (non-uniqueness) occurs when z is large.

It remains **open problem** whether there is a sharp activity threshold separating intervals of uniqueness and non-uniqueness. That is, a z_c such that for $z>z_c$ there are more than one extremal Gibbs measure and for $z<z_c$ there is only one Gibbs measure.

Proposition *For the continuum Ising model we have $\#\mathcal{G}(z)=1$ when z is sufficiently small.*

Proof. Let P, $P'\in\mathcal{G}(z)$. We will show that $P=P'$ when z is small enough.

Let R be the range of J, i.e., $J(x)=0$ when $|x|>R$.

Divide $\mathbb{R}^d$ into cubic cells $\Delta(i)$, $i\in\mathbb{Z}^d$, of linear size R.

Let p_c^* be the Bernoulli **site** percolation threshold of the graph with vertex set $\mathbb{Z}^d$ and edges between all points having distance 1 in the max-norm.

Consider the Poisson measure $Q^z=\pi^z\times\pi^z$ on $\Omega=\mathcal{X}^2$.

Let ξ ,ξ' be two independent realizations of Q^z, and suppose z is so small that $Q^z\times Q^z(N_i+N_i'\ge 1) < p_c^*$, where N_i and N_i' are the numbers of particles (plus or minus) in $\xi\cap\Delta(i)$, respectively $\xi'\cap\Delta(i)$.

Then, for any finite union Λ of cells we have

$$Q^z \times Q^z(\Lambda \overset{\geq 1}{\longleftrightarrow} \infty) = 0,$$

where $\{\Lambda \overset{\geq 1}{\longleftrightarrow} \infty\}$ denotes the event that a cell in Λ belongs to an infinite connected set of cells $\Delta(i)$ containing at least one particle in either ξ or ξ'.

Holley-Preston imply that $P \times P' \leq Q^z \times Q^z$. Hence

$$P \times P'(\Lambda \overset{\geq 1}{\longleftrightarrow} \infty) = 0.$$

In other words, given two independent realizations ξ and ξ' of P and P' there exists a **random corridor** of width R around Λ which is completely **free of particles**.

Given $\Delta \supset \Lambda$, denote $K_\Lambda(\Delta)$ the set of (ξ, ξ') such that "there is an empty corridor of width R in Δ around Λ and there is no $\Delta' \subset \Delta$ with this property".

Now, suppose that for any local set B depending on Λ and boundary conditions ξ and ξ', we have

$$\begin{aligned} &G_\Delta \times G_\Delta(B \times \mathcal{X} \cap K_\Lambda(\Delta)|(\xi,\xi')_{\Delta^c}) \\ &\qquad = G_\Delta \times G_\Delta(\mathcal{X} \times B \cap K_\Lambda(\Delta)|(\xi,\xi')_{\Delta^c}) \end{aligned} \tag{4.33}$$

then, integrate with respect to $P \times P'$ to obtain

$$(P \times P')(B \times \mathcal{X} \cap K_\Lambda(\Delta)) = (P \times P')(\mathcal{X} \times B \cap K_\Lambda(\Delta)) \tag{4.34}$$

Summing over all choices of $K_\Lambda(\Delta)$, we get $P(B) = P'(B)$, for any bounded measurable B in Ω. This implies $P = P'$.

It remains to show (4.33).

If Δ does not contain a corridor of width R around Λ free of ξ and ξ' particles, then both sides of (4.33) are 0.

If Δ does contain such corridor, the boundary conditions $(\xi, \xi')_{\Delta^c}$ have the same effect as the empty boundary conditions. Hence, we can replace ξ and ξ' by empty configurations. Since the event $K_\Lambda(\Delta)$ and the specifications $(G_\Delta \times G_\Delta)(\cdot|\emptyset \times \emptyset)$ are invariant under the map $(\xi, \xi') \to (\xi', \xi)$, the equation (4.33) holds. □

Existence of phase transition We will see that there are at least two Gibbs measures for z large, using a coupling with a *random-cluster model* at high density.

Gibbs distribution with + boundary conditions.

$$G_\Lambda^+ = \int \pi_{\Lambda^c}^z(dY_{\Lambda^c}^+)\, G_\Lambda(\cdot|Y_{\Lambda^c}^+, \emptyset) \tag{4.35}$$

This is a mixture of specifications with a boundary conditions: Poisson(z) of plus-particles and no minus-particles in Λ^c.

Random-cluster model: Fix the activity z. Define the probability measure χ_Λ^z on $\mathcal{X} \times \mathcal{E}$ describing random graphs (Y, E):

$$\chi_\Lambda^z(dY, dE) = Z_{\Lambda|Y_{\Lambda^c}}^{-1}\, 2^{k(Y,E)}\, \pi^z(dY)\, \mu_Y^{p,\Lambda}(dE).$$

where $k(Y, E)$ is the number of clusters of the graph (Y, E);

$$Z_{\Lambda|Y_{\Lambda^c}} := \int 2^{k(Y,E)}\, \pi^z(dY_\Lambda)\, \mu_Y^{p,\Lambda}(dE).$$

is the normalization factor and $\mu_Y^{p,\Lambda}$ is the probability measure on E for which the edges $e = \{x, y\} \subset Y$ are drawn independently with probability

$$p(x-y) = \begin{cases} 1 - e^{-J(x-y)} & \text{if } e = \{x, y\} \not\subset Y_{\Lambda^c} \\ 1 & \text{otherwise.} \end{cases} \tag{4.36}$$

χ_Λ is called the *continuum random-cluster* distribution in Λ with connection probability function p and **wired** boundary condition. This means that all points connected to Y_{Λ^c} belong to the same cluster. See [24] for a discrete version of the random cluster model.

If we eliminate the factor $2^{k(Y,E)}$ we have just the continuum percolation model with a unique cluster containing Y_{Λ^c}.

Dependent Boolean percolation In the case of a hard-core interspecies repulsion we have the deterministic rule

$$p(x-y) = \mathbf{1}\{\|x-y\| < r\}. \tag{4.37}$$

In this case χ_Λ^z describes a dependent Boolean percolation model. The dependency comes from the factor $2^{k(Y,E)}$.

4.4.1 Coupling random-cluster and continuum Ising

Let P_Λ be the distribution on $\mathcal{X} \times \mathcal{X} \times \mathcal{E}$ given by

$$P_\Lambda(dX^+, dX^-, dE) = \pi^z(dX^+)\, \pi_\Lambda^z(dX^-)\, \mu_{X^+\cup X^-}^{p,\Lambda}(dE)$$

with p given by (4.36) with $Y = X^+ \cup X^-$. In words, the model consists of two independent Poisson processes with random edges connecting points regardless of the sign of the connected points.

In the hard core case the distribution of the edges is deterministic. Just connect all pair of points that are at distance less than r.

Let A_Λ be the set of configurations with no connections between particles of different sign:

$$A_\Lambda := \{(X^+, X^-, E) \in \mathcal{X} \times \mathcal{X} \times \mathcal{E} \,:\, E \in \mathcal{E}(X^+ \cup X^-) \tag{4.38}$$

$$\text{and } E \not\ni \{x, y\} \text{ for } x \in X^+,\, y \in X^-\} \tag{4.39}$$

Configurations contained in A have the property that any connected component C is either contained in X^+ or contained in X^-. Define Q_Λ as the conditioned measure

$$Q_\Lambda = P_\Lambda(\cdot|A_\Lambda) \tag{4.40}$$

$$Q_\Lambda(dX^+, dX^-, dE) = \frac{1}{P_\Lambda(A_\Lambda)} \mathbf{1}\{(X^+, X^-, E) \in A_\Lambda\} \tag{4.41}$$

$$\times\, \pi^z(dX^+)\, \pi^z_\Lambda(dX^-)\, \mu^{p,\Lambda}_{X^+\cup X^-}(dE) \tag{4.42}$$

To obtain a sample of Q_Λ in the hard core case (4.37), sample Poisson processes X^+ and X^-. If $\|x - y\| > r$ for all $x \in X^+$, $y \in X^-$, then accept the sample. Otherwise, reject and sample again.

Proposition 4.9. *For any bounded box Λ in $\mathbb{R}^d$, let (X^+, X^-, E) be distributed with Q_Λ. Then,*

1. *The the point processes (X^+, X^-) has marginal distribution G^+_Λ.*
2. *The random graph $(X^+ \cup X^-, E)$ has distribution χ^z_Λ.*

Proof. **1.**

$$\begin{aligned}
&Q_\Lambda(dX^+, dX^-)\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+) \pi^z_\Lambda(dX^-) \int_{E:(X^+,X^-,E)\in A_\Lambda} \mu^{p,\Lambda}_{X^+\cup X^-}(dE)\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+) \pi^z_\Lambda(dX^-) \prod_{x\in X^+, y\in X^-} (1 - p_\Lambda(\{x, y\}))\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+) \pi^z_\Lambda(dX^-) \prod_{x\in X^+, y\in X^-} \exp(-J(x - y))\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+) \pi^z_\Lambda(dX^-) \exp(-H_\Lambda(X^+, X^-))
\end{aligned}$$

$$= G^+_\Lambda(\mathrm{d}X^+, \mathrm{d}X^-) \tag{4.43}$$

In the hard core case:

$$\begin{aligned}
&Q_\Lambda(dX^+, dX^-)\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+)\pi^z_\Lambda(dX^-) \int_{E:(X^+,X^-,E)\in A_\Lambda} \mu^{p,\Lambda}_{X^+\cup X^-}(dE)\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+)\pi^z_\Lambda(dX^-) \prod_{x\in X^+, y\in X^-} \mathbf{1}\{\|x-y\| > r\}\\
&= G^+_\Lambda(\mathrm{d}X^+, \mathrm{d}X^-)
\end{aligned} \tag{4.44}$$

2. Let $f : \mathcal{X} \times \mathcal{E}$ be a test function. Then, denoting $\mathbb{E}$ the expectation associated to P_Λ, and abusing notation, considering X^+, X^-, E as integration variables in the first line and as random variables in the others, we get

$$\begin{aligned}
&\int_{\mathcal{X}\times\mathcal{X}\times\mathcal{E}} f(X^+\cup X^-, E)\, Q_\Lambda(dX^+, dX^-, dE)\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \mathbb{E}\big(f(X^+\cup X^-, E)\ \mathbf{1}\{(X^+, X^-, E) \in A_\Lambda\}\big)\\
&= \frac{1}{P_\Lambda(A_\Lambda)} \mathbb{E}\big[\mathbb{E}\big(f(X^+\cup X^-, E)\ \mathbf{1}\{(X^+, X^-, E) \in A_\Lambda\} \,|\, (\tilde{Y}, E)\big)\big]
\end{aligned} \tag{4.45}$$

where $\tilde{Y} := X^+ \cup X^-$ is a Poisson process π^{z_Λ} in $\mathbb{R}^d$ with intensity $z_\Lambda(x) = z(1 + \mathbf{1}\{x \in \Lambda\})$.

$$= \frac{1}{P_\Lambda(A_\Lambda)} \mathbb{E}\big[f(\tilde{Y}, E)\ P_\Lambda((X^+, X^-, E) \in A_\Lambda \,|\, (\tilde{Y}, E))\big] \tag{4.46}$$

$$= \frac{1}{P_\Lambda(A_\Lambda)} \mathbb{E}\big[f(\tilde{Y}, E)\ \frac{2^{k(\tilde{Y},E)}}{2^{\#\tilde{Y}}}\big] \tag{4.47}$$

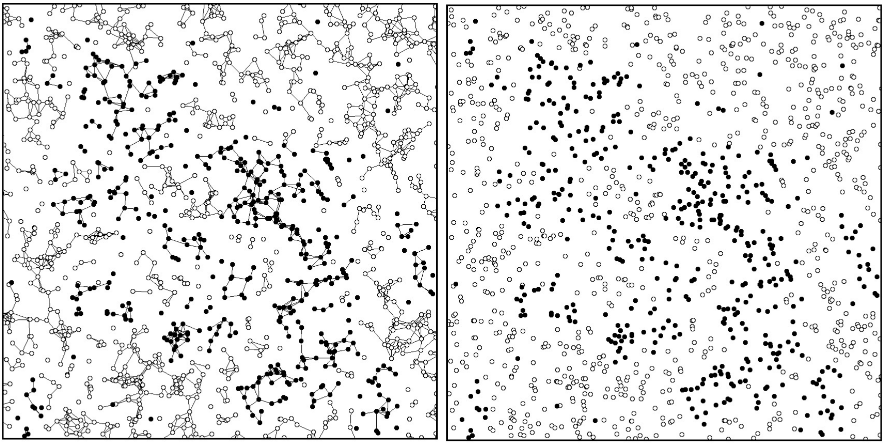
because the set $\{(X^+, X^-) : X^+ \cup X^- = \tilde{Y} \text{ and } (X^+, X^-, E) \in A_\Lambda\}$ has $2^{k(\tilde{Y},E)}$ elements, and each element has probability $2^{-\#\tilde{Y}}$, by the color theorem. The above expression equals

$$= \frac{1}{ZP_\Lambda(A_\Lambda)} \mathbb{E}\big[f(Y, E)\ 2^{k(Y,E)}\big] \tag{4.48}$$

where Y is a Poisson process with law π^z (exercise). The above expression equals

$$= \int f(Y, E)\, \chi^z_\Lambda(dY, dE). \tag{4.49}$$

The normalization comes for free because we started with a probability. □



Figures from Georgii's web page. Left picture: a sample of the measure Q_Λ. Right picture: the Ising marginal (erasing the edges). If we erase the colors in the left picture we get a sample of the random cluster model.

Proposition 4.10. *3. The conditioned law of* $(X^+ \cup X^-, E)$ *given* $\xi = (X^+, X^-)$ *is the following. Independently for all* $e = \{x, y\} \in E(Y)$ *let* $e \in E$ *with probability*

$$p_{\Lambda,X^+,X^-}(e) = \begin{cases} 1 - e^{-J(x-y)}, & e \subset X^+ \text{ or } e \subset X^-,\ e \not\subset \Lambda^c, \\ 1, & e \subset \Lambda^c, \\ 0, & \text{otherwise.} \end{cases} \tag{4.50}$$

In the hard core case,

$$p_{\Lambda,X^+,X^-}(e) \tag{4.51}$$

$$= \begin{cases} 1\{\|x - y\| < r\}, & e \subset X^+ \text{ or } e \subset X^-,\ e \not\subset \Lambda^c, \\ 1, & e \subset \Lambda^c, \\ 0, & \text{otherwise.} \end{cases} \tag{4.52}$$

That is, put the edge $\{x, y\}$ *if* $\|x - y\| < r$ *and* $\{x, y\} \subset X^+$ *or* $\{x, y\} \subset X^-$.

4. The conditional distribution of (X^+, X^-) *given* (Y, E),

$$Q_\Lambda(\mathrm{d}X^+, \mathrm{d}X^- | (Y, E)),$$

is the following. For each finite cluster C *of* (Y, E) *let* $C \subset X^+$ *or* $C \subset X^-$ *according to independent flips of a fair coin; the unique infinite cluster of* (Y, E) *containing* Y_{Λ^c} *is included into* X^+.

Proof. **3.** Fix E and denote $p(x,y) = Q_\Lambda(\{x,y\} \in E | X^+, X^-)$. Since $(X^+, X^-, E) \in A_\Lambda$, we have that $p(x,y) = 0$ if $x \in X^+$ and $y \in X^-$, so that E does not contain vertices with different sign. The other edges intersecting Λ may take any value while those contained in Λ^c must be 1. Hence

$$p(x,y) = p_{\Lambda,X^+,X^-}(x,y),$$

as defined in (4.50) and

$$Q_\Lambda((X^+ \cup X^-, E) \,|\, X^+, X^-) = \prod_{\{x,y\}\in E} p(x,y). \tag{4.53}$$

In the **hard core case**, connect all pair of points (with same sign) distant less than r, see (4.51). Indeed, in this case

$$p(x,y) = \mathbf{1}\{\|x-y\| < r\}, \quad x,y \in X^+ \cup X^-. \tag{4.54}$$

4. Given (Y,E) the only possible configurations (X^+, X^-) satisfying that any cluster C is contained either in X^+ or in X^-. Since each cluster has the same probability to be in either of those sets, we have

$$Q_\Lambda(X^+, X^- \,|\, (Y,E)) \quad = \prod_{C \text{ cluster of } (Y,E)} \frac{1}{2} = \Big(\frac{1}{2}\Big)^{k(Y,E)}. \qquad \square$$

Percolation and phase transition

The random-cluster representation gives the following key identity: For any finite box $\Delta \subset \Lambda$,

$$\int [\#X^+_\Delta - \#X^-_\Delta] G^+_\Lambda(dX^+, dX^-) = \int \#\{x \in Y_\Delta : x \leftrightarrow Y_{\Lambda^c}\} \chi_\Lambda(dY, dE);$$

where $x \leftrightarrow Y_{\Lambda^c}$ means that x is connected to a point of Y_{Λ^c} in the graph (Y,E). This is because the finite clusters have the same probability to belong to X^+ and to X^- and the points connected to the infinite cluster belong to X^+.

The **difference between the mean number of plus- and minus-particles** in Δ is the same as expected number of points in Δ connected to the infinite cluster in χ_Λ.

Percolation in the random cluster model The point marginal distribution of χ_Λ is given by

$$\nu_\Lambda f := \int f(Y)\, \chi_\Lambda(dY, dE), \tag{4.55}$$

for test functions $f : \mathcal{X} \to \mathbb{R}$.

Then, item 4 of Proposition 4.9 implies

$$\chi_\Lambda(dY, dE) = \nu_\Lambda(dY)\varphi_{\Lambda,Y}(dE)$$

with

$$\varphi_{\Lambda,Y}(dE) = \frac{1}{Z_\Lambda} 2^{k(Y,E)} \mu_Y^{p,\Lambda}(dE)$$

In the hard core case:

$$\varphi_{\Lambda,Y}(dE) = \frac{1}{Z_\Lambda} 2^{k(Y,E)} \prod_{\{x,y\}\in E} \mathbf{1}\{\|x-y\| < r\}\, dE$$

E is deterministic, depending only on Y in this case.

Proposition 4.11 (Edge domination)**.** *$\varphi_{\Lambda,Y}$ is stochastically larger than the Bernoulli edge measure $\mu_Y^{\tilde{p}}$ for which edges are drawn independently between points x, $y \in Y$ with probability*

$$\tilde{p}(x-y) = \frac{1-e^{-\delta}}{(1-e^{-\delta}) + 2e^{-\delta}}, \text{ for } |x-y| \leq 2r, \tag{4.56}$$

and with probability 0 otherwise, where δ and r satisfy (4.28).

Proof. We fix $Y \sim \nu$ and consider edges distributed with $\mu(dE) := \varphi_{\Lambda,Y}(dE)$ on one hand and with $\mu'(dE)\mu_Y^{\tilde{p}}$ on the other hand. In order to apply Holley inequality compute

$$\begin{aligned} \mathrm{p}(e) &:= \mu(\eta(e) = 1 | \eta_{E(Y)\setminus\{e\}}) \\ &= \frac{p(x,y) 2^{k(Y,E(\eta^1))} \mu_Y^p(\eta_{E(Y)\setminus\{x,y\}})}{\left[p(x,y)2^{k(Y,E(\eta^1))} + (1-p(x,y))2^{k(Y,E(\eta^0))}\right] \mu_Y^p(\eta_{E(Y)\setminus\{x,y\}})} \end{aligned}$$

where $E(\eta) = \{e \in E(Y) : \eta(e) = 1\}$ and

$$\begin{aligned} \eta^1(\{x,y\}) = 1 \text{ and } \eta^1(e) = \eta(e) \text{ for } e \neq \{x,y\}, \\ \eta^0(\{x,y\}) = 0 \text{ and } \eta^0(e) = \eta(e) \text{ for } e \neq \{x,y\} \end{aligned} \tag{4.57}$$

The numerator is the probability under μ_Y^p of the configuration $\eta_{E(Y)\setminus\{e\}}$ off e and $\eta(e) = 1$. In the denominator we have the marginal distribution of $\eta_{E(Y)\setminus\{e\}}$ under the same measure.

We have

$$k(Y, E(\eta^0)) \leq k(Y, E(\eta^1)) + 1, \tag{4.58}$$

because connecting x and y decreases at most by one the number of clusters. Hence,

$$\mathrm{p}(x,y) \geq \frac{p(x,y)}{p(x,y)+2(1-p(x,y))} \geq \frac{1-e^{-\delta}}{(1-e^{-\delta})+2e^{-\delta}} = \tilde{\mathrm{p}}(x,y).$$

if $|x-y|>2r$, by hypothesis. Apply Holley inequality to conclude. □

Proposition 4.12 (Point domination)**.** *There exists $\alpha > 0$ such that the point marginal ν_Λ of χ_Λ dominates a Poisson process of rate αz:*

$$\nu \geq \pi^{\alpha z}. \tag{4.59}$$

Proof. The point marginal ν_Λ has Papangelou intensity

$$\gamma(x|Y) = \frac{z \displaystyle\int 2^{k(Y\cup x,E)} \phi_{\Lambda,Y\cup x}(dE)}{\displaystyle\int 2^{k(Y,E)} \phi_{\Lambda,Y}(dE)}.$$

In the hard rod case, E is deterministically determined by Y and we get the simplified expression

$$\gamma(x|Y) = \frac{z\, 2^{k(Y\cup x,E)}}{2^{k(Y,E)}}.$$

To get a lower estimate for $\gamma(x|Y)$ one has to compare the effect on the number of clusters in (Y,E) when a particle at x and corresponding edges are added. In principle, this procedure could connect a large number of distinct clusters lying close to x, so that $k(Y\cup x,\cdot)$ was much smaller than $k(Y,\cdot)$. However, one can show that this occurs only with small probability, so that

$$\gamma(x|Y) \geq \alpha z \text{ for some } \alpha > 0. \tag{4.60}$$

for instance for $\mathbb{R}^2$ hard core, we can connect at most 6 disjoint clusters (exercise) and we get

$$\gamma(x|Y) \geq \frac{z}{2^6} > 0. \tag{4.61}$$

So, it works for $\alpha = 2^{-6}$. □

Conclusion: The previous propositions imply that χ_Λ **is stochastically larger than the Poisson random-edge measure** $P_{\alpha z,\tilde{p}}$ defined in (11).

Hence,

$$\int \#\{x \in Y_\Delta : x \leftrightarrow Y_{\Lambda^c}\}\, \chi_\Lambda(dY, dE) \geq \ell(\Delta)\, \theta(\alpha z, \tilde{p}; \mathbb{R}^d)$$

Finally, since $G_\Lambda^+ \leq \pi^z \times \pi^z$ by (17), the Gibbs distributions G_Λ^+ have a cluster point $P^+ \in \mathcal{G}(z)$, a Gibbs measure satisfying

$$\int [\#X_\Delta^+ - \#X_\Delta^-] P^+(dX^+, dX^-) \geq \theta(\alpha z, \tilde{p}; \mathbb{R}^d).$$

By spatial averaging one can achieve that P^+ is in addition translation invariant. Together with the continuum percolation Theorem, saying $\theta(z, p; \mathbb{R}^d) > 0$ for sufficiently large zp, this leads to the following theorem.

Theorem *For the continuum Ising model on $\mathbb{R}^d$, $d \geq 2$, with Hamiltonian* (4.27) *and sufficiently large activity z there exist two translation invariant Gibbs measures P^+ and P^- having a majority of plus- resp. minus-particles and related to each other by the plus-minus interchange.*

This result is due to Georgii and Haggstrom (1996). In the special case of the Widom-Rowlinson model it has been derived independently in the same way by Chayes, Chayes, and Kotecky (1995). The first proof of phase transition in the Widom-Rowlinson model was found by Ruelle in 1971, and for a soft but strong repulsion by Lebowitz and Lieb in 1972. Gruber and Griffiths (1986) used a direct comparison with the lattice Ising model in the case of a species-independent background hard core.

As a matter of fact, one can make further use of stochastic monotonicity. (In contrast to the preceding theorem, this only works in the present case of two particle types.) Introduce a partial order '$\leq$ ' on $\Omega = \mathcal{X}^2$ by writing $(X^+, X^-) \leq (Y^+, Y^-)$ when $X^+ \subset Y^+$ and $X^- \supset Y^-$. (21). A straightforward extension of Holley-Preston then shows that the measures G_Λ^+ decrease stochastically relative to this order when Λ increases. (This can be also deduced from the couplings obtained by perfect simulation, see the paper of Georgii, Section 4.4.) It follows that P^+ is in fact the limit of these measures, and is in particular translation invariant. Moreover, one can see that P^+ is stochastically maximal in $\mathcal{G}$ in this order.

Corollary *For the continuum Ising model with any activity $z > 0$, a phase transition occurs if and only if*

$$\int \hat{P}^+(dX^+, dX^-) \mu_{p,X^+}(0 \stackrel{+}{\longleftrightarrow} \infty) > 0;$$

here $\hat{P}^+$ is the Palm measure of P^+, and the relation $0 \stackrel{+}{\longleftrightarrow} \infty$ means that the origin belongs to an infinite cluster in the graph with vertex set X^+ and random edges drawn according to the probability function $p = 1 - e^{-J}$.

It is not known whether P^+ and P^- are the only extremal elements of $\mathcal{G}(z)$ when $d = 2$, as it is the case in the lattice Ising model. However, using a technique known in physics as the Mermin–Wagner theorem one can show the following.

Theorem *If J is twice continuously differentiable then each $P \in \mathcal{G}(z)$ is translation invariant.*

A proof can be found in Georgii (1999). The existence of non-translation invariant Gibbs measures in dimensions $d \geq 3$ is an open problem.

5 Perfect simulation of point processes

This section is based on joint work with Fernandez and Garcia [13]. Here we propose a perfect simulation algorithm and construction of a hard core measures locally absolutely continuous with respect to a Poisson process. They will be invariant measures of birth-and-death processes of points interacting by exclusion. We obtain perfect samples of finite windows of the *infinite-volume* measure

Background. See Wilson page for a complete list of references on *perfect simulation*: `http://www.dbwilson.com/exact/`. Spatial processes: Kendall and Moeller. All the above apply for finite state space or finite regions (*coupling with finite coalescence time*).

5.1 Free birth and death spatial process in $\mathbb{R}^d$

Let $\mathbb{X} = \mathbb{R}^d$ and consider a mean measure w on $\mathbb{X}$.

We construct a birth and death process $(X_t)_{t\in\mathbb{R}}$, $X_t \in \mathcal{X}$, the set of countable subsets of $\mathbb{X}$; that is, the evolution of a point process.

Let $\mathbf{C}$ be a Poisson process on $\mathbb{X} \times \mathbb{R} \times \mathbb{R}^+$ with intensity

$$w(x)\, dx\, dt\, e^{-s} ds \tag{5.1}$$

An element of $\mathbf{C}$ is a triplet $C = (x, T, S)$, with $x \in \mathbb{X}$, $T \in \mathbb{R}$, $S \in \mathbb{R}^+$. We identify the point $C = (x, T, S)$ with the space-time *cilinder*

$$C = x \times [T, T + S) \qquad \text{(abusing notation)} \tag{5.2}$$

and interpret x as the *basis*, T as the *birth-time*, S as the *life-time* and $T+S$ as the *death-time* of a the basis x of the cylinder C.

We say that $C \in \mathbf{C}$ is *alive* at time t if $T \le t < T+S$. Define the process

$$\tilde{X}_t \;:=\; \big\{\text{Basis}(C) \,:\, C \in \mathbf{C},\, C \text{ alive at } t\big\}. \tag{5.3}$$

Call $(\tilde{X}_t)_{t\in\mathbb{R}}$ the *stationary free birth death process.*

Semigroup and generator. For a function $f : \mathbb{X} \to \mathbb{R}$ with compact support, we define

$$\tilde{S}_t f(X) := E(f(\tilde{X}_t)|\tilde{X}_0 = X)). \tag{5.4}$$

$\tilde{S}_t$ is a semigroup with *generator* $\tilde{L}$, an operator defined on test f with bounded support, by

$$\widetilde{L}f(X) = \sum_{x\in X}[f(X\setminus\{x\}) - f(X)] + \int_{\mathbb{X}} dx\, w(x)\, [f(X\cup\{x\}) - f(X)]. \tag{5.5}$$

That is, it satisfies the Kolmogorov equations

$$\frac{d}{dt}\tilde{S}_t f(X) = \tilde{L}\tilde{S}_t f(X) \qquad \text{Backward equations} \tag{5.6}$$

$$\frac{d}{dt}\tilde{S}_t f(X) = \tilde{S}_t\tilde{L} f(X) \qquad \text{Forward equations} \tag{5.7}$$

Proposition 5.1 (Stationary measure)**.** *Denote $\tilde{\mu}$ the marginal law of $\tilde{X}_t$. Then $\tilde{\mu}$ is a Poisson process on $\mathbb{X}$ of intensity w.*

Proof. We show that X_0 has distribution $\tilde{\mu}$. Let $\Phi(x,t,s) := x\,\mathbf{1}\{-s < t < 0\}$. By the mapping theorem, $X_0 = \Phi(\mathbf{C})$ is a Poisson process with intensity

$$\int_{-\infty}^{0} dt \int_{-t}^{\infty} e^{-s} ds\, w(x) = w(x). \tag{5.8}$$

Details and general t as an exercise. □

The process in finite time intervals. We define the free process for *positive* times in $[0,\infty)$ with initial configuration $X_0 = X \subset \mathbb{X}$ by including *initial cylinders*

$$\mathbf{C}_0(X) = \Big\{\big(x, 0, S_x\big) : x \in X\Big\}, \tag{5.9}$$

where S_x are independent exp(1) variables. The sets of cylinders of $\mathbf{C}$ born at positive times are denoted by

$$\mathbf{C}_+ := \{C \in \mathbf{C} \,:\, \text{Birth}(C) \ge 0\}. \tag{5.10}$$

Define

$$\tilde{X}_t^X \ =: \ \Big\{\text{Basis}(C) \ : \ C \in \mathbf{C}_+ \cup \mathbf{C}_0(X), \, C \text{ alive at time } t\Big\}$$

Then $(\tilde{X}_t^X)_{t\geq 0}$ has initial configuration X and generator $\tilde{L}$.

5.2 Birth and death spatial process with exclusions

We define now a measure with exclusions and then a birth death process having the measure as invariant. Let Λ be a bounded Borel subset of $\mathbb{X}$.

A hard core measure Define the set

$$\mathcal{A}^\Lambda = \{X \subset \mathbb{X} : \|x - x'\| > r, \ x, x' \in X \cap \Lambda\}. \tag{5.11}$$

The condition $\|x - x'\| > r$ can be thought of as $B(x) \cap B(x') = \emptyset$, where $B(x)$ is a ball of radious $\frac{r}{2}$ and center x. Denote

$$\mu^\Lambda = \tilde{\mu}(\cdot|\mathcal{A}^\Lambda). \tag{5.12}$$

μ^Λ is the distribution of the Poisson process $\tilde{\mu}$, conditioned to the hard core exclusion rule: "no pair of points stay at distance less than r". The Papangelou intensity of μ^Λ is given by

$$\gamma^\Lambda(x|X) = w(x)\, \mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\}, \quad x \in \Lambda. \tag{5.13}$$

We will see that the process with birth rate $\gamma^\Lambda(x|X)$ and rate of death of x equal 1 has μ^Λ as invariant.

Construction of a birth and death process with exclusion Recall the Poisson process $\mathbf{C}^\Lambda$ on $\Lambda \times \mathbb{R} \times \mathbb{R}_{\geq 0}$, with intensity (5.1) and the sets $\mathbf{C}_+^\Lambda$ of cylinders born after time 0 and, for an $X \in \mathcal{A}^\Lambda$, the set of cylinders born at time 0 with bases X, denoted $\mathbf{C}_0^\Lambda(X)$, see (5.10) (5.9).

Recall that the intensity of $\mathbf{C}^\Lambda$ is locally integrable, and that there are a finite number of cylinders in $\mathbf{C}$.

We say that a cylinder $C = (x, t, s)$ is *compatible* with a given configuration of cylinders $\mathbf{C}'$ if

$$\|x - x'\| > r, \text{ for all cylinder } (x', t', s') \in \mathbf{C}', \text{ alive at time } t. \tag{5.14}$$

Let $X \in \mathcal{A}^\Lambda$ and define iteratively the set of *kept* cylinders $\mathbf{K}_0^\Lambda = \mathbf{K}_0^\Lambda(X)$, as follows. Define $t_0 = 0$ and $\mathbf{K}_0 := \mathbf{C}_0^\Lambda(X)$ and, assuming we have defined t_k and $\mathbf{K}_k$, let

$$t_{k+1} := \inf\{t > t_k : (x,t,s) \in \mathbf{C}_+^\Lambda \cup \mathbf{C}_0^\Lambda(X)\},$$

and, denoting $C = (x,t,s)$ the cylinder realizing the infimum, that is, with $t = t_{k+1}$, let

$$\mathbf{K}_{k+1} := \begin{cases} \mathbf{K}_k \cup \{C\} & \text{if } C \text{ is compatible with } \mathbf{K}_k \\ \mathbf{K}_k & \text{otherwise.} \end{cases}$$

This corresponds to iteratively erase cylinders (x,t,s) with $\gamma(x|X_{t-}) = 0$. Denote by

$$\mathbf{K}_0^\Lambda := \cup_{k\geq 0}\mathbf{K}_k, \tag{5.15}$$

the resulting set of kept cylinders.

Define

$$X_t^\Lambda := \big\{x \,:\, (x',t',s') \in \mathbf{K}_0^\Lambda(X),\, (x',t',s') \text{ alive at } t\big\}, \quad t \geq 0\,. \tag{5.16}$$

Proposition 5.2. *The process* $(X_t^\Lambda)_{t\geq 0}$ *defined in* (5.16) *is Markov with generator*

$$Lf(X) := \sum_{x\in X}[f(X\setminus\{x\}) - f(X)] + \int_{\mathbb{X}} dx\, \gamma(x\,|\,X)\,[f(X\cup\{x\}) - f(X)]. \tag{5.17}$$

when applied to test functions f.

Proof. Hint to show (5.17). Denote $\mathbf{K} = \mathbf{K}_0^\Lambda$ and write

$$E[f(X_h)|X_0 = X] \tag{5.18}$$

$$= E[\sum_{(x,0,s)\in\mathbf{K}:s<h} f(X\setminus\{x\})] \tag{5.19}$$

$$+ E[\sum_{(x,t,s)\in\mathbf{K}:t<h} f(X\cup\{x\})] \tag{5.20}$$

$$+ f(X)\big(1 - E\mathbf{1}\big\{\{(x,0,s)\in\mathbf{K} : s<h\} \cup \{(x,t,s)\in\mathbf{K} : t<h\} = \emptyset)\}\big\}\big) \tag{5.21}$$

$$+ E(\mathbf{1}\{\text{other things}\}), \tag{5.22}$$

where the set "other things" has intensity $o(h|\Lambda|)$. □

5.2.1 Time-stationary construction in finite-volume

For each bounded box $\Lambda \subset \mathbb{X}$, there are random times $\{\tau_j(\mathbf{C}) \in \mathbb{R} : j \in \mathbb{Z}\}$, such that

(a) $\tau_j \to \pm\infty$ for $j \to \pm\infty$ and

(b) There is no $C \in \mathbf{C}^\Lambda$ alive at time τ_j, for all $j \in \mathbb{Z}$.

We obtain the kept cylinders by using the algorithm (5.16) in each interval $[\tau_i, \tau_{i+1})$. Denote by

$$\mathbf{K}^\Lambda := \text{ the (time stationary random) set of kept cylinders.} \tag{5.23}$$

$$X_t^\Lambda := \big\{\text{Basis}(C) \,:\, C \in \mathbf{K}^\Lambda,\, C \text{ alive at } t\big\} \tag{5.24}$$

By construction, X_t^Λ is a stationary process. By Proposition 5.2, it is Markov with generator L. Denote μ^Λ the law of X_t^Λ.

Proposition 5.3. *The measure μ^Λ is invariant for this dynamics.*

Proof. We drop the dependence of Λ in the notation and write μ instead of μ^Λ. It suffices to show that $\mu L f = 0$ for test functions f. That is,

$$\int \mu(dX) \sum_{x\in X} [f(X \setminus \{x\}) - f(X)] \tag{5.25}$$

$$+ \int \mu(dX) \int_{\mathbb{X}} \gamma(x|X)\; [f(X \cup \{x\}) - f(X)]\, dx \;=\; 0. \tag{5.26}$$

Recalling (5.12), $\mu(dX) = \tilde{\mu}(dX)\, \mathbf{1}\{X \in \mathcal{A}^\Lambda\}(\tilde{\mu}(\mathcal{A}^\Lambda))^{-1}$. Since the factor $(\tilde{\mu}(\mathcal{A}^\Lambda))^{-1}$ is in both terms of (5.26), it suffices to show

$$\int \tilde{\mu}(dX)\, \mathbf{1}\{X \in \mathcal{A}^\Lambda\} \sum_{x\in X} [f(X \setminus \{x\}) - f(X)] \tag{5.27}$$

$$+ \int \tilde{\mu}(dX)\, \mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\}\, w(x)\, [f(X \cup \{x\}) - f(X)] = 0, \tag{5.28}$$

because $\gamma(x|X) = w(x)\, \mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\}$ and $\mathbf{1}\{X \in \mathcal{A}^\Lambda\} \geq \mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\}$.

Recall Slivniak-Mecke Theorem 1.11: for $h : \mathbb{X} \times \mathcal{X} \to \mathbb{R}$ and $\tilde{\mu}$ Poisson process with intensity w:

$$\int \tilde{\mu}(dX) \Big(\sum_{x\in X} h(x; X \setminus \{x\}) \Big) = \int_{\mathbb{X}} \int \tilde{\mu}(dX) h(x; X) w(x)\, \mathrm{d}x. \tag{5.29}$$

Taking $h(x; X) := \mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\}[f(X) - f(X \cup \{x\}]$, we have

$$h(x; X \setminus \{x\}) = \mathbf{1}\{X \in \mathcal{A}^\Lambda\} \, [f(X \setminus \{x\}) - f(X)]. \tag{5.30}$$

Using (5.29), we get that (5.27) equals

$$\int_{\mathbb{X}} \int \tilde{\mu}(dX) \mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\} \, w(x) \, [f(X) - f(X \cup \{x\})] \tag{5.31}$$

which cancels (5.28). □

The proof confirms the following heuristic: for $X' = X \cup \{x\}$,

$$(\text{weight of } X') \times (\text{rate of } X' \to X) = (\text{weight of } X) \times (\text{rate of } X \to X') \tag{5.32}$$

that is,

$$\tilde{\mu}(dX') \mathbf{1}\{X' \in \mathcal{A}^\Lambda\} \times 1 = \tilde{\mu}(dX) \times w(x) \mathbf{1}\{X' \in \mathcal{A}^\Lambda\} \tag{5.33}$$

and $\tilde{\mu}(dX') = w(x)\tilde{\mu}(dX)$.

5.2.2 Infinite-volume construction

The goal is to construct an infinite volume version of the process X_t^Λ via the identification of a set of kept cylinders $\mathbf{K}$. The problem is that there is no cylinder C in $\mathbf{C}$ to start the algorithm (5.16). We then propose to explore backwards in time if a cylinder $C \in \mathbf{C}$ belongs or not to the set of kept cylinders.

Consider an arbitrary cylinder configuration $\mathbf{C}$. The first generation of *ancestors* of a cylinder C is the set of cylinders C' born before C and alive at the birth-time of C whose basis intersect the basis of C.

$\mathbf{A}_1^C$: First generation of ancestors of C.

$\mathbf{A}_n^C$: n-th generation of ancestors of $C :=$ union of first generation of ancestors of the cylinders belonging to the $(n-1)$-th generation of C:

$$\mathbf{A}_n^C = \bigcup_{C' \in \mathbf{A}_{n-1}^C} \mathbf{A}_1^{C'} \tag{5.34}$$

Notice that a cylinder C' may belong to more than one generation of ancestors of C. The *Clan of ancestors* of C is defined by

$$\mathbf{A}^C := \cup_{n \geq 1} \mathbf{A}_n^C. \tag{5.35}$$

When necessary, we write $\mathbf{A}^C[\mathbf{C}]$, to stress that the ancestors are taken from the set $\mathbf{C}$.

The construction of the set $\mathbf{K}$ of kept cylinders works if **the clan of ancestors of C is finite** for all $C \in \mathbf{C}$. We apply the algorithm (5.16) to this finite set, pretending there are no other cylinders. The set of kept cylinders in the clan of C is denoted $\mathbf{K}(C)$.

Denote $\mathbf{K}[\mathbf{C}] := \cup_{C\in\mathbf{C}}\mathbf{K}(C) \subset \mathbf{C}$, the set of *kept* cylinders of $\mathbf{C}$.

Denote $\mathbf{C}_0(X)$ the set of cylinders $(x, 0, S_x)$ with basis $x \in X$, all born at time zero and with iid exponential life times S_x. Denote $\mathbf{C}_{(0,t]}$ the set of cylinders born in the interval $(0, t]$ and

$$\mathbf{C}_{[0,t]}(X) := \mathbf{C}_{(0,t]} \cup \mathbf{C}_0(X). \tag{5.36}$$

Theorem 5.4 (Infinite volume birth and death. Existence). *Let $X \in \mathcal{A}_{\mathbb{X}}$, $\mathbf{C}$ a Poisson process with intensity* (5.1) *and assume $\mathbf{A}^C[\mathbf{C}_{[0,t]}(X)]$ is almost surely finite for all $C \in \mathbf{C}_{[0,t]}(X)$. Then, the process*

$$X_t \;=\; \big\{\text{Basis}(C) \,:\, C \in \mathbf{K}[\mathbf{C}_{[0,t]}(X)],\, C \textit{ alive at } t\big\}$$

is Markov with generator L.

Theorem 5.5 (Infinite volume birth and death. Time invariance). *If $\mathbf{A}^C$ is almost surely finite for all $C \in \mathbf{C}$, then*

$$X_t = \big\{\text{Basis}(C) \,:\, C \in \mathbf{K}[\mathbf{C}],\, C \textit{ alive at } t\big\}$$

is Markov with generator L and stationary.

The marginal law of X_t, denoted μ is the unique invariant measure for the process. Furthermore, for all bounded supported $f : \mathbb{X} \to \mathbb{R}$, we have

$$\lim_{\Lambda\nearrow\mathbb{X}} \mu^\Lambda f = \mu f. \tag{5.37}$$

So that μ is Gibbs for the specification $(\mu^\Lambda)_\Lambda$.

We now give sufficient conditions on w to satisfy the conditions of Theorem 5.4 and Theorem 5.5.

Define

$$\alpha \;:=\; \sup_{x\in\mathbb{X}} \int_{\mathbb{X}} \mathbf{1}\{\|y-x\| < r\}\, w(y)\, dy. \tag{5.38}$$

Theorem 5.6 (Finiteness of ancestor clan). *Let $\mathbf{C}$ be a Poisson process with intensity* (5.1).

(i) Let $X \in \mathcal{A}_{\mathbb{X}}$. If $\alpha < \infty$ then $\mathbf{A}^C[\mathbf{C}_{[0,t]}(X)]$ is almost surely finite for all $C \in \mathbf{C}_{[0,t]}(X)$.

(ii) If $\alpha < 1$, then $\mathbf{A}^C[\mathbf{C}]$ is almost surely finite for all $C \in \mathbf{C}$.

We prove this theorem in the next section, by dominating the clan of ancestors of C by a multitype branching process with offspring distribution $\mathbf{A}_1^C$.

5.2.3 Oriented percolation and branching

Oriented percolation Define the random graph $\mathcal{G} := (\mathbf{C}, \mathcal{E})$ with vertices $\mathbf{C}$ and oriented edges

$$\mathcal{E}(\mathbf{C}) = \{(C, C') \in \mathbf{C}^2 : C' \in \mathbf{A}_1^C\}; \tag{5.39}$$

that is, the edge (C, C') is present if C' is an ancestor of C. We say that there is *oriented percolation* in $\mathbf{C}$ if some $C \in \mathbf{C}$ has an infinite number of ancestors: $\#\mathbf{A}^C = \infty$. The goal of this subsection is to show that under the conditions of Theorem 5.5, there is no percolation in $\mathbf{C}$, almost surely.

Multitype branching process Given a cylinder D, define $\mathbf{B}_0^D := D$ and inductively

$$\mathbf{B}_n^D := \cup_{C \in \mathbf{B}_{n-1}^D} \mathbf{B}_1^C, \tag{5.40}$$

such that the conditioned distribution of the family $(\mathbf{B}_1^C : C \in \mathbf{B}_{n-1}^D)$ given $\mathbf{B}_{n-1}^D$, consists on independent clans $\mathbf{B}_1^C$ with the same distribution as $\mathbf{A}_1^C$. The total progeny of D is denoted

$$\mathbf{B}^D := \cup_{C \in \mathbf{B}_{n-1}^D} \mathbf{B}_n^D. \tag{5.41}$$

This is just the ecology model we have studied in the Cox process section. The difference is that now the points are called cylinders and we have only one cylinder D in the initial configuration. Then each cylinder C has daughters $\mathbf{B}_1^C$ independent of the other points. We say that $(\mathbf{B}_n^D)_{n \geq 0}$ is a *multitype branching process* with offspring distribution $\mathbf{B}_1^C$, a Poisson process with mean measure given for $C = (x, T, S)$ by

$$\mathbf{1}\{\|y - x\| < r, t < T, s > T - t\}\, w(y)\, e^{-s}\, dydtds. \tag{5.42}$$

Proposition 5.7 (Coupling percolation and branching)**.** *For any cylinder D, there exist a coupling $(\hat{\mathbf{A}}^D, \hat{\mathbf{B}}^D)$ satisfying*

$$\hat{\mathbf{A}}^D \subset \hat{\mathbf{B}}^D, \tag{5.43}$$

where the marginal $\hat{\mathbf{B}}^D$ *is distributed as the multitype branching process* $\mathbf{B}^D$ *defined in* (5.41) *and the marginal* $\hat{\mathbf{A}}^D$ *is distributed as the clan of ancestors* $\mathbf{A}^D$ *defined in* (5.35).

New proof. In this proof we construct $\hat{\mathbf{B}}^D$ using independent Poisson processes as in (5.40). Then we give a rule to erase cylinders $B \in \hat{\mathbf{B}}^D$ to obtain a clan $\hat{\mathbf{A}}^D$ contained in $\hat{\mathbf{B}}^D$.

Denote $\mathbf{B} = \hat{\mathbf{B}}^D$. Let $n(B) = n$ if and only if $B \in \mathbf{B}_n$, the generation number of B. For $B, B' \in \mathbf{B}$, we say that B' precedes B, denoted $B' \prec B$, if $n(B') < n(B)$ or $n(B') = n(B)$ and $\text{Birth}(B') > \text{Birth}(B)$. Since $\mathbf{B}$ with $\prec$ is a well ordered set, we can label the elements of $\mathbf{B}$ to have $\mathbf{B} = \{B_1, B_2, \dots\}$ with $B_1 = D$ and $B_i \prec B_{i+1}$, $i \geq 1$.

We say that B' is the *mother* of B if $B \in \mathbf{B}_1^{B'}$. We say that $B' = (x', t', s')$ is a *potential mother* of $B = (x, t, s)$ if B' is not the mother of B and $\|x - x'\| < r$ and $t' \in (t, t + s)$.

Color the cylinders of $\mathbf{B}$ iteratively as follows. Paint B_1 blue. For $k \leq \#\mathbf{B}$, assume $B_1, \dots, B_{k-1}$ are colored. If the mother of B_k is blue and she precedes all blue potential mothers of B_k, then paint B_k blue; otherwise, paint B_k red.

Denote $\hat{\mathbf{A}}^D :=$ set of blue cylinders. By construction $\hat{\mathbf{A}}^D \subset \hat{\mathbf{B}}^D$ and $\hat{\mathbf{A}}^D$ has the same distribution as $\mathbf{A}^D$ (exercise). □

Old proof. In this proof we construct first the clan $\mathbf{A}^D$ of a given cylinder $D \in \mathbf{C}$ and add new (random) cylinders B to obtain $\mathbf{B}^D$.

Let $n(C) = \min\{\ell \geq 0 : C \in \mathbf{A}_\ell^C\}$. For $C, C' \in \mathbf{A}^D$, we say that C' precedes C, denoted $C' \prec C$, if $n(C') < n(C)$ or $n(C') = n(C)$ and $\text{Birth}(C') > \text{Birth}(C)$. For each $C \in \mathbf{A}^D$ define

$$\check{\mathbf{A}}_1^C := \mathbf{A}_1^C \cap \left(\cup_{C' \prec C} \mathbf{A}_1^{C'}\right), \qquad \widetilde{\mathbf{A}}_1^C := \mathbf{A}_1^C \setminus \check{\mathbf{A}}_1^C. \tag{5.44}$$

Cylinders in $\widetilde{\mathbf{A}}_1^C$ are not direct ancestors of those $C' \prec C$. Given C and $\{C' \in \mathbf{A}^D : C' \prec C\}$, the random sets $\widetilde{\mathbf{A}}_1^C$ and $\check{\mathbf{A}}_1^C$ are independent Poisson processes (with intensity depending on C and $C' \prec C$). We have

$$\mathbf{A}_1^C = \widetilde{\mathbf{A}}_1^C \,\dot\cup\, \check{\mathbf{A}}_1^C, \qquad \mathbf{A}_n^D = \cup_{C \in \mathbf{A}_{n-1}^D} \widetilde{\mathbf{A}}_1^C. \tag{5.45}$$

Introduce a family of independent random sets $(\check{\mathbf{B}}_1^C)_{C \in \mathbf{A}^D}$, independent of $(\mathbf{A}_1^C)_{C \in \mathbf{A}^D}$, such that, given $\{C' \in \mathbf{A}^D : C' \prec C\}$, $\check{\mathbf{B}}_1^C$ has the same distribution as $\check{\mathbf{A}}_1^C$. Define

$$\mathbf{B}_1^C := \widetilde{\mathbf{A}}_1^C \cup \check{\mathbf{B}}_1^C. \tag{5.46}$$

By construction, $\mathbf{B}_1^C$ has the same law as $\mathbf{A}_1^C$, and, given $\mathbf{A}_n^D$, the sets $(\mathbf{B}_1^C : C \in \mathbf{A}_n^D)$ are independent. Introduce

$$\hat{\mathbf{B}}_n^D := (\cup_{C\in\mathbf{A}_{n-1}^D}\check{\mathbf{B}}_1^C) \cup \big(\cup_{B\in\hat{\mathbf{B}}_{n-1}^D}\mathbf{B}_1^B\big), \quad \hat{\mathbf{B}}_0^D := \emptyset, \tag{5.47}$$

where, given $\hat{\mathbf{B}}_{n-1}^D$ the random sets $(\mathbf{B}_1^B : B \in \hat{\mathbf{B}}_{n-1}^D)$ are independent Poisson processes with intensity (5.40). In words, $\hat{\mathbf{B}}_n^D$ consists on the progeny of cylinders B created at times $k < n$. Finally define the process $(\mathbf{B}_n^D)_{n\ge 0}$ by $\mathbf{B}_0^D = \{D\}$ and, iteratively

$$\mathbf{B}_n^D := \big(\cup_{C\in\mathbf{A}_{n-1}^D}\widetilde{\mathbf{A}}_1^C\big) \cup \big(\cup_{B\in\hat{\mathbf{B}}_{n-1}^D}\mathbf{B}_1^B\big), \tag{5.48}$$

By construction, $\mathbf{B}_n^D$ is a branching process with mean measure (5.40). The total progeny is

$$\mathbf{B}^D := \cup_{n\ge 1}\mathbf{B}_n^D \tag{5.49}$$

Finally (5.43) follows from (5.45) and (5.48). □

The ecology model in $\mathbb{X}$ The bases of the cylinders in the branching process $\mathbf{B}^D$ induce a continuous time branching process in $\mathbb{X}$. For a cylinder D with basis x and alive at time 0, define $\mathbf{b}_n^x \subset \mathbb{X}$ as the basis of the cylinders in the nth generation of ancestors of D:

$$\mathbf{b}_n^x = \big\{\text{Basis}(C) : C \in \mathbf{B}_n^D\big\}. \tag{5.50}$$

Denote $I(x,y) := \mathbf{1}\{\|x-y\| < r\}$. Recall $\mathbf{B}_1^C$ is a Poisson process on $\mathbb{X} \times \mathbb{R} \times \mathbb{R}_+$ with intensity $w(y)\, I(x,y)\, \mathbf{1}\{t < T\}\, dt\, \mathbf{1}\{s > t\}\, dx$. The Mapping theorem says then that $\mathbf{b}_1^x$ is a Poisson process with mean measure

$$m(x,y)\, dy = w(y) I(x,y) \Big[\int_{-\infty}^0 dt \int_t^\infty ds\, e^{-s}\Big] dy \;=\; w(y) I(x,y) dy. \tag{5.51}$$

Define $m^1(x,y) := m(x,y)$ and for $n \ge 1$, iteratively,

$$m^n(x,y) := \int dz\;\, m^{n-1}(x,z)\, m(z,y). \tag{5.52}$$

Lemma 5.8. *We have*

$$E\#\mathbf{b}_n^x = \int m^n(x,y)\, dy \le \alpha^n, \tag{5.53}$$

where α is defined in (5.38)*. If $\alpha < 1$, then*

$$E\#\mathbf{b}^x = \sum_{n\ge 0}\int_{\mathbb{X}} m^n(x,y)\, dy \;\le\; \sum_{n\ge 0}\alpha^n = \frac{1}{1-\alpha}. \tag{5.54}$$

Proof. The first identity is left as an exercise. By definition,

$$\int_{\mathbb{X}} m^n(x,y)\,dy = \int w(x_1)I(x,x_1)\int w(x_2)I(x_1,x_2) \quad \dots \tag{5.55}$$
$$\times \int w(y)\,I(x_{n-1},y)\,dx_1\dots dx_{n-1}dy \;\le\; \alpha^n. \qquad \square$$

This Lemma shows, in particular, that the multitype branching process $\mathbf{b}_n$ is subcritical if $\alpha < 1$.

5.3 Continuous-time branching process

Let C be a cylinder with basis a and alive at time 0. Exploring the clan $\mathbf{B}^C$ backwards in time, we define a continuous-time process Y_t^a with initial state $Y_0^a = a$, and

$$Y_t^a = \{\text{Basis}(B) : B \in \mathbf{B}^C,\, B \text{ alive at time } -t, \text{Birth}(\text{Mom}(B)) > -t\} \tag{5.56}$$

where $\text{Mom}(B)$ is the unique $B' \in \mathbf{B}^C$ such that $B \in \mathbf{B}_1^{B'}$.

Each individual x waits an exponential time of parameter 1 after which she dies and produces offsprings distributed as a Poisson process μ^x with mean measure $w(x)I(x,y)dy$. The generator is

$$Lf(Y) = \sum_{x\in Y}\int \mu^x(dZ)[f(Y\setminus\{x\}\cup Z) - f(Y)] \tag{5.57}$$

Let

$$R_t(a) := E(\#Y_t^a), \tag{5.58}$$

the expected number of points at time t.

Lemma 5.9 (Mean number of individuals)**.** *The mean number of individuals at time t, $R_t(a)$ satisfies*

$$R_t(a) \;\le\; e^{(\alpha-1)t}. \tag{5.59}$$

Proof. Let $p(j) = e^{-\alpha}\alpha^j/j!$, the law of a Poisson random variable with mean α. Let y_t be a jump Markov process in $\mathbb{N}$ with starting point $y_0 = 1$ and rates Q, given for $j \ge 1$ by

$$q(j,\ell) = j\,p(\ell+1-j), \quad \text{for } \ell \ge j-1,\ \ell \ne j. \tag{5.60}$$

When the process is at state k, at rate k it chooses a j with law Poisson(α), and increments its value by $j-1$; the state 0 is absorbing. There is a coupling $(\#Y_t^a, y_t)$ such that $\#Y_t^a \le y_t$, implying that for $r_t := Ey_t$, we have

$$R_t(a) \le r_t. \tag{5.61}$$

We have $r_t = P_t f(1)$ for $f(j) := j$ and $P_t = e^{Qt}$. By (2.25) and (2.28),

$$\begin{aligned}
\frac{d}{dt} P_t f(1) &= \sum_{j\ge 1} \sum_{\ell \ge j-1} p_t(1,j)\, q(j,\ell)\, [\ell - j] && (5.62)\\
&= \sum_{j\ge 1} p_t(1,j)\, j \sum_{\ell\ge j-1} p(\ell + 1 - j)\, [\ell - j] && (5.63)\\
&= \sum_{j\ge 1} p_t(1,j)\, j \sum_{\ell'\ge 0} p(\ell')\, (\ell' - 1) \qquad (\ell' = \ell - j + 1) && (5.64)\\
&= r_t\, (\alpha - 1). && (5.65)
\end{aligned}$$

Since $r_0 = 1$, the solution is $r_t = \exp(t(\alpha - 1))$. This and (5.61) imply (5.59). □

The above construction gives bounds for the total number of cylinders alive at some time in $[-t, 0]$. Let

$$\hat{Y}_t^a = \{\text{Basis}(B) : B \in \mathbf{B}^C,\ B \text{ alive at time } -t',\ \text{for some } -t < t' < 0\} \tag{5.66}$$

this is a growth process where there are no deaths. The generator is

$$\hat{L} f(Y) = \sum_{x\in Y} \int \mu^x(dZ)[f(Y \cup Z) - f(Y)] \tag{5.67}$$

Let

$$\hat{R}_t(a) := E(\#\hat{Y}_t^a), \tag{5.68}$$

the expected number of points until time t.

Lemma 5.10. *We have*

$$\hat{R}_t(a) \le e^{\alpha t}. \tag{5.69}$$

Proof. Exercise. □

5.4 Time length and space width

Define the time length and space width of a finite cluster $\mathbf{A}^C$, for C alive at time τ, by

$$\mathrm{TL}(\mathbf{A}^C) := \sup\{\mathrm{Birth}(D) : D \in \mathbf{A}^C\} - \tau, \qquad \mathrm{SW}(\mathbf{A}^C) := r\,\#\mathbf{A}^C. \quad (5.70)$$

The branching process Y_t^a allows us to estimate the time-length of a clan, due to the fact:

$$Y_t^a = \emptyset \text{ implies } \mathrm{TL}(\mathbf{B}^C) < t\,. \quad (5.71)$$

Theorem 5.11 (Percolation, time length and space width)**.**

(i) If $\alpha < \infty$, then for C alive at time t,

$$E\#\mathbf{A}^C[\mathbf{C}_{[0,t]}(X)] \le 2e^{\alpha t}. \quad (5.72)$$

(ii) If $\alpha < 1$, then the probability of backward oriented percolation is zero.

(iii) If $\alpha < 1$, then for any positive t,

$$\mathbb{P}\big(\mathrm{TL}(\mathbf{A}^C) > t\big) \;\le\; e^{-(1-\alpha)t} \quad (5.73)$$

(iv) If $\alpha < 1$, then

$$\mathbb{E}\big(\mathrm{SW}(\mathbf{A}^C)\big) \;\le\; \frac{r}{1-\alpha} \quad (5.74)$$

Proof. (i) The total number of cylinders in $\mathbf{A}^D[\mathbf{C}_{[0,t]}(X)]$ is dominated by the number of cylinders in $\mathbf{A}^D$ alive at some time in $[0,t]$ plus the number of cylinders $C \in \mathbf{C}_0(X)$ such that C is ancestor of some cylinder C' alive at some point of $[0,t]$.

The number of cylinders alive at some point of $[0,t]$ has the same distribution as those alive at some point of $[-t,0]$ in the clan of a D alive at time t. This is dominated by $\hat{Y}_t^x$, whose expectation is $\hat{R}_t \le e^{\alpha t}$, by Lemma 5.10. There are at most $\hat{Y}_t^x$ cylinders at time 0 that can intersect those in $\mathbf{A}^D \subset \mathbf{B}^D$ alive at some point of $[0,t]$. This is the origin of the "2" in (5.72).

(ii) We follow Hall (1985). For $C = (x,T,S) \in \mathbf{C}$, recall the domination $\mathbf{A}^C \subset \mathbf{B}^C$ in (5.43) and the definition (9.1) saying that $\mathbf{b}^x$ consists of the bases of $\mathbf{B}^C$. So, $E\#\mathbf{b}^x < 1/(1-\alpha)$ in (5.54) shows part (i).

(iii) By definition, if $C = (x,T,S) \in \mathbf{C}$ is alive at time 0, we have

$$Y_t^x = \emptyset \text{ implies } \mathrm{TL}(\mathbf{A}^C) \le t. \quad (5.75)$$

Hence,

$$P(\mathrm{TL}(\mathbf{A}^C) > t) \ \le \ P(Y_t^x = \emptyset) \le e^{-(1-\alpha)t} \tag{5.76}$$

by the rightmost inequality in (5.59).

(iv) We find upperbounds for the space diameter of the backwards percolation clan through upperbounds for the total number of occupied points by the multitype branching process $\mathbf{b}_n$ defined by (9.1). In fact, for $C = (a, T, S)$,

$$\mathrm{SW}(\mathbf{A}^C) \le r\,\#\mathbf{b}^a \tag{5.77}$$

By (5.54)

$$E\,\mathrm{SW}(\mathbf{A}^C) \le r\,E\#\mathbf{b}^a \le \frac{r}{\alpha}. \qquad \square$$

Proof. **Proof of Theorem 5.6** The first part follows from (i) of previous theorem. The second part follows from (ii). Complete the details as an exercise. □

5.5 Thermodynamic limit

Let $\alpha < 1$ and $\mathbf{C}$ the Poisson process with mean measure (5.1). Let $\mathbf{K}^\Lambda$ be the set of kept cylinders when the we consider ancestors only cylinders with basis in Λ and $\mathbf{K}$ the set of kept cylinders in the infinite volume. Let X^Λ be the set of basis of cylinders in $\mathbf{K}^\Lambda$ alive at time 0 and X be the set of basis of cylinders in $\mathbf{K}$ alive at time 0. Let $\tilde{X}$ be the set of bases of cylinders in $\mathbf{C}$ alive at time 0. Then, for every $x \in \tilde{X}$,

$$\lim_{\Lambda \nearrow \mathbb{X}} \mathbf{1}\{x \in X^\Lambda\} \ = \ \mathbf{1}\{x \in X\}, \qquad \text{a.s.} \tag{5.78}$$

Let $\Delta \subset \mathbb{X}$ be a bounded box. The above limit implies that, taking $\Lambda \supset \Delta$,

$$\lim_{\Lambda \nearrow \mathbb{X}} X^\Lambda \cap \Delta = X \cap \Delta, \qquad \text{a.s..} \tag{5.79}$$

This implies

$$\lim_{\Lambda \nearrow \mathbb{X}} \int \mu^\Lambda(dX) f(X \cap \Delta) = \int \mu(dX) f(X \cap \Delta). \tag{5.80}$$

which is the thermodynamic limit.

5.6 Perfect simulation

Assume $\alpha < 1$. We describe an algorithm to simulate $X \cap \Delta$, for X distributed with the Gibbs measure μ and $\Delta \subset \mathbb{X}$ finite.

1. Construct $\mathbf{A}^C$ for C alive at time 0 and with basis in Δ. These clans are finite as $\alpha < 1$.

2. Cleaning algorithm: Decide, using the exclusion rule from younger to older which cylinders are kept or erased inside each clan.

   Call $\mathbf{K}^C$ the set of *kept* cylinders in $\mathbf{A}^C$.

3. Define

$$X \cap \Delta := \{\text{Basis}(C) \,:\, C \in \mathbf{K}^C,\, C \text{ alive at time } 0\}$$

   for $x \in \Delta$.

This configuration has the marginal distribution of the infinite-volume measure μ on the set $\Delta \subset \mathbb{X}$.

5.7 Construction of Gibbs measures

If the measure is locally absolutely continuous with respect to $\widetilde{\mu}$ when restricted to a finite region:

$$\mu^\Lambda(d\eta) = \frac{1}{Z} e^{-H_\Lambda(\eta)} \tilde{\mu}(d\eta)$$

that is, the Hamiltonian acts only inside Λ. We are interested in Gibbs measures with respect to the specification (μ^Λ : Λ bounded subset of $\mathbb{X}$).

We propose a dynamics with rate of birth:

$$w(dx) \times \frac{e^{-H(\eta \cup x)}}{e^{-H(\eta)}}$$

which is the Papangelou intensity measure for μ, and rate of death 1.

The generator is now

$$Lf(\eta) = \sum_{x \in \eta} [f(\eta \setminus x) - f(\eta)] + \int_{\mathbb{X}} w(dx) \, \frac{e^{-H(\eta \cup x)}}{e^{-H(\eta)}} \, [f(\eta \cup \{x\}) - f(\eta)]$$

and the semigroup

$$S_t f(\eta) := E(f(\eta_t)|\eta_0 = \eta)). \tag{5.81}$$

We say that μ is reversible for (η_t) if for any test function f, g, we have

$$\int d\mu\, g\, S_t f = \int d\mu\, f\, S_t g$$

Under suitable conditions, the next identity implies μ reversible.

$$\int d\mu\, g\, Lf = \int d\mu\, f\, Lg$$

These equations are consequence of the following heuristic microscopics:

$$= (\text{weight of } \eta \cup x) \times (\text{rate of } \eta \cup x \to \eta).$$
$$(\text{weight of } \eta) \times (\text{rate of } \eta \to \eta \cup x)$$

which gives:

$$e^{-H(\eta\cup x)} \times 1 = e^{-H(\eta)} \times \frac{e^{-H(\eta\cup x)}}{e^{-H(\eta)}}$$

We construct the dynamics in function of the Poisson process $\mathbf{C}$ on $\mathbb{X} \times \mathbb{R} \times \mathbb{R}^+ \times [0,1]$ with intensity

$$w(x)dx\, dt\, e^{-s}\, ds\, du. \tag{5.82}$$

A point of this Poisson process is a *flagged cylinder*

$$C = (x, T, S, U) \tag{5.83}$$

where $T = \text{birth}(C) \in \mathbb{R}$
$S = \text{life}(C) \sim$ exponential(1)
$U = \text{flag}(C) \sim$ Uniform$[0,1]$

We perform the same construction as before but now we include $C(x, T, S, U)$ in $\mathbf{K}$ if

$$U \leq M(x|\eta_{T-}) \tag{5.84}$$

where

$$M(x|\eta) = \frac{e^{-H(\eta\cup x)}}{e^{-H(\eta)}}. \tag{5.85}$$

6 Ballistic particles and hard rods

This section is based on work in preparation with Dante Grevino and Herbert Spohn. The original model comes from a paper by Boldriguini, Dobrushin and Soukhov [6]

6.1 Ideal gas dynamics

An element $(q, v, \ell) \in \mathbb{X} := \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is called *particle* with position q, velocity v and length ℓ. Elements of $\mathcal{X}$ are called *configurations.*

Under the *ideal gas* dynamics, each particle $\mathsf{x} = (q, v, \ell) \in \mathsf{X}$ moves at speed v, conserving speed and length, with no interaction with other particles. Define

$$T_t\mathsf{X} := \{(q, v, \ell) \in \mathbb{X} : (q - vt, v, \ell) \in \mathsf{X}\}. \tag{6.1}$$

This is a deterministic dynamics. Denote

$$\mathcal{X} := \big\{\mathsf{X} \subset \mathbb{X} : T_t\mathsf{X} \text{ is locally finite for all } t \in \mathbb{R}\big\} \tag{6.2}$$

We will consider random initial configurations on $\mathcal{X}$.

6.1.1 Macroscopic gas evolution

Assume F is absolutely continuous with density f and let

$$\Gamma := \big\{\gamma : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : \textstyle\iint \gamma(v, \ell)dvd\ell < \infty\big\} \tag{6.3}$$

$$\mathcal{F} := \big\{f : \mathbb{X} \to \mathbb{R}_{\geq 0} : \text{there is } \gamma \in \Gamma \text{ such that } \sup_q f(q, v, \ell) \leq \gamma(v, \ell), \forall q, v\big\}$$

Define the *ideal gas* time evolution as the operator $\mathcal{T}_t : \mathcal{F} \to \mathcal{F}$ given by $\mathcal{T}_t f := f \circ T_{-t}$:

$$\mathcal{T}_t f(q, v, \ell) := f(q - vt, v, \ell) \tag{6.4}$$

This operator is a bijection with inverse $\mathcal{T}_{-t}$. Notice that $\mathcal{T}_t$ conserves $\mathcal{F}$:

$$f \in \mathcal{F} \quad \text{if and only if} \quad \mathcal{T}_t f \in \mathcal{F}. \tag{6.5}$$

6.1.2 Random initial configuration

Let P denote a probability on $\mathcal{X}$ and E the expectation with respect to P. The first moment measure of P is the measure on $\mathbb{X}$ defined by $F(\cdot) := \mathsf{E}\sum_{\mathsf{x}\in\mathsf{X}} \mathbf{1}\{\mathsf{x} \in \cdot\}$ (mean measure).

Lemma 6.1. *Assume* X *is a point process on* $\mathcal{X}$*. If* X *has mean measure* F *with density* $f \in \mathcal{F}$*, then* $T_t\mathsf{X}$ *has mean measure* $F \circ T^{-1}$ *with density* $\mathcal{T}_t f$*.*

Lemma 6.2. *If* X *is Poisson with intensity* $f \in \mathcal{F}$*, then* $T_t\mathsf{X}$ *is Poisson with intensity* $\mathcal{T}_t f$*.*

Proof. Mapping theorem. $\square$

Invariant measures Define the shift operator $S_a : \mathcal{X} \to \mathcal{X}$ by $S_a\mathsf{X} = \{(q-a, v, \ell) : (q, v, \ell) \in \mathsf{X}\}$ (configuration as seen from a). A measure P is *shift invariant* or S-invariant if $S_a\mathsf{X}$ has distribution P for all $a \in \mathbb{R}$.

A measure P on $\mathcal{X}$ is T*-invariant* if $T_t\mathsf{X}$ has distribution P for all t.

If P is shift invariant and F is the mean measure of P, with density $f \in \mathcal{F}$, then f takes the form

$$f(q, v, \ell) = \rho \tilde{f}(v, \ell), \tag{6.6}$$

where ρ is a positive constant and $\tilde{f}$ is a positive measure on $\mathbb{R}\times\mathbb{R}_{\geq 0}$.

Lemma 6.3. *Let* X *be a shift invariant Poisson process on* $\mathcal{X}$ *with distribution* P *and intensity* $f \in \mathcal{F}$*. Then,* P *is* T*-invariant.*

Proof. By Lemma 6.2, $T_t\mathsf{X}$ is a Poisson process with intensity $\mathcal{T}_t f = f \circ T_{-t}$. Using (6.6),

$$\begin{aligned} \int \varphi\, \mathcal{T}_t f &= \iiint \varphi(q + vt, v, \ell)\, \rho_F\, dq\, \tilde{F}(dv, d\ell) \qquad (6.7) \\ &= \iiint \varphi(q, v, \ell)\, \rho_F\, dq\, \tilde{F}(dv, d\ell) = \int \varphi\, dF. \end{aligned}$$ $\square$

We say that a shift invariant measure P on $\mathcal{X}$ is space *mixing* if

$$\lim_{a\to\infty} \big| \mathsf{E}[\alpha_1(\mathsf{X})\, \alpha_2(S_a\mathsf{X})] - \mathsf{E}(\alpha_1(\mathsf{X}))\, \mathsf{E}(\alpha_2(\mathsf{X})) \big| = 0, \tag{6.8}$$

for test functions $\alpha_i : \mathcal{X} \to \mathbb{R}_{\geq 0}$ with finite expectation and compact space support.

Let X be a point process in $\mathcal{X}$ with distribution P, such that the n-point correlation function of P is absolutely continuous with density f_n, $n \geq 1$. That is, $f_n : \mathbb{X}^n \to \mathbb{R}_{\geq 0}$ satisfy

$$\mathsf{E}\big[\kappa_{\mathsf{X}}(A_1) \dots \kappa_{\mathsf{X}}(A_n)\big] = \int_{A_1\times\cdots\times A_n} f_n(\mathsf{x}_1, \dots, \mathsf{x}_n)\, d\mathsf{x}_1 \dots d\mathsf{x}_n, \tag{6.9}$$

for any collection of bounded, pairwise disjoint Borel sets $A_1, \ldots, A_n \subset \mathbb{X}$, where $\kappa_{\mathsf{X}}(A)$ is the number of elements in $\mathsf{X} \cap A$.

Proposition 6.4. *Let* X *be a point process in* $\mathcal{X}$ *with distribution* P*, with correlation densities* f_n *satisfying (6.9). If* P *is* T*-invariant and mixing, then* P *is a Poisson process with intensity* f_1*.*

Sketch proof. Since P is T-invariant, the one-point correlation function P must be shift invariant: $f(q, v, \ell) = f(q + vt, v, \ell)$. By T-invariance, the n-correlation function f_n satisfies

$$f_n(\mathsf{x}_1, \ldots, \mathsf{x}_n) = f_n(S_{v_1 t}\mathsf{x}_1, \ldots, S_{v_n t}\mathsf{x}_n), \tag{6.10}$$

for $t \in \mathbb{R}$, $\mathsf{x}_i = (q_i, v_i, \ell_i) \in \mathbb{X}$, where v_i are distinct speed values. Since P is mixing,

$$\lim_{t\to\infty} f_n(S_{v_1 t}\mathsf{x}_1, \ldots, S_{v_n t}\mathsf{x}_n) = f(\mathsf{x}_1) \cdots f(\mathsf{x}_n), \tag{6.11}$$

which is the n-correlation function of a Poisson process with intensity f. A result by Lenard 1973 implies that X is a Poisson process with intensity f. □

Superposition of particle configurations with the same speed Examples of invariant measures that are not absolutely continuous include superposition of particle configurations with the same speed.

Let W be a finite or countable set of speeds and for each $w \in W$, denote $\mathbb{X}'_w := \{(q, v, \ell) \in \mathbb{X} : v = w\}$ the set of configurations with speed w. Let $\mathbb{X}' := \cup_{w \in W} \mathbb{X}'_w$ and consider a point process $\mathsf{X} \in \mathcal{X}' := \{\mathsf{X} \in \mathcal{X} : \mathsf{X} \subset \mathbb{X}'\}$ with distribution P. Denote P_w the distribution of $\mathsf{X}_w := \mathsf{X} \cap \mathbb{X}'_w$. In this case we say that P_w is the w-marginal of P.

Proposition 6.5. *Let* P *be a probability on* $\mathcal{X}'$ *with shift invariant marginals* $(\mathsf{P}_w)_{w \in W}$*. (a) If the* w*-marginals are independent then* P *is* T*-invariant. (b) If* P *is mixing and* T*-invariant, then the* w*-marginals are independent.*

Proof. (a) Under the free gas dynamics the w-marginal at time t is a translation by wt of the time zero marginal, which has law P_w for all t. Shifted marginals are also independent, so the superposition of the marginals at time t has law P, showing P is T-invariant.

(b) Let $\mathsf{X} = \cup_w \mathsf{X}_w$ have T-invariant and mixing distribution P, where X_w is the w-marginal configuration. Note that the marginal law of X_w is also mixing. Take measurable sets $A_w \subset \mathcal{X}_w$, all depending on the same finite interval of positions. For any finite number of speeds w, use T-invariance to get

$$\mathsf{P}\big(\bigcap_w \{X_w \in A_w\}\big) = \mathsf{P}\big(\bigcap_w \{X_w \in S_{tw} A_w\}\big) \underset{t\to\infty}{\to} \prod_w \mathsf{P}\big(\{X_w \in A_w\}\big),$$

by shift invariance and mixing. □

6.1.3 Ideal gas hydrodynamics

Let $f \in \mathcal{F}$, and denote

$$f^\varepsilon(q, v, \ell) := f(\varepsilon q, v, \ell). \tag{6.12}$$

$$\mathsf{X}^\varepsilon := \text{Poisson processes on } \mathbb{X} \text{ with intensity } f^\varepsilon, \tag{6.13}$$

Denote P, E the distribution of the family $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$. Denote κ^ε the counting empirical measure associated to X^ε, defined on test functions $\varphi : \mathbb{X} \to \mathbb{R}_{\geq 0}$, by

$$\kappa^\varepsilon \varphi := \varepsilon \sum_{(q,v,\ell)\in\mathsf{X}^\varepsilon} \varphi(\varepsilon q, v, \ell). \tag{6.14}$$

Since φ is nonnegative, the expectation of $\kappa^\varepsilon \varphi$ is well defined by

$$\mathsf{E}(\kappa^\varepsilon \varphi) = \varepsilon \iiint \varphi(\varepsilon q, v, \ell)\, f(\varepsilon q, v, \ell)\, dq dv d\ell = \int \varphi f \leq \infty. \tag{6.15}$$

The law of large numbers in this subsection holds for any nonnegative φ. If $\int \varphi f < \infty$, then the convergence is to that value. Otherwise, the convergence is to ∞.

It is convenient to have a joint construction of $(\mathsf{X}^\varepsilon)_{\varepsilon^{-1}\in\mathbb{N}}$. Take $\varepsilon^{-1} \in \mathbb{N}$, consider a family $(\mathsf{X}_i)_{i\geq 1}$ of i.i.d. Poisson processes with intensity f, and define

$$\mathsf{X}^\epsilon := \cup_{i=1}^{\varepsilon^{-1}} \{(q, v, \ell) : (\varepsilon q, v, \ell) \in \mathsf{X}_i\}. \tag{6.16}$$

By the superposition Theorem, X^ε as defined in (6.16) is a Poisson process with intensity f^ε. This is consistent with the former definition (6.13).

Lemma 6.6 (LLN). *Let X^ε be the Poisson process defined in (6.16). For any $f \in \mathcal{F}$ and test function $\varphi \geq 0$, we have*

$$\lim_{\varepsilon\to 0} \kappa^\varepsilon \varphi = \int \varphi f, \qquad a.s. \tag{6.17}$$

Proof. From the construction (6.16), we can write

$$\kappa^\varepsilon \varphi = \varepsilon \sum_{i=1}^{\varepsilon^{-1}} \kappa_i \varphi, \qquad \kappa_i \varphi := \sum_{(q,v,\ell)\in \mathsf{X}_i} \varphi(q,v,\ell). \tag{6.18}$$

Since $\mathsf{E}(\kappa_i\varphi) = \int \varphi\, dF$, the result for $\varepsilon^{-1} \in \mathbb{N}$ follows from the strong law of large numbers for iid nonnegative random variables, see [10], for instance. □

The empirical counting measure κ^ε_t for the ideal gas at time t is defined by

$$\kappa^\varepsilon_t \varphi \;:=\; \varepsilon \sum_{(q,v,\ell)\in T_{\varepsilon^{-1}t}\mathsf{X}^\varepsilon} \varphi(\varepsilon q, v, \ell) \tag{6.19}$$

Lemma 6.7 (LLN at time t)**.** *Let X^ε be a Poisson process with intensity f^ε. For any test function $\varphi \geq 0$, we have*

$$\lim_{\varepsilon\to 0} \kappa^\varepsilon_t \varphi = \int \varphi\, \mathcal{T}_t f, \qquad a.s., \quad t \in \mathbb{R}. \tag{6.20}$$

Proof. By Lemma 6.2, $T_{\varepsilon^{-1}t}\mathsf{X}^\varepsilon$ is a Poisson process with intensity $\mathcal{T}_{\varepsilon^{-1}t} f^\varepsilon = f^\varepsilon \circ T_{-\varepsilon^{-1}t}$. Hence, the mean of $\kappa^\varepsilon_t\varphi$ is

$$\mathsf{E}\kappa^\varepsilon_t \varphi = \varepsilon \iiint \varphi(\varepsilon(q + v\varepsilon^{-1}t), v, \ell)\, f(\varepsilon q, v, t)\, dq dv d\ell \tag{6.21}$$

$$= \iiint \varphi(q + vt, v, \ell)\, f(q, v, t)\, dq dv d\ell = \int \varphi\, \mathcal{T}_t f. \tag{6.22}$$

To conclude use Lemma 6.6. □

Mass convergence Let the empirical mass (length) measure be defined on nonnegative test functions φ by

$$\lambda^\varepsilon_t \varphi := \varepsilon \sum_{(q,v,\ell)\in T_{\varepsilon^{-1}t}\mathsf{X}^\varepsilon} \ell\, \varphi(\varepsilon q, v, \ell). \tag{6.23}$$

The following Lemma follows from Lemmas 6.6 and 6.7.

Lemma 6.8. *Let $\varphi \geq 0$ be a test function. Then,*

$$\lim_{\varepsilon\to 0} \lambda^\varepsilon_t \varphi = \iiint \ell\varphi(q,v,\ell)\, \mathcal{T}_t f(q,v,\ell)\, dq dv d\ell, \qquad a.s., \quad t \in \mathbb{R}. \tag{6.24}$$

6.1.4 Flow convergence

Recall the definition of flows (6.33) and for each process X^ε, q and v define the ε-*scaled mass flows* along the trajectory $(q+vs)_{s\in[0,t]}$ by

$$j^+_\varepsilon(q,v,t) := \varepsilon\, j^+_{\mathsf{X}^\varepsilon}(\varepsilon^{-1}q, v, \varepsilon^{-1}t), \qquad j^-_\varepsilon(q,v,t) := \varepsilon\, j^-_{\mathsf{X}^\varepsilon}(\varepsilon^{-1}q, v, \varepsilon^{-1}t).$$
$$j_\varepsilon(q,v,t) := j^+_\varepsilon(q,v,t) - j^-_\varepsilon(q,v,t).$$

Lemma 6.9. *Let $f\in\mathcal{F}$ and $q,v\in\mathbb{R}$. Then,*

$$\lim_{\varepsilon\to 0} j^+_\varepsilon(q,v,t) = j^+_f(q,v,t), \qquad \lim_{\varepsilon\to 0} j^-_\varepsilon(q,v,t) = j^-_f(q,v,t).$$
$$\lim_{\varepsilon\to 0} j_\varepsilon(q,v,t) = j_f(q,v,t) \tag{6.25}$$

where the macroscopic flows $j^\pm_f, j_f$ are defined in (6.65).

Proof. Let φ^+ be the indicator of the set of points (a,w,ℓ) whose ideal trajectory $(a+ws)_{s\in[0,t]}$ cross the trajectory $(q+vs)_{s\in[0,t]}$ from right to left:

$$\varphi^+(a,w,\ell) := \mathbf{1}\{a>q \text{ and } a+wt<q+vt\}. \tag{6.26}$$

Then,

$$j^+_\varepsilon(q,v,t) = \varepsilon \sum_{(a,w,\ell)\in\mathsf{X}^\varepsilon} \ell\,\varphi^+(\varepsilon a, w, \ell) = \lambda^\varepsilon_0\varphi^+, \tag{6.27}$$

and $\mathsf{E}\lambda^\varepsilon_0\varphi^+ = \mathsf{E}j^+_\varepsilon(q,v,t) = j^+_f(q,v,t) < \infty$, for $f\in\mathcal{F}$, see (6.67). Use Lemma 6.8 to get the first limit in (6.25). Use the same argument for the second limit and the identity $j = j^+ - j^-$ for the third limit. □

Corollary 6.10. *Let $f\in\mathcal{F}$, f^ε and X^ε as in (6.13) and (6.12). $o_t(\mathsf{X}) := j_{\mathsf{X}}(0,0,t)$, as defined in (6.35). Then,*

$$\lim_{eps\to 0} \varepsilon o_{\varepsilon^{-1}t}(\mathsf{X}^\varepsilon) = o_t = j_f(0,0,t), \tag{6.28}$$

as defined in (6.70).

6.2 Hard rods

The *mass* of a configuration X is the sum of the lengths of its particles:

$$m(\mathsf{X}) := \sum_{(q,v,\ell)\in\mathsf{X}} \ell. \tag{6.29}$$

Finite configurations have finite mass but infinite configurations may have infinite mass. The signed mass of a configuration X between real points a and b is defined by

$$m_a^b(\mathsf{X}) := m((q,v,\ell) \in \mathsf{X} : a \le q < b) - m((q,v,\ell) \in \mathsf{X} : b \le q < a), \quad (6.30)$$

that is, the mass of the particles with positions in $[a, b)$ if $a < b$ or minus the mass of the particles with positions in $[b, a)$, otherwise. If $a = b$, then $m_a^b(\mathsf{X}) = 0$.

Consider the lexicographic order between particles and for any configuration X, time t in $\mathbb{R}$ and trajectory $(q + sv)_{s\in\mathbb{R}}$, define

$$J^+_{q,v,t}\mathsf{X} := \{(a, w, \ell) \in \mathsf{X} : a > q \text{ and } a + wt < q + vt\} \quad (6.31)$$

$$J^-_{q,v,t}\mathsf{X} := \{(a, w, \ell) \in \mathsf{X} : a < q \text{ and } a + wt > q + vt\} \quad (6.32)$$

These are the particles of X whose ideal gas trajectories intersect the trajectory of $(q + sv)_{s\in\mathbb{R}}$ from right to left and from left to right, respectively, up to time t. The masses of these sets are called *mass flows* and denoted by

$$j^+_{\mathsf{X}}(q, v, t) := m(J^+_{q,v,t}\mathsf{X}), \qquad j^-_{\mathsf{X}}(q, v, t) := m(J^-_{q,v,t}\mathsf{X}). \quad (6.33)$$

The set of configurations X with finite mass flows is defined by

$$\mathcal{X} := \{\mathsf{X} \subset \mathbb{X} : j^+_{\mathsf{X}}(q, v, t) < +\infty \text{ and } j^-_{\mathsf{X}}(q, v, t) < +\infty, \text{ for all } q, v, t \in \mathbb{R}\}. \quad (6.34)$$

For $\mathsf{X} \in \mathcal{X}$, the *net mass flow* is defined by

$$j_{\mathsf{X}}(q, v, t) := j^+_{\mathsf{X}}(q, v, t) - j^-_{\mathsf{X}}(q, v, t). \quad (6.35)$$

6.2.1 Dilation and contraction

The hard rod configuration space $\mathcal{Y}$ is defined by

$$\mathcal{Y} := \{\mathsf{Y} \in \mathcal{X} : (q, q + \ell) \cap (q', q' + \ell') = \emptyset,\ (q, v, \ell), (q', v', \ell') \in \mathsf{Y}\}. \quad (6.36)$$

The expression *hard rods* refers to "rods cannot intersect", which is the exclusion condition characterizing $\mathcal{Y}$. The set of configurations in $\mathcal{Y}$ with no rod containing $a \in \mathbb{R}$ is denoted by

$$\mathcal{Y}_a := \{\mathsf{Y} \in \mathcal{Y} : a \notin (q, q + \ell),\ (q, v, \ell) \in \mathsf{Y}\}. \quad (6.37)$$

For every a in $\mathbb{R}$, define the *dilation* (w.r.t. a) map $D_a : \mathcal{X} \to \mathcal{Y}_a$, by

$$D_a\mathsf{X} := \big\{\big(q + m_a^q(\mathsf{X}), v, \ell\big) : (q, v, \ell) \in \mathsf{X}\big\}, \quad (6.38)$$

see (6.30) for the definition of the signed mass $m_a^q(\mathsf{X})$. The map D_a is a bijection and its inverse is the *contraction* (w.r.t. a) map $C_a : \mathcal{Y}_a \to \mathcal{X}$, given by

$$C_a\mathsf{Y} := \big\{\big(q - m_a^q(\mathsf{Y}), v, \ell\big) : (q, v, \ell) \in \mathsf{Y}\big\}. \tag{6.39}$$

Define the mapped position of a point b in the dilation with respect to a of X and in the contraction with respect to a of $\mathsf{Y} \in \mathcal{Y}_a$, by

$$D_{\mathsf{X},a}(b) := b + m_a^b(\mathsf{X}), \quad C_{\mathsf{Y},a}(b) := b - m_a^b(\mathsf{Y}). \tag{6.40}$$

The operators $D_{\mathsf{X},a}$ and $C_{\mathsf{Y},a}$ conserve mass: for $a < b$ we have

$$m_a^b(\mathsf{X}) = m_a^{D_{\mathsf{X},a}(b)}(D_a\mathsf{X}) = m^b_{D_{\mathsf{X},a}(b)}(D_b\mathsf{X}), \quad \mathsf{X} \in \mathcal{X}, \tag{6.41}$$

$$m_a^b(\mathsf{Y}) = m_a^{C_{\mathsf{Y},a}(b)}(C_a\mathsf{Y}) = m^b_{C_{\mathsf{Y},a}(b)}(C_b\mathsf{Y}), \tag{6.42}$$

where we assume $\mathsf{Y} \in \mathcal{Y}_a$ in the first identity of (6.42) and $\mathsf{Y} \in \mathcal{Y}_b$ in the second one.

6.2.2 Hard rod dynamics

The evolution of a hard rod configuration is deterministic: each rod travels ballistically at its own speed until there is a collision: if at time $t-$ we have rods located at (q, v, ℓ) and (q', v', ℓ') with $q' = q + \ell$ and $v > v'$, then, at time t each rod is shifted by the length of the other rod, interchanging order:

$$\begin{array}{cc} \text{Before collision, at time } t- & \text{After collision, at time } t \\ (q, v, \ell), (q', v', \ell') & (q + \ell', v, \ell), (q' - \ell, v', \ell') \end{array} \tag{6.43}$$

After collision each rod continues at its speed until the next collision. The dynamics is well defined for any finite number of initial rods.

We define now the dynamics when there are infinitely many rods. The *hard rod dynamics as seen from a zero-length zero-speed hard rod sitting initially at the origin* is the operator $(\hat{U}_t)_{t\in\mathbb{R}}$ defined by

$$\hat{U}_t\mathsf{Y} := D_0T_tC_0\mathsf{Y}, \qquad \mathsf{Y} \in \mathcal{Y}_0. \tag{6.44}$$

Denote S_a the shift operator, acting on real numbers, particles and configurations by

$$S_ab = b - a, \quad S_a(q, v, \ell) = (S_aq, v, \ell), \quad S_a\mathsf{X} = \{S_a\mathsf{x} : \mathsf{x} \in \mathsf{X}\}. \tag{6.45}$$

$S_a\mathsf{X}$ is the configuration as seen from a.

Define the *hard rod dynamics* for configurations $\mathsf{Y} \in \mathcal{Y}_0$ by

$$U_t\mathsf{Y} := S_{-o_t(C_0\mathsf{Y})}\hat{U}_t\mathsf{Y}, \qquad \mathsf{Y} \in \mathcal{Y}_0, \tag{6.46}$$

$$o_t(\mathsf{X}) := j_{\mathsf{X}}(0,0,t), \qquad \mathsf{X} \in \mathcal{X}, \tag{6.47}$$

see (6.35). $o_t(C_0\mathsf{Y})$ coincides with the position at time t of the a hard rod $o := (0,0,0)$ added to the configuration Y at time 0. The addition of the particle o at an empty site does not modify the evolution of the rods of Y.

If $\mathsf{Y} \in \mathcal{Y} \setminus \mathcal{Y}_0$, then Y has a rod (q, v, ℓ) containing the origin, that is, with $q < 0 < q + \ell$. Since $S_q\mathsf{Y} \in \mathcal{Y}_0$, we can use (6.46) to define

$$U_t\mathsf{Y} := S_{-q}U_tS_q\mathsf{Y}, \qquad \mathsf{Y} \in \mathcal{Y} \setminus \mathcal{Y}_0. \tag{6.48}$$

Lemma 6.11. *For finite* $\mathsf{Y} \in \mathcal{Y}$*, this dynamics coincides with the one described by (6.43).*

Proof. By pictures. □

Tagged rod motion Given a hard rod configuration $\mathsf{Y} \in \mathcal{Y}_q$, that is with no rod containing q, we study the motion of a tagged rod with arbitrary length inserted at time zero in q with speed v and look at its position $u_{\mathsf{Y},v;t}(q)$ at time t, defined by

$$u_{\mathsf{Y},v;t}(q) := q + vt + j_{C_q\mathsf{Y}}(q, v, t), \tag{6.49}$$

in particular, for $\mathsf{Y} \in \mathcal{Y}_0$, $o_t(\mathsf{Y}) = u_{\mathsf{Y},0;t}(0)$. The definition makes sense as the tagged particle jumps to the right by ℓ when it is crossed by a size ℓ rod from right to left, and by $-\ell$, if the crossing is from left to right. The rods crossing the tagged rod are the same in the compressed or uncompressed dynamics.

6.2.3 Palm measures and invariance for hard rods

Palm measures Recall $\mathcal{Y}_0$ is the set of hard rod configurations with no rod containing the origin, as defined in (6.37). Assume P is a shift invariant measure on $\mathcal{Y}$. Define the Palm measure $\hat{\mathsf{P}}$ by

$$\hat{\mathsf{P}} = \mathsf{P}(\,\cdot\,|\mathcal{Y}_0), \tag{6.50}$$

that is, $\hat{\mathsf{P}}$ is the measure P conditioned to the event "no rod contains the origin".

Denote $D_0\mathsf{P}(A) := \mathsf{P}(D_0^{-1}A)$ and write $\overset{D}{=}$ to mean "same distribution"

Proposition 6.12 (Ideal gas and hard rod invariance)**.** *If* P *is shift invariant on* $\mathcal{X}$*, then*

$$\mathsf{P} \textit{ is } T\textit{-invariant} \quad \textit{if and only if} \quad D_0\mathsf{P} \textit{ is } \hat{U}\textit{-invariant.} \tag{6.51}$$

Proof. Let X be random with law P and $\mathsf{Y} := D_0\mathsf{X}$. If P is T-invariant, then $\hat{U}_t\mathsf{Y} = D_0T_t\mathsf{X} \overset{D}{=} D_0\mathsf{X} = \mathsf{Y}$. Hence $D_0\mathsf{P}$ is $\hat{U}$-invariant, as Y has law $D_0\mathsf{P}$. Reciprocally, $T_t\mathsf{X} = T_tC_0\mathsf{Y} = C_0\hat{U}_t\mathsf{Y} \overset{D}{=} C_0\mathsf{Y} = \mathsf{X}$. □

Empty shift and anti-Palm For $\mathsf{Y} \in \mathcal{Y}_0$ and a a real number, define

$$b(a,\mathsf{Y}) := D_{C_0\mathsf{Y},0}(a); \tag{6.52}$$

so, the empty space in Y between 0 and $b(a,\mathsf{Y})$ is a, if a is positive and $-a$ otherwise. Define the *empty shift by* a by

$$\hat{S}_a\mathsf{Y} := S_{b(a,\mathsf{Y})}\mathsf{Y} = D_0S_aC_0\mathsf{Y}, \quad \mathsf{Y} \in \mathcal{Y}_0,\ a \in \mathbb{R}. \tag{6.53}$$

We say that a measure $\hat{\mathsf{P}}$ on $\mathcal{Y}_0$ is empty-shift invariant if it is $\hat{S}_a$-invariant for all a:

$$\hat{\mathsf{E}}\varphi(\mathsf{X}) = \hat{\mathsf{E}}\varphi(\hat{S}_a\mathsf{X}), \tag{6.54}$$

where $\hat{\mathsf{E}}$ is expectation with respect to $\hat{\mathsf{P}}$ and φ is a test function.

Given an empty-shift invariant measure $\hat{\mathsf{P}}$ on $\mathcal{Y}_0$, define the inverse-Palm measure P, E on $\mathcal{Y}$ by

$$\mathsf{E}\varphi(\mathsf{Y}) := \frac{1}{\hat{\mathsf{E}}b(a,\mathsf{Y})}\,\hat{\mathsf{E}}\Big(\int_0^{b(a,\mathsf{Y})} \varphi(S_x\mathsf{Y})dx\Big), \quad a \neq 0. \tag{6.55}$$

(See formula (4.14^o) in Thorisson [48].) A sample of this measure is obtained by sampling a configuration Y with law $\frac{1}{\hat{\mathsf{E}}b(a,\mathsf{Y})}b(a,\mathsf{Y})\hat{\mathsf{P}}(d\mathsf{Y})$ (size biased by $b(a,\mathsf{Y})$) and then make a random shift uniformly distributed in the interval $(0, b(a,\mathsf{X}))$. To see this interpretation, just multiply and divide by $b(a,\mathsf{X})$ the expression (6.55) to get:

$$\frac{1}{\hat{\mathsf{E}}b(a,\mathsf{Y})}\,\hat{\mathsf{E}}\Big(b(a,\mathsf{Y})\frac{1}{b(a,\mathsf{Y})}\int_0^{b(a,\mathsf{Y})} \varphi(S_x\mathsf{Y})dx\Big). \tag{6.56}$$

The resulting mesure P is shift invariant and its definition does not depend on the choice of $a \neq 0$.

Proposition 6.13 (Harris [27]). *Let* P *be a shift invariant measure and* $\hat{\mathsf{P}}$ *its Palm measure. Then*

$$\mathsf{P} \text{ is } U\text{-invariant if and only if } \hat{\mathsf{P}} \text{ is } \hat{U}\text{-invariant} \tag{6.57}$$

For a proof of this proposition see also [42] and [15].

The above results can be condensated in the following theorem

Theorem 6.14. *Let* P *be a shift invariant and* U*-invariant measure on* $\mathcal{Y}$*. Let* $\hat{\mathsf{P}} := \mathsf{P}(\cdot|\mathcal{Y}_0)$ *and assume that* $C_0\hat{\mathsf{P}}$ *is mixing. Then* $C_0\hat{\mathsf{P}}$ *is a Poisson process.*

Proof. Since P is shift invariant and U-invariant, Harris Theorem 6.13 implies $\hat{\mathsf{P}}$ is $\hat{U}_t$ invariant which, in turn, implies $C_0\hat{\mathsf{P}}$ is T-invariant, by Proposition 6.12. Since by hypothesis $C_0\hat{\mathsf{P}}$ is mixing, part (b) of Proposition 6.5 implies $C_0\hat{\mathsf{P}}$ is a Poisson process. □

6.3 Macroscopic dynamics

6.3.1 Admissible functions, dilations and contractions

Let f be a density and define its mass and momentum functions as

$$\sigma_f(q) := \iint \ell\, f(q,v,\ell)\, dv d\ell \tag{6.58}$$

$$\zeta_f(q) := \iint v\,\ell\, f(q,v,\ell)\, dv d\ell \tag{6.59}$$

Define the set $\mathcal{F}$ of densities with uniformly bounded mass and momentum:

$$\mathcal{F} := \left\{ f : \mathbb{X} \in \mathbb{R}_{\geq 0} : \max\{\|\dot{\zeta}_f\|_\infty, \|\zeta_f\|_\infty, \|\sigma_f\|_\infty, \|\dot{\sigma}_f\|_\infty\} < \infty \right\} \tag{6.60}$$

Dilation and contraction Define the set of dilated macroscopic densities as

$$\mathcal{G} := \{ f \in \mathcal{F} : \|\sigma_f\|_\infty < 1 \}.$$

For $a \in \mathbb{R}$, $f \in \mathcal{F}$ and $g \in \mathcal{G}$, define the point dilation and contraction, by

$$D_{f,a}(b) := b + \int_a^b \sigma_f(q) dq, \qquad C_{g,a}(b) := b - \int_a^b \sigma_g(q) dq, \qquad a, b \in \mathbb{R}.$$

Define the dilation and contraction (w.r.t. a) operators as the bijections $D_a : \mathcal{F} \to \mathcal{G}$, $C_a : \mathcal{G} \to \mathcal{F}$ by

$$D_a f(q,v,\ell) = \frac{f(D_{f,a}^{-1}(q),v,\ell)}{1+\sigma_f(D_{f,a}^{-1}(q))}, \qquad C_a g(q,v,\ell) = \frac{g(C_{g,a}^{-1}(q),v,\ell)}{1-\sigma_g(C_{g,a}^{-1}(q))}. \tag{6.61}$$

The derivatives are given by

$$\frac{d}{dq}D_{f,a}(q) = 1+\sigma_f(q) \in [1,\infty), \quad \frac{d}{dq}C_{g,a}(q) = 1-\sigma_g(q) \in (0,1]. \tag{6.62}$$

In particular, $\frac{d}{dq}C_{g,a}(q) \geq 1 - \|\sigma_g\|_\infty > 0$. Thus, both functions are diffeomorphisms, the former is a dilation and the later is a contraction.

Both operators conserve mass:

$$\int_{D_{f,a}(b)}^{D_{f,a}(c)} \sigma_{D_a f}(q)dq = \int_b^c \sigma_f(q)dq, \qquad \int_{C_{g,a}(b)}^{C_{g,a}(c)} \sigma_{C_a g}(q)dq = \int_b^c \sigma_g(q)dq.$$

By additivity of the integral, it suffices to consider $a = b$. Consider each expression as a function of $q = c$, check each equality for $q = a$ and conclude by checking the equality of the derivatives.

Shift The shift operator S_a applied to $f \in \mathcal{F}$ of $g \in \mathcal{G}$ is given by

$$S_a f(q,v,\ell) := f(q-a,v,\ell). \tag{6.63}$$

The shift conserves density, mass and momentum. For f in $\mathcal{F}$, a, b, c, v, ℓ, q in $\mathbb{R}$, we have

$$\int_{b+a}^{c+a} S_a f(q,v,\ell)dq = \int_b^c f(q,v,\ell)dq,$$

$$\sigma_{S_a f}(q) = \sigma_f(q-a), \quad \zeta_{S_a f}(q) = \zeta_f(q-a).$$

6.3.2 Macroscopic evolution of hard rods

Mass Flows The macroscopic counterpart of the mass flows defined in (6.35) are defined by

$$j_f^+(q,v,t) := \int_0^\infty \int_{-\infty}^v \int_q^{q+(v-w)t} \ell\, f(a,w,\ell)\, da\, dw\, d\ell \tag{6.64}$$

$$j_f^-(q,v,t) := \int_0^\infty \int_v^\infty \int_{q+(v-w)t}^q \ell\, f(a,w,\ell)\, da\, dw\, d\ell, \tag{6.65}$$

$$j_f(q,v,t) := j_f^+(q,v,t) - j_f^-(q,v,t). \tag{6.66}$$

The right-left *flow* $j_f^+(q,v,t)$ gives the mass crossing from right to left the trajectory $(q+vs)_{s\in\mathbb{R}}$ in the time interval $[0,t]$, when the initial density is f. Analogously, $j_f^-(q,v,t)$ collects the left-to-right flow. The net flow is the difference of those.

Notice that if $f \in \mathcal{F}_\gamma$, for some $\gamma \in \mathcal{L}$, then

$$0 \le j_f^+(q,v,t) \le \int_{-\infty}^{v} \int_0^\infty (v-w)\ell\,\gamma(w,\ell)\,d\ell\,dw \;<\; \infty, \tag{6.67}$$

by definition of $\mathcal{L}$. Analogously $j_f^-(q,v,t) < \infty$. We can conclude that

$$j_f^+(q,v,t) < \infty,\;\; j_f^+(q,v,t) < \infty,\;\; |j_f(q,v,t)| < \infty, \qquad f\in\mathcal{F}, \quad q,v,t\in\mathbb{R}. \tag{6.68}$$

Hard-rod evolution Define the operator $\mathcal{U}_t : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{U}_t g(q,v,\ell) := S_{-o_t} D_0 \mathcal{T}_t C_0 g(q,v,\ell) \tag{6.69}$$

$$o_t := j_{C_0 g}(0,0,t). \tag{6.70}$$

As in the microscopic case, $D_0\mathcal{T}_tC_0g$ is the macroscopic hard rod evolution as seen from a zero-speed, zero-length particle $(0,0,0)$, added at the origin at time 0, and o_t is the position of this test particle at time t. The shift by $-o_t$ is performed in order to obtain the hard rod evolution of g (as seen from the origin).

Lemma 6.15 (Group property)**.** *The family $(\mathcal{U}_t)_{t\in\mathbb{R}}$ is a group. In particular*

$$\mathcal{U}_{t+s} = \mathcal{U}_t\mathcal{U}_s, \qquad t,s\in\mathbb{R}. \tag{6.71}$$

Tagged rod evolution For every g in $\mathcal{G}$, we define the macroscopic position at time t of a hard rod (q,v,ℓ), as the bijection $u_{g,v,t} : \mathbb{R}\to\mathbb{R}$

$$u_{g,v,t}(q) := q + vt + j_{C_0g}(q,v,t). \tag{6.72}$$

Notice that this definition does not depend on the length ℓ of the hard rod.

6.4 Hydrodynamics of hard rods

6.4.1 Law of large numbers for dilations

We prove now law of large numbers for dilations of Poisson processes.

Lemma 6.16 (LLN for dilations). *Let $f \in \mathcal{F}$ and X^ε a Poisson process with intensity f^ε. Then,*

$$\lim_{\varepsilon\to 0} \varepsilon D_{\mathsf{X}^\varepsilon,0}(\varepsilon^{-1}a) = D_{f,0}(a), \qquad a.s.. \tag{6.73}$$

Proof. Take $a > 0$ and recall (6.40) to get

$$\lim_{\varepsilon\to 0} \varepsilon D_{\mathsf{X}^\varepsilon,0}(\varepsilon^{-1}a) = \lim_{\varepsilon\to 0} \varepsilon \sum_{(q,v,\ell)\in\mathsf{X}^\varepsilon} \ell\, \mathbf{1}\{\varepsilon q \in [0,a)\} \tag{6.74}$$

$$= \iiint \ell\, \mathbf{1}\{\varepsilon q \in [0,a)\}\, f(q,v,\ell)\, dqdvd\ell, \quad \text{a.s.} \tag{6.75}$$

$$= D_{f,0}(a), \tag{6.76}$$

where the limit was proven in Lemma 6.6. ☐

Lemma 6.17 (LLN for empirical measure of dilations). *Let F, G and H be measures on $\mathbb{X}$ absolutely continuous with densities $f \in \mathcal{F}$, and $g, h \in \mathcal{G}$, respectively, satisfying $g = D_0 f$ and $h(q,v,\ell) = \ell g(q,v,\ell)$. Let X^ε be a Poisson processes on $\mathcal{X}$ with intensity f^ε and let $\mathsf{Y}^\varepsilon := D_0\mathsf{X}^\varepsilon$. For any bounded interval $A \subset \mathbb{R}$ and $\tilde\varphi : \mathbb{R}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, denote*

$$\varphi(q,v,\ell) = \mathbf{1}\{q \in A\}\, \tilde\varphi(v,\ell). \tag{6.77}$$

Then,

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{(q,v,\ell)\in\mathsf{Y}^\varepsilon} \ell\varphi(\varepsilon q, v, \ell) = \int \varphi\, h, \quad a.s. \tag{6.78}$$

Proof. The limit (6.78) is equivalent to

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{(q,v,\ell)\in\mathsf{X}^\varepsilon} \ell\varphi(\varepsilon D_{\mathsf{X}^\varepsilon,0}(q), v, \ell) \tag{6.79}$$

$$= \iiint \ell\varphi(D_{f,0}(q), v, \ell)\, f(q,v,\ell)\, dqdvd\ell, \quad \text{a.s..} \tag{6.80}$$

This follows from (a) the sum in (6.79) equals the sum in (6.78), by definition of D_0, and (b) by definition, $\int \varphi\, h = \iiint \ell\varphi(q,v,\ell)\, D_0 f(q,v,\ell)\, dqdvd\ell$. Change variables $q = D_{f,0}(\hat q)$ and use (6.61) and (6.62) to get (6.80).

We now show (6.80) for $A = [0,b]$, for $b > 0$, the general case follows from this case. Denote

$$\varphi_b(q,v,\ell) := \mathbf{1}\{q \in [0,b]\}\, \tilde\varphi(v,\ell). \tag{6.81}$$

Denote $a := D_{f,0}^{-1}(b)$ and a', a'' satisfying $0 < a' < a < a''$ and $D_{f,0}(a') < D_{f,0}(a) < D_{f,0}(a'')$. By (6.73), there is $\varepsilon^* > 0$ such that,

$$\varepsilon D_{\mathsf{X}^\varepsilon,0}(\varepsilon^{-1}a') \leq b, \tag{6.82}$$

for $\varepsilon < \varepsilon^*$, and

$$\mathbf{1}\{\varepsilon q \in [0, a']\} = \mathbf{1}\{\varepsilon D_{\mathsf{X}^\varepsilon,0}(q) \in [0, \varepsilon D_{\mathsf{X}^\varepsilon,0}(\varepsilon^{-1}a')]\} \leq \mathbf{1}\{\varepsilon D_{\mathsf{X}^\varepsilon,0}(q) \in [0, b]\},$$

for all $(q, v, \ell) \in \mathsf{X}^\varepsilon$, for $\varepsilon < \varepsilon^*$. Since $\ell\tilde{\varphi}(q, v, \ell) \geq 0$, the above inequality implies

$$\begin{aligned} I^\varepsilon(a') &:= \varepsilon \sum_{(q,v,\ell)\in\mathsf{X}^\varepsilon} \ell\varphi_{a'}(\varepsilon q, v, \ell) && (6.83)\\ &\leq \varepsilon \sum_{(q,v,\ell)\in\mathsf{X}^\varepsilon} \ell\tilde{\varphi}(v, \ell)\, \mathbf{1}\{\varepsilon D_{\mathsf{X}^\varepsilon,0}(q) \in [0, b]\} =: \tilde{I}^\varepsilon(b), && (6.84)\end{aligned}$$

for $\varepsilon < \varepsilon^*$. Use Lemma 6.8 with $t = 0$ to obtain that $I^\varepsilon(a')$ converges to $I(a')$, defined below, implying $I(a') \leq \liminf \tilde{I}^\varepsilon(b) \leq \limsup \tilde{I}^\varepsilon(b) \leq I^\varepsilon(a'')$, using the same argument for the upperbound. Since this holds for all $a' < a < a''$, we can conclude that

$$\begin{aligned} &\lim_{\varepsilon\to0} \varepsilon \sum_{(q,v,\ell)\in\mathsf{X}^\varepsilon} \ell\tilde{\varphi}(v, \ell)\, \mathbf{1}\{\varepsilon D_{\mathsf{X}^\varepsilon,0}(q) \in [0, b]\} && (6.85)\\ &\quad = \iiint \ell\varphi_a(q, v, \ell)\, f(q, v, \ell)\, dq dv d\ell \;\; =: \;\; I(a) && (6.86)\end{aligned}$$

Since

$$\begin{aligned} \varphi_a(q, v, \ell) &= \mathbf{1}\{q \leq a\}\tilde{\varphi}(v, \ell) && (6.87)\\ &= \mathbf{1}\{D_{f,0}(q) \leq D_{f,0}(a)\}\tilde{\varphi}(v, \ell) = \varphi_b(D_{f,0}(q), v, \ell), && (6.88)\end{aligned}$$

as $b = D_{f,0}(a)$, we have that $I(a)$ equals the right hand side of (6.80) for $\varphi = \varphi_b$. □

6.4.2 Hard rod hydrodynamics

Let $(\mathsf{Y}^\varepsilon)_{\epsilon>0}$ be a family of point processes on the hard rod space $\mathcal{Y}$. Define the *ε-scaled empirical mass measure for hard rods* associated to this family by

$$\pi_t^\varepsilon \varphi := \varepsilon \sum_{(q,v,\ell)\in U_{\varepsilon^{-1}t}\mathsf{Y}^\varepsilon} \ell\varphi(\varepsilon q, v, \ell). \tag{6.89}$$

Theorem 6.18 (Hard rod hydrodynamics)**.** *Let F and G be absolutely continuous measures on $\mathbb{X}$ with densities $f \in \mathcal{F}$ and $g \in \mathcal{G}$, respectively, satisfying $g = D_0 f$. Denote by H_t the absolutely continuous measure on $\mathbb{X}$ with density $h_t(q,v,\ell) := \ell\,\mathcal{U}_t g(q,v,\ell)$.*

For each $\varepsilon > 0$, let f^ε be defined by $f^\varepsilon(q,v,\ell) := f(\varepsilon q, v, \ell)$. Let X^ε be a Poisson processes on $\mathcal{X}$ with intensity f^ε, and let $\mathsf{Y}^\varepsilon := D_0\mathsf{X}^\varepsilon$. Let π_t^ε be the empirical mass measure associated to Y^ε, defined in (6.89).

For any bounded interval $A \subset \mathbb{R}$ and $\tilde\varphi : \mathbb{R}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, denote $\varphi(q,v,\ell) := \mathbf{1}\{q \in A\}\,\tilde\varphi(v,\ell)$. Then,

$$\lim_{\varepsilon\to 0} \pi_t^\varepsilon \varphi = \int \varphi\, h_t, \qquad a.s. \tag{6.90}$$

Proof. Since $\mathsf{Y}^\varepsilon = D_0\mathsf{X}^\varepsilon \in \mathcal{Y}_0$, by definition (6.46) we have

$$U_{\varepsilon^{-1}t}\mathsf{Y}^\varepsilon = S_{-o_t^\varepsilon} D_0 \mathsf{X}_t^\varepsilon, \tag{6.91}$$

where $o_t^\varepsilon := o_{\varepsilon^{-1}t}(\mathsf{X}^\varepsilon)$ and $\mathsf{X}_t^\varepsilon := T_{t\varepsilon^{-1}}\mathsf{X}^\varepsilon$. Substituting in (6.89), we have

$$\pi_t^\varepsilon\,\varphi = \varepsilon \sum_{(q,v,\ell)\in S_{-o_t^\varepsilon} D_0\mathsf{X}_t^\varepsilon} \ell\varphi(\varepsilon q, v, \ell) = \varepsilon \sum_{(q,v,\ell)\in \mathsf{X}_t^\varepsilon} \ell\varphi\big(\varepsilon(D_{X_t^\varepsilon,0}(q) + o_t^\varepsilon), v, \ell\big).$$

Take $b > 0$ and consider φ_b, as defined in (6.81), with the convention that for negative b, $\varphi_b(q,v,\ell) := \mathbf{1}\{b < q < 0\}\,\tilde\varphi(v,\ell)$. Take $b > 0$ to get

$$\pi_t^\varepsilon\,\varphi_b = \varepsilon \sum_{(q,v,\ell)\in \mathsf{X}_t^\varepsilon} \ell\mathbf{1}\big\{0 \leq \varepsilon(D_{X_t^\varepsilon,0}(q) + o_t^\varepsilon) \leq b\big\}\,\tilde\varphi(v,\ell) \tag{6.92}$$

The laws of large numbers (6.28) and (6.78) imply

$$\lim_{\varepsilon\to 0} \varepsilon\, o_t^\varepsilon = o_t, \qquad \lim_{\varepsilon\to 0} \varepsilon D_{X_t^\varepsilon,0}(q) = D_{\mathcal{T}_t f,0}(q), \tag{6.93}$$

where $o_t = j_f(0,0,t)$, as defined in (6.70). The second limit holds because X_t^ε is a Poisson process with intensity $(\mathcal{T}_t f)^\varepsilon$.

An argument as in (6.83) and (6.86) gives, for $-o_t > 0$,

$$\lim_{\varepsilon\to 0} \pi_t^\varepsilon\,\varphi_b = \iiint \ell\,\varphi_b\big(D_{\mathcal{T}_t f,0}(q) + o_t, v, \ell\big)\,\mathcal{T}_t f(q,v,\ell)\,dq dv d\ell \tag{6.94}$$

$$= \iiint \ell\,\varphi_b(q,v,\ell)\,S_{-o(t)} D_o \mathcal{T}_t f(q,v,\ell)\,dq dv d\ell \tag{6.95}$$

$$= \iiint \ell\,\varphi_b(q,v,\ell)\,\mathcal{U}_t g(q,v,\ell)\ dq dv d\ell = \int \varphi_b\,dH_t. \qquad \square$$

6.4.3 Density evolution equation

Theorem 6.19 (Density evolution equation)**.** *Let $\tilde{g}$ in $\mathcal{G}$ and $\hat{g}(q, v, \ell; t) := \mathcal{U}_t\tilde{g}(q, v, \ell)$. Then $\hat{g}(q, v, \ell; t)$ is the unique solution of the following Cauchy problem for functions $\hat{g} : \mathbb{X}\times\mathbb{R} \to \mathbb{R}$ such that $\hat{g}(\cdot; t) \in \mathcal{G}$ for all t.*

$$\begin{aligned}
&\partial_t\hat{g}(q, v, \ell; t) + v\, \partial_q\hat{g}(q, v, \ell; t) \\
&\qquad + \partial_q\left(\hat{g}(q, v, \ell; t)\, \frac{\int_0^{+\infty}\int_{-\infty}^{+\infty} \ell'\,(v - w)\,\hat{g}(q, w, \ell'; t)\, dw\, d\ell'}{1 - \int_0^{+\infty} \ell' \int_{-\infty}^{+\infty} \hat{g}(q, w, \ell'; t)\, dw\, d\ell'}\right) = 0 \\
&\hat{g}(q, v, \ell; 0) = \tilde{g}(q, v, \ell)
\end{aligned}$$

Recall the microscopic tagged rod evolution (6.49) and macroscopic (6.72).

Corollary 6.20. *We have the following alternative formulation of the hard rod evolution:*

$$\mathcal{U}_t g(q, v, \ell) = g\big(u^{-1}_{g,v,t}(q), v, \ell\big)\, \frac{d}{dq} u^{-1}_{g,v,t}(q) \tag{6.96}$$

Define the evolving effective velocity field associated to $\hat{g}$ as the function

$$v^{\text{eff}}_{\hat{g}} := v + \frac{\int_0^{+\infty} \ell' \int_{-\infty}^{+\infty} (v - w)\hat{g}(q, w, \ell'; t) dw d\ell'}{1 - \sigma_h(q, t)} \in \mathbb{R}$$

We have the following equivalent formulation,

$$v^{\text{eff}}_{\hat{g}}(q, v, t) = \frac{v - \zeta_{\hat{g}}(q, t)}{1 - \sigma_{\hat{g}}(q, t)} \tag{6.97}$$

Now we can consider the following Cauchy problem

$$\begin{cases} \dot{q}_{\hat{g},q,v} = v^{\text{eff}}_{\hat{g}}(q_{\hat{g},q,v}(t), v, t) \\ q_{\hat{g},q,v}(0) = q \end{cases} \tag{6.98}$$

Since $\hat{g}$ is in $\mathcal{G}$, for every v in $\mathbb{R}$ we have that $v^{\text{eff}}_{\hat{g}}(\cdot, v, \cdot)$ and $\partial_q v^{\text{eff}}_{\hat{g}}(\cdot, v, \cdot)$ are in $\mathcal{C}^2(\mathbb{R}^2)$. Then, by the general theory of ODEs, for each (q, v) in $\mathbb{R}^2$, there exists a global solution to the problem (6.98). Let us define

$$u_{\hat{g},v,t}(q) = q_{\hat{g},q,v}(t).$$

which is a diffeomorphism.

Proposition 6.21. *For every $(q, v, \ell; t)$ in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$, we have*

$$\begin{cases} \hat{g}(q, v, \ell; t) = g_0(u^{-1}_{\hat{g},v,t}(q), v, \ell)\frac{d}{dq} u^{-1}_{\hat{g},v,t}(q) \\ u_{\hat{g},v,t} = u_{\tilde{g},v,t} \end{cases} \tag{6.99}$$

7 Poisson line processes

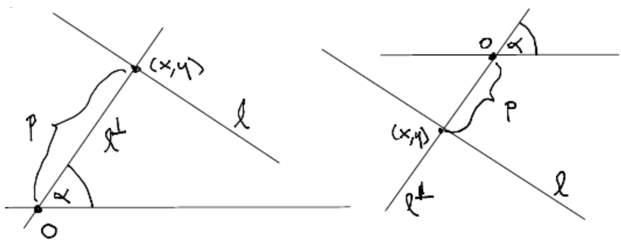
Poisson line processes is a well known subject in point processes (Kingman [43], Daley and Vere Jones [9]) and Stochastic Geometry (Chiu, Stoyan, Kendall and Mecke [8]) and geostatistics (Lantuéjoul [31, 32, 33]).

7.1 Line processes in $\mathbb{R}^2$

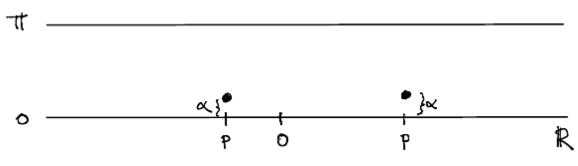
Let $\mathbb{L}$ be the set of infinite, undirected straight lines in $\mathbb{R}^2$. A countable random subset of $\mathbb{L}$ is called *line process.* To construct a Poisson line process on $\mathbb{L}$, it suffices to define a mean measure μ on $\mathbb{L}$. To do that it is convenient to use convenient line representations.

Mapping $\mathbb{L}$ to a band $\mathbb{A}$ Given $\ell \in \mathbb{L}$, let $\ell^{\perp}$ be the unique line perpendicular to ℓ containing the origin. Let $x = (x, y) := \ell \cap \ell^{\perp}$ the intersection point of these lines. Let p be the signed distance $(x^2 + y^2)^{1/2}\text{Sign}(y)$ and $\alpha := \arctan \frac{y}{x}$, the angle between $\ell^{\perp}$ and the positive oriented x-axis, $0 \le \alpha < \pi$.

Denote $\mathbb{A} := \mathbb{R} \times [0, \pi)$. The map $\phi : \ell \mapsto (p, \alpha)$ is a bijection between $\mathbb{L}$ and $\mathbb{A}$.



The lines of the figures above map to the points below:



The line ℓ has the following parametrization

$$x \cos \alpha + y \sin \alpha = p \tag{7.1}$$

Any measure μ on $\mathbb{L}$ induces a measure on $\mathbb{A}$, and vice-versa. We do not distinguish these two measures.

The set of lines in $\mathbb{L}$ intersecting the ball $B(0,r) = \{(x,y) : x^2 + y^2 \leq r\}$ is the set$\{(p,\alpha) \in \mathbb{A} : |p| \leq r\}$.

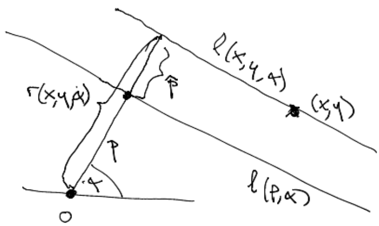
We say that a set of lines is locally finite if only if a finite number of lines intersect any bounded Borel set. A locally finite set $L \subset \mathbb{L}$ maps into a locally finite set $\phi(L) \subset \mathbb{A}$.

7.2 Uniform Poisson line process

If μ is a locally integrable measure on $\mathbb{A}$, then the existence theorem shows that there is a Poisson process with mean measure μ on $\mathbb{A}$ and $\mathbb{L}$; we use the same symbol μ for the measure in any of those sets.

When μ_λ is Lebesgue measure on $\mathbb{A}$ of constant intensity λ, the corresponding Poisson process is called *uniform Poisson line process* with intensity λ.

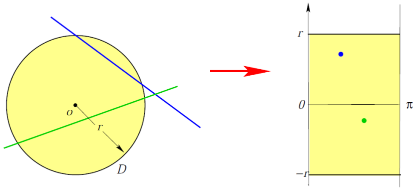
Lemma 7.1 (Invariance by isometries)**.** *The Poisson line process is invariant by translations, rotations and reflexions.*



Proof. Exercise. Hint for translation invariance: use the line parametrization formula (7.1). Denote $(\hat{p}, \hat{\alpha}) = S_{(x,y)}(p,\alpha)$, the line (p,α), as seen from (x,y). Clearly $\hat{\alpha} = \alpha$. To conclude it suffices to show that $\hat{p} = r((x,y),\alpha) - p$, where $r((x,y),\alpha)$ is the distance between $\ell((x,y),\alpha)$, the line of angle α containing the point (x,y), and the origin. $\square$

Lines intersecting a ball The number of lines hitting the ball $B(0,r)$ follows a Poisson distribution with mean

$$\lambda \, \pi \, 2r = \lambda \, \text{Perimeter}(B(0,r)).$$



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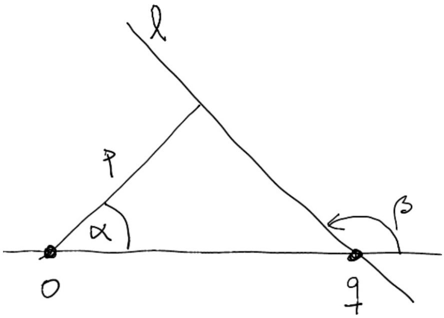
Crossing line process We now parametrize lines according to their intersection with the abscissa axis. Mapl a line $\ell \in \mathbb{L}$ to the pair (q, β), where q is the intersection of ℓ with the abscissa axis and β is the incidence angle of ℓ with the positively oriented abscissa axis. The parametrizations (p, α) and (q, β) are related by the bijection

$$\psi(p, \alpha) := \Big(\frac{p}{\cos\alpha}, \pi - \alpha\Big), \qquad \psi^{-1}(q, \beta) = (q \sin\beta, \pi - \beta). \tag{7.2}$$



Lemma 7.2 (Crossing line process)**.** *Let* L *be the uniform Poisson line process. Then, the random set*

$$\mathsf{Z} := \big\{\big(q(\ell), \beta(\ell)\big) : \ell \in \mathsf{L}\big\} \tag{7.3}$$

is a Poisson process in $\mathbb{R} \times [0, \pi)$ *with mean measure*

$$dq \sin\beta \, d\beta. \tag{7.4}$$

Furthermore, Z *can be seen as a homogeneous Marked Poisson process on* $\mathbb{R}$ *with intensity 2 and marks* β *distributed with density*

$$f(\beta) = \frac{1}{2}\sin\beta, \qquad \beta \in [0, \pi). \tag{7.5}$$

Proof. We have $\mathsf{Z} = \psi(\mathsf{L})$, where $\psi : \mathbb{A} \to \mathbb{A}$ is given by (7.2). Hence, Z is a Poisson process with mean measure

$$d\mu(\psi^{-1}(q, \beta)) \;=\; dq \sin\beta \, d\beta, \tag{7.6}$$

showing the first part of the lemma. For the second part, multiply and divide the mean measure (7.4) by 2, to get that Z is rate 2 homogeneous Poisson process in $\mathbb{R}$ with mark distribution (7.5). □

Corollary 7.3. *Let $\hat\ell = (\hat p, \hat\alpha)$ be a fixed line and L the uniform Poisson line process. The set of lines of L crossing $\hat\ell$ is a rate 2 Poisson process in $\hat\ell$ with mark distribution*

$$\frac{\sin|\hat\alpha - \alpha|}{2}\, d\alpha, \qquad 0 < \alpha < \pi. \tag{7.7}$$

Proof. Exercise of isometry invariance. □

Corollary 7.4. *Under the conditions of Lemma 7.2, the mean number of lines of L crossing a segment I contained in $(\hat p, \hat\alpha)$ is $2|I|$, for any line $(\hat p, \hat\alpha)$.*

Corollary 7.5 (Crossing line construction)**.** *Let Y be a Poisson process with mean measure $2\lambda\, dq\, \frac{|\sin\beta|}{2}\, d\beta$ in $\mathbb{R}\times[0,\pi)$. Recall ψ defined in (7.2) and let*

$$\mathsf{L} := \{\psi^{-1}(\ell) : \ell \in \mathsf{Y}\} \tag{7.8}$$

Show that this process is the uniform line process of intensity λ in $\mathbb{L}$.

This corollary tells that one way to construct the uniform process L is to take a rate 2λ Poisson process in $\mathbb{R}$ and to each point attach a line with random incident angle β with distribution $\frac{\sin\beta}{2}$.

Line processes and ideal gas There is a correspondence between ideal gas configurations X and line configurations L by the map

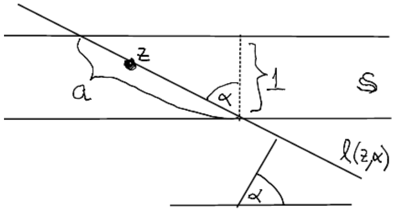
$$(q, v, l) \mapsto \ell = (q + vt)_{t\in\mathbb{R}} \tag{7.9}$$

for the moment l plays no role in this map.

Exercise: find the distribution of X producing a isometry invariant Poisson line process L.

Strip construction Let $\mathbb{S}$ be the strip $\mathbb{R}\times[0,1]$ and Y be a uniform Poisson process on $\mathbb{S}\times[0,\pi)\times[0,1)$. Let

$$\mathsf{L} = \{(z,\alpha) \in \mathbb{S}\times[0,\pi) \,:\, (z,\alpha,u)\in\mathsf{Y},\ u < \cos\alpha\}. \tag{7.10}$$



Since the set of points in the intersection $\ell(z,\alpha) \cap \mathbb{S}$ has length $a = \frac{1}{\cos\alpha}$, we need to divide by a to get intensity 1 for each line. Instead, we include a Uniform$[0,1]$ mark u as an accept/reject test: the line (z,α) is accepted if $u < \frac{1}{a} = \cos\alpha$.

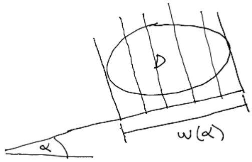
Lemma 7.6 (Strip construction)**.** *The process* L *defined in (7.10) is a uniform Poisson line process on* $\mathbb{L}$*.*

Proof. Exercise. Show that the intersection $\mathsf{L} \cap \{(x,y) : y = 0\} \subset \mathbb{R}$ is a Poisson process of intensity 2 and that the angles are distributed as in (7.7). □

Lines hitting a convex set Let $D \subset \mathbb{R}^2$ be a bounded convex set and L a uniform Poisson Line process. Define the Poisson line process crossing D by

$$\mathsf{L}_D := \{\ell \in \mathsf{L} : \ell \cap D \neq \emptyset\}. \tag{7.11}$$

For each angle α, there is an interval $I(\alpha)$ of length $w(\alpha)$ such that $\ell(p,\alpha) \in \mathsf{L}_D$ if and only if $p \in I(\alpha)$:



Lemma 7.7 (Perimeter lemma)**.** *Let* L *be a uniform Poisson line process of intensity* λ*. Then, the mean number of lines crossing a bounded convex set* $D \subset \mathbb{R}^2$ *is*

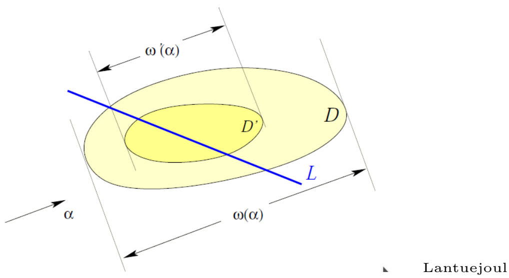
$$\int_0^\pi w(\alpha)\mathrm{d}\alpha = \lambda \,\mathrm{Perimeter}(D), \tag{7.12}$$

Proof. See Chapter 3 in Santalo [44] and Proposition 15.3.IV of Daley-Vere Jones, [9]. The idea in [9] is to first show the result for a convex polygon with vertices in the boundary of D and then perform a limit when the maximal side of the polygon goes to 0. By Corollary 7.4, the mean number of lines crossing each side I of the polygon is $2\lambda|I|$. Since each line crossing D crosses 2 sides, the mean number of crossings is λ times the sum of the side lengths, that is, the perimeter. Exercise: perform the limit to get the result for any convex set. □

Angle of lines crossing a convex set The direction of a line hitting a convex set D is not uniform. The measure of the lines crossing D with angle α is the size of the projection of D on the line having angle α with the absice axis, $w(\alpha)$ in the figure. Hence, the probability density for a line of angle α hit D is

$$f(\alpha) = \frac{w(\alpha)}{\int_0^\pi w(\alpha)\mathrm{d}\alpha} = \frac{w(\alpha)}{\text{Perimeter}(D)} \tag{7.13}$$

Let $D' \subset D$ convex sets. Then, given that a line ℓ crossed D, the conditional probability that ℓ also crossed D' is given by



$$P(\ell \cap D' \neq \emptyset \,|\, \ell \cap D \neq \emptyset) = \frac{\text{Perimeter}(D')}{\text{Perimeter}(D)} \tag{7.14}$$

Debiasing construction Consider a Poisson process Y in $\mathbb{R}^2 \times [0,\pi) \times \mathbb{R}_{\geq 0}$ with Lebesgue intensity 1, that is, with mean measure

$$\mathrm{d}z\, \mathrm{d}\alpha\, \mathrm{d}u \tag{7.15}$$

A point $\mathsf{y} \in \mathsf{Y}$ has 3 components: $\mathsf{y} = (z, \alpha, u) \in \mathbb{R}^2 \times [0,\pi) \times \mathbb{R}_{\geq 0}$. Let A be a bounded Borel set and $\mathbb{L}_A$ the set of lines intersecting A.

For $z \in \mathbb{R}^2$, $\alpha \in [0,\pi]$, let $\ell(z,\alpha)$ be the unique line with perpendicular angle α containing z. Let $|\ell \cap A|$ be the length of $\ell \cap A$. Define

$$\mathsf{L}_A := \big\{\ell(z,\alpha) \,:\, (z,\alpha,u) \in \mathsf{Y},\; u \le |\ell(z,\alpha) \cap A|^{-1} < \infty\big\}. \tag{7.16}$$

Proposition 7.8 (Debiasing construction in a bounded convex set)**.** *Let D be a convex set and L_D the process defined in (7.16). Then L_D is a uniform Poisson line process of rate 1 intersecting D.*

Proof. (with Julio Rossi) The mean number of lines intersecting D is

$$\int_D \int_0^\pi \int_0^\infty \mathbf{1}\Big\{u < \frac{1}{|\ell(z,\alpha)\cap D|}\Big\}\, du\, d\alpha\, dz \tag{7.17}$$

$$= \int_D \int_0^\pi \frac{1}{|\ell(z,\alpha)\cap D|} d\alpha\, dz \tag{7.18}$$

Fubini and a change of variables (see Figure 2), gives

$$= \int_0^\pi \int_0^{w(\alpha)} \int_0^{b(u,\alpha)} \frac{1}{b(u,\alpha)} dv\, du\, d\alpha \;=\; \int_0^\pi w(\alpha) d\alpha. \qquad \square$$

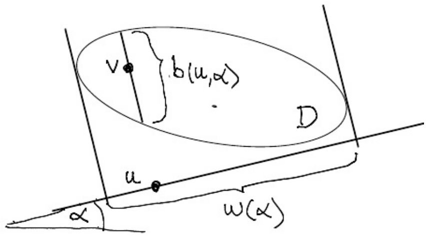


Figure 2: Change of variables

Theorem 7.9 (Debiasing construction in $\mathbb{R}^2$)**.** *Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be a partition of $\mathbb{R}^2$ with bounded Borel elements. Let $\mathbb{L}_A := \{\ell \in \mathbb{L} : \ell \cap A \neq \emptyset\}$ and define*

$$\mathsf{L} := \cup_{k\ge 1}\big[\mathsf{L}_{A_k} \setminus (\mathbb{L}_{A_1} \cup \dots \cup \mathbb{L}_{A_{k-1}})\big]. \tag{7.19}$$

The process L defined in (7.19) is the uniform Poisson line process in $\mathbb{R}^2$.

Proof. Exercise. $\square$

Uniform Poisson oriented line process The uniform Poisson oriented line process $\vec{\mathsf{L}}$ is the usual process, but now each line has an independent random orientation, chosen with probability $\frac{1}{2}$.

Proposition 7.10. *Let $\vec{\mathsf{L}}$ be the uniform oriented line Poisson process. Let ∂D be the perimeter of a convex bounded set D. Then the set of lines of $\vec{\mathsf{L}}$ entering D is a marked Poisson process on $\partial \mathsf{L} \times [0, \pi)$ with nonhomogeneous mean measure*

$$du \, \frac{\cos|\alpha(u) - \alpha|)}{2} \, d\alpha \tag{7.20}$$

where $\alpha(w)$ is the angle of the tangent line to ∂D at the point $w \in \partial D$.

Proof. Mapping theorem. □

Corollary 7.11 (Lines crossing a convex set)**.** *Let w be a point in the boundary of a convex set D. The intensity of lines in $\vec{\mathsf{L}}$ entering D and containing w is 1.*

Proof. Let $\alpha(w)$ be the angle of the tangent of the boundary at w. For $\alpha \neq \alpha(w)$, let $w(b, \alpha)$ be the length of $\ell(z, \alpha) \cap D$. The intensity of the line $\ell(w, \alpha)$ entering D is $\frac{\cos(\alpha - \alpha(w))}{2}$. Hence, the total intensity of lines entering D at w is:

$$\int_0^\pi \frac{\cos|\alpha - \alpha(w)|}{2} d\alpha = 1.$$ □

As a corollary of this lemma, we have an alternative proof to the perimeter lemma.

Alternative proof of the Perimeter Lemma. The intensity of lines entering D is the integral along the perimeter of the line intensity at each point of the perimeter, which is 1, by Corollary 7.11. □

7.3 Chentsov's model

This model is taken from [31, 32].

Let $\vec{\mathsf{L}}$ be a uniform Poisson oriented-line process with intensity λ. An oriented line ℓ divides $\mathbb{R}^2$ in the right and left semiplanes. Consider the function

$V_\ell(z) = 1$ for z in the right semiplane of ℓ, and $V_\ell(z) = -1$ for z in the left semiplane of ℓ. Define

$$\xi(z) = \sum_{\ell \in \vec{\mathsf{L}}} (V_\ell(z) - V_\ell(0)), \qquad z \in \mathbb{R}^2 \tag{7.21}$$

where 0 is the origin of $\mathbb{R}^2$.

Notice that $\xi(z) =$ signed number of crossings of $\vec{\mathsf{L}}$ of $\overline{0z}$, the interval with extremes in 0 and z. The height difference $\xi(z) - \xi(z')$ depends only on crossings of the interval $\overline{zz'}$.

Lemma 7.12 (Height differences along lines are random walks)**.** *For any $z, w \in \mathbb{R}^2$ the process $(\xi(z + ws) - \xi(z))_{s\in\mathbb{R}}$ is a continuous time nearest neighbor random walk with rate λ.*

Proof. Under construction. □

Corollary 7.13 (Covariances)**.**

$$\mathrm{Cov}(\xi(z + us), \xi(z)) = \mathrm{Var}(\xi(z)). \tag{7.22}$$

Lemma 7.14 (Brownian motion)**.** *Denote*

$$\xi^\varepsilon_t := \varepsilon\big(\xi(z) - \xi(z + u\varepsilon^{-2}t)\big) \tag{7.23}$$

Then, as $\varepsilon \to 0$, the process ξ^ε_t converges weakly to Brownian motion.

7.3.1 Chentsov's model and hard rods

Consider a hard rod Poisson process X with mean measure μ and the marked Poisson oriented line process $\vec{\mathsf{L}}$ given by

$$\vec{\mathsf{L}} = \{((q + vs)_{s\in\mathbb{R}}, l) : (q, v, l) \in \mathsf{X}\} \tag{7.24}$$

Consider that all lines in $\vec{\mathsf{L}}$ are oriented in the positive time direction.

Define Chentsov functions $V_{\ell,l}$ and $\xi(z)$ by

$$V_{\ell,l}(z) := l\,\mathbf{1}\{z \text{ is in the right semiplane of } \ell\} \tag{7.25}$$

$$\xi(z) := \sum_{\ell,l \in \vec{\mathsf{L}}} (V_{\ell,l}(z) - V_{\ell,l}(0)), \qquad z \in \mathbb{R}^2 \tag{7.26}$$

In particular, if $z = (0, q)$,

$$\xi((0, x)) = m(\{(q, v, l) \in \mathsf{X} : 0 < q < x\}). \tag{7.27}$$

is the mass of the hard rods in the interval $(0, x)$.

Hard rods with negative mass Let X be a Poisson process on $\mathbb{X} := \mathbb{R}^3$ and $\vec{\mathsf{L}}$ as in (7.24). Observe that we are also considering hard rods (q, v, l) with negative l. Define

$$V_{\ell,l}(z) := l\,\big(\mathbf{1}\{z \text{ is in the right semiplane of } \ell \text{ and } l > 0\} \tag{7.28}$$

$$+\,\mathbf{1}\{z \text{ is in the left semiplane of } \ell \text{ and } l < 0\} \tag{7.29}$$

$$\xi(z) := \sum_{\ell,l\in\vec{\mathsf{L}}} (V_{\ell,l}(z) - V_{\ell,l}(0)), \qquad z \in \mathbb{R}^2 \tag{7.30}$$

We can compute all quantities of hard rods, using this definition. For instance, if $z_0 = (q, 0)$ and $z_t = (q + vt, t)$, the difference between the heights at z_t and z_0 gives the mass flow of the ideal gas along the segment (z_0, z_t):

$$\xi(z_t) - \xi(z_0) = j_{\mathsf{X}}(q, v, t).$$

8 The free Bose gas

This section is based on joint work with Armendáriz and Yuhjtman [3].

Finite-volume spatial random permutation In 1953 Feynman [16] proposes the following partition function for the *free Bose Gas*:

$$Z_{\Lambda,n} = \frac{(\alpha/\pi)^{nd/2}}{n!} \sum_{\sigma\in S_n} \int_{\Lambda^n} e^{-\alpha \sum_i \|x_i - x_{\sigma(i)}\|^2}\, dx_1 \dots dx_n, \tag{8.1}$$

where $\alpha > 0$ is the temperature, S_n is the set of permutations of $\{1, \dots, n\}$ and Λ is a compact subset of $\mathbb{R}^d$.

Let the *Hamiltonian* $H : \cup_{n\ge1}(\Lambda^n \times S_n) \to \mathbb{R}$ be defined by

$$H(\underline{x}, \sigma) := \sum_{i=1}^{n} \|x_i - x_{\sigma(i)}\|^2. \tag{8.2}$$

For each compact Λ and $n \ge 1$, define the measure $G_{\Lambda,n}$ on $\Lambda^n \times S_n$, by

$$G_{\Lambda,n} g := \frac{(\alpha/\pi)^{nd/2}}{Z_{\Lambda,n}} \frac{1}{n!} \sum_{\sigma\in S_n} \int_{\Lambda^n} g(\underline{x}, \sigma)\, e^{-\alpha H(\underline{x},\sigma)} d\underline{x}, \tag{8.3}$$

where $Z_{\Lambda,n}$ is the partition function defined in (8.1), $\underline{x} = (x_1, \dots, x_n)$ and g is a test function. The measure $G_{\Lambda,n}$ is the *canonical measure* associated to H and *empty* boundary conditions. Canonical means that the measure concentrates on configuration with a fixed number of points.

Infinite-volume spatial random permutations We want to construct a random spatial permutation (X, σ), where X is a point process in $\mathbb{R}^d$ and $\sigma : \mathsf{X} \to \mathsf{X}$ is a bijection (permutation). For each $\rho > 0$, we require that the law of (X, σ) be translation-invariant, with *point density* ρ and *Gibbs* for the specification induced by $G_{\Lambda,n}$, to be defined later.

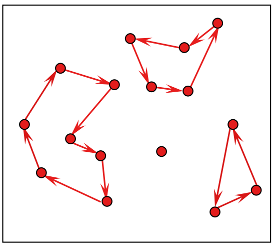
Loops and spatial permutations Denote the space of *unrooted loops* of size k by $\mathbb{D}_k := (\mathbb{R}^d)^k/_\sim$, where the equivalence relation $\sim$ is defined by $(x_1, \dots, x_k) \sim (x_2, \dots, x_k, x_1)$. The class of equivalence of $(x_1, \dots, x_k)$ is denoted $[x_1, \dots, x_k]$. We denote the space of loops by $\mathbb{D} := \cup_{k \geq 1} \mathbb{D}_k$.

The support of γ is denoted by $\{\gamma\} := \{x_1, \dots, x_k\}$. We look at γ as a spatial permutation $(\{\gamma\}, \gamma)$, by abusing notation and denoting $\gamma(x_i) = x_{i+1}$ and $\gamma(x_k) = x_1$.

For finite X, there is a bijection between spatial permutations and loop configurations

$$(\mathsf{X}, \sigma) \mapsto \Gamma, \tag{8.4}$$

where $\gamma \in \Gamma$ if γ is an unrooted loop satisfying $\{\gamma\} \subset \mathsf{X}$ and $\gamma(x) = \sigma(x)$ for all $x \in \{\gamma\}$. Reciprocally, $\Gamma \mapsto (\mathsf{X}, \sigma)$, the spatial permutation defined by $\mathsf{X} = \cup_{\gamma \in \Gamma} \{\gamma\}$; $\sigma(x) = \gamma(x)$ for $x \in \{\gamma\}$, $\gamma \in \Gamma$.



Loops induced by a spatial random permutation. An arrow from x to y means $y = \sigma(x)$. An isolated dot at x means $x = \sigma(x)$, a loop of length 1.

8.1 Gaussian loop soup in $\mathbb{R}^d$

Loop factorization For finite X, the weight of (X, σ) in (8.3) is the product of the weights of its loops:

$$e^{-\alpha \sum_{x \in \mathsf{X}} \|\sigma(x) - x\|^2} = \prod_{\gamma \in \sigma} e^{-\alpha \sum_{x \in \{\gamma\}} \|\gamma(x) - x\|^2}$$

This observation by Sütő [45] suggests that a spatial random permutation can be seen as a Poisson process of loops. We will define a Poisson process in $\mathbb{D}$, called *loop soup* and show that it is a Gibbs measure for the Hamiltonian H.

Loop soup mean measure Fix an activity parameter $\lambda \in (0,1]$, denote $\gamma = [x_1, \ldots, x_k] \in \mathbb{D}$ and define the measure Q_λ on $\mathbb{D}$, by

$$Q_\lambda(d[x_1, \ldots, x_k]) := \tfrac{\lambda^k}{k} \left(\tfrac{\alpha}{\pi}\right)^{kd/2} e^{-\alpha \sum_{i=1}^k \|x_i - \gamma(x_i)\|^2} \, dx_1 \ldots dx_k .$$

The denominator k compensates the fact that each rooted k-loop is counted k times.

The mean density of points belonging to k-loops is defined as

$$\rho_k(A) := \frac{1}{|A|} \int_{\mathbb{D}_k} \textstyle\sum_{j=1}^k \mathbf{1}_A(x_j)\; Q_\lambda(d[x_1, \ldots, x_k])\,. \tag{8.5}$$

Proposition 8.1. *For any nonempty compact set $A \subset \mathbb{R}^d$, we have*

$$\rho_k(A) = \left(\frac{\alpha}{\pi}\right)^{d/2} \frac{\lambda^k}{k^{d/2}} =: \rho_k\,. \tag{8.6}$$

Proof. Denote

$$p(x,y) := \left(\frac{\alpha}{\pi}\right)^{d/2} \exp\big(-\alpha\|x-y\|^2\big).$$

the density of a Gaussian random variable in $\mathbb{R}^d$ with mean x and covariance matrix $(2/\alpha)\mathrm{Id}$, where Id is the identity matrix. We have

$$\begin{aligned}
\rho_k(A) &= \frac{\lambda^k}{k|A|} \int_{(\mathbb{R}^d)^k} \Big(\sum_{j=1}^k \mathbf{1}_A(x_j)\Big) \prod_{i=1}^k p(x_i, x_{i+1})\, dx_1 \ldots dx_k \\
&= \frac{\lambda^k}{k|A|} \sum_{j=1}^k \int_{(\mathbb{R}^d)^k} \mathbf{1}_A(x_j) \prod_{i=1}^k p(x_i, x_{i+1})\, dx_1 \ldots dx_k \\
&= \frac{\lambda^k}{k|A|} \sum_{j=1}^k \int_{\mathbb{R}^d} \mathbf{1}_A(x_j) \left(\frac{\alpha}{\pi k}\right)^{d/2} dx_j = \rho_k.
\end{aligned}$$

□

The point density of Q_λ is defined by

$$\rho(\lambda) := \sum_{k \geq 1} \rho_k\,. \tag{8.7}$$

By (8.6), we have

$$\rho(\lambda) = \left(\frac{\alpha}{\pi}\right)^{d/2} \sum_{k\geq 1} \frac{\lambda^k}{k^{d/2}} < \infty \iff \begin{cases} d \leq 2 \text{ and } 0 \leq \lambda < 1, \text{ or} \\ d \geq 3 \text{ and } 0 \leq \lambda \leq 1\,. \end{cases} \tag{8.8}$$

For $d \geq 3$ define the *critical density* $\rho_c := \rho(1)$, so that

$$\rho_c = \left(\frac{\alpha}{\pi}\right)^{d/2} \sum_{k \geq 1} \frac{1}{k^{d/2}}. \tag{8.9}$$

The function $\rho : [0,1] \to [0, \rho_c]$ is invertible for $d \geq 3$ and $\rho : [0,1) \to [0, \infty)$ is invertible for $d = 1, 2$. Let $\lambda(\rho)$ be its inverse.

Proposition 8.2. *Let d and λ be as in (8.8). Then Q_λ is σ-finite.*

Proof. It suffices to show that there is a countable partition of $\mathbb{D}$ in sets with finite Q_λ measure. A partition of $\mathbb{R}^d$ in bounded Borel sets A_i produces the following partition of $\mathbb{D}$: $D_0 = \emptyset$ and

$$D_i := \big\{\gamma \in \mathbb{D} : \{\gamma\} \cap A_i \neq \emptyset\big\} \setminus (D_0 \cup \cdots \cup D_{i-1}), \quad i \geq 1. \tag{8.10}$$

Since the number of loops in a compact set is bounded by the number of points in the support of those loops,

$$Q_\lambda(D_i) \leq \rho(\lambda)\, |A_i| < \infty, \quad \text{for all } i,$$

by (8.5) and Proposition 8.1. □

Loop soup Let $\mathcal{D}$ be the set of locally finite loop soup configurations,

$$\mathcal{D} := \Big\{\Gamma \subset \mathbb{D} : \sum_{\gamma \in \Gamma} \mathbf{1}\big\{\{\gamma\} \cap A \neq \emptyset\big\} < \infty, \forall \text{ compact } A \subset \mathbb{R}^d\Big\}. \tag{8.11}$$

Let d and λ satisfy (8.8). Define the *loop soup* at fugacity λ by

$$\Gamma_\lambda := \text{ Poisson process on } \mathbb{D} \text{ with intensity } Q_\lambda, \tag{8.12}$$
$$\mu_\lambda := \text{ Law of } \Gamma_\lambda. \tag{8.13}$$

This process is analogous to the Brownian loop soup introduced by Lawler and Werner [35], and the random walk loop soup of Lawler and Trujillo Ferreras [34], see also Le Jan [36].

A sample Γ of a Gaussian loop soup is a countable collection of unrooted Gaussian loops in $\mathbb{R}^d$, with the property that any compact set contains finitely many points in the support of the loops.

Loop soup density Given a compact $A \subset \mathbb{R}^d$ let the set of loops (with support) contained in A and configurations with loops contained in A be, respectively

$$\mathbb{D}_\Lambda := \big\{\gamma \in \mathbb{D} : \{\gamma\} \subset \Lambda\big\}, \qquad \mathcal{D}_\Lambda := \{\Gamma \in \mathcal{D} : \Gamma \subset \mathbb{D}_\Lambda\}. \tag{8.14}$$

Since $Q_\lambda(\mathbb{D}_A) < \infty$, by definition of Poisson process, for measurable, bounded $g : \mathcal{D}_A \to \mathbb{R}$,

$$\mu_\lambda g = e^{-Q_\lambda(\mathbb{D}_A)} \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{\mathbb{D}_A} \cdots \int_{\mathbb{D}_A} g\big(\{\gamma_1, \dots, \gamma_\ell\}\big)\, Q_\lambda(d\gamma_1) \dots Q_\lambda(d\gamma_\ell) \tag{8.15}$$

$$= \sum_{\ell \geq 0} \frac{e^{-Q_\lambda(\mathbb{D}_A)}}{\ell!} \int_{\mathbb{D}_A} \cdots \int_{\mathbb{D}_A} g\big(\{\gamma_1, \dots, \gamma_\ell\}\big)\, f_\lambda\big(\{\gamma_1, \dots, \gamma_\ell\}\big)\, d\gamma_1 \dots d\gamma_\ell, \tag{8.16}$$

where

$$f_\lambda\big(\{\gamma_1, \dots, \gamma_\ell\}\big) := \textstyle\prod_{i=1}^{\ell} w_\lambda(\gamma_i), \tag{8.17}$$

$$w_\lambda\big([x_1, \dots, x_k]\big) := \lambda^k \textstyle\prod_{i=1}^{k} p(x_i, x_{i+1}), \quad \text{with } x_{k+1} = x_1, \tag{8.18}$$

and where, for any bounded measurable $h : \mathbb{D}_A \to \mathbb{R}$,

$$\int_{\mathbb{D}} h(\gamma)\, d\gamma := \sum_{k \geq 1} \frac{1}{k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h\big([x_1, \dots, x_k]\big)\, dx_1 \dots dx_k, \tag{8.19}$$

if the right hand side is well defined. Taking $h(\gamma) = h_1(\gamma)\, w_\lambda(\gamma) \mathbf{1}_{\{\gamma \in \mathbb{D}_A\}}$ for some bounded $h_1 : \mathbb{D}_A \to \mathbb{R}$ and recalling the assumption $Q_\lambda(\mathbb{D}_A) < \infty$, we conclude that (8.19) is well defined for this h and it is bounded by $\|h_1\|_\infty Q_\lambda(\mathbb{D}_A)$.

The function $e^{-Q_\lambda(\mathbb{D}_A)} f_\lambda$ is the *density* of the Gaussian loop soup at fugacity λ in the set $\mathbb{D}_A$; in particular, if $g \equiv 1$ we have $\mu_\lambda g = 1$.

8.1.1 Finite-volume loop soup and spatial permutations

We consider a compact set $\Lambda \subset \mathbb{R}^d$ and show that the Gaussian loop soup in Λ conditioned to have n points has the same law as the canonical spatial random permutation with n points. We then introduce the grand canonical spatial random permutation in Λ and show that it coincides with the Gaussian loop soup in Λ.

Canonical measure Recall the definition (8.3) of the canonical measure $G_{\Lambda,n}$ and denote the space of loop-soup configurations contained in Λ with exactly n points, by

$$\mathcal{D}_{\Lambda,n} := \big\{\Gamma \in \mathcal{D}_\Lambda : \textstyle\sum_{\gamma\in\Gamma} |\{\gamma\}| = n\big\}, \tag{8.20}$$

where $|\{\gamma\}|$ is the number of points in the support of γ.

The Gaussian loop soup restricted to loops contained in Λ is denoted by

$$\Gamma_{\Lambda,\lambda} := \Gamma_\lambda \cap \mathbb{D}_\Lambda, \qquad \mu_{\Lambda,\lambda} := \text{Law of } \Gamma_{\Lambda,\lambda}. \tag{8.21}$$

We now show that $\mu_{\Lambda,\lambda}$ conditioned to have n points in Λ equals the canonical measure $G_{\Lambda,n}$ defined in (8.3).

Proposition 8.3 (Conditioned loop soup and canonical measure)**.** *Let λ, d satisfy (8.8). Then,*

$$\mu_{\Lambda,\lambda}(\,\cdot\,|\mathcal{D}_{\Lambda,n}) = G_{\Lambda,n} g \tag{8.22}$$

Proof. Since there is a bijection between the supports of these probability measures, it suffices to verify that the weights assigned by their densities to any given configuration satisfy a fixed ratio. Let (X,σ) be a spatial permutation such that $\mathsf{X} \subset \Lambda$ and $|\mathsf{X}| = n$. Let Γ be the cycle decomposition of (X,σ); clearly $\Gamma \in \mathcal{D}_{\Lambda,n}$. Then, by the definition of Poisson process, the loop soup conditioned density of $\Gamma \in \mathcal{D}_{\Lambda,n}$ is

$$f_\lambda(\Gamma|\mathcal{D}_{\Lambda,n}) = \frac{e^{-Q_\lambda(D_\Lambda)}}{\mu_\lambda(\mathcal{D}_{\Lambda,n})} \prod_{\gamma\in\Gamma} w_\lambda(\gamma) = \frac{\lambda^n e^{-Q_\lambda(D_\Lambda)}}{\mu_\lambda(\mathcal{D}_{\Lambda,n})} \prod_{\gamma\in\Gamma} w_1(\gamma), \tag{8.23}$$

where w_λ was defined in (8.18).

On the other hand, the density of the canonical measure $G_{\Lambda,n}$ can be written as a function of the cycle decomposition of σ by

$$\frac{1}{Z_{\Lambda,n}} f_{\Lambda,n}(\mathsf{X},\sigma) = \frac{1}{Z_{\Lambda,n}}\, e^{-\alpha H(\mathsf{X},\sigma)} = \frac{1}{Z_{\Lambda,n}} \prod_{\gamma\in(\mathsf{X},\sigma)} w_1(\gamma). \qquad \square$$

Grand-canonical measure Denote

$$\tilde\lambda := (\alpha/\pi)^{d/2}\lambda. \tag{8.24}$$

The grand-canonical spatial random permutation at fugacity $\lambda \le 1$ associated to the canonical density (8.3) is defined by

$$G_{\Lambda,\lambda}\, g := \frac{1}{Z_{\Lambda,\lambda}} \sum_{n\ge 0} \frac{\tilde\lambda^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{\Lambda^n} g(\underline{x},\sigma) e^{-\alpha H(\underline{x},\sigma)} d\underline{x}, \tag{8.25}$$

where $\underline{x} = (x_1, \ldots, x_n)$, $H(\underline{x}, \sigma) := \sum_{i=1}^n \|x_i - x_{\sigma(i)}\|^2$, and

$$Z_{\Lambda,\lambda} := \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{\Lambda^n} e^{-\alpha H(\underline{x},\sigma)} d\underline{x}. \tag{8.26}$$

In terms of the canonical measures, we have

$$G_{\Lambda,\lambda}\, g = \frac{1}{Z_{\Lambda,\lambda}} \sum_{n\geq 0} \tilde{\lambda}^n\, Z_{\Lambda,n}\, G_{\Lambda,n} g, \tag{8.27}$$

$$Z_{\Lambda,\lambda} = \sum_{n\geq 0} \tilde{\lambda}^n\, Z_{\Lambda,n}. \tag{8.28}$$

In particular, if we take $g_{\Lambda,n}(\underline{x},\sigma) := \mathbf{1}_{\{\underline{x}\in\Lambda^n\}}$, that is, the indicator function that the total number of points is n, we get

$$G_{\Lambda,\lambda} g_{\Lambda,n} = \frac{\tilde{\lambda}^n\, Z_{\Lambda,n}}{Z_{\Lambda,\lambda}}. \tag{8.29}$$

Compute

$$\begin{aligned}
\mu_{\Lambda,\lambda} g_{\Lambda,n} &= \int_{\mathcal{D}_\Lambda} \mu_\lambda(d\Gamma) g_{\Lambda,n}(\Gamma) \\
&= P(\textstyle\sum_k k\,\xi_k \;=\; n), \\
&= \frac{1}{n!} \sum_{\sigma\in\mathcal{S}_n} \prod_{\gamma\in\sigma} \lambda_{\Lambda,|\gamma|} \\
&= \frac{e^{-Q_\lambda(\mathbb{D}_\Lambda)}}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} \lambda_{|I|}
\end{aligned} \tag{8.30}$$

where ξ_k = number of loops of size k in Λ. ξ_k are independent Poisson random variables with mean

$$\lambda_{\Lambda,k} := \frac{\tilde{\lambda}^k}{k} \textstyle\int_\Lambda \cdots \int_\Lambda e^{-H([x_1,\ldots,x_k])}\, dx_1 \ldots dx_k. \tag{8.31}$$

Proposition 8.4 (Loop soup and grand canonical measure)**.** *Let λ and d satisfy (8.8), the critical or subcritical region. The Gaussian loop soup restricted to Λ defined in (8.21) and the grand-canonical measure (8.25) are equivalent:*

$$\mu_{\Lambda,\lambda} = G_{\Lambda,\lambda}. \tag{8.32}$$

Proof. We look at both measures as point processes in $\mathcal{D}_A$. To prove the proposition it suffices to show that the measures have the same Laplace functionals. Given $\psi : \mathbb{D}_\Lambda \to \mathbb{R}_+$, define $g : \mathcal{D}_\Lambda \to \mathbb{R}$ as

$$g(\Gamma) := \exp\Big(-\textstyle\sum_{\gamma\in\Gamma}\psi(\gamma)\Big).$$

By Campbell's theorem,

$$\begin{aligned}\mu_{\Lambda,\lambda} g &= \int_{\mathcal{D}_\Lambda} \mu_\lambda(d\Gamma) e^{-\sum_{\gamma\in\Gamma}\psi(\gamma)} = \exp\Big(\int_{\mathbb{D}_\Lambda} (e^{-\psi(\gamma)} - 1) Q_\lambda(d\gamma)\Big)\\ &= e^{-Q_\lambda(\mathbb{D}_\Lambda)} \exp\Big(\sum_{k\geq 1} \frac{1}{k!} a_k\Big),\end{aligned}$$

where, using the definition of Q_λ,

$$\begin{aligned}a_k &:= \tilde{\lambda}^k (k-1)! \int_{\Lambda^k} dx_1 \dots dx_k \, e^{-\alpha H([x_1,\dots,x_k])} \, e^{-\psi([x_1,\dots,x_k])}\\ &= \tilde{\lambda}^k \sum_{\gamma\in\mathcal{C}_k} \int_{\Lambda^k} dx_1 \dots dx_k \, e^{-\alpha H(\underline{x},\gamma)} \, e^{-\psi(\underline{x},\gamma)}\end{aligned}$$

where $\mathcal{C}_k$ is the set of cycles of size k with elements $\{1,\dots,k\}$, $\underline{x} = (x_1,\dots,x_k)$, and $(\underline{x},\gamma) := [x_1, x_{\gamma(1)}, \dots, x_{\gamma^{k-1}(1)}]$; notice that $\mathcal{C}_k$ has cardinality $(k-1)!$. On the other hand, by Lemma 8.5 below we can write

$$\begin{aligned}\mu_{\Lambda,\lambda} g &= e^{-Q_\lambda(\mathbb{D}_\Lambda)} \sum_{n\geq 0} \frac{1}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} a_{|I|}\\ &= e^{-Q_\lambda(\mathbb{D}_\Lambda)} \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} \sum_{\gamma\in\mathcal{C}_{|I|}} \int_{\Lambda^{|I|}} dx_1 \dots dx_{|I|}\\ &\qquad\qquad \times e^{-\alpha H([x_1,\dots,x_{|I|}])} \, e^{-\psi([x_1,\dots,x_{|I|}])}\\ &= e^{-Q_\lambda(\mathbb{D}_\Lambda)} \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \prod_{\gamma\in\sigma} \int_{\Lambda^{|\gamma|}} dx_1 \dots dx_{|\gamma|} \, e^{-\alpha H(\underline{x},\gamma)} e^{-\psi(\underline{x},\gamma)}\\ &= e^{-Q_\lambda(\mathbb{D}_\Lambda)} \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{\Lambda^n} dx_1 \dots dx_n \, e^{-\alpha H(\underline{x},\sigma)} \prod_{\gamma\in\sigma} e^{-\psi((x_i : i\in\{\gamma\}),\gamma)}\\ &= G_{\Lambda,\lambda} g,\end{aligned}$$

where $(\underline{x},\sigma)$ is the spatial permutation that maps x_i to $x_{\sigma(i)}$ and $\{\gamma\}$ is the set of indices that appear in the cycle γ. □

Lemma 8.5 (Combinatorial lemma)**.** *Let* $(a_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$ *be such that*

$$\sum_{n\geq 1} \frac{1}{n!} |a_n| < \infty.$$

Then

$$\exp\Big(\sum_{n\geq 1}\frac{1}{n!}\,a_n\Big) = 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{P\in\mathcal{P}_n}\prod_{I\in P}a_{|I|}, \tag{8.33}$$

where $\mathcal{P}_n$ is the set of partitions of $\{1,\ldots,n\}$ into non-empty sets,

$$\mathcal{P}_n = \Big\{P \text{ partition of } \{1,\ldots,n\} : \emptyset\notin P\Big\}, \tag{8.34}$$

and $|I|$ stands for the cardinality of the set I.

Proof. By the series expansion of the exponential function

$$\begin{aligned}
\exp\Big(\sum_{n\geq 1}\frac{1}{n!}\,a_n\Big) &= \sum_{j\geq 0}\frac{1}{j!}\Big(\sum_{\ell\geq 1}\frac{1}{\ell!}a_l\Big)^j\\
&= 1+\sum_{j\geq 1}\frac{1}{j!}\sum_{\substack{i_1,\ldots,i_j\\ i_\ell\geq 1}}\frac{a_{i_1}}{i_1!}\cdots\frac{a_{i_j}}{i_j!}\\
&= 1+\sum_{j\geq 1}\frac{1}{j!}\sum_{n\geq 1}\sum_{\substack{i_1+\cdots+i_j=n\\ i_\ell\geq 1}}\frac{a_{i_1}}{i_1!}\cdots\frac{a_{i_j}}{i_j!}\\
&= 1+\sum_{j\geq 1}\frac{1}{j!}\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{i_1+\cdots+i_j=n\\ i_\ell\geq 1}}\binom{n}{i_1\ldots i_j}\,a_{i_1}\ldots a_{i_j}\\
&= 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{j\geq 1}\frac{1}{j!}\sum_{\substack{i_1+\cdots+i_j=n\\ i_\ell\geq 1}}\binom{n}{i_1\ldots i_j}\,a_{i_1}\ldots a_{i_j}\\
&= 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{j\geq 1}\sum_{P=\{I_1,\ldots,I_j\}\in\mathcal{P}_n}a_{|I_1|}\ldots a_{|I_j|}\\
&= 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{P\in\mathcal{P}_n}\prod_{I\in P}a_{|I|}.
\end{aligned}$$

The change of the order of summation in the fifth line above is justified by the absolute convergence of $\sum_{n\geq 1}\frac{1}{n!}\,a_n$. □

8.1.2 Point correlations of the loop soup

Point correlations The n-point correlations of a finite point process X defined on $\mathbb{X}$ with density f with respect to a Poisson process of rate 1 is

$$\varphi(x_1,\ldots,x_n) := E(f(\mathsf{X}\cup\{x_1,\ldots,x_n\})) \tag{8.35}$$

for any n and pairwise different $x_1, \ldots, x_n$ in $\mathbb{X}$. Intuitively, this is the limit, as $\varepsilon \to 0$ of the probability to observe one point in each ball $B(x_i, \varepsilon)$, divided by $|B(x_i, \varepsilon)|^n$. For pairwise disjoint $A_i \subset \mathbb{X}$, we have

$$E(n_{A_1}(\mathsf{X}) \ldots n_{A_n}(\mathsf{X})) = \int_{A_1} \cdots \int_{A_n} \varphi(x_1, \ldots, x_n)\, dx_1 \ldots dx_n. \tag{8.36}$$

where $n_A(\mathsf{X})$ is the number of points in $A \cap \mathsf{X}$.

Point correlations of the loop soup We will compute the n-point correlations of the point-marginal of the loop soup. Define

$$K_\lambda(x, y) := \sum_{k \geq 1} \left(\frac{\alpha}{\pi k}\right)^{d/2} \lambda^k\, e^{-\frac{\alpha}{k}\|x-y\|^2}. \tag{8.37}$$

and denote the number of points in A of a loop soup configuration Γ, by

$$n_A(\Gamma) = \sum_{\gamma \in \Gamma} n_A(\{\gamma\}), \tag{8.38}$$

where recall $\{\gamma\}$ is the set of points in the support of γ.

Proposition 8.6 (Point correlations of the loop soup)**.** *Let Γ_λ be the loop soup defined in (8.12) and (X, σ) the corresponding spatial permutation, as defined in (8.4). The n-point correlation density of X is given by*

$$\varphi_\lambda(x_1, \ldots, x_n) = \mathrm{Perm}\big(K_\lambda(x_i, x_j)\big)_{i,\,j=1}^{n}, \tag{8.39}$$

where $\mathrm{Perm}(K)$ is the permanent of the matrix $K \in \mathbb{R}^{n \times n}$.

Sketch of the proof. We compute the 3-point correlation density. To simplify notation, denote $\mu = \mu_\lambda$, $Q = Q_\lambda$ and $K_{xy} = K_\lambda(x, y)$. Given pairwise disjoint bounded Borel sets $A, B, C \subset \mathbb{R}^d$, the third moment measure for the point marginal ν_λ over $A \times B \times C$ is given by

$$\begin{aligned}
&\int n_A(\Gamma)\, n_B(\Gamma)\, n_C(\Gamma)\ \mu(d\Gamma) \qquad (8.40)\\
&= \int \textstyle\sum_{\gamma \in \Gamma} n_A(\gamma) \sum_{\gamma' \in \Gamma} n_B(\gamma') \sum_{\gamma'' \in \Gamma} n_C(\gamma'')\, \mu(d\Gamma)\\
&= \int \Big(\textstyle\sum_{\gamma \in \Gamma} n_A(\gamma)\, n_B(\gamma)\, n_C(\gamma)\\
&\quad + \textstyle\sum_{\gamma \in \Gamma} n_A(\gamma) \sum_{\gamma' \in \Gamma,\, \gamma' \neq \gamma} n_B(\gamma')\, n_C(\gamma')\\
&\quad + \textstyle\sum_{\gamma \in \Gamma} n_B(\gamma) \sum_{\gamma' \in \Gamma,\, \gamma' \neq \gamma} n_A(\gamma')\, n_C(\gamma')
\end{aligned}$$

$$+ \textstyle\sum_{\gamma\in\Gamma} n_C(\gamma) \sum_{\gamma'\in\Gamma,\, \gamma'\neq\gamma} n_A(\gamma')\, n_B(\gamma')$$

$$+ \textstyle\sum_{\gamma\in\Gamma} n_A(\gamma) \sum_{\gamma'\in\Gamma,\, \gamma'\neq\gamma} n_B(\gamma') \sum_{\gamma''\in\Gamma\setminus\{\gamma,\gamma'\}} n_C(\gamma'')\Big)\mu(d\Gamma)$$

$$= Q(n_A\, n_B\, n_C) + Q(n_A)\, Q(n_B\, n_C) \tag{8.41}$$

$$+ Q(n_B)\, Q(n_A\, n_C) + Q(n_C)\, Q(n_A\, n_B) + Q(n_A)\, Q(n_B)\, Q(n_C).$$

To get identity (8.41) we use that if μ is a Poisson process of loops, then

(a) the expectation of the product of functions of different loops factorize (Theorem 3.2 in [39]), and

(b) by Campbell's theorem, $\int_{\mathcal{D}} \sum_{\gamma\in\Gamma} g(\gamma)\mu(d\Gamma) = \int_{\mathbb{D}} g(\gamma)Q(d\gamma) =: Q(g)$.

Define

$$\langle a_1 \dots a_k\rangle := \big\{\gamma \in D : \gamma \text{ goes through } a_1, \dots, a_k \text{ in this order}\big\} \tag{8.42}$$

and compute

$$Q(n_A\, n_B\, n_C) = \int \textstyle\sum_{a\in\gamma} \mathbf{1}_A(a) \sum_{b\in\gamma} \mathbf{1}_B(b) \sum_{c\in\gamma} \mathbf{1}_C(c)\, Q(d\gamma) \tag{8.43}$$

$$= \int \sum_{\substack{\{a,b,c\}\subset\{\gamma\} \\ a\in A, b\in B, c\in C}} \big(\mathbf{1}_{\langle abc\rangle}(\gamma) + \mathbf{1}_{\langle acb\rangle}(\gamma)\big)\, Q(d\gamma) \tag{8.44}$$

$$= \int_A \int_B \int_C (K_{ab}K_{bc}K_{ca} + K_{ac}K_{cb}K_{ba})\, dc\, db\, da, \tag{8.45}$$

where (8.44) follows from partitioning the set of cycles that go through a, b, c according to the order in which they visit the points, and (8.45) follows from the argument in the proof of Proposition 8.1.

Using the same argument to compute the other terms in (8.41), we conclude that the third moment measure (8.40) is absolutely continuous with respect to Lebesgue measure in $(\mathbb{R}^d)^3$ with Radon-Nikodym derivative

$$\varphi_\lambda(x, y, z) = K_{xx}K_{yy}K_{zz} + K_{xx}K_{yz}K_{zy} + K_{xy}K_{yx}K_{zz}$$
$$+ K_{xy}K_{yz}K_{zx} + K_{xz}K_{yx}K_{zy} + K_{xz}K_{yy}K_{zx},$$

which proves (8.39) for $n = 3$; see Fig. 3. We leave the proof of the general case to the reader. □

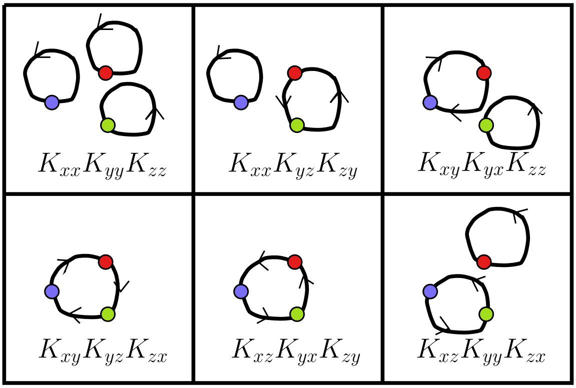


Figure 3: Gaussian loop soup 3-point correlations. A directed lace between two points means that the loop goes through the points in the indicated order. Point x is blue, y is red and z is green.

8.2 Gaussian interlacements

In this subsection we consider $d \geq 3$. The space of doubly infinite trajectories of a walk in $\mathbb{R}^d$ is defined by

$$\mathbb{W} := \left\{\omega : \mathbb{Z} \to \mathbb{R}^d,\ \lim_{n\to\pm\infty} \|\omega(n)\| = \infty\right\}$$

Define $X_n(\omega) := \omega(n)$, the position of the walk at time n and denote P^x the distribution of a double infinite random walk with Gaussian increments, starting at x. We have $P^x(X_0 = x) = 1$ and, under P^x, $X_n - X_{n-1}$ are iid centered Normal random variables with covariance matrix $\frac{1}{2\alpha}\mathrm{Id}$.

Intensity via capacity (Sznitman) Define the entrance time of ω in a compact set A by:

$$T_A(\omega) := \inf\left\{n \in \mathbb{Z},\ X_n(\omega) \in A\right\} \in\ (-\infty, \infty].$$

For a test function $g : \mathbb{W} \to \mathbb{R}$, define

$$Q_A^{\mathrm{cap}} g := \int_A E^x\left[g\,\mathbf{1}_{\{T_A=0\}}\right] dx.$$

The measure $e_A(x) := P^x[T_A = 0]$ is called *equilibrium measure* and $\int_A e_A(x)\,dx$ is called *Capacity* of A.

Intensity via visit debiasing Denote the number of visits to A by

$$n_A(\omega) := \sum_{n\in\mathbb{Z}} \mathbf{1}_A(X_n(\omega))$$

Define:

$$Q_A^{\mathrm{unif}} g := \int_A E^x\left[\frac{g}{n_A}\right] dx.$$

Under this measure, the weight of a trajectory intersectig A is inversely proportional to the number of visits to A.

Equivalence Define the time shift θ on $\mathbb{W}$ by $[\theta\omega](k) := \omega(k+1)$; the time shift acts on functions by $(\theta g)(\omega) := g(\theta\omega)$. Since the Lebesgue measure is reversible for the random walk, we have

$$\int_{\mathbb{R}^d} E^x[g]\, dx = \int_{\mathbb{R}^d} E^x[\theta^i g]\, dx, \quad \text{for any } i \in \mathbb{Z}, \tag{8.46}$$

for any bounded measurable test function $g : \mathbb{W} \to \mathbb{R}$.

Proposition 8.7 (Equivalence between Q_A^{unif} and Q_A^{cap})**.** *For any bounded set $A \subset \mathbb{R}^d$ and measurable bounded function $g : \mathbb{W} \to \mathbb{R}$ invariant under time shifts, $g = \theta g$, we have*

$$Q_A^{\mathrm{unif}} g = Q_A^{\mathrm{cap}} g. \tag{8.47}$$

Proof. Write

$$\begin{aligned}
Q_A^{\mathrm{unif}} g &= \int_A dx\, E^x\left[\frac{g}{n_A}\sum_{i\le 0} \mathbf{1}_{\{T_A=i\}}\right] \\
&= \sum_{i\le 0}\int_{\mathbb{R}^d} dx\, E^x\left[\mathbf{1}_A(X_0)\mathbf{1}_{\{T_A=i\}}\,\frac{g}{n_A}\right] && \text{by Fubini} \\
&= \sum_{i\ge 0}\int_{\mathbb{R}^d} dx\, E^x\left[\mathbf{1}_A(X_i)\mathbf{1}_{\{T_A=0\}}\,\theta^i\Big(\frac{g}{n_A}\Big)\right] && \text{by (8.46)} \\
&= \int_A dx\, E^x\left[\mathbf{1}_{\{T_A=0\}}\,\frac{g}{n_A}\sum_{i\ge 0} \mathbf{1}_A(X_i)\right] && \text{since } \theta^i\Big(\frac{g}{n_A}\Big) = \frac{g}{n_A} \\
&= \int_A dx\, E^x\left[\mathbf{1}_{\{T_A=0\}}\, g\right] = Q_A^{\mathrm{cap}} g.
\end{aligned}$$

□

We also have that the measures Q_A^{cap} and Q_A^{unif} are finite:

$$Q_A^{\mathrm{unif}}(\mathbb{W}) = Q_A^{\mathrm{cap}}(\mathbb{W}) = \mathrm{cap}(A) \le |A|. \tag{8.48}$$

Lemma 8.8 (Compatibility and additivity)**.** *Let $A \subset B$ be bounded sets of $\mathbb{R}^d$, and let g be a test function that is invariant under time shifts, $g = \theta g$. Then*

$$Q_B^{\mathrm{cap}} g \mathbf{1}_{\{T_A < \infty\}} = Q_A^{\mathrm{cap}} g \qquad \textit{(compatibility)} \tag{8.49}$$

$$Q_B^{\mathrm{cap}} g = Q_A^{\mathrm{cap}} g + Q_{B\setminus A}^{\mathrm{cap}} g \mathbf{1}_{\{T_A = \infty\}} \qquad \textit{(additivity)}, \tag{8.50}$$

The same holds for Q^{unif}.

Proof. Writing $\mathbf{1}_{\{T_A<\infty\}} = \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{T_A=i\}}$ we have

$$\begin{aligned} Q_B^{\mathrm{cap}} g \mathbf{1}_{\{T_A<\infty\}} &= \sum_{i\ge 0} \int dx\, \mathbb{E}^x \big[\mathbf{1}_{\{T_B=0\}} \mathbf{1}_{\{T_A=i\}}\, g\big] \\ &= \sum_{i\ge 0} \int dx\, \mathbb{E}^x \big[\theta^i \big(\mathbf{1}_{\{T_B=-i\}} \mathbf{1}_{\{T_A=0\}}\, g\big)\big] \\ &= \sum_{j\le 0} \int dx\, \mathbb{E}^x \big[\mathbf{1}_{\{T_B=j\}} \mathbf{1}_{\{T_A=0\}}\, g\big] \\ &= \int dx\, \mathbb{E}^x \Big[\mathbf{1}_{\{T_A=0\}}\, g \sum_{i\le 0} \mathbf{1}_{\{T_B=i\}}\Big] = Q_A^{\mathrm{cap}} g, \end{aligned}$$

since $\mathbf{1}_{\{T_A=0\}} \sum_{i\le 0} \mathbf{1}_{\{T_B=i\}} = \mathbf{1}_{\{T_A=0\}}$. This proves (8.49). To get (8.50) write

$$\begin{aligned} Q_B^{\mathrm{cap}} g &= Q_B^{\mathrm{cap}} g (\mathbf{1}_{\{T_A<\infty\}} + \mathbf{1}_{\{T_A=\infty\}}) \\ &= Q_A^{\mathrm{cap}} g + Q_B^{\mathrm{cap}} g \mathbf{1}_{\{T_A=\infty\}} = Q_A^{\mathrm{cap}} g + Q_{B\setminus A}^{\mathrm{cap}} g \mathbf{1}_{\{T_A=\infty\}}. \end{aligned}$$

Since $g\mathbf{1}_{\{T_A<\infty\}} = \theta(g\mathbf{1}_{\{T_A<\infty\}})$, $g = \theta g$ and $g\mathbf{1}_{\{T_A=\infty\}} = \theta(g\mathbf{1}_{\{T_A=\infty\}})$, Proposition 8.7 implies that (8.49) and (8.50) hold for Q^{unif} as well. □

Consider the equivalence relation $\sim$ defined by $\omega \sim \theta\omega$ and let

$$\widetilde{\mathbb{W}} := \mathbb{W}/\sim \tag{8.51}$$

be the space of trajectories modulo time shift and denote $\pi : \mathbb{W} \to \widetilde{\mathbb{W}}$, the projection. Given $A \subset \mathbb{R}^d$ let

$$\mathbb{W}_A := \{\omega \in \mathbb{W} : T_A(\omega) < \infty\}, \quad \text{trajectories intersecting } A$$

$$\widetilde{\mathbb{W}}_A := \pi(\mathbb{W}_A), \quad \text{classes of trajectories intersecting } A$$

If $g : \mathbb{W} \to \mathbb{R}$ is shift invariant then it can be extended to $\tilde{g} : \widetilde{\mathbb{W}} \to \mathbb{R}$ by $g(\tilde{\omega}) = g(\omega)$, for any choice of representative $\omega \in \pi^{-1}(\tilde{\omega})$. Let $\pi_* Q_A^{\mathrm{cap}}$ and $\pi_* Q_A^{\mathrm{unif}}$ denote the push-forward measures, defined by

$$(\pi_* Q_A^{\mathrm{cap}})\tilde{g} := Q_A^{\mathrm{cap}}(\tilde{g} \circ \pi). \tag{8.52}$$

Proposition 8.9 (Infinite volume mean measure for interlacements)**.** *There exists a unique σ-finite measure Q^{ri} on $\widetilde{\mathbb{W}}$ such that for each bounded set $A \subset \mathbb{R}^d$*

$$\mathbf{1}_{\widetilde{\mathbb{W}}_A} Q^{\mathrm{ri}} = \pi_* Q_A^{\mathrm{cap}} = \pi_* Q_A^{\mathrm{unif}}. \tag{8.53}$$

Proof. Let $\tilde{g} : \widetilde{\mathbb{W}} \to \mathbb{R}$ and define $g : \mathbb{W} \to \mathbb{R}$ by $g = \tilde{g} \circ \pi$. Then $\theta g = g$ and

$$\pi_* Q_A^{\mathrm{cap}} \tilde{g} = Q_A^{\mathrm{cap}} \tilde{g} \circ \pi = Q_A^{\mathrm{cap}} g = Q_A^{\mathrm{unif}} g = Q_A^{\mathrm{unif}} \tilde{g} \circ \pi = \pi_* Q_A^{\mathrm{unif}} \tilde{g},$$

by Proposition 8.7. This proves the second equality in (8.53).

Let $\{A_n\}_{n \geq 1}$ be an increasing sequence of bounded Borel sets in $\mathbb{R}^d$ such that $A_n \nearrow_{n \to \infty} \mathbb{R}^d$. Then $\widetilde{\mathbb{W}} = \bigcup_{n \geq 1} \widetilde{\mathbb{W}}_{A_n}$ and uniqueness of the measure satisfying (8.53) follows. Define Q^{ri} on $\widetilde{\mathbb{W}}_{A_n}$ by

$$\mathbf{1}_{\widetilde{\mathbb{W}}_{A_n}} Q^{\mathrm{ri}} := \pi_* Q_{A_n}^{\mathrm{cap}}. \tag{8.54}$$

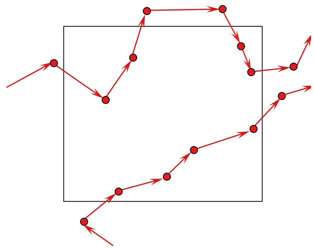
Let A be a bounded set and take n sufficiently large such that $A_n \supset A$. Then

$$\mathbf{1}_{\widetilde{\mathbb{W}}_A} \mathbf{1}_{\widetilde{\mathbb{W}}_{A_n}} \left(\pi_* Q_{A_n}^{\mathrm{cap}} \right) = \mathbf{1}_{\widetilde{\mathbb{W}}_A} \left(\pi_* Q_{A_n}^{\mathrm{cap}} \right) = \pi_* \mathbf{1}_{\mathbb{W}_A} Q_{A_n}^{\mathrm{cap}} = \pi_* Q_A^{\mathrm{cap}}, \tag{8.55}$$

where the last identity follows from (8.49). When $A = A_m$ for some $m < n$, (8.55) proves that the definition (8.54) is consistent and that the measure Q^{ri} defined in (8.54) satisfies (8.53). By (8.53), $Q^{\mathrm{ri}}(\widetilde{\mathbb{W}}_{A_n}) = Q_{A_n}^{\mathrm{cap}}(\mathbb{W}) = \mathrm{cap}(A_n) < \infty$, which proves the σ-finite property. □

Gaussian interlacements Let $d \geq 3$ and $\beta > 0$. The Gaussian random interlacements process at point density ρ is

$$\Gamma_\rho^{\mathrm{ri}} := \text{ Poisson process on } \widetilde{\mathbb{W}} \text{ with intensity } \rho Q^{\mathrm{ri}},$$
$$\mu_\rho^{\mathrm{ri}} := \text{ Law of } \Gamma_\rho^{\mathrm{ri}}.$$



This is the discrete counterpart of Sznitman [46] Brownian interlacements. In fact, if one take a Brownian trajectory and look at integer times, one obtains a Gaussian random walk trajectory.

Construction of Gaussian interlacements at density ρ

1. Sample a homogeneous Poisson *point* process X_0 on $\mathbb{R}^d$ of intensity ρ.
2. Take a bounded box Λ
3. To each point $x \in \mathsf{X}_0 \cap \Lambda$ sample a double-infinity Gaussian walk with law P^x.
4. Accept the walk with probability 1 over number of visits to Λ.

4' (Alternative to item 4.) Accept the walk if $T_\Lambda(\omega) = 0$.

5. The accepted walks will be a sample of the random interlacement intersecting Λ.
6. To sample in $\mathbb{R}^d$, consider a partition $(\Lambda_j)_{j\geq 1}$ of $\mathbb{R}^d$ with Λ_j bounded.
7. Perform the procedure (1) to (5) in each $\Lambda_1, \Lambda_2, \dots$ successively.
8. Reject walks with starting point in Λ_j that have points in previous visited boxes $\Lambda_1, \dots, \Lambda_{j-1}$.

8.2.1 Point marginal of Gaussian interlacements

Correlations

$$\nu_\rho^{\mathrm{ri}} := \text{Point marginal of Gaussian interlacements } \mu_\rho^{\mathrm{ri}}$$

Correlations:

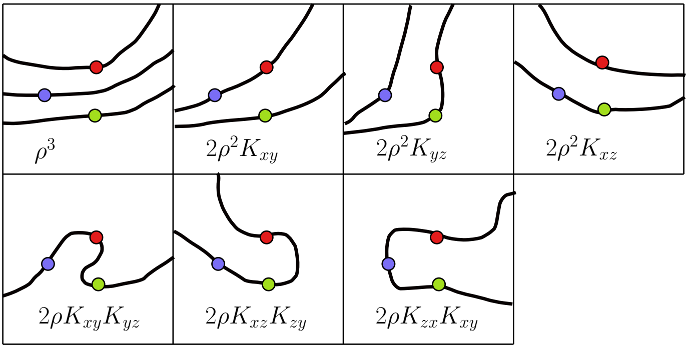
$$\varphi_\rho^{\mathrm{ri}}(x_1, \dots, x_n) = \sum_{P \in \mathcal{P}_n} \prod_{I \in P} \sum_{\sigma \in \mathcal{S}_I} V_\rho(x_{\sigma(i_1)}, \dots, x_{\sigma(i_{|I|})}). \tag{8.56}$$

$\mathcal{P}_n :=$ partitions of $\{1, \dots, n\}$ with nonempty sets,
$\mathcal{S}_I :=$ permutations of I,
$(i_1, \dots, i_{|I|})$ arbitrary order of I and

$$V_\rho(x_1, \dots, x_\ell) := \rho K_{x_1,x_2} \dots K_{x_{\ell-1},x_\ell}$$

(Here $\lambda = 1$ and $K_{xy} = K_1(x, y)$)

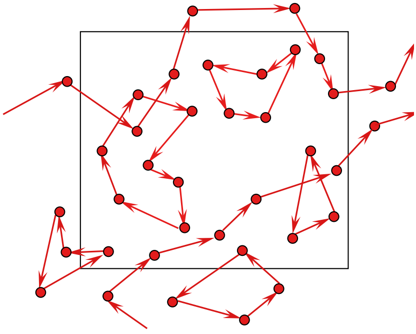
$$\varphi_\rho^{\text{ri}}(x, y, z) = \rho^3 + 2\rho^2 K_{xy} + 2\rho^2 K_{yz} + 2\rho^2 K_{xz} \\ +2\rho K_{xy} K_{yz} + 2\rho K_{xz} K_{zy} + 2\rho K_{zx} K_{xy}.$$



8.3 Infinite volume spatial Gaussian permutation

Let $\rho > 0$ and $\Gamma_{\lambda(\rho\wedge\rho_c)}$, $\Gamma^{\text{ri}}_{(\rho-\rho_c)^+}$ be independent realizations of the loop soup and Gaussian interlacements, respectively. Define

$$(\mathsf{X}, \sigma)_\rho := \Gamma_{\lambda(\rho\wedge\rho_c)} \cup \Gamma^{\text{ri}}_{(\rho-\rho_c)^+}$$



A superposition of independent realizations of the Gaussian loup soup at density $\min\{\rho, \rho_c\}$ and Gaussian interlacement at density $(\rho - \rho_c)^+$.

8.3.1 Markov property

$\Lambda \subset \mathbb{R}^d$ and a spatial permutation $\Gamma = (\mathsf{Y}, \kappa) \in \mathcal{X}$,

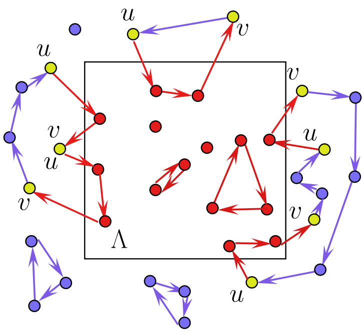
$$\begin{aligned} I_\Lambda \mathsf{Y} &:= \mathsf{Y} \cap \Lambda, && \text{red points} \\ O_\Lambda \mathsf{Y} &:= \mathsf{Y} \cap \Lambda^c, && \text{purple and yellow points} \\ U_\Lambda \mathsf{Y} &:= \{u \in \mathsf{Y} \cap \Lambda^c : \kappa(u) \in \Lambda\}, && \text{yellow} \\ V_\Lambda \mathsf{Y} &:= \{v \in \mathsf{Y} \cap \Lambda^c : \kappa^{-1}(v) \in \Lambda\}, && \text{yellow} \end{aligned}$$

and the maps

$$\begin{aligned} I_\Lambda \kappa &: I_\Lambda \mathsf{Y} \cup U_\Lambda \mathsf{Y} \to I_\Lambda \mathsf{Y} \cup V_\Lambda \mathsf{Y}, \quad & I_\Lambda \kappa(x) = \kappa(x) \quad & \text{red arrows,} \\ O_\Lambda \kappa &: O_\Lambda \mathsf{Y} \setminus U_\Lambda \mathsf{Y} \to O_\Lambda \mathsf{Y} \setminus V_\Lambda \mathsf{Y}, \quad & O_\Lambda \kappa(x) = \kappa(x) \quad & \text{purple arrows.} \end{aligned}$$

Define the inside and outside projections (with respect to Λ) by

$$I_\Lambda(\mathsf{Y}, \kappa) := (I_\Lambda \mathsf{Y}, I_\Lambda \kappa), \qquad O_\Lambda(\mathsf{Y}, \kappa) := (O_\Lambda \mathsf{Y}, O_\Lambda \kappa).$$



Decomposition of a loop soup intersecting Λ.

Inside = red
Outside = purple + yellow

Proposition 8.10 (Markov property)**.** *The Gaussian random permutation μ_ρ is Markov:*

$$\begin{aligned} &\mu_\rho\big(dI_\Lambda(\Gamma) \big| \, O_\Lambda(\Gamma) \text{ occurs outside } \Lambda\big) \\ &\qquad = \mu_\rho\big(dI_\Lambda(\Gamma) \big| \, (U_\Lambda, V_\Lambda\big) \text{ occur outside } \Lambda). \end{aligned}$$

Proof. We start with the subcritical and critical case; that is, with the loop soup. Conditioning on purple and yellow, the law of red points and arrows depends only on the labeled yellow points. Conditioned on labeled yellow points, purple points and arrows are independent of red points and arrows.

Recall the notation $\mathbb{D}_\Lambda := \{\gamma \in \mathbb{D} : \{\gamma\} \subset \Lambda\}$, and denote $\mathbb{D}_{\partial\Lambda} := \big(\mathbb{D}_\Lambda \cup \mathbb{D}_{\Lambda^c}\big)^c$. Note that the restricted Gaussian loop soups

$$\Gamma_\lambda \cap \mathbb{D}_\Lambda \quad \text{(loops contained in } \Lambda\text{)},$$

$$\begin{aligned} &\Gamma_\lambda \cap \mathbb{D}_{\Lambda^c} \quad \text{(loops contained in } \Lambda^c), \\ &\Gamma_\lambda \cap \mathbb{D}_{\partial\Lambda} \quad \text{(loops intersecting } \Lambda \text{ and } \Lambda^c) \end{aligned} \tag{8.57}$$

are independent Poisson processes with intensity measures $Q_\lambda \mathbf{1}_{\mathbb{D}_\Lambda}$, $Q_\lambda \mathbf{1}_{\mathbb{D}_{\Lambda^c}}$, $Q_\lambda \mathbf{1}_{\mathbb{D}_{\partial\Lambda}}$, respectively, and form a partition of Γ_λ.

Due to the independence of the partition (8.57), the inside and outside components of Γ_λ are partitioned into independent pieces as follows,

$$I_\Lambda(\Gamma_\lambda) = \big(\Gamma_\lambda \cap \mathbb{D}_\Lambda\big) \,\dot\cup\, \partial I_\Lambda(\Gamma_\lambda), \tag{8.58}$$

$$O_\Lambda(\Gamma_\lambda) = \big(\Gamma_\lambda \cap \mathbb{D}_{\Lambda^c}\big) \,\dot\cup\, \partial O_\Lambda(\Gamma_\lambda), \tag{8.59}$$

where

$$\begin{aligned} \partial I_\Lambda(\Gamma) := \big\{\eta = (u, x_1, \dots, x_{\ell(\eta)}, v) :& \\ u \in U_\Lambda(\Gamma),\ x_i = \kappa^i(u) \in \Lambda,\ v = \kappa^{\ell(\eta)+1}(u) \in V_\Lambda(\Gamma)\big\},& \quad (8.60) \\ \partial O_\Lambda(\Gamma) := \big\{\eta' = (v, y_1, \dots, y_{\ell(\eta')}, u) :& \\ v \in V_\Lambda(\Gamma),\ y_i = \kappa^i(v) \in \Lambda^c,\ u = \kappa^{\ell(\eta')+1}(v) \in U_\Lambda(\Gamma)\big\},& \quad (8.61) \end{aligned}$$

where $\ell(\eta) = \min\{\ell \geq 1 : \kappa^{\ell+1}(u) \in V_\Lambda(\Gamma)\}$ and $\ell(\eta') = \min\{\ell \geq 0 : \kappa^{\ell+1}(v) \in U_\Lambda(\Gamma)\}$. These numbers count the number of points visited by the associated path, excluding the endpoints u and v.

In the figure, an element of $\partial I_\Lambda(\Gamma)$ is given by a red path linking two yellow points with labels u and v respectively, while an element of $\partial O_\Lambda(\Gamma)$ is a purple path that links two yellow points v and u. Each path in the inside boundary $\partial I_\Lambda(\Gamma)$ contains at least one point in Λ, so that $\ell(\eta) \geq 1$, while the outside boundary $\partial O_\Lambda(\Gamma)$ might contain a path (v, u) with $v \in V_\Lambda$, $u \in U_\Lambda$; $\ell(\eta') = 0$ in this case. There is no path when $u = v \in U_\Lambda \cap V_\Lambda$.

Given $\eta = (u, x_1, \dots, x_\ell, v) \in \partial I_\Lambda(\Gamma)$ and $\eta' = (v, y_1, \dots, y_\ell, u) \in \partial O_\Lambda(\Gamma)$, consider the weights

$$\omega(\eta) := p(u, x_1)\, p(x_\ell, v) \prod_{i=1}^{\ell-1} p(x_i, x_{i+1}) \qquad \ell \geq 1, \tag{8.62}$$

$$\omega(\eta') := p(v, y_1)\, p(y_\ell, u) \prod_{i=1}^{\ell-1} p(y_i, y_{i+1}) \qquad \ell \geq 1, \tag{8.63}$$

$$\omega(v, u) := p(v, u) \qquad \ell = 0, \tag{8.64}$$

where p the Brownian transition density at time $\frac{2}{\alpha}$. By (8.18), the weight of a cycle $\gamma = [x_0, \dots, x_{n-1}]$ is given by

$$\omega_\lambda(\gamma) = \lambda^n \prod_{i=0}^{n-1} p(x_i, x_{i+1}), \quad \text{with } x_n = x_0. \tag{8.65}$$

If γ intersects both Λ and Λ^c, this weight factorizes as

$$\omega_\lambda(\gamma) = \lambda^n \prod_{\eta \in \partial I_\Lambda(\gamma)} \omega(\eta) \prod_{\eta' \in \partial O_\Lambda(\gamma)} \omega(\eta'). \tag{8.66}$$

Replacing (8.66) in (8.17) we obtain

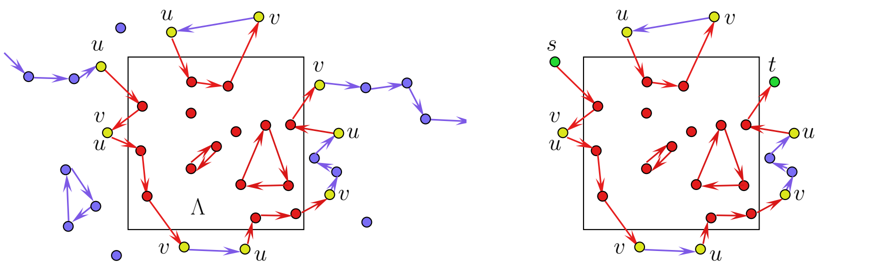
$$\begin{aligned} f_\lambda(\Gamma \cap \partial \mathbb{D}_\Lambda) &= e^{-Q_\lambda(\partial \mathbb{D}_\Lambda)} \, \lambda^{|U_\Lambda(\Gamma) \cup V_\Lambda(\Gamma)|} \\ &\qquad \prod_{\gamma \in \Gamma \cap \partial \mathbb{D}_\Lambda} \Big[\prod_{\eta \in \partial I_\Lambda(\gamma)} \omega(\eta) \, \lambda^{\ell(\eta)} \prod_{\eta' \in \partial O_\Lambda(\gamma)} \omega(\eta') \lambda^{\ell(\eta')} \Big] \\ &= e^{-Q_\lambda(\partial \mathbb{D}_\Lambda)} \, \lambda^{|U_\Lambda(\Gamma) \cup V_\Lambda(\Gamma)|} \\ &\qquad \prod_{\eta \in \partial I_\Lambda(\Gamma)} \lambda^{\ell(\eta)} \omega(\eta) \prod_{\eta' \in \partial O_\Lambda(\Gamma)} \lambda^{\ell(\eta')} \omega(\eta') \, . \end{aligned} \tag{8.67}$$

In view of the partition into independent processes (8.57) and the representation (8.67) above, we conclude that

$$\begin{aligned} \mu_\rho\big(dI_\Lambda(\Gamma) \big| \, O_\Lambda(\Gamma)\big) &= \frac{1}{Z} \, f_\lambda^{ls}(\Gamma \cap \mathbb{D}_\Lambda) \, dx_1 \ldots dx_{|\mathsf{Y} \cap \Lambda|} \\ &\qquad \prod_{\eta \in \partial I_\Lambda(\Gamma)} \lambda^{\ell(\eta)} \omega(\eta) \, dx_1^\eta \ldots dx_{\ell(\eta)}^\eta , \end{aligned} \tag{8.68}$$

where Z is a normalizing constant $Z\big(\alpha, \lambda, \Lambda, U_\Lambda(\Gamma), V_\Lambda(\Gamma)\big)$. Identity (8.68) above implies that the conditioned measure on the left only depends on the sets $U_\Lambda(\Gamma)$ and $V_\Lambda(\Gamma)$, proving the Markov property in the critical and subcritical cases $\rho \leq \rho_c$.

Now we consider the supercritical case. Cutting the part of the infinite trajectories that are outside Λ and not directly connected to Λ, we can proceeds in the same way as in the loop-soup.



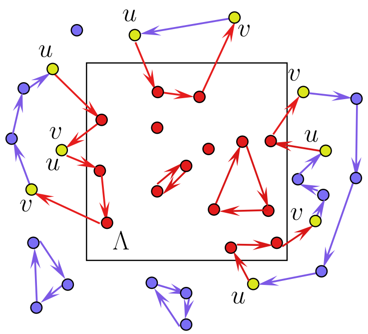
Since the fugacity $\lambda = 1$, the pieces η inside Λ have the same law for both loops and infinite trajectories. $\square$

8.3.2 Gibbs property

Define Λ-compatibility between infinite volume permutations by

$$(\mathsf{X},\sigma) \sim_\Lambda (\mathsf{Y},\kappa)$$

if and only if $O_\Lambda(\mathsf{X},\sigma) = O_\Lambda(\mathsf{Y},\kappa)$, see §8.3.1 for the definition of O_Λ.



Two spatial permutations are Λ-compatible if they have same yellow points and purple points and arrows.

Conditioned Hamiltonian: For $(\mathsf{X},\sigma) \sim_\Lambda (\mathsf{Y},\kappa)$

$$H_\Lambda((\mathsf{X},\sigma)|(\mathsf{Y},\kappa)) := \sum_{x\in[\mathsf{X}\cap\Lambda]\cup[\kappa^{-1}(\mathsf{Y}\cap\Lambda)\setminus\Lambda]} \|x-\sigma(x)\|^2.$$

Fix yellow and purple and sum over red points and arrows.

Specifications $G_{\Lambda,\lambda}(\cdot|(\mathsf{Y},\kappa)) :=$ law of red points and arrows.

Specification: measure $G_{\Lambda,\lambda}(\cdot|(\mathsf{Y},\kappa))$ with density

$$f_{\Lambda,\lambda}((\mathsf{X},\sigma)|(\mathsf{Y},\kappa)) := \frac{1}{Z_{\Lambda,\lambda}(\mathsf{Y},\kappa)}\lambda^{|\mathsf{X}\cap\Lambda|}\exp\big(-\alpha H_\Lambda((\mathsf{X},\sigma)|(\mathsf{Y},\kappa))\big),$$

where

$$Z_{\Lambda,\lambda}(\mathsf{Y},\kappa) := \sum_{n\geq|\kappa(\mathsf{Y}\cap\Lambda)\cap\Lambda^c)|}\frac{\lambda^n}{n!}\int_{\Lambda^n}\mathbf{1}\{\mathsf{X}=\{x_1,\dots,x_n\}\cup[\mathsf{Y}\setminus\Lambda]\}$$
$$\times\sum_{\sigma:(\mathsf{X},\sigma)\sim_\Lambda(\mathsf{Y},\kappa)}\exp\big(-H_\Lambda((\mathsf{X},\sigma)|(\mathsf{Y},\kappa))\big)dx_1,\dots dx_n.$$

Denote $G_{\Lambda,\lambda}(\cdot|(\mathsf{Y},\kappa))$ the measure with density $f_{\Lambda,\lambda}((\mathsf{X},\sigma)|(\mathsf{Y},\kappa))$. That is, for bounded measurable $g:(\mathsf{X},\sigma)\mapsto g(\mathsf{X},\sigma)$:

$$G_{\Lambda,\lambda}(g|(\mathsf{Y},\kappa)) \tag{8.69}$$

$$:= \sum_{n \geq |\kappa(\mathsf{Y}\cap\Lambda)\cap\Lambda^c)|} \frac{\lambda^n}{n!} \int_{\Lambda^n} \mathbf{1}\{\mathsf{X} = \{x_1, \dots, x_n\} \cup [\mathsf{Y} \setminus \Lambda]\}$$
$$\times \sum_{\sigma:(\mathsf{X},\sigma)\sim_\Lambda(\mathsf{Y},\kappa)} g(\mathsf{X},\sigma) f_{\Lambda,\lambda}((\mathsf{X},\sigma)|(\mathsf{Y},\kappa)) dx_1, \dots dx_n.$$

Theorem 8.11 (Gaussian random permutation on $\mathbb{R}^d$ is Gibbs)**.** *For $d \geq 3$ and $\lambda \leq 1$ the loop soup measure μ_λ is Gibbs for the specifications $(G_{\Lambda,\lambda} : \Lambda$ compact):*

$$\mu_\lambda g = \int d\mu_\lambda(\mathsf{Y},\kappa)\, G_{\Lambda,\lambda}(g|(\mathsf{Y},\kappa)) \qquad \textit{DLR equations}$$

For all $\rho \geq \rho_c$ the measure

$\mu_1 * \mu^{\mathrm{ri}}_{\rho-\rho_c}$ is Gibbs for the specifications $(G_{\Lambda,1} : \Lambda$ compact).

Corollary 8.12 (Point and permutation marginals)**.** *Point and permutation marginals can be computed explicitely.*

8.3.3 Thermodynamic limit of Point and permutation marginals

Theorem 8.13 (Thermodynamic limit of point marginal)**.** *Fix density $\rho >$ 0.*

$G^{\mathrm{point}}_{\Lambda,|\Lambda|\rho} :=$ *law of point-marginal with $|\Lambda|\rho$ points.*

Subcritical $\rho \leq \rho_c$ or $d \leq 2$ Fichtner 1991; Tamura-Ito 2006.

$$G^{\mathrm{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{\mathrm{TI}}_\rho \textit{ as } \Lambda \nearrow \mathbb{R}^d.$$

Supercritical $\rho > \rho_c$ and $d \geq 3$ Tamura-Ito 2007.

$$G^{\mathrm{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{point}_\rho = \nu^{TI}_{\rho_c} * \nu^\infty_{\rho-\rho_c}.$$

Corollary 8.14 (Point marginal of Gaussian random permutation coincides with thermodynamic limit)**.** *The Point marginal of Gaussian random permutation coincide with thermodynamic limit:*

$\nu^{\mathrm{TI}}_\rho = \nu_{\lambda(\rho)}$, *point marginal of loop soup at fugacity $\lambda(\rho)$.*

$\nu^\infty_\rho = \nu^{\mathrm{ri}}_\rho$, *point marginal of Gaussian interlacements at ρ.*

Partial "Thermodynamic limit" of permutation marginal

$G^{\text{permut}}_{\Lambda,|\Lambda|\rho} := \sigma$-marginal of $G_{\Lambda,|\Lambda|\rho}$

$$G^{\text{permut}}_{\Lambda,\rho} \Rightarrow \nu^{\text{permut}}_{\rho} \text{ for } \textit{cycle-size distribution}.$$

Macroscopic cycles: cycles with size bigger than $\varepsilon|\Lambda|$.

Subcritical case. $\rho \le \rho_c$ or $d = 1, 2$

The expected fraction of points in macroscopic cycles is zero. BU 2011

Supercritical case. $d \ge 3$ and $\rho > \rho_c$

(a) expected fraction of points in macroscopic cycles is $\frac{\rho-\rho_c}{\rho}$.

(b) Rescaled macroscopic cycles have random lenght:
Benfatto, Cassandro, Merola Presutti 2005.

Poisson-Dirichlet distribution (as uniform permutations): Betz-Ueltschi 2011.

Current problems Thermodynamic limit of canonical measure. That is, the Gaussian random permutation in a box Λ should converge to the infinite volume GRP constructed here.

Extensions: Poisson process on $\mathbb{Z}^d$

Other interactions besides Gaussian.

Quantum case, when the Brownian trajectories from x to y interact. (BCMP 2005 treated the mean field case).

9 Factor graphs

A *factor graph* of a point configuration X is a measurable function which maps X on a graph $\mathsf{G} = (\mathsf{X}, E)$ in an equivariant way. This means that

$$\mathsf{X} \mapsto (\mathsf{X}, E) \text{ if and only if } \gamma\mathsf{X} \mapsto (\gamma\mathsf{X}, \gamma E),$$

for any isometry γ. The set of edges E is a subset of $\{\{x, y\} : x,\, y \in \mathsf{X}\}$. If $\{x, y\} \in E$ we say that x and y are neighbors.

A graph is *locally finite* if all vertex has a finite number of neighbors.

A *tree* is a graph with no cycles. A tree is *one-ended* if any pair of infinite self-avoiding paths coalesce.

Theorem 9.1 (Tree factors of point processes). *Let X be a Poisson process on $\mathbb{R}^d$. Then there exists a locally finite one-ended tree factor on X.*

A weaker version of this theorem was proven by [12], for $d = 2$ and $d = 3$ in a translation equivariant way. Holroyd and Peres [28] proved it in $\mathbb{R}^d$ and Timar [49] for ergodic point processes with an index function.

9.1 Factor trees

FLT approach [12] We describe a translation invariant construction of a graph with vertices in a Poisson process. When $d = 2, 3$, the resulting graph is a one-ended connected tree. When $d \geq 4$ the graph will be a forest (union of disjoint trees).

Let $d \geq 2$; and X be a homogeneous Poisson process in $\mathbb{R}^d$ with intensity 1.

$s = (s_1, \ldots, s_d) \in \mathsf{X}$.

Obstacles: Let $u(s) = (s_1, \ldots, s_{d-1})$ and

$$B(s) = \{(u', s_d) : u' \in \mathbb{R}^{d-1}; \|u' - u(s)\|_{d-1} \leq 1\}, \qquad B(\mathsf{X}) = \cup_{s \in \mathsf{X}} B(s).$$

$B(s)$ is a $d-1$ dimensional disc of radious 1, perpendicular to the axis $\{s : u(s) = 0\}$.

Each point emits a laser ray in the positive dth coordinate. Define the first time that the ray of s meets an obstacle by

$$\tau(s, \mathsf{X}) := \inf\{t > s_d \,:\, (u(s), t) \cap B(\mathsf{X}) \neq \phi\}.$$

Define the *mother of* s as the center of the obstacle:

$$\alpha(s) \;:=\; s' \in \mathsf{X} \quad \text{if} \quad \tau(s, \mathsf{X}) = s'_d \tag{9.1}$$

s is a *daughter* of $\alpha(s)$

Random directed graph

$$\mathsf{G} := (\mathsf{X}, E) \text{ with edges } E = \{(s, \alpha(s)) \,:\, s \in \mathsf{X}\}.$$

Ancestors $\alpha^0(s) = s$ and iteratively, for $n \geq 1$, $\alpha^n(s) = \alpha(\alpha^{n-1}(s))$ the $(n+1)$th grand mother of s.

$$\begin{aligned} D^1(s) &:= \{s' \in \mathsf{X} \,:\, \alpha(s') = s\}, \\ D^n(s) &:= \{s' \in \mathsf{X} \,:\, \alpha(s') = s'', \text{ for some } s'' \in D^{n-1}(s)\}, \end{aligned}$$

$$D(s) \quad := \quad \bigcup_{n\geq 0} D^n(s),$$

The daughters, the nth generation of *descendents* and the *branch* of s.

Let G^o be the graph obtained from $\mathsf{X}^o = \mathsf{X} \cup \{0\}$; it is distributed with the Palm measure of the law of X.

Theorem 9.2. *For* G *and* G^o *it holds a.s.:*

- *(a)* G *is well defined.*
- *(b) In* $d = 2, 3$*,* G *is a one-ended locally finite tree.*
- *(c) In* $d \geq 4$*,* G *is a forest of one-ended locally finite trees.*
- *(d) All branches of* G *are finite.*
- *(e) All vertex has a mother.*
- *(f) Each vertex has an ancestor with a younger sister.*

Sketch proof. (a) it suffices to see that for almost all realizations of X and X^o each point has a unique mother.

(b) Coalescing random walks in dimension 1 and 2. Recurrence.

(c) Coalescing random walks in dimension $d \geq 3$. Transcience.

(d) Exercise. There is a proof with a particle system. Look for a proof with the mass transport principle.

(e) (f) Exercise. □

Timar Approach [49] Consider a *isometry invariant ergodic point process* X on $\mathbb{R}^d$ with finite intensity and with an index function.

Isometry is a map that preserves distance.

A function $f : \{(s, \mathsf{X}) \in \mathbb{R}^d \times \mathcal{X} \ : \ s \in \mathbb{X}\} \to \mathbb{R}$ is *isometry-equivariant* if $f(\gamma(s), \gamma\mathsf{X}) = f(s, \mathsf{X})$ for any isometry $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ (translations, rotations, reflections, etc).

An *index function* is a one-to-one isometry-equivariant function

$$\mathtt{i} : \{(s, \mathsf{X}) \in \mathbb{R}^d \times \mathcal{X} \ : \ s \in \mathsf{X}\} \to \mathbb{R}.$$

For instance, size of the Voronoi cell, the sum of the distance to the Delaunay neighbors.

In a *non-equidistant process*, the distance between any two points is different. For those processes there are many index functions.

Theorem 9.3 (Timar [49]). *Let* X *be a isometry invariant point process with an index function. Then there exists a locally finite one-ended tree factor on* X.

The mass transport principle Call a measure μ on $\mathbb{R}^d \times \mathbb{R}^d$ *diagonally invariant* if for any isometry $\gamma : \mathbb{R}^d \to \mathbb{R}^d$

$$\mu(\gamma A \times \gamma B) = \mu(A \times B)$$

for any measurable sets $A, B \subset \mathbb{R}^d$.

A *Borel measure* is a sigma finite measure defined in the Borel sigma algebra (sigma finite means that there is a measurable partition of the space with finite measure parts). ℓ is the Lebesgue measure.

Lemma 9.4 (Mass transport principle). *Let* μ *be a nonnegative, diagonally invariant Borel measure on* $\mathbb{R}^d \times \mathbb{R}^d$. *Suppose that* $\mu(A \times \mathbb{R}^d) < \infty$ *for some nonempty open* $A \subset \mathbb{R}^d$. *Then there is a constant* c *such that* $\mu(B \times \mathbb{R}^d) = c\,\ell(B)$. *Moreover,*

$$\mu(B \times \mathbb{R}^d) = \mu(\mathbb{R}^d \times B),$$

for all Borel $B \subset \mathbb{R}^d$.

Proof. The Borel measure $\nu_1 := \mu(\cdot \times \mathbb{R}^d)$ on $\mathbb{R}^d$ is invariant under isometries and since $\nu_1(A) < \infty$, it must be sigma finite. Hence ν_1 is a multiple of the Lebesgue measure: $\nu_1 = c\ell$ for some constant c.

Take $b > 0$ and let $B = [0, b)^d$; let $B_z = B + bz$, where z is a d-dimensional integer. Then

$$\nu_1(B) = \mu(B \times \mathbb{R}^d) = \sum_z \mu(B \times B_z) \tag{9.2}$$

$$= \sum_z \mu(B_{-z} \times B) = \mu(\mathbb{R}^d \times B) =: \nu_2(B) \tag{9.3}$$

Since the Borel measures ν_1 and ν_2 on $\mathbb{R}^d$ coincide in arbitrary hypercubes, they must be the same measure. □

Corollary 9.5 (Density version of mass transport). *Let* μ *be a nonnegative, diagonally invariant Borel measure on* $\mathbb{R}^d \times \mathbb{R}^d$ *and assume* μ *is absolutely* *continuous with respect to Lebesgue measure. Let f_μ, g_1, g_2 be the Radon-Nikodym derivative of μ and the marginals of μ, respectively:*

$$\mu(A \times A') = \int_A \int_{A'} f_\mu(x,y)\, dx\, dy, \tag{9.4}$$

$$g_1(y) := \int_{\mathbb{R}^d} f_\mu(x,y)\, dx \qquad g_2(y) = \int_{\mathbb{R}^d} f_\mu(y,x)\; dx \tag{9.5}$$

Then $g_1(y) = g_2(y) = c$ for almost every y, with the constant c as in the previous lemma.

Proof. The Lemma says that for all Borel set A

$$\int_A g_1(y)dy = \int_A g_2(y)dy = \int_A c\, dy.$$

But if the integrals are equal on every Borel set then the three functions are equal almost everywhere. □

Lemma 9.6 (Point process mass transport). *Let $T : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^+$ be a nonnegative, measurable "mass transport function", such that $T(x,y,\mathsf{X}) = T(\gamma x, \gamma y, \gamma \mathsf{X})$ for any isometry γ. Define*

$$f(x,y) := \mathbb{E}T(x,y,\mathsf{X})$$

and assume that $\int_A (\int_{\mathbb{R}^d} f(x,y)\, dx)\, dy \; < \infty$ for some open $A \subset \mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} f(x,y)\, dx = \int_{\mathbb{R}^d} f(y,x)\, dx, \quad a.s.$$

$T(x,y,\mathsf{X})$ is the amount of mass sent from x to y if the configuration is X. Then $\int_{\mathbb{R}^d} f(x,y)\, dx$ and $\int_{\mathbb{R}^d} f(y,x)\, dx$ are the expected amount of mass sent into or sent out of y, respectively.

Proof. Let μ be the Borel measure on $\mathbb{R}^d \times \mathbb{R}^d$ determined by the measure on products of Borel sets B, B' by

$$\mu(B \times B') := \mathbb{E} \int_B \int_{B'} T(x,y,\mathsf{X})\, dx\, dy$$

By definition both μ and f are isometry-invariant. The function f is the Radon-Nikodym derivative of μ:

$$\int_B \int_{B'} f(x,y)\, dx\, dy = \int_B \int_{B'} \mathbb{E}T(x,y,\mathsf{X})\, dx\, dy = \mu(B \times B'),$$

by Fubini as the functions are nonnegative. By hypothesis there is an open set A with $\mu(A \times \mathbb{R}^d) < \infty$. Then Lemma 9.4 and Corollary 9.5 hold and, in particular, $f_\mu = f$ gives $\int_{\mathbb{R}^d} f(x,y)\, dx = \int_{\mathbb{R}^d} f(y,x)\, dx$ a.s.. □

Clumping A particion of a set is associated to an equivalence relation such that each set in the partition is a *class of equivalence* for the relation.

A *locally finite clumping* is a sequence of coarser partitions of X, defined on X in an isometry-equivariant way, so that in every partition all the classes are finite.

A partition $\mathcal{P}$ of X is coarser than a partition $\mathcal{P}'$ if each element of $\mathcal{P}$ is the union of elements of $\mathcal{P}'$.

An element of a partition is called a *clump*. A clumping is *connected* if any two vertices belong to the same clump for some partition (and hence all but finitely many of them).

Proposition 9.7 (Clumping and one-ended tree)**.** *If* X *has a connected locally finite clumping, then* X *has a locally finite tree with one end.*

Proof. Start with the first partition. Connect every vertex to the vertex of the highest index in its clump to get a forest in each clump of the second partition.

For each tree in this forest, connect the vertex of highest index in the tree to the vertex of highest index in the whole clump to get a tree in each clump of the second partition and a forest in each clump or the third particion.

Continue the process this way.

Tree. The graph we get is clearly a forest, constructed in an isometry-equivariant way. It is also a tree, by connectedness of the clumping.

One end. The only path starting from a vertex v to infinity is the one that goes through the vertices of greatest index in each clump which contains v. Since the clmping is connected, any two points belong to the same clump, eventually. This means they share the path to infinity from this moment.

Locally finite. Otherwise, define a mass transport so that each vertex v sends 1 mass to its neighbour that is on the path connecting v to infinity. The mass sent out is 1 for each point, and so, by the finite intensity of X, it is finite for a fixed unit cube K. If there are vertices of infinite degree in a configuration, they receive infinite mass. If there are such vertices with positive probability, then the event that K contains a vertex of infinite degree also has positive probability. Hence the expected mass received by K is infinite, giving a contradiction. □

Construction of a clumping Let `i` be the index function of the point process X.

Lemma 9.8 (Uniform index function). *If the point process* X *has an index function* $\mathtt{j}$*, then there is an index function* $s \mapsto \mathtt{i}(s) \in [0,1]$ *such that in the Palm version of* X*,* $\mathtt{i}(0)$ *has uniform distribution:* $\hat{\mathbb{P}}(\mathtt{i}(0) \in [0,a]) = a$*.*

Proof. Let $\mathtt{j}$ be an index function of X and $F : \mathbb{R} \to [0,1]$ be the function defined by

$$F(r) = \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\#\{s \in \mathsf{X} \cap \Lambda : \mathtt{j}(s) \leq r\}}{\ell(\Lambda)}$$

The limit exists by ergodicity of the process. By definition of index function, F is continuous. Let $F^{-1}(u) = \sup\{r : F(r) = u\}$. Since there are no points with indexes in the regions where F is constant, we can substitute $\mathtt{j}$ by $\mathtt{i} = F(\mathtt{j})$. The function $\mathtt{i}$ so defined is an index function and belongs to $[0,1]$. Furthermore, the marginal distribution of the index function of the origin (in the Palm version of X) is the uniform distribution in $[0,1]$. □

A subset of the vertex set of a graph G is called *independent*, if no two of its elements are adjacent.

Lemma 9.9 (Independent vertex sets). *Let* X *be a point process and* G *be a locally finite graph on the vertex set* X*, defined in an isometry-equivariant way. Then, there is* $\mathsf{Y} \subset \mathsf{X}$*, defined in an equivariant way such that* Y *is an independent set of* G*.*

Proof. Let $\mathtt{i}$ be a uniform index function; it exists by Lemma 9.8. Let

$$\mathsf{Y} := \{v \in \mathsf{X} : \mathtt{i}(v) < \mathtt{i}(w), \text{ for all } w \text{ neighbor of } v\}.$$

By construction, Y is an independent set. □

Proposition 9.10 (Independent set with minimal distances 2^k). *For all* k*, there is a nonempty subset* $\mathsf{V}_k \subset \mathsf{X}$*, chosen in an equivariant way, such that the distance between any two vertices in* V_k *is at least* 2^k*.*

Proof. Connect two points of X if their distance is less than 2^k and apply Lemma 9.9. □

Voronoi cells Define the *Voronoi cell* of a point $s \in \mathsf{X}$ as the set of sites in $\mathbb{R}^d$ closer to s than to any other point of X:

$$V(s) = \{x \in \mathbb{R}^d : |x - s| \leq |x - s'| \text{ for all } s' \in \mathsf{X} \setminus \{s\}\}$$

Lemma 9.11 (Voronoi cells are bounded)**.** *For an isometry invariant point process, the volume of the Voronoi cell containing some fixed point x in $\mathbb{R}^d$ is a.s. finite. As a consequence every Voronoi cell is bounded a.s.*

Proof. By contradiction. Define a mass-transport function $T(x, y, \mathsf{X})$ to be 1 if x and y are in the same Voronoi cell (say, the one corresponding to an X-point s) and if y is in the ball of volume 1 around s:

$$T(x, y, \mathsf{X}) = \mathbf{1}\{x \in V_y,\, y \in B_y\} = \mathbf{1}\{y \in V_x,\, y \in B_y\}$$

where V_y is the Voronoi cell containing y and B_y is the intersection of V_y and the ball of volume 1 around s.

Defining $f(x, y) := \mathbb{E}\, T(x, y, \mathsf{X})$, we have $\int_{\mathbb{R}^d} f(x, y)\, dy \le 1$. This implies also that the assumption of Lemma 9.6 holds. However, if the volume of the Voronoi cell containing x is infinite with positive probability then

$$\begin{aligned}
\int f(y, x)\, dy &= \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}\{y \in V_x\}\mathbf{1}\{x \in B_x\} dy \\
&= \mathbb{E}\Big(\mathbf{1}\{x \in B_x\} \int_{\mathbb{R}^d} \mathbf{1}\{y \in V_x\} dy\Big) \\
&= \mathbb{E}\Big(\mathbf{1}\{x \in B_x\}\ell(V_x)\Big) \qquad (9.6)
\end{aligned}$$

This contradicts Lemma 9.6.

A Voronoi cell is a convex polyhedron. Lemma 9.12 below shows that in convex polyhedrons, the surface area is bounded by a constant times the volume. Convex polyhedrons with finite surface area are bounded. □

Lemma 9.12 (Surface area and volume of convex polyhedrons)**.** *Let $K \subset \mathbb{R}^d$ be a convex polyhedron that contains a ball of radius r. Then the volume of K divided by the surface area of K is at least $c\,r$, where $c > 0$ is a constant depending only on d.*

Proof. Connect the center P of the ball to each vertex, thus subdividing the polygon to "pyramids", whose apices are P. The altitudes of the pyramids from P are at least r by the hypothesis, and this gives the claim. This is because the area of the bases sum up to the surface area, and the volume of the pyramids to the volume of K. □

Lemma 9.13. *Let O be a fixed point of $\mathbb{R}^d$. Suppose there is an equivariant measurable partition $\mathcal{P}(\mathsf{X}) = \mathcal{P}$ of $\mathbb{R}^d$ such that all the parts are bounded with probability 1, and suppose that for each part P in $\mathcal{P}$ a measurable subset of it is given by a measurable mapping $\phi = \phi(\mathcal{P}, P)$. Suppose that $(\mathcal{P}, \phi(\mathcal{P}, .))$ is* *invariant under isometries of* $\mathbb{R}^d$. *Assume further, that for each* $P \in \mathcal{P}(\mathsf{X})$, *Vol*$(\phi(\mathcal{P}, P))/$*Vol*$(P) \le p$. *Then the probability that* O *lies in* $\cup_{P\in\mathcal{P}}\phi(\mathcal{P}, P)$ *is at most* p.

Proof. Define $T(x, y, \mathsf{X})$ to be $1/\ell(P)\,(>0)$ if y is in $P \in \mathcal{P}$ and x is in $\phi(\mathcal{P}, P)$. Let $T(x, y, \mathsf{X})$ be 0 otherwise. Denote, as usual, $f(x, y) := \mathbb{E}\,T(x, y, \mathsf{X})$. The expected mass sent out from O is

$$\begin{aligned}
\int_{\mathbb{R}^d} f(O, y)\,dy &= \mathbb{E}\int dy \sum_{P\in\mathcal{P}} \frac{1}{\ell(P)}\mathbf{1}\{O \in \phi(\mathcal{P}, P),\, y \in P\} \\
&= \mathbb{E}\sum_{P\in\mathcal{P}} \mathbf{1}\{O \in \phi(\mathcal{P}, P)\}\,\frac{1}{\ell(P)}\int dy\mathbf{1}\{y \in P\} \\
&= \mathbb{E}\sum_{P\in\mathcal{P}} \mathbf{1}\{O \in \phi(\mathcal{P}, P)\} \\
&= \mathbb{E}\mathbf{1}\{O \in \cup_{P\in\mathcal{P}}\phi(\mathcal{P}, P)\} \\
&= \mathbb{P}(O \in \cup_{P\in\mathcal{P}}\phi(\mathcal{P}, P) \qquad (9.7)
\end{aligned}$$

The expected mass coming into O is

$$\begin{aligned}
\int_{\mathbb{R}^d} f(y, O)\,dy &= \mathbb{E}\int dy \sum_{P\in\mathcal{P}} \frac{1}{\ell(P)}\mathbf{1}\{y \in \phi(\mathcal{P}, P),\, O \in P\} \\
&= \mathbb{E}\sum_{P\in\mathcal{P}} \mathbf{1}\{O \in P\}\,\frac{\ell(\phi(\mathcal{P}, P))}{\ell(P)} \\
&\le p\,\mathbb{E}\sum_{P\in\mathcal{P}} \mathbf{1}\{O \in P\} \\
&= p. \qquad (9.8)
\end{aligned}$$

So by the mass transport principle, $\mathbf{P}[O \in \cup_{P\in\mathcal{P}}\phi(\mathcal{P}, P)] \le p$. □

Hence, there is a sequence $\mathsf{Y}_k \subset \mathsf{X}$, constructed in an equivariant way, such that the minimal distance between any two points of Y_k is at least 2^k.

Let B_k be the union of the boundaries of the Voronoi cells on Y_k. Those cells are a.s. bounded.

Define a partition $\mathcal{P}_k$ of X by saying that $x, y \in \mathsf{X}$ are in the same clump of $\mathcal{P}_k$ iff they are in the same component of $\mathbb{R}^d \setminus \cup_{i=k}^{\infty} B_i$. This clumping is locally finite with probability 1 by finite intensity and the fact that the cells defining $\mathcal{P}_i$ are bounded a.s.

Otherwise define a mass transport so that every vertex in an infinite clump sends one mass to the element of Y_i whose cell contains infinitely many components, where i is chosen to be minimal. The expected mass sent out is 1, but the expected received is infinity, by the indirect assumption.

Proposition 9.14 (Connected locally finite clumping)**.** *The $\mathcal{P}_k$ define a connected, locally finite clumping on* X*.*

Proof. We have to prove that the clumping is connected. That is, any fixed ball Q in $\mathbb{R}^d$ is intersected by only a finite number of the B_k's almost always, and so any two X-points inside Q are in the same clump of $\mathcal{P}_k$, if k is large enough.

Denote by δ the diameter of Q. Now let N_k be the set of points in $\mathbb{R}^d$ of distance less than δ from B_k, the union of the *thickened boundaries* of the Voronoi cells of Y_k.

The volume of the thickened boundary of a cell is bounded from above by a times the surface area of the cell, where $a(\delta)$ is a constant.

It can be proven that there exists a constant c such that for every Voronoi cell V of Y_k,

$$\mathrm{Volume}(V) \;\geq\; c\,2^k\,\mathrm{Volume}(N_k \cap V).$$

Q is intersected by B_k only if N_k contains the center O of Q. So it suffices to prove that for any fixed point O, the expected number of N_k's that contain O is finite.

In Lemma 9.13, put $\mathcal{P}$ to be the Voronoi cells on Y_k (k fixed), and $\phi(P)$ to be the intersection of the Voronoi cell P with the thickened boundary. The lemma combined with Lemma 9.12 says that the probability that O is contained in N_k is at most $2^{-k}/c$ and the expected number of N_k's containing O is at most $1/c$, and we are done. □

Proof of Theorem 9.3. Proposition 9.7 says that if X has a locally finite connected clumping, then there is a locally finite tree with one-end. Proposition 9.14 guarantees the existence of a locally finite connected clumping. □

9.2 Two ended path factor

A *two ended path* is a directed graph isomorphic to the directed graph $(\mathbb{Z}, E_1)$, where $E_1 = \{(x, x+1) : x \in \mathbb{Z}\}$.

Theorem 9.15 (Two-ended path factor)**.** *If* X *has a one ended tree factor, then* X *has a two-ended path factor.*

Points in the tree as mothers, daughters and sisters

age order among sisters determined by the lexicographic order.

If the point $X_0 = 0 \in \mathsf{X}^o$ has a daughter, let X_1 be the eldest daughter

If it does not have a daughter but has a younger sister, let X_1 be the eldest among its older sisters.

If it does not have a daughter and not a younger sister, move down the tree until you hit the first point that has a younger sister and let X_1 be the eldest among its younger sisters.

The chosen point X_1 is the *successor* of x. Iteratively $X_2, \dots$

Inverse:

If the point $X_0 = 0 \in \mathsf{X}^o$ has no older sister, choose the mother.

If the point at the origin has an older sister, move to the youngest among its older sisters and then move from her up the tree choosing the younger daughter in each step until you come to a point with no daughter; choose that point.

The chosen point X_{-1} is the *predecessor* of X_0. Iteratively $X_{-2}, \dots$

Succession lines

Let s, s' vertices of a tree. *succession line* from s to s' if there exists a finite sequence of vertices $s = s_0, \dots, s_k = s'$ such that $s_{\ell-1}$ is successor of s_ℓ for $\ell = 1, \dots, k$.

infinite succession line if all vertex has a predecessor and a successor and

unique infinite succession line if furthermore for all couple of vertices s, s' there is a succession line either from s to s' or from s' to s.

A tree G has a unique connected component, finite branches and all vertex has a mother and an ancestor with a younger sister if and only if G has a (translation invariant) unique infinite succession line.

10 Spanning trees

This section is based on the lecture notes of Wigderson [50] and the book of Lyons and Peres [37].

Consider a finite graph $\mathsf{G} = (V, E)$. A tree is a connected graph with no cycles. A spanning tree of G is a tree $\mathsf{T} = (V, E')$ of G with $E' \subset E$. That is a tree with all vertices and edges contained in the set of edges of sfG.

10.1 Wilson Algorithm

We introduce now a method to sample a uniform spanning tree of G. Uniform means that it is chosen uniformly among all possible spanning trees.

A walk is a sequence of vertices $W = v_0, v_1, \ldots, v_n$ such that v_i is neighbor of v_{i+1}.

Given a walk $W = v_0, v_1, \ldots, v_n$, the loop-erased walk LW is the walk obtained by erasing all the cycles of W. More precisely, if W has no cycles, do nothing; otherwise, let j be the smallest index such that $v_j = v_i$ for some $i < j$. Then, delete $v_i, v_{i+1}, \ldots, v_{j-1}$; and iterate this process to obtain a walk LW with no loops.

A rooted tree is a tree T together with a specified vertex r (called the root). We orient all the edges of T towards r.

Wilson algorithm Given a connected graph G and a vertex r, define a growing sequence of rooted trees T_i, by defining $\mathsf{T}_0 = \{r\}$ and if T_{i-1} is a spanning tree of G, stop. Otherwise, pick an arbitrary vertex v not contained in T_{i-1} and start a simple random walk W at v until it hit a vertex in T_{i-1}. LW is then a loopless path from v to T_{i-1}.

Define $\mathsf{T}_i := \mathsf{T}_{i-1} \cup LW$.

Each step will take finite time a.s., since G is finite, and each T_i is a tree, since we are adding a pendent loopless path to the previous tree at each step.

The algorithm finishes after all vertices have been incorporated. So we get a (rooted) spanning tree at the end of the process.

Proposition 10.1. *Wilson's Algorithm produces a uniformly random rooted spanning tree (T, r). The marginal distribution of T is a uniformly random spanning tree of V.*

To prove this result we need another construction of Wilson algorithm.

Stack construction of Wilson algorithm For each non-root vertex v, consider an infinite sequence of independent random variables $S_v := (S_v(i))_{i\geq 1}$, where each $S_v(i)$ is uniformly distributed on the neighbors of v. Denote $S := (S_v)_{v\in V\setminus\{r\}}$, and assume that the members of S are mutually independent.

Use S to perform Wilson algorithm as follows. The first time v is visited by a walk, use $S_v(1)$ to simulate the next jump of the walk. The second time v is visited, use $S_v(2)$, and so on. Since each time that v is visited, we use a fresh random variable to decide the jump, the tree obtained with this algorithm has the same distribution as the one obtained with Wilson algorithm. We denote the resulting tree $\mathsf{T}(S)$.

Place the random variables $S_v(i)$ in a stack below each vertex v. The top of the stack at v shows one of its neighbors; we can think of an arrow from v to its neighbor $S_v(i)$. That is, the top of the stack at all vertices induces a directed graph. If this directed graph has no cycles, we stop the process.

Otherwise, pick one of the cycles and "pop" it from the stack. That is, eliminate the top element $S_v(i)$ of the stack for each vertex v in this cycle and uncover the next move of the stack, $S_v(i+1)$. This gives a new directed graph. Repeat this process.

A possible configuration of the top of the stack after an iteration shows directed edges $(v, S_v(i))$; we wan to to keep track of the color (height) i of the arrow, so that we look at the top-stack configuration $(v, S_v(i), i)$. Then by a colored cycle, we just mean a cycle in one of these directed graphs, together with the marking of which level in the stack each of its edges came from.

Lemma 10.2. *In the above process, it doesn't matter what order we pop the cycles in. More formally, suppose each vertex comes with an (arbitrary) infinite stack, and we perform the above process. Then one of the following two cases occurs:*

1. Any order of popping will never terminate (i.e. infinitely many cycles will be popped).

2. In any order of popping, the same set of colored cycles will be popped.

Proof. Suppose that C is any colored cycle that can be popped at some point in some popping order. That means that there is a sequence of colored cycles $C_1, \dots, C_k = C$ such that we can pop the cycles in that order. Let $C' \neq C_1$ be any other cycle that can be popped at the beginning. Then we will show that either $C' = C$ or else C can still be popped via a sequence that begins with C' . This suffices, since it implies that any alternative choice of popping sequence either preserves the infinite number of poppings or preserves the (finite) set of cycles to be popped.

If C' is vertex-disjoint from $C_1 \cup \dots \cup C_k$, then popping it first will not affect any of the stacks in the sequence $C_1, \dots, C_k$, so we will still be able to pop that sequence. Therefore, let $1 \leq m \leq k$ be the first index for which $C' \cap C_m \neq \emptyset$. Let x be some vertex shared by C' and C_m . Then since C'

can be popped at the first stage, all its edges must have color 1. Since x was not in the sequence $C_1, \ldots, C_{m-1}$, and since C_m can be popped after all of those, the color on the edge leaving x in C_m must also be 1, since x was never popped. Thus, the successor of x in C_m is the same as its successor in C'. Applying this same argument to the successor of x, and then to its successor, and so on, implies that $C' = C_m$. But then C_m is vertex-disjoint from $C_1, \ldots, C_{m-1}$, so either $C' = C$ or else we can pop these cycles in the order $C_m, C_1, \ldots, C_{m-1}, C_{m+1}, \ldots, C_k$. □

Proof of Proposition 10.1. The construction of $\mathsf{T}(S)$ corresponds to a method for popping the stacks in some order (namely loop-erasing the random walks). So Wilson's Algorithm will produce the same spanning tree distribution as any other method of popping the stacks, since we just proved that the specific method doesn't matter.

Since Wilson's Algorithm terminates in finite time a.s., we see that we only need to pop finitely many cycles. Thus, we can think of our collection of stacks as a union of finitely many colored cycles O sitting on top of a spanning tree T, and below it, all the unused $S_v(i)$ realizations, which we'll never see since we terminate the process once we hit T.

So, we have the map $S \mapsto (O, T)$ and that T is obtained at a "hitting time" for the popping cycle dynamics. So that T is independent of O. Since any T is equally likely, the distribution of the final spanning tree is uniform. □

For any given root r, $(T(S), r)$ has distribution

$$\frac{1}{Z} \prod_{v \neq r} \frac{1}{\operatorname{degr}(v)} \mathbf{1}\{\mathsf{G}(s) \text{ is a tree with root } r\}$$

where s is a realization of $(S_v(1))_{v \in V}$, $\mathsf{G}(s)$ is the graph obtained with the arrows s and Z is normalization (sum over s of the above product).

Hence, to get a uniform tree, we need to weight the r's with its degree. Check.

Uniform spanning trees in Z^d See Pemantle [40].

10.2 Minimal Euclidean spanning trees

Let X be a homogeneous Poisson process in $\mathbb{R}^d$ with intensity 1.

Let Λ be a finite box and consider the set of points $V = \mathsf{X} \cap \Lambda$.

Let (V, E) be a graph. The *cost* of the graph is

$$\mathrm{cost}(V, E) = \sum_{e \in E} \|e\|. \tag{10.1}$$

where $\|(x, y)\|$ is the length of $x - y$. The *Euclidean minimal spanning tree* of V is a spanning tree $\mathsf{T} = (V, E)$ where

$$\mathrm{cost}(V, E) = \min_{E' \in \mathcal{E}} \mathrm{cost}(V, E') \tag{10.2}$$

where $\mathcal{E}'$ is the set of edge configurations E' satisfying (V, E') is a spanning tree of V.

References Aldous Steele [1], Alexander [2].

Kesten Lee algorithm for MST for weighted graphs From [30] [7]. This algorithm is also called *Add and delete algorithm.* It is an iterative algorithm for constructing an MST on a connected graph starting from an MST on a connected subgraph. It can be applied to the EMST of Poisson process with weight $w(e) = \|e\|$.

(i) Addition of an edge: Suppose $G_1 = (V, E_1, w)$ is a finite connected weighted graph and $G_0 = (V, E_0, w)$ is a connected subgraph of G_1 such that $E_1 = E_0 \cup e_0$, that is, G_1 has the same vertex set and one extra edge e_0. Suppose T_0 is an MST on G_0 . Consider the graph $T_0 \cup e_0$, that is, add the edge e_0 to T_0. Then $T_0 \cup e_0$ has a unique cycle C. Let e be an edge in C such that $w(e) = \max_{e' \in C} w(e')$, and set $T_1 = T_0 \cup e_0 \setminus e$. Thus, we are removing an edge in C that has the maximal edge-weight in C.

(ii) Addition of a vertex: Suppose $G_1 = (V_1, E_1, w)$ is a finite connected weighted graph and $G_0 = (V_0, E_0, w)$ is a connected subgraph of G_1 such that $V_1 = V_0 \cup v_0$ and $E_1 = E_0 \cup e_0$. Thus G_1 has one extra vertex v_0 and one extra edge e_0. Since G_1 is connected, v_0 is necessarily an endpoint of e_0. Suppose T_0 is an MST on G_0. Set $T_1 = T_0 \cup e_0$.

Proposition 10.3. *(Proposition 2). The tree T_1 constructed in (i) or (ii) is an MST on G_1 .*

We can start from an MST on a connected graph and use the add and delete algorithm inductively to construct an MST on any larger finite connected graph.

Kruskal's greedy algorithm For the Poisson process $\mathsf{X} \cap \Lambda$. See Aldous and Steele [1].

1. Start at a point $x_0 \in V$.
2. Pick the closest point to the origin, call it x_1 and connect it with the edge $e_1 = (x_0, x_1)$. Denote $(\mathsf{X}_1, E_1) := (\{x_0, x_1\}, (x_0, x_1))$.
3. For $k < n$ proceed iteratively.

   Let x_{k+1} be the point in $\mathsf{X} \setminus \mathsf{X}_k$ closest to X_k. Let y_k be the point in X_k realizing the minimal distance to x_{k+1}. Let $e_{k+1} := (y_k, x_{k+1})$ and

$$\mathsf{X}_{k+1} = \mathsf{X}_k \cup \{x_{k+1}\} \qquad E_{k+1} = E_k \cup \{e_{k+1}\}. \tag{10.3}$$

Denote $E_n := \{e_1, \ldots, e_n\}$ the set of edges identified by the algorithm. The graph (X_n, E_n) is a MST denoted T_n of X_n.

For $x \in \mathsf{X}$, denote $\mathsf{T}_n(x, X)$ the tree obtained when the initial point is x instead of the origin.

Let $\mathsf{T}(x, \mathsf{X}) := \cup_n \mathsf{T}_n(x, X)$.

Lemma 10.4 (Aldous and Steele [1])**.** *Let $\mathsf{G} = \mathsf{G}(\mathsf{X})$ be the graph with vertices X defined by taking (x, y) as an edge in G if (x, y) is an edge in either $\mathsf{T}(x, \mathsf{X})$ or in $\mathsf{T}(y, \mathsf{X})$. Then the graph G is a forest and each component of G is an infinite tree.*

Aldous and Steele: "It is natural to conjecture that G is in fact a.s. a tree, but this seems to be related to deep issues in continuum percolation."

10.3 Spanning trees and Delaunay triangulations

Under construction.

Based on the book Computational Geometry [29].

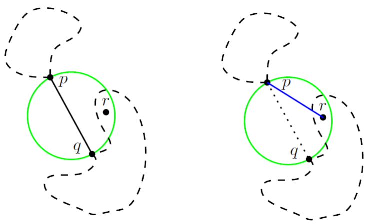
Lemma 10.5. *The Euclidean Minimum Spanning Tree does not have cycles (it really is a tree)*

Proof. Suppose G is the shortest connected graph and it has a cycle. Removing one edge from the cycle makes a new graph G0 that is still connected but which is shorter. Contradiction □

Lemma 10.6. *Every edge of the Euclidean Minimum Spanning Tree is an edge in the Delaunay graph*

Proof. Suppose T is an EMST with an edge e = pq that is not Delaunay Consider the circle C that has e as its diameter. Since e is not Delaunay, C must contain another point r in P (different from p and q)

Either the path in T from r to p passes through q, or vice versa. The cases are symmetric, so we can assume the former case



Pictures from [29]

□

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11 Exercises

1 Poisson random variables

Una variable aleatoria X es Poisson(μ) si $P(X = k) = e^{-\mu}\mu^k/k!$, para $k \geq 0$ entero.

1. Demuestre que si X es Poisson(μ), entonces $EX = \mu$, $E(X^2 - \mu) = \mu$.
2. Calcule la función generadora de momentos $E(z^X)$ para $0 < z < 1$.
3. Calcule $E(X!)$ para $\mu < 1$.
4. Calcule $\frac{dP(X=k)}{d\mu}$ y $\frac{d\sum_{k=0}^{n} P(X=k)}{d\mu}$.
5. Sea $X \sim$ Poisson(μ) e $Y \sim$ Poisson(λ), independientes. Calcule la distribución de $X + Y$.
6. Teorema de sumas numerables. Sean $\mu_1, \mu_2, \dots$ una sucesión de parámetros positivos y $X_1, X_2, \dots$ variables independientes con $X_i \sim$ Poisson(μ_i). Demuestre que si $\sigma = \sum_i \mu_i < \infty$, entonces
$$S := \sum_{i=1}^{\infty} X_i \tag{11.1}$$
converge casi seguramente a S y S es una variable Poisson(σ).

   Si $\sigma = \infty$, entonces S_n converge a ∞ casi seguramente.
7. En las condiciones del teorema anterior, demuestre que si $k_1 + \cdots + k_n = k$ (todos no negativos), entonces
$$P(X_1 = k_1, \dots, X_n = k_n \mid S_n = k) = \frac{k!}{k_1! \dots k_n!} \Big(\frac{\mu_1}{\mu}\Big)^{k_1} \dots \Big(\frac{\mu_n}{\mu}\Big)^{k_n} \tag{11.2}$$
donde $\mu := \mu_1 + \cdots + \mu_n$. Es decir, distribución Multinomial($k; \frac{\mu_1}{\mu}, \dots, \frac{\mu_n}{\mu}$).
8. Recíproca. Supongamos que $(X_{k,1}, \dots, X_{k,n}) \sim$ Multinomial($k; p_1, \dots, p_n$) y $K \sim$ Poisson(λ). Entonces
$$X_{K,1}, \dots, X_{K,n} \text{ son variables independientes con } X_{K,i} \sim \text{Poisson}(\lambda p_i). \tag{11.3}$$
9. Prove the Disjointness Lemma. Let S_1 and S_2 independent Poisson processes on R and A measurable with $\mu_1(A)$ and $\mu_2(A)$ finite. Then $P(S_1 \cap S_2 \cap A = \emptyset) = 1$.

2 Poisson processes

1. Construct a homogeneous Poisson process in $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+$ as a union of Poisson processes on B_i, a partition of $\mathbb{R}^d$ with $|B_i| < \infty$.

2. Let μ be a mean measure on $\mathbb{R}^d$ with intensity $\lambda(x)$ and $\hat{S}$ be a homogeneous Poisson process of intensity 1 on $\mathbb{R}^d \times \mathbb{R}_+$. Define

$$S = \{x \in \mathbb{R}^d : (x, y) \in \hat{S},\ 0 \leq y \leq \lambda(x)\} \tag{11.4}$$

Prove that S is a Poisson process on $\mathbb{R}^d$ with intensity $\lambda(\cdot)$.

3. Prove the restriction theorem: Let S be a Poisson process on the space R with mean measure μ and $B \subset R$. Then $S_B := S \cap B$ is a Poisson Process with mean measure $\mu_B(A) = \mu(A \cap B)$.

4. Prove the mapping theorem: Let S be a Poisson process on the space R with mean measure μ. Let $f : R \to T$ be a measurable function such that

$$\mu^*(B) := \mu(f^{-1}(B)) \tag{11.5}$$

has no atoms. Then $S^* := f(S)$ is a Poisson process on T with mean measure μ^*.

5. Let S be homogeneous Poisson process with mean measure $\mu(A) = |A|$. $\lambda : \mathbb{R} \to \mathbb{R}_+$ a positive intensity. Take $M(x) = \int_0^x \lambda(y)dy$. Show that

$$S^* = M^{-1}(S) \quad \text{is a Poisson process with intensity } \lambda.$$

6. Let S be a PP with constant intensity λ on $\mathbb{R}^2$. Let

$$f(x, y) = ((x^2 + y^2)^{1/2}, \tan^{-1}(y/x)), \tag{11.6}$$

Prove that $S^* = \{f(s) : s \in S\}$ is a Poisson process on $\{(r, \theta) : r > 0,\ 0 \leq \theta < 2\pi\}$ with intensity $\lambda^*(r, \theta) = \lambda r$.

7. Sea S un proceso de Poisson de intensidad constante λ en $\mathbb{R}^2$. Numere los puntos de S por orden de distancia al orígen. Calcule la distribución conjunta de (Y_1, Y_2), donde $Y_i =$ distancia del i-ésimo punto.

8. Assume Campbell formula holds for positive f. Prove it for all f by decomposing $f = f^+ - f^-$.

9. Let S be a Poisson process in $\mathbb{R}^d$ with intensity $\lambda(\cdot)$. Let $(X_s : s \in S)$ be a sequence of iid random variables. Show that the process

$$\tilde{S} := \{s + X_s : s \in S\} \tag{11.7}$$

is a Poisson process with absolutely continuous measure $\tilde{\mu}$. Compute the intensity of $\tilde{\mu}$.

Show that if $\lambda(x) \equiv \lambda$, then $\tilde{S} \overset{D}{=} S$.

10. Let $(X_i : i \in \mathbb{Z}^d)$ be a family of iid random variables with distribution $\mathcal{N}(0, \sigma^2 I)$, where I is the identity matrix. Let

$$S_\sigma := \{i + X_i : i \in \mathbb{Z}^d\} \tag{11.8}$$

Prove that as $\sigma \to \infty$, S_σ converges to a homogeneous Poisson process with intensity 1.

3 Poisson processes on $\mathbb{R}$

1. Let $S = \{X_i : i \in \mathbb{Z}\}$ be a homogeneous Poisson process in $\mathbb{R}$ with

$$\cdots < X_{-2} < X_{-1} < X_0 < 0 < X_1 < X_2 < \dots \tag{11.9}$$

Show that $(X_1, X_2, \dots)$ has the same distribution as $(-X_0, -X_{-1}, \dots)$ and that the two processes are independent.

2. Let S be a homogeneous Poisson process on $\mathbb{R}^d$. Show that

$$\lim_{|B| \nearrow \infty} \frac{N(B)}{|B|} = \lambda. \quad \text{a.s.} \tag{11.10}$$

3. Give a recipy to simulate a homogeneous Poisson process on $\mathbb{R}$ with rate λ, having as input a sequence of iid Uniform$[0, 1]$ random variables.

4. Give a recipy to simulate a non-homogeneous Poisson process on $\mathbb{R}$ with rate $(\lambda(t))_{t \in \mathbb{R}}$, having as input a sequence of iid Uniform$[0, 1]$ random variables.

5. Prove that for a PP(λ), X_n has law Gamma(n, λ).

6. Let $\lambda(t) = \lambda \sum_{m \in \mathbb{Z}} \mathbf{1}\{2m \le t < 2m + 1\}$. Choose a point X at random in $[-T, T]$ and consider the interval $[X, X + 1]$. Compute the asymptotic probability that $[X, X + 1]$ contains n points, as $T \to \infty$.

7. Points of a PP(λ) are distributed in a connected Borel subset of $\mathbb{R}^2$ with infinite area. Start from an arbitrary origin in the region and search the region in increasing concentric circles. Let A_k be the area searched before encountering the k-th point. Prove that A_1, $A_2 - A_1$, $A_3 - A_1$,... are iid exponential(λ) random variables. Generalize to $\mathbb{R}^d$.

8. Let $\lambda < 1$ and consider the independent processes A a PP(λ) and S a PP(1). Let $W : \mathbb{R} \to \mathbb{Z}$ and $Q : \mathbb{R} \to Z_{\geq 0}$ be defined by

$$W(t) - W(r) = \sum_{a \in A} \mathbf{1}\{r < a \leq t\} - \sum_{s \in S} \mathbf{1}\{r < s \leq t\} \tag{11.11}$$

$$Q(t) = W(t) - \min_{r<t} W(r). \tag{11.12}$$

Define

$$D := \{s \in S : Q(s-) > 0\}. \tag{11.13}$$

A arrivals. S services, D departures. Q is a stationary $M/M/1$ queue. Show that D is a PP(λ).

9. Denote $\pi_k(\mu) := \frac{e^{-\mu}\mu^k}{k!}$. Prove that

$$\sum_{k=0}^{n} \pi_k(\mu) = 1 - \int_0^{\mu} \pi_n(\lambda) d\lambda. \tag{11.14}$$

10. Let $(X_i)_{i\in\mathbb{Z}}$ be a Poisson process with rate λ in $\mathbb{R}$. Use (11.14) to show that $P(X_n \leq x) = \int_0^x \lambda\pi_{n-1}(\lambda u)du$. Conclude that X_n is a Gamma(n, λ) random variable.

4 Marked Processes and LLN

1. Prove Slivnyak-Mecke Theorem: Let S be a Poisson process with intensity $\lambda(\cdot)$ on $R = \mathbb{R}^d$. Let $h : (x_1, \dots, x_n; S) \mapsto h(x_1, \dots, x_n; S) \in \mathbb{R}$, where X is a denumerable set of R. Then

$$E\Big(\sum_{s_1,\dots,s_n \in S} h(s_1, \dots, s_n; S \setminus \{s_1, \dots, s_n\}) \Big) \tag{11.15}$$
$$= \int_R \cdots \int_R Eh(x_1, \dots, x_n; S)\lambda(x_1) \dots \lambda(x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

where the sum is over distinct $s_1, \dots, s_n$. See Moeller-Waagpetersen, Theorem 3.3.

2. Show the strong law of large numbers for the empirical process of an inhomogeneous Poisson process. S^ε is family of Poisson processes with mean measure $\varepsilon^{-1}\mu$, and denote N^ε its counting measure. The empirical measure of a Borel set A is the random variable

$$\pi^\varepsilon(A) := \varepsilon N^\varepsilon(A). \tag{11.16}$$

Show that

$$\lim_{\varepsilon\to 0} \pi^\varepsilon(A) = \mu(A). \quad \text{a.s.} \tag{11.17}$$

3. Show the coloring theorem.

4. Show that if the probability that if the arrivals to a testing facility is Poisson(λ) and the probability that each arriving person will be positive with probability α. What is the probability that there are 3 positive between the 5 first arrivals. What is the expected time until the arrival of the 4th positive tested person.

5. Under the conditions of the Marking Theorem. If $\mu(R) < \infty$, the points $\{m_s : s \in S\}$ form a Poisson process on M with mean measure μ_m given by

$$\mu_m(B) := \int_R \int_B \mu(\mathrm{d}x)\, p(x, \mathrm{d}m), \qquad B \subset M. \tag{11.18}$$

6. Under the conditions of the Marking Theorem. If m takes countable values denoted $1, 2, \dots$, the point process S_i with i-marks is $\mathrm{PP}(\mu_i)$, where

$$\mu_i(A) = \int_A \mu(\mathrm{d}x)\, p(x, \{m_i\}) \tag{11.19}$$

and $S_1, S_2, \dots$ are independent. Notice that the color distribution may depend on the position of the point x.

7. Optative. Prove the Boolean percolation theorem.

8. Ideal gas invariance. Show that if $f(q, v) = \lambda\, f_2(v)$ such that $\int |v|^2 f_2(v) dv < \infty$, then

$$S \text{ is a PP}(F) \text{ if and only if } T_t S \text{ is a PP}(F). \tag{11.20}$$

This is a process with homogeneous spatial intensity λ and with independent speeds, satisfying a second moment condition.

5 Cox processes

1. Compute the mean, variance and covariances of the Gravitational field of a Galaxy with density $\lambda(x)$ and mass density $\rho(x, m)$ satisfying

$$\int_{\mathbb{R}^3} m\lambda(x)\rho(x, m) dx\, dm < \infty.$$

2. Let S be a Cox process in $\mathbb{R}^d$ with random mean measure $\Lambda(x)dx$, where $\Lambda(x) \equiv L$ and L is a random variable on $\mathbb{R}_+$ with distribution $f(\ell)\, d\ell$. Compute the correlations $\phi(x, y)$ for distinct points $x, y \in \mathbb{R}^d$. The definition of ϕ is given by: for disjoint bounded $A, B \subset \mathbb{R}^d$, we can express $E(N(A)\, N(B)) = \int_A \int_B \phi(x, y) dx\, dy$.

3. Compute the n-point correlations of the Neyman-Scott process.

4. Matern cluster process. Compute the correlation functions for the Neyman-Scott process with kernel $k(x) = \mathbf{1}\{\|x\| \leq r\}/|B(0, r)|$; that is, uniform in the ball $B(0, r)$. Compute the density λ of the process and the 2-point correlations.

5. Thomas cluster process. Compute the correlation functions for the Neyman-Scott process with kernel $k(x) = \exp\{-\|x\|^2/2w^2\}\, (2\pi w^2)^{d/2}$; that is, Normal$(0, w^2 I)$. Compute the density λ of the process and the 2-point correlations.

6 Percolation and stochastic domination

1. Theorem percolation phase transition. Prove that $\theta(z, p; \mathbb{R}^d)$ is increasing as function of z and $p(\cdot)$.

2. Theorem percolation phase transition. Prove that $\theta(z, p; \mathbb{R}^d) = 0$ when $z \int p(x) dx$ is sufficiently small. Use a branching argument. Notice that the argument of Grimmett for discrete percolation using self avoiding walks does not work here. Why?

3. Stochastic domination. Let X, Y be random variables taking values in $\mathbb{R}$. Prove that X is stochastically dominated by Y if and only if for all nondecreasing $f : \mathbb{R} \to \mathbb{R}$ we have

$$Ef(X) \leq Ef(Y). \tag{11.21}$$

4. Show that the Papangelou conditional intensity is $\lambda(x)$ for a Poisson process of intensity $\lambda(\cdot)$ in $\mathbb{R}^d$.

5. Compute the Papangelou conditional intensity for a Cox process in $\mathbb{R}^d$.

6. Show that if X_i are Poisson processes with intensities z_i, $z_1 \le z_2$, then X_1 is stochastically dominated by X_2.

7. Show that in the Ising model, the measure $G_\Lambda(\cdot|\xi_{\Lambda^c})$ has Papangelou intensity

$$\gamma(x|X^+,X^-) = \left\{ \begin{array}{ll} z\exp[-\sum_{y\in X^-} J(x-y)] & \text{if } x\in\Lambda^+ \\ z\exp[-\sum_{y\in X^+} J(x-y)] & \text{if } x\in\Lambda^- \end{array} \right\} \le z.$$

8. Use Holley Preston inequality to show that the Ising model measure G_Λ is stochastically dominated by a Poisson process.

9. Show that if P is a limit of $G_\Lambda(\cdot|\xi_{\Lambda^c})$, as $\Lambda \nearrow \mathbb{R}^d$, then P a Gibbs measure for those specifications. Hint: Use (4.32).

10. Let π^z be a Poisson process with constant intensity z and π^{z_Λ} be the superposition of π^z and π^z_Λ, a Poisson process with rate z in Λ. Show that π^{z_Λ} is a Poisson process with non-homogeneous intensity $z_\Lambda(x) = z(1+\mathbf{1}\{x\in\Lambda\})$ and the following identity holds:

$$\pi^{z_\Lambda}(dY) = \frac{1}{Z} 2^{\#Y_\Lambda} \pi^z(dY) \tag{11.22}$$

where Z is the normalization.

7 Jump Markov processes

1. Show that for the counting measure $N(t)$ of a one-dimensional homogeneous Poisson process satisfies $P(N(t+h)-N(t)\ge 2) = o(h)$.

2. *Fila con un servidor y espacio limitado de espera.* Construya un proceso Markoviano de saltos X_t en $\mathbb{X} = \{0,1,2\}$, donde X_t es el número de clientes en el sistema en el instante t. Los clientes llegan a tasa λ y los servicios son exponenciales a tasa μ. Los clientes que llegan cuando el sistema está saturado con dos clientes, se retiran. Las tasas son:

$$q(0,1) = q(1,2) = \lambda \tag{11.23}$$
$$q(1,0) = \mu; \quad q(2,1) = 2\mu \tag{11.24}$$
$$q(x,y) = 0, \text{ en los otros casos.} \tag{11.25}$$

3. La fila M/M/1 es el proceso X_t en $\{0,1,2,\dots\}$ con tasas $\lambda,\mu\in\mathbb{R}_{\ge 0}$ dadas para $x\ge 0$ por

$$q(x,x+1) = \lambda, \tag{11.26}$$

$$q(x+1,x) = \mu. \quad (11.27)$$

Construya el proceso usando dos procesos de Poisson homogeneos independientes A y S, de intensidades λ y μ, respectivamente.

4. * Demuestre las ecuaciones de Kolmogorov.

5. De condiciones en λ y μ para que la fila M/M/1 tenga una medida invariante. Calcule esa medida.

6. Demuestre el Teorema 2.4.

7. * Demuestre que el proceso Z_t definido en (2.34) es de Markov con matriz Q. Demuestre que es estacionario.

8. Calcule la distribución de $Z_{\tau(t)}$. Demuestre que si $\tau(t) \neq \tau(t')$, entonces $Z_{\tau(t)}$ y $Z_{\tau(t')}$ son independientes.

9. Demuestre que si $s < \tau(t)$, entonces $X_t^x = Z_t$ para todo x.

10. Demuestre el teorema de convergencia al equilibrio 2.6 en el caso que $\gamma(Q) > 0$ (recurrencia de Harris).

11. Demuestre el corolario 2.8.

12. * Simule manualmente el proceso Z_t para la matriz (2.35). Para un t dado, diga el valor de $\tau(t)$ y $u(t)$.

8 Birth death evolution of point processes

1. Show that the process $\tilde{X}_t$ is a Poisson process of intensity w. Complete the details in the proof of Proposition 5.1 and extend the proof to all $t \in \mathbb{R}$.

2. Prove that the operator $\tilde{L}$ defined in (5.5) satisfies the Kolmogorov equations (5.6) for the semigroup (5.4).

3. * For each bounded box $\Lambda \subset \mathbb{X}$, find random times $\{\tau_j(\mathbf{C}) \in \mathbb{R} : j \in \mathbb{Z}\}$, such that

   (a) $\tau_j \to \pm\infty$ for $j \to \pm\infty$ and

   (b) There is no $C \in \mathbf{C}^\Lambda$ alive at time τ_j, for all $j \in \mathbb{Z}$.

4. * Show that the Papangelou intensity of μ^Λ defined in (5.12) is given by

$$\gamma^\Lambda(x|X) = w(x)\,\mathbf{1}\{X \cup \{x\} \in \mathcal{A}^\Lambda\}, \quad x \in \Lambda. \quad (11.28)$$

5. Show that the process $(X_t^\Lambda)_{t\geq 0}$ defined in (5.16) is Markov with generator (5.17).

6. Prove Lemma 5.10.

7. Prove Theorem 5.6 using Theorem 5.11.

9 Ideal gas and hard rods

1. * Find a locally finite configuration X such that $T_t\mathsf{X}$ is not locally finite, for some t.

2. Show that $\mathcal{T}_t$ conserves $\mathcal{F}$.

3. Assume X is a random point process on $\mathcal{X}$. If X has mean measure F with density $f \in \mathcal{F}$, then $T_t\mathsf{X}$ has mean measure $F \circ T^{-1}$ with density $\mathcal{T}_t f$.

4. * Assume the conditions of Lemma 6.17. Show that

$$\int \varphi h = \iiint \ell\varphi(D_{f,0}(q), v, \ell)\, f(q, v, \ell)\, dq dv d\ell \tag{11.29}$$

10 Poisson line process

1. Prove that the uniform Poisson line process in $\mathbb{R}^2$ is translation, rotation and reflection invariant.

2. Prove that the distribution of angles of the lines of L crossing the line (p_0, θ) is given by (7.7). Complete the details of Lemma 7.2.

3. * Find the distribution of an ideal gas Poisson process X such that the map $(q, v, l) \mapsto \ell = (q + vt)_{t\in\mathbb{R}}$ producs an isometry invariant Poisson line process L.

4. Given a convex set $D \subset \mathbb{R}^2$ give an expression for the set $\phi(L(D))$, where $\phi(\ell) = (p, \alpha)$ and $L(D)$ is the set of lines intersecting D.

5. Prove the perimeter Lemma.

6. * Compute the distribution of the number of lines that cross $B(0, r)$.

7. * Compute the distribution of the number of lines that cross a square.

8. Compute the distribution of the number of lines that cross a convex set.

9. How would you define a Poisson process of planes in $\mathbb{R}^3$?

10. How would you define a Poisson process of lines in $\mathbb{R}^3$?

11 Bose

1. Give an algorithm to simulate a Gaussian loop γ of length 3 and law

$$Q_\lambda(d[x_1,\dots,x_k]) := \tfrac{\lambda^k}{k}\left(\tfrac{\alpha}{\pi}\right)^{kd/2} e^{-\alpha\sum_{i=1}^k \|x_i-\gamma(x_i)\|^2}\, dx_1\dots dx_k.$$

with $x_1 =$ origin.

2. Choose a density $\rho > 0$ and simulate a loop soup in dimension $d = 2$, intersecting a bounded box $\Lambda \subset \mathbb{R}^2$, at point density ρ.

3. Prove that $Q_\lambda(n_A\, n_B) = \int_A \int_B K_{ab}K_{ba}\, da\, db$. See Proposition 8.6.

4. Prove that for any test function $g : \mathbb{W} \to \mathbb{R}$ invariant by time shifts $(\theta g = g)$ we have $\int_{\mathbb{R}^d} E^x g\, dx = \int_{\mathbb{R}^d} E^x[\theta g]\, dx$.

5. Simulate a sample of the random Gaussian interlacements intersecting a box Λ.

12 Factor graphs

1. Show that for an isometry invariant point process, the volume of the Voronoi cell containing some fixed point x in $\mathbb{R}^d$ is a.s. finite.

13 Spanning trees

1. Perfom a simulation of Wilson algorithm for a graph of 6 points.

2. Perform the stack construction for the same tree of the previous exercise. Show in the example that the loop popping pops the same colored loops for any order of popping.

3. Prove that O is independent of T in the proof of Proposition 10.1.

4. Prove that Kruskal algorithm produce a minimal spanning tree in the set of points $\mathsf{X} \cap \Lambda$.