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Procesos puntuales

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Examples

A point process is a random countable subset S of a measurable space R.

Times of arrival of a bus. $\mathbb{R}$

Location of earthquakes during last year. S^1 (sphere).

$\mathbb{Z}^d$ is a point process on $\mathbb{R}^d$.

$\{x + Y_x : x \in \mathbb{Z}^d,\, Y_x \text{ iid in } \mathbb{R}^d\}$.

$\{x \in \mathbb{Z}^d : Y_x = 1,\, Y_x \text{ iid in } \{0,1\}\}$ Binomial or Bernoulli process.

Point processes

From the book *Poisson processes* by J.F.C. Kingman. Oxford Studies in Probability. Claredon Press. 1993, Reprinted 2002.

We consider a space R and a subset of $\mathcal{P}(R)$ of *measurable* sets, satisfying

1) empty set is measurable.

2) complement of measurable set is measurable

3) countable union of measurable sets is measurable

This is a σ-field or σ-algebra.

We want that for a point process S and for all measurable $A \subset R$,

$$N_S(A) := \sum_{s\in S} \mathbf{1}\{s \in A\} \tag{1}$$

is a random variable and want that the points in R are measurable. We ask that the *diagonal* $\{(x, y) \in R \times R : x = y\}$ is measurable. This implies $\{x\}$ is measurable for all $x \in R$.

We can think N_S as a random counting measure on R.

Usually $R = \mathbb{R}^d$ and the σ-algebra as the Borel sets. The diagonal in $\mathbb{R}^d \times \mathbb{R}^d$ is closed, so measurable. Later we consider Poisson processes of straight lines and of trajectories of random walks.

A *point process* is a random countable subset S of R.

1 Poisson process

Want a point process S such that for $N = N_S$:

(1) for disjoint $A_1, A_2, \dots, A_n$ we have $N(A_i)$ are independent.

(2) $N(A) \sim \text{Poisson}(\mu(A))$ where $\mu : A \mapsto \mu(A)$ satisfying $0 \le \mu(A) \le \infty$ is a measure on R called *mean measure*.

If N satisfies (1) and (2) we have

$$EN(A) = \mu(A) \tag{2}$$

and if A_i is a partition of A with $\mu(A_i) < \infty$, then

$$\sum_i N(A_i) = N(A) \tag{3}$$

and

$$\sum_i EN(A_i) = EN(A), \text{ that is, } \sum_i \mu(A_i) = \mu(A) \tag{4}$$

so that μ is a measure on R.

Joint distribution of $N(A_i) : i = 1, \dots, n$: Consider the family

$$\{B = A_1^* \cap \dots \cap A_i^* : A_i^* \in \{A, A^c\}\} \tag{5}$$

Sets in this family are disjoint. Denote $B_1, \dots, B_{2^n}$ the elements of that set. Then,

$$A_j = \cup_{i \in \gamma_j} B_i \tag{6}$$

and

$$N(A_j) = \sum_{i \in \gamma_j} N(B_i) \tag{7}$$

By (2) $N(B_i)$ are Poisson($\mu(B_i)$) and since B_i are disjoint, by (1), B_i are independent. So we can write the joint distribution of A_i:

$$(N(A_1), \dots, N(A_n)) = (\sum_{i \in \gamma_1} N(B_1), \dots, \sum_{i \in \gamma_n} N(B_n)). \tag{8}$$

For instance, for $n = 2$, we want the joint distribution of (A_1, A_2):

$$B_1 = A_1 \cap A_2, \quad B_2 = A_1 \cap A_2^c, \quad B_3 = A_1^c \cap A_2, \quad B_3 = A_1^c \cap A_2^c$$

and

$$A_1 = B_1 \cup B_2, \quad A_2 = B_1 \cup B_3. \tag{9}$$

The expectation of the product is given by

$$E(N(A_1)\, N(A_2)) = \dots = \mu(A_1)\mu(A_2) + \mu(A_1 \cap A_2). \tag{10}$$

And the covariance is given by

$$E(N(A_1)\, N(A_2)) - EN(A_1)\, EN(A_2) = \mu(A_1 \cap A_2). \tag{11}$$

Exercise. Show that for a Poisson process the mean measure μ must be nonatomic: $\mu(\{x\}) = 0$ for all x.

Intensity. If μ is absolutely continuous with respect to Lebesgue,

$$\mu(A) = \int_A \lambda(x) \mathrm{d}x \tag{12}$$

where $\mathrm{d}x = \mathrm{d}x_1 \dots \mathrm{d}x_n$. $\lambda : \mathbb{R}^d \to \mathbb{R}_+$ is called the *intensity measure.*

If λ is continuous at x and $A \ni x$,

$$\mu(A) \approx \lambda(x)|A| \tag{13}$$

When $\lambda(x) \equiv \lambda \in \mathbb{R}_+$, we say that the Poisson process is *homogeneous.*

Superposition Theorem

Theorem 1.1 (Superposition of Poisson processes)**.**

Let $S_1, S_2, \dots$ be Poisson processes with mean measures $\mu_1, \mu_2, \dots$ respectively. Then $S := \cup_n S_n$ is a Poisson process with mean measure $\mu := \sum_n \mu_n$.

Proof. Let $N_n(A) :=$ number of points in A for the Poisson process S_n (with intensity μ_n). Then, by the countably additivity theorem,

$$N(A) = \sum_n N_n(A) \tag{14}$$

has distribution Poisson($\sum_n \mu_n$). If $\mu_n(A) = \infty$ for some n, then $N_n(A) = N(A) = \infty$.

To show independence of $N(A_i)$ for disjoint A_i, it suffices to observe that de double array of variables $N_n(A_i)$ arfe independent. ☐

Theorem 1.2 (Restriction Theorem). *Let S be a Poisson process on the space R with mean measure μ and $B \subset R$. Then $S_B := S \cap B$ is a Poisson Process with mean measure $\mu_B(A) = \mu(A \cap B)$.*

Proof. Exercise. ☐

Theorem 1.3 (Mapping). *Let S be a Poisson process on the space R with mean measure μ. Let $f : R \to T$ be a measurable function such that*

$$\mu^*(B) := \mu(f^{-1}(B)) \tag{15}$$

has no atoms. Then $S^* := f(S)$ is a Poisson process on T with mean measure μ^*.

Proof. Exercise. ☐

Example. S homogeneous Poisson process with mean measure $\mu(A) = |A|$. $\lambda : \mathbb{R} \to \mathbb{R}_+$ an intensity. Take $f = \lambda^{-1}$. Then

$$S^* = f(S) \quad \text{is a Poisson process with intensity } \lambda. \tag{16}$$

Projection of Poisson process Let μ be a mean measure on $\mathbb{R}^d$ with intensity $\lambda(x)$ and $\hat{S}$ be a homogeneous Poisson process of intensity 1 on $\mathbb{R}^d \times \mathbb{R}_+$. Define

$$S = \{x \in \mathbb{R}^d : (x, y) \in \hat{S},\ 0 \le y \le \lambda(x)\} \tag{17}$$

Then S is a Poisson process on $\mathbb{R}^d$ with intensity $\lambda(\cdot)$.

Proof. For bounded $A \subset \mathbb{R}^d$, let

$$\hat{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 \le y \le \tilde{\lambda}(x)\}.$$

Then

$$N(A) = \hat{N}(\hat{A}) \tag{18}$$

Easy to check the properties now. Have to use the disjointness lemma. ☐

Polar coordinates Let S be a PP with constant intensity λ on $\mathbb{R}^2$. Let

$$f(x, y) = ((x^2 + y^2)^{1/2}, \tan^{-1}(y/x)), \tag{19}$$

and

$$\mu^*(B) = \int\int_{f^{-1}(B)} \lambda \mathrm{d}x\mathrm{d}u = \int\int_B \lambda\, r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{20}$$

Then (r, θ) form a Poisson process in the strip $\{(r, \theta) : r > 0,\ 0 \le \theta < 2\pi\}$ with rate function $\lambda^*(r, \theta) = \lambda r$. The r-projection gives a Poisson process S^1 in $\mathbb{R}$ with intensity

$$\lambda^1(r) = \int_0^{2\pi} \lambda^*(r, \theta)\mathrm{d}\theta = 2\pi\lambda r. \tag{21}$$

1.1 The Bernoulli process

Let S be a Poisson process with measure μ on R and condition S to have n points. What is the distribution of those points? Let $A_1, \dots, A_n$ be disjoint subsets of R, then

$$\begin{aligned} &P(N_S(A_1) = n_1, \dots, N_S(A_k) = n_k \mid N_S(R) = n) \\ &= \frac{\prod_{j=1}^{k} e^{-\mu(A_j)}(\mu(A_j))^{n_j}(n_j!)^{-1}}{e^{-\mu(R)}(\mu(R))^n(n!)^{-1}} \\ &= \frac{n!}{n_1! \dots n_k!}\Big(\frac{\mu(A_0)}{\mu(R)}\Big)^{n_0} \dots \Big(\frac{\mu(A_k)}{\mu(R)}\Big)^{n_k} \end{aligned}$$

where $n_0 = n - (n_1 + \dots + n_k)$ and $A_0 = R \setminus (A_1 \cup \dots \cup A_k)$.

Definition 1.4. *A random process S on R with n points and such that*

$$P(N_S(A_1) = n_1, \dots, N_S(A_k) = n_k) = \frac{n!}{n_1! \dots n_k!}\Big(\tfrac{\mu(A_0)}{\mu(R)}\Big)^{n_0} \dots \Big(\tfrac{\mu(A_k)}{\mu(R)}\Big)^{n_k}.$$

is called Bernoulli process *with parameters n and $p(A) = \frac{\mu(A)}{\mu(R)}$.*

Notice that p is a probability on R.

Construction of a Bernoulli process. Take $X_1, \dots, X_n$ iid random variables with values on R and distribution p:

$$P(X_i \in A) = p(A).$$

Then $S := \{X_1, \dots, X_n\}$ (as a set) is a Bernoulli process with counting measure

$$N_S(A) = \sum_{i=1}^{n} \mathbf{1}\{X_i \in A\}. \tag{22}$$

Existence theorem Let μ be a non-atomic measure on R that can be decomposed as

$$\mu = \sum_n \mu_n \tag{23}$$

where $\mu_n(R) < \infty$ for all n. Define N_n and $X_{n,j}$, $j \ge 1$ independent random variables with

$$N_n \sim \text{Poisson}(\mu_n(R)), \quad X_{n,j} \sim p_n \tag{24}$$

where $p_n(A) = \frac{\mu_n(A)}{\mu_n(R)}$.

Theorem 1.5 (Existence Theorem)**.** *The process*

$$S := \cup_n \{X_{n,1}, \dots, X_{n,N_n}\}$$

is a Poisson process with mean measure μ.

Proof. First we prove that the process $S_n := \{X_{n,1}, \dots, X_{n,N_n}\}$ is a Poisson process; this is a random number of points on R with distribution p_n. Observe that, given $N_n = m$, the process is Bernoulli:

$$\begin{aligned} &P(N_n(A_1) = m_1, \dots, N_n(A_k) = m_k \mid N_n = m) \\ &= \frac{m!}{m_0! \dots m_k!}\Big(\tfrac{\mu_n(A_0)}{\mu_n(R)}\Big)^{m_0} \dots \Big(\tfrac{\mu_n(A_k)}{\mu_n(R)}\Big)^{m_k}. \end{aligned} \tag{25}$$

where $m_0 = m - (m_1 + \dots + m_k)$ and $A_0 = R \setminus (A_1 \cup \dots \cup A_k)$. Multiplying both terms by $P(N_n(R) = m) = e^{-\mu_n(R)}(\mu_n(R))^m/m!$ and summing over m, we get

$$P(N_n(A_1) = m_1, \dots, N_n(A_k) = m_k)$$

$$
\begin{aligned}
&= \sum_{m\geq 0} \frac{e^{-\mu_n(R)}(\mu_n(R))^m}{m!}\frac{m!}{m_0!\dots m_k!}\left(\tfrac{\mu_n(A_0)}{\mu_n(R)}\right)^{m_0}\cdots\left(\tfrac{\mu_n(A_k)}{\mu_n(R)}\right)^{m_k} \\
&= \sum_{m\geq 0} \frac{e^{-\mu_n(A_0)}(\mu_n(A_0))^{m_0}}{m_0!}\frac{e^{-\mu_n(A_1)}(\mu_n(A_1))^{m_1}}{m_1!}\cdots\frac{e^{-\mu_n(A_k)}(\mu_n(A_k))^{m_k}}{m_k!} \\
&= \frac{e^{-\mu_n(A_1)}(\mu_n(A_1))^{m_1}}{m_1!}\cdots\frac{e^{-\mu_n(A_k)}(\mu_n(A_k))^{m_k}}{m_k!}
\end{aligned}
$$

Hence, the process $S_n := \{X_{n,1},\dots,X_{n,N_n}\}$ is a Poisson process. Since S_n are independent, the superposition Theorem implies that S is a Poisson process with mean measure μ. $\square$

1.2 Campbell Theorem

Want to study variables of the form

$$\Sigma := \sum_{s\in S} f(s) \tag{26}$$

Examples. (a) Radioactiviry. Suppose each point of $S\subset\mathbb{R}$ produces an effect that decays exponentially. The cumulated effect for a site $x\in\mathbb{R}$ can be computed with Σ with the function $f_x(s)=\exp(x-s)\mathbf{1}\{s<x\}$.

(b) The gravitational field in $\mathbb{R}^3$: assuming all stars $s\in S$ have the same mass, $f_x(s)=\frac{1}{\|x-s\|}$.

Theorem 1.6 (Campbell Theorem)**.** *Let S be a Poisson process on R with mean measure μ. Let $f: R\to\mathbb{R}$ be measurable. Then, the sum*

$$\Sigma := \sum_{s\in S} f(s) \tag{27}$$

is absolutely convergent with probability one if and only if

$$\int_R \min(|f(x)|,1)\mu(\mathrm{d}x) < \infty. \tag{28}$$

Under this condition,

$$E(e^{\theta\Sigma}) = \exp\Big\{\int_R \big(e^{\theta f(x)}-1\big)\mu(\mathrm{d}x)\Big\}. \tag{29}$$

for any θ complex when the integral on the right converges or for purely imaginary θ. Moreover,

$$E\Sigma = \int_R f(x)\mu(\mathrm{d}x) \tag{30}$$

meaning that the expectation exists if and only if the integral converges, in which case they are equal. If (30) *converges, then*

$$V\Sigma = \int_R (f(x))^2\mu(\mathrm{d}x). \tag{31}$$

where V is variance.

Proof. We prove first for *simple functions*, that is, a function that takes only a finite number of values and vanishes outside a set of finite μ measure. Let $A_1,\dots,A_k$ be disjoint measurable subsets of R with $m_j := \mu(A_j) < \infty$ and let $f(x)=a_j$ for $x\in A_j$, with $f(x)=0$ if $x\notin\cup_j A_j$. We have then that $N_j := N_S(A_j)$ are independent with law Poisson(m_j) and

$$\Sigma = \sum_{x\in S} f(x) = \sum_j a_j N_j. \tag{32}$$

Since we know the characteristic function of the Poisson random variable, for real or complex θ we have

$$Ee^{\theta\Sigma} = Ee^{\theta\sum_j a_j N_j} = \prod_j Ee^{\theta a_j N_j} \quad \text{(independence of PP)}$$

$$
\begin{aligned}
&= \prod_j \exp\left(e^{(\theta a_j - 1)} m_j\right) \quad \text{(characteristic of Poisson rv)} \\
&= \exp\Big(\sum_j \int_{A_j} (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)\Big) \\
&= \exp\Big(\int_R (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)\Big) \qquad (33)
\end{aligned}
$$

If $f(x) \geq 0$, it is better to consider $\theta = -u$ for real $u \geq 0$. In this case,

$$
Ee^{-u\Sigma} = \exp\Big(\int_R (e^{-uf(x)} - 1)\mu(\mathrm{d}x)\Big)
$$

We know that any non-negative function f can be expressed as the limit $f = \lim_j f_j$ where f_j is an increasing family of simple functions, that is, $f_i \leq f_{i+1}$. Taking $\theta = -u$ for real $u \geq 0$, and setting $\Sigma_j = \sum_{s \in S} f_j(s)$, we have

$$
\begin{aligned}
E(e^{-u\Sigma}) &= \lim_j E(e^{-u\Sigma_j}) \\
&= \lim_j \exp\Big(\int_R (e^{-uf_j(x)} - 1)\mu(\mathrm{d}x)\Big)) \\
&= \exp\Big(\int_R (e^{-uf(x)} - 1)\mu(\mathrm{d}x)\Big). \qquad (34)
\end{aligned}
$$

by monotone convergence theorem. Under condition (28) the integral above converges and goes to zero as $u \to 0$. This implies Σ is a random variable; otherwise, if $\Sigma = \infty$ with positive probability, $E(e^{-u\Sigma}) \equiv 1$.

Expanding (33) and denoting $\mu f := \int f(x)\mu(\mathrm{d}x)$, we get

$$
\int_R (e^{\theta f(x)} - 1)\mu(\mathrm{d}x) = \theta\mu f + \frac{\theta^2}{2!}\mu f^2 + \dots
$$

taking the exponential of both sides and expanding again,

$$
\exp(\int_R (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)) = 1 + \theta\mu f + \frac{\theta^2}{2!}((\mu f)^2 + \mu f^2) + \dots \qquad (35)
$$

On the other hand,

$$
Ee^{\theta\Sigma} = 1 + \theta E\Sigma + \frac{\theta^2}{2!}E\Sigma^2 + \dots \qquad (36)
$$

We conclude then

$$
E\Sigma = \mu f, \qquad V\Sigma = \mu f^2. \qquad (37)
$$

This proves the formula for positive f. We leave as an exercise to prove for all f. □

1.3 Slivnyak-Mecke Theorem

Theorem 1.7 (Slivnyak-Mecke)**.** *Let S be a Poisson process with intensity $\lambda(\cdot)$ on $R = \mathbb{R}^d$. Let $h : (x_1, \dots, x_n; X) \mapsto h(x_1, \dots, x_n; X) \in \mathbb{R}$, where X is a denumerable set of R. Then*

$$
\begin{aligned}
&E\Big(\sum_{s_1,\dots,s_n \in S} h(s_1, \dots, s_n; S \setminus \{s_1, \dots, s_n\})\Big) \\
&\quad = \int_R \cdots \int_R Eh(x_1, \dots, x_n; S)\lambda(x_1)\dots\lambda(x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n.
\end{aligned}
$$

where the sum is over distinct $s_1, \dots, s_n$.

See Møller-Waagpetersen [17], Theorem 3.3. for a proof.

1.4 The characteristic functional

Taking Campbell formula for $f \geq 0$ and $\theta = -1$ we get

$$Ee^{-\Sigma_f} = \exp\Big\{-\int_R (1 - e^{-f(x)})\mu(\mathrm{d}x)\Big\} \tag{38}$$

where $\Sigma_f := \sum_{s\in S} f(s)$.

Proposition 1.8 (Characterizing functional characterizes process)**.** *If S satisfies* (38) *for f non-negative and simple, then S is a Poisson process.*

Proof. Let $A_1, \dots, A_k$ disjoint with $m_i := \mu(A_i) < \infty$. Let $f(s) = \sum_i a_i \mathbf{1}\{s \in A_i\}$. Then $\Sigma_f = \sum_j a_j N(A_j)$ and

$$Ee^{-\Sigma_f} = \exp\Big\{-\sum_{j=1}^{k}(1 - e^{-a_j})m_j\Big\} \tag{39}$$

Hence, taking $z_j := e^{-a_j}$ we have

$$E(z_1^{N(A_1)} \dots z_k^{N(A_k)}) = \prod_j e^{m_j(z_j-1)}. \tag{40}$$

but this is the product of the moment generation functions of Poisson(m_j) random variables calculated at arbitray points $z_j \in (0,1)$. This implies that $N(A_j)$ are independent Poisson(m_j) random variables, which in turn implies S is a PP(μ). □

1.5 Avoidance functions characterize point processes

Given a point process S define the *avoidance function* $\alpha : \mathcal{F} \to [0,1]$ by

$$\alpha(A) := P(S \cap A = \emptyset) \tag{41}$$

Clearly, if S is a PP(μ), we have $\alpha(A) = e^{-\mu(A)}$. Avoidance function are sometimes called *void probabilities.*

For any $a \leq b \in \mathbb{R}^d$, define the *rectangle* $(a,b] \subset \mathbb{R}^d$ by

$$(a,b] := \{(x_1,\dots,x_d) \in \mathbb{R}^d : a_i < x_i \leq b_i, \text{ for all } i\}$$

Theorem 1.9 (Rényi Theorem)**.** *Let μ be a non-atomic measure on $\mathbb{R}^d$ with $\mu(A) < \infty$ if A is bounded. Let S be a point process on $\mathbb{R}^d$ such that for any finite union of rectangles A,*

$$P(S \cap A = \emptyset) = e^{-\mu(A)} \tag{42}$$

Then S is a Poisson process with mean measure μ.

Two ways of using the theorem. (1) if one knows that for finite union of rectangles A, $N_S(A)$ is Poisson$(\mu(A))$, we have (42) and the theorem implies S is PP(μ). Notice that here we have not assumed independence.

(2) if one knows that $\alpha(A) > 0$ for bounded A and we know that $\{S \cap A = \emptyset\}$ and $\{S \cap B = \emptyset\}$ are independent for disjoint A, B, then

$$\alpha(A \cup B) = P(\{S \cap A = \emptyset\} \cap \{S \cap B = \emptyset\}) = \alpha(A)\,\alpha(B) \tag{43}$$

so that $\mu(A) := -\log \alpha(A)$ is finitely additive and non-atomic and the Theorem implies that S is a Poisson process with mean measure μ. Here we have not assumed Poisson distribution for $N(A)$.

Sketch proof. A k-cube is a cube with $a_i = z_i 2^{-k}$, $b_i = (z_i + 1)2^{-k}$ for integers z_i. For each k, k-cubes are a partition of $\mathbb{R}^d$. For k-cubes $C_1, \dots, C_n$ we have

$$P(\cap_r \{S \cap C_r = \emptyset\}) = P(S \cap \cup_r C_r = \emptyset)$$

$$
\begin{aligned}
&= e^{-\mu(\cup_r C_r)} \quad \text{by hipothesis}\\
&= e^{-\sum_r \mu(C_r)} \quad (C_r \text{ disjoint and } \mu \text{ measure})\\
&= \prod_r e^{-\mu(C_r)}.
\end{aligned} \tag{44}
$$

Let G be open bounded set and

$$
\begin{aligned}
N_k(G) &:= \#\{k\text{-cubes } C \text{ contained in } G \text{ with } S \cap C_r \neq \emptyset\}\\
&= \sum_{C \subset G} (1 - \mathbf{1}\{S \cap C = \emptyset\})
\end{aligned} \tag{45}
$$

where the sum is over k-cubes. We have

$$
N(G) = \lim_k N_k(G) \tag{46}
$$

By (44), the events $\{S \cap C_r = \emptyset\}$ are independent, hence the generating function of the variable $N_k(G)$ is product of generating functions of Bernoulli:

$$
E(z^{N_k(G)}) = \prod_{C \subset G} (e^{-\mu(C)} + (1 - e^{-\mu(C)})\, z)
$$

in $|z| < 1$, where the product is over k-cubes. Last expression equals

$$
\begin{aligned}
&= \prod_{C \subset G} (z + (1-z)e^{-\mu(C)}) \approx \prod_{C \subset G} e^{-(1-z)\mu(C)}\\
&= e^{-(1-z)\sum_C \mu(C)} = e^{-(1-z)\mu(G)}
\end{aligned} \tag{47}
$$

where the $\approx$ is a bit technical, see Kingman. It is motivated by the expansion of the exponentials:

$$
\begin{aligned}
z + (1-z)e^{-\mu} &= z + (1-z)(1 - \mu + \mu^2/2 + \dots)\\
&= 1 - (1-z)\mu + (1-z)\mu^2/2 + \dots\\
e^{-(1-z)\mu} &= 1 - (1-z)\mu + ((1-z)\mu)^2/2 + \dots
\end{aligned} \tag{48}
$$

Identity (47) shows that $N(G)$ is a Poisson($\mu(G)$) random variable. The same argument works to show independence of $N(G_1), \dots, N(G_k)$ for mutually disjoint $G_1, \dots, G_k$. This shows that S is Poisson process with mean measure μ. □

An alternative proof can be obtained via coupling. See Section 5 of Chapter 1 of Thorisson.

Coupling Let U Uniform$[0,1]$,

$$
X = \mathbf{1}\{U \geq e^{-\mu}\} \tag{49}
$$

$$
Y = \sum_{j \geq 0} j \mathbf{1}\{U \in I_j\}, \quad \text{where } I_j = e^{-\mu}\Big[\frac{\mu^j}{j!}, \frac{\mu^{j+1}}{(j+1)!}\Big) \tag{50}
$$

This is a *coupling* between those variables. That is, a vector (X, Y) defined in the space where U is defined and with marginals

$X \sim$ Bernoulli($e^{-\mu}$) and $Y \sim$ Poisson(μ).

We have

$$
\begin{aligned}
P(X = 0) &= P(Y = 0) = P(U \leq e^{-\mu})\\
P(X = 1) &= P(U \in I_1 \cup I_2 \cup I_3 \cup \dots)\\
P(Y = 1) &= P(U \in I_1)
\end{aligned} \tag{51}
$$

Hence

$$
P(X \neq Y) = P(U \in I_2 \cup I_3 \cup \dots) \leq \frac{\mu^2}{2}. \tag{52}
$$

Proof of $N_k(G)$ converges to Poisson($\mu(G)$) via coupling.

For each k-cube C define

$X_C := \mathbf{1}\{N_S(C) \neq 0\}$ and $Y_C \sim$Poisson($\mu(C)$).

$$X := \sum_C X_C, \quad Y := \sum_C Y_C \tag{53}$$

where the sum is over k-cubes. By the coupling,

$$P(X \neq Y) \leq \sum_{C \subset G} P(X_C \neq Y_C) \leq \frac{1}{2} \sum_C \mu(C)^2 \tag{54}$$

If $\mu(C) = \int_{C \subset G} \lambda(x) dx$ y $\lambda(x) \leq M$, using $|C| = 2^{-dk}$

$$\sum_{C \subset G} \mu(C)^2 \leq M^2 \sum_{C \subset G} |C|^2 = M^2 \, |G| \, 2^{-dk}, \tag{55}$$

where the sum is over k-cubes. This implies $P(X \neq Y, \text{ i.v.}) = 0$. $\square$

2 Poisson processes on $\mathbb{R}$

S process on $\mathbb{R}$ with mean measure $\mu(A) = \lambda|A|$. Homogeneous Poisson process with *rate* λ. In $d = 1$ the intensity is called rate. We enumerate the points of S as follows: $S = \{X_i : i \in \mathbb{Z}\}$ with

$$\ldots < X_{-2} < X_{-1} < X_0 < 0 < X_1 < X_2 < \ldots \tag{56}$$

X_i are random variables, depending on N. For instance,

$$\{X_n \leq x\} = \{N(0, x) \geq n\}. \tag{57}$$

Notice that $(X_1, X_2, \ldots)$ has the same law as $(-X_0, -X_{-1}, \ldots)$.

Theorem 2.1 (Interval theorem)**.** *Let $S = \{X_i : i \in \mathbb{Z}\}$ be a PP(λ). The random variables $Y_1 = X_1$, $Y_n = X_n - X_{n-1}$ are independent with exponential(λ) distribution.*

Proof. Consider the process

$$S' := \{X_2 - X_1, X_3 - X_1, X_4 - X_1, \ldots\} \tag{58}$$

We want to show that X_1 is independent of S' and that S' is PP(λ). It suffices to show that for some f, the characteristic functionals

$$\Sigma = \sum_{n \geq 2} f(X_n), \qquad \Sigma' = \sum_{n \geq 2} f(X_n - X_1) \tag{59}$$

have the same distribution.

Denote $\xi_k = 2^{-k} \lceil 2^k X_1 \rceil$ the least integer multiple of 2^{-k} greater than X_1. Then ξ_k is a random variable converging to X_1: $\xi_k \searrow_k X_1$. We have

$$\Sigma' = \lim_k \Sigma^k, \qquad \Sigma^k := \sum_{n \geq 2} f(X_n - \xi_k), \tag{60}$$

Now for any z, x,

$$P(\Sigma^k \leq z, \, X_1 \leq x) = \sum_\ell P(\Sigma^k \leq z, \, X_1 \leq x, \, \xi_k = l2^{-k}). \tag{61}$$

When $\xi_k = 2^{-k}$, the points X_n in (le^{-k}, ∞) form a Poisson process and are independent of $\{X_1 < l2^{-k}\}$. Hence

$$\Sigma^k = \sum_{n \geq 2} f(X_n - l2^{-k}) \text{ has the same law as } \Sigma \tag{62}$$

and

$$P(\Sigma^k \leq z,\, X_1 \leq x,\, \xi_k = l2^{-k}) = P(\Sigma \leq z)\, P(X_1 \leq x,\, \xi_k = l2^{-k}) \tag{63}$$

Substituting in (63),

$$P(\Sigma^k \leq z,\, X_1 \leq x) = P(\Sigma \leq z)\, P(X_1 \leq x) = P(\Sigma \leq z)\, P(N(0,x] \geq 1) \tag{64}$$

Letting $k \to \infty$, Σ^k converges almost surely to Σ' and we are done.

Applying induction, we prove that $Y_1, \dots, Y_m$ are independent and independent of $S^m = \{X_{m+1} - X_m, X_{m+2} - X_m, \dots\}$. □

Waiting time paradox We saw that $X_{i+1} - X_i$, $i \geq 1$ are iid exponential (λ). However $X_1 - X_0$ has distribution Gamma$(2, \lambda)$. The density function of $X_1 - X_0$ is $g_2(x) = \lambda x g(x)$, where g is the density of the "typical" interval. This is a general result for stationary point processes.

2.1 Law of large numbers

From the interval theorem, the distribution of X_n is Gamma(n, λ); indeed it is the sum of n exponential(λ):

$$X_n = Y_1 + \cdots + Y_n. \tag{65}$$

Y_j iid exponential(λ).

Theorem 2.2 (law of large numbers)**.**

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\lambda}. \tag{66}$$

Proof. $EY_n = \frac{1}{\lambda}$ and $VY_n = \frac{1}{\lambda^2}$, hence by the lln for iid with finite mean and variance, we get (66). □

As a corollary we get a lln for $N(0,t]$. Since

$$N(0,t] \to_t \infty, \tag{67}$$

it follows that

$$\lim_t \frac{X_{N(0,t]}}{N(0,t]} = \frac{1}{\lambda}. \tag{68}$$

and since

$$X_{N(0,t]} \leq t < X_{N(0,t]+1} \tag{69}$$

we get

$$\lim_t \frac{t}{N(0,t]} = \frac{1}{\lambda}. \tag{70}$$

We conclude

$$\lim_t \frac{N(0,t]}{t} = \lambda. \qquad \text{a.s.} \tag{71}$$

We give another proof of (71) without using the interval theorem.

Theorem 2.3. *Let S be a PP(λ) on $(0,\infty)$. Then*

$$\lim_t \frac{N(0,t]}{t} = \lambda. \qquad \textit{a.s.} \tag{72}$$

Proof. First proof. Using Chevichev:

$$P(|\frac{1}{t}N(0,t] - \lambda t| > \varepsilon) \leq \frac{\lambda}{\varepsilon^2 t} \tag{73}$$

taking $t_k = k^2$:

$$\sum_k P(|\frac{1}{k^2}N(0,k^2] - \lambda t| > \varepsilon) \leq \sum_k \frac{\lambda}{\varepsilon^2 k^2} < \infty \tag{74}$$

Hence, by Borel Cantelli,

$$\lim_k \frac{1}{k^2}N(0,k^2] = \lambda, \quad \text{a.s.} \tag{75}$$

Taking $k = k(t) = \lfloor \sqrt{t} \rfloor$,

$$N(0,k^2] \leq N(0,t] \leq N(0,(k+1)^2] \tag{76}$$

which dividing by $(k+1)^2 > t \geq k^2$ implies

$$\frac{N(0,k^2]}{(k+1)^2} \leq \frac{N(0,t]}{t} \leq \frac{N(0,(k+1)^2]}{k^2} \tag{77}$$

Since $k^2/(k+1)^2$ converges to 1, we get the result.

Second proof. It suffices to see that $N_k := N(k,k+1]$ are iid and that

$$N(0,t] = \sum_{i=0}^{\lfloor t \rfloor} N_k + N(\lfloor t \rfloor, t - \lfloor t \rfloor] \tag{78}$$

and use the lln for iid. □

This extends to Poisson processes in general spaces. For instance, for homogeneous Poisson processes in $\mathbb{R}^d$, we can use the same proof to show that if $B(0,k)$ is the ball of center at the origin and radious k, we have

$$\lim_{k\to\infty} \frac{N(B(0,k))}{|B(0,k)|} = \lambda. \tag{79}$$

2.2 Non homogeneous processes in $\mathbb{R}$

Let S be a PP(μ) in $\mathbb{R}$ and define $M(0) = 0$ and for $r < t$,

$$M(t) - M(r) = \mu(r,t]. \tag{80}$$

μ is non-atomic if and only if M is continuous. We have seen that

$$\tilde{S} := \{M(s) : s \in S\} \tag{81}$$

is a homogeneous PP(1) on the interval $(M(-\infty), M(\infty))$.

Notice that the inter-point distances $X_{k+1} - X_k$ are not independent.

Let S_2 be a homogeneous PP(λ) in $\mathbb{R}^2$ and let $S = \{\|s\| : s \in S_2\} \subset \mathbb{R}_+$, the set of distances to the origin of the points in S_2. Then S is a Poisson process of density $\lambda(x) = 2\pi\lambda x$, so that

$$M(x) = \pi\lambda x^2. \tag{82}$$

and $\tilde{S} = M(S)$ is a Poisson process of rate 1 on $\mathbb{R}_+$. So that if $S = \{X_1, X_2, \dots\}$ with $X_i < X_{i+1}$, then $\tilde{S} = \{\tilde{X}_1, \tilde{X}_2, \dots\}$ with $\tilde{X}_i = \pi\lambda X_i^2$ is a PP(1).

Export law of large numbers Let $S = \{X_i : i \in Z\}$ be a non-homogeneous process with cumulated density $M(x)$. We know that $\tilde{S} = \{M(X_i) : i \in Z\}$ is a homogeneous Poisson Process in $(-M(-\infty), M(\infty))$ and that if $M(\infty) = \infty$, then

$$\lim_n \frac{\tilde{X}_n}{n} = 1; \quad \lim_t \frac{\tilde{N}(0,t]}{t} = 1 \tag{83}$$

Using

$$\tilde{N}(0, M(t)] = N(0,t] \tag{84}$$

we get

$$1 = \lim_t \frac{\tilde{N}(0, M(t)]}{M(t)} = \lim_t \frac{N(0,t]}{M(t)} \tag{85}$$

3 Marked Poisson processes

Theorem 3.1 (Coloring Theorem). *Let S be a PP(μ). To each point of S give a color chosen randomly between colors $\{1, \dots, k\}$ with probability p_i to choose color i. Let S_i be the set of points with color i. Then $S_1, \dots, S_k$ are independent Poisson processes with mean measure $\mu_i := p_i \mu$, respectively.*

Proof. Exercise. It comes from the conditional distribution of colors given $N(A) = n$. ☐

In general. Let S be a PP(μ) in R and for some space M, let $(m_s, s \in S)$ be a family of independent random variables in M satisfying that m_s may depend on s but it is independent of $S \setminus \{s\}$. Consider the random set

$$S^* = \{(s, m_s) : s \in S\} \tag{86}$$

To construct S^* we start with S and then we have a family of distributions $p(x, \cdot)$ on M, so that for each $s \in S$ we choose m_s independently of $S \setminus s$ with law $p(s, \cdot)$.

Theorem 3.2 (Marking theorem). *S^* is a Poisson process on $S \times M$ with mean measure*

$$\mu^*(C) = \int\int_C \mu(\mathrm{d}x)\, p(x, \mathrm{d}m), \qquad C \subset S \times M. \tag{87}$$

Proof. Characteristic functional. $f : S \times M \to \mathbb{R}$. $\Sigma^* := \sum_{(s,m_s) \in S^*} f(s, m_s)$.

$$\begin{aligned} E(e^{-\Sigma^*}|S) &= \prod_{s \in S} E(e^{-f(s,m_s)}|S) \\ &= \prod_{s \in S} \int_M e^{-f(s,m)}\, p(s, dm) \end{aligned}$$

Call $f^* := -\log \int_M e^{-f(s,m)}\, p(s, dm)$ and use Campbell formula to get

$$\begin{aligned} E(e^{-\Sigma^*}) &= \exp\Big(-\int_R (1 - e^{-f^*}) \mathrm{d}\mu\Big) \\ &= \exp\Big(-\int_R \int_M (1 - e^{-f(x,m)}) \mu(\mathrm{d}x)\, p(x, \mathrm{d}m)\Big) \\ &= \exp\Big(-\int_{R \times M} (1 - e^{-f}) \mathrm{d}\mu^*\Big). \end{aligned}$$ ☐

Corollaries:

The points $\{m_s : s \in S\}$ form a Poisson process on M with mean measure μ_m given by

$$\mu_m(B) := \int_R \int_B \mu(\mathrm{d}x)\, p(x, \mathrm{d}m), \qquad B \subset M. \tag{88}$$

If m takes only k values, the point process S_i with i-marks is PP(μ_i), where

$$\mu_i(A) = \int_A \mu(\mathrm{d}x)\, p(x, \{m_i\}) \tag{89}$$

and S_i are independent. This allow the color distribution to depend on the position of the point x.

Rain It rains over $R = \mathbb{R}^2$. Drops are in the space-time $R \times M$, where $M = \mathbb{R}_+$ corresponds to the time (= distance) a given drop will touch the pavement R. The point process S^* is a Poisson process with mean measure $\lambda dx dt$. The points $((s, t_s) \in S^* : t_s \leq t)$ form a Poisson process S_t with mean measure $\lambda t \mathrm{d}x$, if the rain is uniform.

Wetting. Each fallen drop s has a random radious $r_s \geq 0$ and we consider the Poisson process with points (s, t_s, r_s); that is, the marks have 2 coordinates.

A typical problem in percolation is to give a distribution to r_s with some finite mean ρ and compute the probability of wetting in function of λ, t and ρ. Here *wetting* is the set of configurations defined by "the origin belongs to an infinite connected wet component".

Theorem 3.3 (Boolean percolation; Penrose, Peter Hall, Georgii)**.** *For fixed ρ, there are $0 < \lambda_1 \leq \lambda_2 < \infty$ such that*

$$\begin{aligned} P(\mathit{wetting}) > 0 \quad & \mathit{if} \quad \lambda > \lambda_2, \\ P(\mathit{wetting}) = 0 \quad & \mathit{if} \quad \lambda < \lambda_1. \end{aligned} \tag{90}$$

Galaxies Galaxies are points of a Poisson process S with masses m_s with density $\rho(s, m)$ in $m > 0$. $S^* = \{(s, m_s)\}$ is a Poisson process with density

$$\lambda^*(x, m) = \lambda(x)\rho(x, m). \tag{91}$$

The above means $\mu^*(\mathrm{d}(x, m)) = \lambda(x)\, \mathrm{d}x\, \rho(x, m)\, \mathrm{d}m$.

The gravitatory field of a the origin $0 \in \mathbb{R}^3$ is the vector (F_1, F_2, F_3) given by

$$F_j = \sum_{x \in S} \frac{G m_s s_j}{(s_1^2 + s_2^2 + s_3^2)^{3/2}} \tag{92}$$

where G is the gravitational constant. The characteristic functional of S^* applied to $t_1 F_1 + t_2 F_2 + t_3 F_3$ gives

$$E(e^{\mathrm{i}t_1 F_1 + \mathrm{i}t_2 F_2 + \mathrm{i}t_3 F_3}) = \exp\Big(\int_R^3 \int_0^\infty (e^{\mathrm{i}Gmt \cdot \psi(x)} - 1)\lambda(x)\rho(x, m)\, \mathrm{d}x, \mathrm{d}m \Big) \tag{93}$$

where

$$\psi_j(x) = (x_1^2 + x_2^2 + x_3^2)^{-3/2} x_j \tag{94}$$

and $t \cdot \psi$ is the scalar product. This holds if (by Campbell Theorem)

$$\int_R^3 m|\psi(x)|\lambda(x)\, \rho(x, m)\, \mathrm{d}x, \mathrm{d}m < \infty \tag{95}$$

If we denote the expected mass of a galaxy at x by

$$\bar{m}(x) := \int m\rho(x, m)\mathrm{d}m \tag{96}$$

(95) is equivalent to

$$\int_R^3 \frac{m|\psi(x)|\lambda(x)}{x_1^2 + x_2^2 + x_3^2} \, \mathrm{d}x < \infty \tag{97}$$

Under (96) we would have the joint distribution of F_1, F_2, F_3.

The problem is that in an uniform universe, where $\bar{m}(x)$ and $\lambda(x)$ are constant, then (97) does not hold.

4 Cox Processes

A Cox process is a "Poisson process with a random mean measure μ".

Assume μ is random in the space of mean measures. For instance,

$$\mu(A) = \int_A \Lambda(x)\mathrm{d}x. \tag{98}$$

$\Lambda(\cdot)$ is a random density. Example: $P(\Lambda(\cdot) = \lambda_i) = p_i$ (a random constant density).

Cox process is a process that conditioned to $\Lambda = \lambda$ is a Poisson process with density λ:

$$P(S \cap A = \emptyset | \Lambda = \lambda) = e^{-\int_A \lambda(x)\mathrm{d}x}. \tag{99}$$

Compute the second moment:

$$\begin{aligned} E(N(A))^2 &= E[E(N(A)^2|\mu)] \\ &= E(\mu(A) + \mu(A)^2) \\ &= E\mu(A) + (E\mu(A))^2 + \mathrm{Var}(\mu(A)) \end{aligned} \tag{100}$$

so that $\mathrm{Var}(N(A)) \geq EN(A)$. The identity only holds if $\mu(A)$ concentrates on a point (degenerate). Cox processes are "over dispersed". $N(A)$ has greater variance than a Poisson random variable with the same mean.

Void probabilities, characteristic functional, correlations The void probabilities of a Cox process are given by

$$\begin{aligned} \alpha(A) &= P(S \cap A = \emptyset) \\ &= E[P(S \cap A = \emptyset | \Lambda)] \\ &= E \exp\Big(-\int_A \Lambda(x)\mathrm{d}x\Big) \end{aligned} \tag{101}$$

this is the expectation of a function of the random density Λ.

The generating functional is given by

$$E^{-\Sigma_f} = E\exp\Big(-\int_R (1 - e^{-f(x)})\Lambda(x)\mathrm{d}x\Big) \tag{102}$$

The intensity function is given by

$$\rho(x) = E\Lambda(x). \tag{103}$$

and the n-point correlation function for distinct $x_1, \dots, x_n$:

$$\psi(x_1, \dots, x_n) = E(\Lambda(x_1) \dots \Lambda(x_n)) \tag{104}$$

so that, for mutually disjoint $A_1, \dots, A_n$

$$E(N(A_1) \dots N(A_n)) = E\Big(\int_{A_1} \cdots \int_{A_n} \Lambda(x_1) \dots \Lambda(x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n\Big). \tag{105}$$

Ecology The members of a population form a non-homogeneous $\mathrm{PP}(\lambda(\cdot))$ denoted S. If we consider λ as a realization of a random density Λ, we have a Cox process.

For instance, take $\phi(x)$ as the mean number of daughters of an individual at site x and $g(x, \cdot)$, the density of the continuous distribution of the position of each daughter.

The daughters of a plant at x are a Poisson process S_x with intensity $\lambda_x(y) = \phi(x)g(x, y - x)$.

Given S, the superposition theorem says that the set of daughters S' is a Poisson process of rate $\lambda'(y) = \sum_{s \in S} \phi(s)\, g(s, y - s)$.

When ϕ and g are independent of x, the set of daughters is a Cox process with random intensity

$$\Lambda'_S(x) = \phi \sum_{s \in S} g(x - s) \tag{106}$$

Exercise. Show that if $E\Lambda(x) = \lambda$, then $E\Lambda'(x) = \lambda\phi$. In particular, if $\phi = 1$ we have the same density in each generation.

Neyman-Scott processes These are special cases of the ecology models of previous paragraph.

Let C be a stationary Poisson Process on $\mathbb{R}^d$ with intensity $\kappa > 0$. Conditioned on C, let X_c be independent Poisson processes on $\mathbb{R}^d$ with intensity function

$$\rho_c(x) = \alpha k(x - c)$$

where the *kernel* $k(\cdot)$ is a density function; that is, $k(x) \geq 0$ and $\int k = 1$.

Then $X = \cup_{c \in C} X_c$ is a *Neyman-Scott* process with cluster centres C and clusters X_c. More general Neyman-Scott processes do not ask C to be Poisson.

X is a Cox process with (random) intensity

$$\Lambda(x) = \sum_{c \in C} \alpha k(x - c). \tag{107}$$

5 Stochastic geometry

5.1 Line processes

R set of infinite, undirected straight lines in $\mathbb{R}^2$. A countable random subset of R is called *line process*. If it is a Poisson process, it will be called *Poisson line process*.

We need a mean measure μ on R.

For each line ℓ there is a unique line $\ell^\perp$ containing the origin perpendicular to ℓ. Let $L := \ell \cap \ell^\perp$ the intersection point of these lines. Let p be the distance $\|L\|\mathrm{Sign}(L_2)$, using the notation $L = (L_1, L_2)$. Let θ be the angle that $\ell^\perp$ makes with the positive x-axis, $0 \leq \theta < \pi$.

Any measure on R induces a measure on

$$\tilde{R} := \{(p, \theta) : -\infty < p < \infty, 0 \leq \theta < \pi\} \tag{108}$$

and vice-versa. We do not distinguish these two measures, called μ.

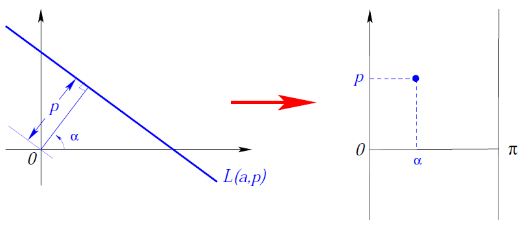
If $B(0, r) = \{(x, y) : x^2 + y^2 \leq r\}$, then ℓ intersects $B(0, r)$ iff $|p| \leq r$. A locally finite set $\tilde{A} \subset \tilde{R}$ maps into a subset $A \subset R$ which is "locally finite" in the sense that only a finite number of lines in $\tilde{A}$ intersect any bounded set. The existence theorem shows that there is a Poisson process with mean measure μ on R and $\tilde{R}$.

5.2 Uniform Poisson line process

Approach and pictures taken from slides *Poisson line process*, by Cristian Lentuejuol, see also [13].

When μ_λ is Lebesgue measure on $\tilde{R}$ of constant intensity λ, the corresponding Poisson process is called *uniform Poisson line process* with intensity λ.

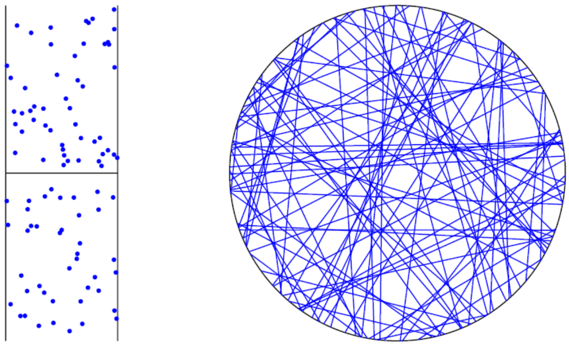
Line parametrization



Equation of a line:

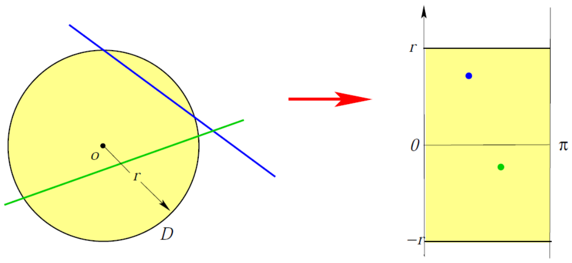
$$x \cos \alpha + y \sin \alpha = p \tag{109}$$

Poisson line process:

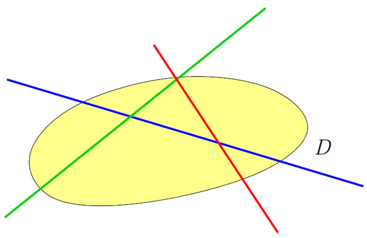


A Poisson line process with intensity λ is parametrized by a Poisson point process with intensity λ on $[0, \pi) \times \mathbb{R}$.

The number of lines hitting the disk $B(0, r)$ follows a Poisson distribution with mean $\lambda \times \pi \times 2r = \lambda\,\text{Perimeter}(B(0, r))$:



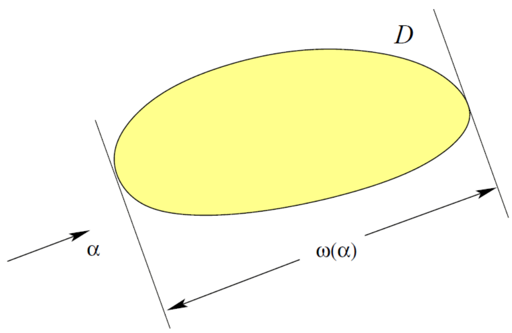
Distribution of lines hitting a convex domain Each Poisson line hitting a convex domain $D \subset \mathbb{R}^2$ comes from a point uniformly located on a set $\tilde{D} \subset \tilde{R}$.



The mean number of lines crossing a convex is

$$\int_0^\pi w(\alpha)\mathrm{d}\alpha = \text{Perimeter}(D), \tag{110}$$

see Chapter 3 in Santalo [18]. Exercise: find an elementary proof of this fact.

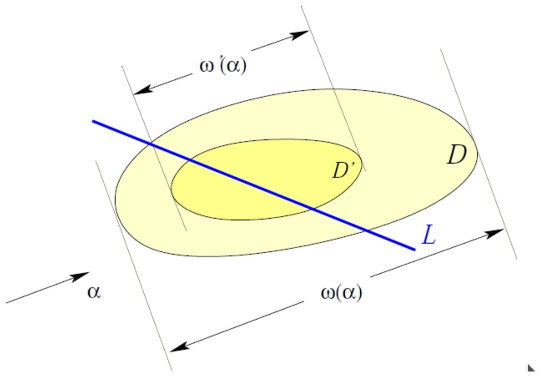


The direction of a line hitting D is not uniform. The rate of a line crossing D with angle α is the diameter of D $w(\alpha)$ in the direction α, see figure. Hence, the probability density for a line of angle α hit

D is

$$f(\alpha) = \frac{w(\alpha)}{\int_0^\pi w(\alpha)\mathrm{d}\alpha} = \frac{w(\alpha)}{\text{Perimeter}(D)} \tag{111}$$

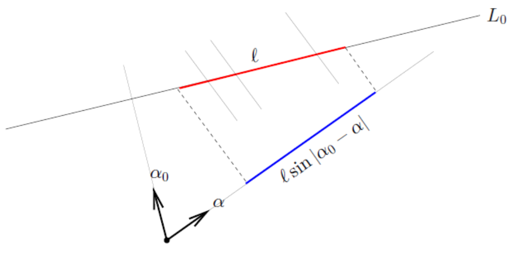
Uniform line hitting a convex domain. Let $D' \subset D$ convex domains. Then, given that a line crossed D, the conditional probability thatit also crossed D' is given by



$$P(\ell \cap D' \neq \emptyset \,|\, \ell \cap D \neq \emptyset) = \frac{\text{Perimeter}(D')}{\text{Perimeter}(D)} \tag{112}$$

Directions of Poisson lines hitting a fixed line

Even if the Poisson line network is isotropic, the distribution of the angles between Poisson lines and a fixed line is not uniform on $[0, \pi)$.



The directions of the Poisson lines hitting L_0 has distribution

$$\frac{\ell \sin(|\alpha_0 - \alpha|}{\int_0^\pi \ell \sin(|\alpha_0 - \alpha|} = \frac{\sin(|\alpha_0 - \alpha|}{2}, \qquad 0 < \alpha < \pi \tag{113}$$

Exercise. Show that the uniform Poisson line process in $\mathbb{R}^2$ is translation, rotation and reflection invariant.

Alternative construction Consider a Poisson process $\tilde{S}$ in $\mathbb{R}^2 \times [0,1] \times [0,\pi)$ with Lebesgue intensity

$$\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}\theta \tag{114}$$

This is a marked Poisson process in $\mathbb{R}^2$ with two marks, a uniform random variable in $[0,1)$ and a uniform random variable in $[0,\pi)$: $\tilde{s} \in \tilde{S}$ has 3 components:

$$\tilde{s} = (s(\tilde{s}), u(\tilde{s}), \theta(\tilde{s})) \in \mathbb{R}^2 \times [0,1] \times [0,\pi).$$

Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be a partition of $\mathbb{R}^2$ with bounded elements. Let R_A be the set of lines that intersect A.

Let $\ell(\tilde{s})$ be the line containing the point $s(\tilde{s})$ with angle $\theta(\tilde{s})$ with the positive x-axis. Let $|\ell \cap A|$ be the length of the intersection of ℓ and A. Define

$$S_A := \left\{\ell(\tilde{s}) : s(\tilde{s}) \in A,\, u(\tilde{s}) \leq |\ell(\tilde{s}) \cap A|^{-1},\, \tilde{s} \in \tilde{S}\right\} \tag{115}$$

Define

$$S := \cup_{k\geq 1}\big[S_{A_k} \setminus (R_{A_1} \cup \cdots \cup R_{A_{k-1}})\big]. \tag{116}$$

Conjecture 5.1 (Line process via size debiasing)**.** *The process S defined in* (116) *is the uniform Poisson line process.*

Proof. Under consideration. □

More on line processes in Stoyan, Kendall and Mecke [4].

6 Gibbsian point processes

This section is based on the paper [11] by Georgii.

6.1 Discrete setting

The set of spin configurations is $\xi = (\xi_i)_{i\in\mathbb{Z}^d}$, where ξ_i belongs to a finite set S. $\Omega = S^{\mathbb{Z}^d}$ is the *configuration space*, provided of the product topology and the Borel sigma-algebra $\mathcal{F}$. We are interested in probability measures on the space $(\Omega, \mathcal{F})$. *Lattice systems.* For $\xi \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ denote $\xi_\Lambda = (\xi_i)_{i\in\Lambda}$; same notation as the projection of Ω on S^Λ.

Specifications Prescribe the probability of a finite set of spins when the other spins are fixed. That is, we look for probability measures P on $(\Omega, \mathcal{F})$ with conditional probabilities

$$G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) \tag{117}$$

for finite Λ.

Examples: 1) Markovian case. (117) depends only on the spins in the boundary of Λ defined by $\partial\Lambda := \{i \notin \Lambda : |i-j| = 1 \text{ for some } j \in \Lambda\}$, that is,

$$G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) = G_\Lambda(\xi_\Lambda|\xi_{\partial\Lambda}) \tag{118}$$

2) Gibbsian case. Hamiltonian H_Λ and Boltzmann-Gibbs formula

$$G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) = Z^{-1}_{\Lambda|\xi_{\Lambda^c}} \exp[-H_\Lambda(\xi)], \tag{119}$$

where $Z^{-1}_{\Lambda|\xi_{\Lambda^c}} = \sum_{\xi':\xi'_\Lambda=\xi_\Lambda} \exp[-H_\Lambda(\xi')]$. (119) are the DLR equations.

See the book of Georgii [10] for a exhaustive account of this matter

Definition 6.1 (Gibbs measures)**.** *A probability measure P on $(\Omega, \mathcal{F})$ is a* Gibbs measure *for $\mathbf{G} = (G_\Lambda)_{\Lambda finite}$ if*

$$P(\xi_\Lambda \text{ occurs in } \Lambda \,|\, \xi_{\Lambda^c} \text{ occurs off } \Lambda) = G_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) \tag{120}$$

for P-almost all ξ and all finite $\Lambda \subset \mathbb{Z}^d$.

Gibbs measures do not exist automatically. However, in the present case of a finite state space S, Gibbs measures do exist whenever G is Markovian in the sense of (2), or almost Markovian in the sense that the conditional probabilities (1) are continuous functions of the outer configuration ξ_{Λ^c}. In this case one can show that any weak limit of $G_\Lambda(\cdot|\xi_{\Lambda^c})$ for fixed ξ $\Lambda \nearrow \mathbb{Z}^d$ is a Gibbs measure.

A basic observation is that the Gibbs measures for a given consistent family G of conditional probabilities form a convex set $\mathcal{G}$. Therefore one is interested in its extremal points, which are characterized in the next theorem.

Let $\mathcal{F}_\Lambda$ be the sigma algebra generated by $\big\{\{\xi_i = k_i\} : i \in \Lambda, k_i \in S\big\}$ and $\mathcal{T} := \cap_{\Lambda:|\Lambda|<\infty}\mathcal{F}_{\Lambda^c}$, the *tail* sigma algebra –generated by sets not depending on the values of any finite set of spins.

Theorem 6.2 (Extremal Gibbs measures)**.** *The following statements hold:*

(a) A Gibbs measure $P \in \mathcal{G}$ is extremal in $\mathcal{G}$ if and only if P is trivial on $\mathcal{T}$, i.e., if and only if any tail measurable real function is P-almost surely constant.

(b) Distinct extremal Gibbs measures are mutually singular on $\mathcal{T}$.

(c) Any non-extremal Gibbs measure is the barycenter of a unique probability weight on the set of extremal Gibbs measures. (Convex combination of extremal measures).

A proof can be found in Georgii [10], Theorems (7.7) and (7.26).

(a) means that the extremal Gibbs measures are macroscopically deterministic: on the macroscopic level all randomness disappears, and an experimenter will get non-fluctuating measurements of macroscopic quantities like magnetization or energy per lattice site.

(b) asserts that distinct extremal Gibbs measures show different macroscopic behavior. So, they can be distinguished by looking at typical realizations of the spin configuration through macroscopic glasses.

(c) implies that any realization which is typical for a non-extremal Gibbs measure is in fact typical for a suitable extremal Gibbs measure. In physical terms: any configuration which can be seen in nature is governed by an extremal Gibbs measure, and the non-extremal Gibbs measures can only be interpreted in a Bayesian way as measures describing the uncertainty of the experimenter. These observations lead us to the following definition.

Definition 6.3 (Phase transition)**.** *Any extremal Gibbs measure is called a* phase *of the corresponding physical system. If distinct phases exist, one says that a* phase transition *occurs.*

So, in terms of this definition the existence of phase transition is equivalent to the nonuniqueness of the Gibbs measure.

One mechanism related to phase transition is the formation of infinite clusters in suitably defined random graphs. Such infinite clusters serve as a link between the local and global behavior of spins, and make visible how the individual spins unite to form a specific collective behavior.

6.1.1 Holley inequality

Suppose the state space S is a subset of $\mathbb{R}$ and thus linearly ordered. Then the configuration space Ω has a natural partial order given by $\xi \leq \xi'$ if and only if $\xi_i \leq \xi'_i$ for all i, and we can speak of increasing real functions $f : \Omega \to \mathbb{R}$.

Let P, P' be two probability measures on Ω. We say that P is stochastically smaller than P', denoted $P \leq P'$, if $\int f dP \leq \int f dP'$ for all measurable bounded increasing f on Ω. This is equivalent to have a coupling $\hat{P}$ on Ω^2 with marginals P and P' such that $\hat{P}(\xi \leq \xi') = 1$.

A sufficient condition for stochastic monotonicity is given in the proposition below. Although this condition refers to the case of finite products (for which stochastic monotonicity is similarly defined), it is also useful in the case of infinite product spaces. This is because (by the very definition) the relation $\leq$ is preserved under weak limits.

Proposition 6.4 (Holley inequality)**.** *Let Λ be a finite index set, and μ, μ' two probability measures on $\{0,1\}^\Lambda$ giving positive weight to each element of $\{0,1\}^\Lambda$. Suppose the single-site conditional probabilities at any $i \in \Lambda$ satisfy*

$$\mu(\xi_i = 1 | \xi_{\Lambda\setminus\{i\}}) \leq \mu'(\xi'_i = 1 | \xi'_{\Lambda\setminus\{i\}}) \text{ for } \xi \leq \xi',$$

then $\mu \leq \mu'$.

Proof. Let

$$p(i, \xi_{\Lambda\setminus\{i\}}) = \mu(\xi(i) = 1 | \xi_{\Lambda\setminus\{i\}} \text{ occurs off } i) \tag{121}$$

the conditional probability that the site i takes value 1 given the configuration $\xi_{\Lambda\setminus\{i\}}$ in the other sites. Then when $\xi_i = 1$ we have

$$\mu(\xi) = p(i, \xi_{\Lambda\setminus\{i\}})\mu(\xi_{\Lambda\setminus\{i\}}) \tag{122}$$

$$\mu(\xi^i) = (1 - p(i, \xi_{\Lambda\setminus\{i\}}))\mu(\xi_{\Lambda\setminus\{i\}}) \tag{123}$$

where $(\xi^i)_j = \xi_j$ if $j \neq i$; $(\xi^i)_i = 1 - \xi_i$, a configuration differing from ξ only at site i. In the above case $(\xi^i)_i = 0$.

Define analogously $p'(i, \xi_{\Lambda\setminus\{i\}})$. By hypothesis, we have

$$p(i, \xi_{\Lambda\setminus\{i\}}) \leq p(i, \xi'_{\Lambda\setminus\{i\}}) \text{ whenever } \xi \leq \xi'. \tag{124}$$

Consider a pure jump Markov process $\xi(t)$ on $\{0,1\}^\Lambda$ with the following evolution. At rate 1 each site i is updated with a Bernoulli distribution with parameter $p(i, \xi_{\Lambda\setminus\{i\}}(t-))$.

The positive entries of the matrix are

$$Q(\xi, \xi^i) = \begin{cases} p(i, \xi_{\Lambda\setminus\{i\}}) & \text{if } \xi_i = 0 \\ 1 - p(i, \xi_{\Lambda\setminus\{i\}}) & \text{if } \xi_i = 1 \end{cases} \tag{125}$$

The measure μ is reversible for the process $\xi(t)$. Indeed, (122)-(123) imply

$$\mu(\xi)\, Q(\xi, \xi^i) = \mu(\xi^i)\, Q(\xi^i, \xi). \tag{126}$$

Let $P_t(\xi, \zeta) = P(\xi(t) = \zeta | \xi(0) = \xi)$ the semigroup associated to Q. Then, the ergodic theorem for finite jump Markov processes says

$$\lim_{t\to\infty} P_t(\xi, \zeta) = \mu(\zeta). \tag{127}$$

Let $N = (N_i : i \in \Lambda)$ be a collection of independent marked Poisson processes in $\mathbb{R} \times [0,1]$ with intensity $dt\, du$.

We construct the process as a function of N, $\xi(t) = (\xi(t))_{t\geq 0}[N]$, as follows. Fix an initial configuration $\xi(0)$ and assume we know $\xi(s)$ up to time $t-$ and $(t,u) \in N_i$. Then at time t update ξ_i as follows:

$$\xi_i(t) = \mathbf{1}\{u \leq p(i, \xi_{\Lambda\setminus\{i\}}(t-))\} \tag{128}$$

The updating does not depend on the value of $\xi_i(t-)$. The process so defined has transition matrix Q. Indeed if $\xi_i = 1$ we have

$$\begin{aligned} &P(\xi(t+\delta) = \xi^i | \xi(t) = \xi) \\ &= P\big(N_i \cap ([t, t+\delta) \times \big[p(i, \xi_{\Lambda\setminus\{i\}}(t)), 1\big) = 1\big) + P(\text{other things}) \\ &= \delta(1 - p(i, \xi_{\Lambda\setminus\{i\}})) + o(\delta). \end{aligned} \tag{129}$$

and analogously when $\xi_i = 0$,

$$P(\xi(t+\delta) = \xi^i | \xi(t) = \xi) = \delta p(i, \xi_{\Lambda\setminus\{i\}}) + o(\delta). \tag{130}$$

Define Q' and P'_t analogously with μ'.

Take $\xi(0) \leq \xi'(0)$ and consider the coupling

$$\big((\xi(t))_{t\geq 0}[N], (\xi'(t))_{t\geq 0}[N]\big) \tag{131}$$

Both marginals use the same marked Poisson processes N.

If $(t,u) \in N_i$ and the configurations are ordered at time $t-$, we have

$$\xi_i(t) = \mathbf{1}\{u \leq p(i, \xi_{\Lambda\setminus\{i\}}(t-))\} \leq \mathbf{1}\{u \leq p(i, \xi'_{\Lambda\setminus\{i\}}(t-))\} = \xi'_i(t)$$

where the inequality comes from (124). Hence, $\xi \leq \xi'$ implies $(\xi(t))_{t\geq 0}[N] \leq (\xi'(t))_{t\geq 0}[N]$ for all t. In turn, this implies that their distributions satisfy $P_t(\xi, \cdot) \leq P'_t(\xi', \cdot)$. Use (127) to conclude that $\mu \leq \mu'$. $\square$

6.1.2 Bernoulli percolation

Consider $\mathbb{Z}^d$, $d \geq 2$ as a graph with vertex set $\mathbb{Z}^d$ and edge set $E(\mathbb{Z}^d) = \{e = \{i,j\} \subset \mathbb{Z}^d : |i-j| = 1\}$.

Parameters $0 \leq p_s, p_b \leq 1$, site and bond probabilities.

Random subgraph $\Gamma = (X, E)$ of $(\mathbb{Z}^d, E(\mathbb{Z}^d))$, where $X = \{i \in \mathbb{Z}^d : \xi_i = 1\}$, $E = \{e \in E(X) : \eta_e = 1\}$, where $E(X) = \{e \in E(\mathbb{Z}^d) : e \subset X\}$ is the set of edges between the sites of X, and $(\xi_i : i \in \mathbb{Z}^d)$, $(\eta_e : e \in E(\mathbb{Z}^d))$ are independent Bernoulli variables with $P(\xi_i = 1) = p_s$, $P(\eta_e = 1) = p_b$.

This is *Bernoulli mixed site-bond percolation.* Setting $p_b = 1$ we obtain pure site percolation, and $p_s = 1$ corresponds to pure bond percolation.

Let $\{0 \leftrightarrow \infty\}$ denote the event that Γ contains an infinite path starting from 0, and $\theta(p_s, p_b; \mathbb{Z}^d) = P(0 \leftrightarrow \infty)$ be its probability.

By Kolmogorov's zero-one law, we have $\theta(p_s, p_b; \mathbb{Z}^d) > 0$ if and only if Γ contains an infinite connected component –called *cluster*– with probability 1. In this case one says that *percolation* occurs.

The following proposition asserts that this happens in a non-trivial region of the parameter square, which is separated by the so-called critical line from the region where all clusters of Γ are almost surely finite. The change of behavior at the critical line is the simplest example of a critical phenomenon.

Proposition 6.5 (Bernoulli percolation)**.** *The function $\theta(p_s, p_b; \mathbb{Z}^d)$ is increasing in p_s, p_b and d. Moreover,*

(1) $\theta(p_s, p_b; \mathbb{Z}^d) = 0$ when $p_s p_b$ is small enough;

(2) $\theta(p_s, p_b; \mathbb{Z}^d) > 0$ when $d \geq 2$ and $p_s p_b$ is sufficiently close to 1.

Proof. The monotonicity in p_s and p_b follows from coupling by defining the $\eta_e(U_e) = \mathbf{1}\{U_e \leq p_b\}$ and $\eta_x(U_x) = \mathbf{1}\{U_x \leq p_s\}$, where U_e U_x are iid Uniform$[0, 1]$ or using Holley inequality; the monotonicity in d follows by noticing that an infinite percolation cluster in $\mathbb{Z}^2$ is contained in an infinite percolation cluster in $\mathbb{Z}^d$ for $d \geq 3$.

The following arguments are taken from Grimmett [12].

(1) A *self avoiding walk* (SAW) is a path that visit no vertex more than once. Let σ_n be the number of SAW starting at the origin with length n. Let N_n be the set of those walks having all edges and sites open (open SAW). Then

$$\theta = P(N_n \geq 1 \text{ for all } n \geq 1).$$

This is because the cluster of the origing has infinitely many points if and only if there are open SAW starting at the origin of all lenghts. The above expresion equals

$$\lim_{n\to\infty} P(N_n \geq 1)$$

Now

$$P(N_n \geq 1) \leq EN_n = (p_s p_b)^n \sigma_n$$

An upperbound for σ_n is

$$\sigma_n \leq (2d)(2d-1)^{n-1},\ n \geq 1$$

because the first step of a SAW can be performed in $2d$ different ways and each subsequent step can be performed at most in $2d - 1$ different ways (since the walk is self avoiding, it cannot come back).

Hence

$$\theta \leq \lim_{n\to\infty} (2d)(2d-1)^{n-1}(p_s p_b)^n$$

But this is 0 if $p_s p_b < (2d - 1)^{-1}$. This gives

$$p_s p_b < \frac{1}{2d-1} \quad \text{implies} \quad \theta = 0.$$

(2) *Planar duality.* Given the graph $\mathbb{Z}^2$ with edges $E(\mathbb{Z}^2)$, construct a *dual* graph $\Gamma^* = (V^*, E(V^*)$ with vertices $V^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ and edges $E(V^*) = \{\{x, y\} \subset V^* : |x - y| = 1\}$. This is the graph $\Gamma = (\mathbb{Z}^2, E(\mathbb{Z}^2))$ translated by $(\frac{1}{2}, \frac{1}{2})$. See Fig. 1

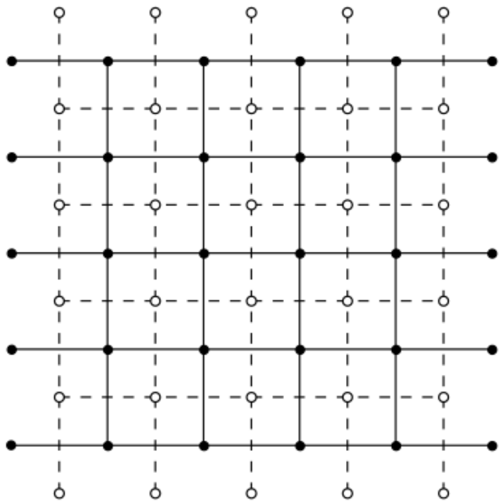


Figure 1: The dual graph.

Each edge $e = \{x, y\}$ of the graph Γ is crossed by an edge e^* of the dual graph Γ^*. We construct a percolation model in Γ^* by declaring

$$e^* \text{ open if and only if } e = \{x, y\} \text{ open and } x \text{ open.}$$

The *Peierls argument* dominates the probability of $\{0 \not\leftrightarrow \infty\}$ by the probability of the existence of a closed circuit in the dual graph containing the origin. In fact, if there exists a closed circuit in the dual graph containing the origin, there cannot be an infinite open path containing the origin, that is,

$$\{0 \not\leftrightarrow \infty\} \subset \{\text{there is a closed circuit of the dual graph } \Gamma^*\}.$$

Let M_n be the number of closed circuits of the dual graph of lenght n sorrounding the origin. Then

$$1 - \theta = P(0 \not\leftrightarrow \infty) = P\Big(\sum_{n\geq 4} M_n \geq 1)\Big) \leq E\Big(\sum_{n\geq 4} M_n\Big),$$

where we used that the shorter circuit sorrounding the origin has lenght 4. The above expression equals

$$\sum_{n\geq 4} EM_n \leq \sum_{n\geq 4} (n4^n)(1 - p_s p_b)^n$$

In fact, the number of circuits of lenght n sorrounding the origin in the dual graph must cross in at least one point the semiline $[0, \infty)$. The leftmost point the circuit intersects this semiline can take at most n values: $\{\frac{1}{2}, \frac{3}{2}, \dots, n + \frac{1}{2}\}$. Starting the circuit from this point, at each step it can take at most 4 different values. Hence the the number of circuits of lenght n sorrounding the origin in the dual graph is dominated by $n4^n$. This circuit is closed if all bonds are closed, hence the probability that it is closed is dominated by $(1 - p_s p_b)^n$. It is possible to take $\varepsilon > 0$ such that

$$\sum_{n\geq 4} n\varepsilon^n < 1$$

Then $p_s p_b \geq 1 - \varepsilon$ implies $\theta > 0$. □

6.2 Continuum percolation

Poisson points connected by Bernoulli edges.

Let $\mathfrak{X}$ = countable subsets X of $\mathbb{R}^d$.

σ-algebra generated by the counting variables $N(B) := \#(X \cap B)$ for bounded Borel sets $B \subset \mathbb{R}^d$; $(X_\Lambda := X \cap \Lambda)$.

$\mathcal{E}$ = locally finite subsets of $E(\mathbb{R}^d) = \{\{x, y\} \subset \mathbb{R}^d : x \neq y\}$. $\mathcal{E}$ = possible edge configurations with an analogous σ-algebra.

For $X \in \mathfrak{X}$ let $E(X) = \{e \in E(\mathbb{R}^d) : e \subset X\}$ = possible edges between the points of X, and $\mathcal{E}(X) = \{E \in \mathcal{E} : E \subset E(X)\}$ = edge configurations between the points of X.

Random graph $\Gamma = (X, E)$ in $\mathbb{R}^d$ as follows.

• Pick a random point configuration $X \in \mathfrak{X}$ according to the Poisson point process π^z on $\mathbb{R}^d$ with intensity $z > 0$.

• For given $X \in \mathfrak{X}$, pick a random edge configuration $E \in \mathcal{E}(X)$ according to the Bernoulli measure μ_X^p on EX for which the events $\{E \ni e\}$, $e = \{x, y\} \in E(X)$, are independent with probability $\mu_X^p(E \ni e) = p(x-y)$; here $p : \mathbb{R}^d \to [0, 1]$ is a given even measurable function.

The distribution of the random graph Γ on $X \times E$ is

$$P^{z,p}(dX, dE) = \pi^z(dX)\mu_X^p(dE) \tag{132}$$

It is called the **Poisson random-edge model**, or **Poisson random-connection model**, Penrose (1991) Meester and Roy (1996).

Boolean model: $p(x - y) = \mathbf{1}\{|x - y| \leq 2r\}$ for some $r > 0$.

x and y are connected if and only if $B(x, r)$ and $B(x, r)$ overlap. Same as random set $\Xi = \cup_{x\in X} B(x, r)$, for random X with distribution π^z.

Percolation probability of a typical point: Writing $x \leftrightarrow \infty$ when x belongs to an infinite cluster of $\Gamma = (X, E)$,

$$\theta(z, p; \mathbb{R}^d) = \int \frac{\#\{x \in X_\Lambda : x \leftrightarrow \infty\}}{|\Lambda|} P^{z,p}(dX, dE)$$

for an arbitrary bounded box Λ with volume $|\Lambda|$. By translation invariance, $\theta(z, p; \mathbb{R}^d)$ does not depend on Λ. In terms of the Palm measure, $\theta(z, p; \mathbb{R}^d) = \hat{P}^{z,p}(0 \leftrightarrow \infty)$.

Theorem [Penrose, 1991] *$\theta(z, p; \mathbb{R}^d)$ is increasing as function of z and $p(\cdot)$. Moreover, $\theta(z, p; \mathbb{R}^d) = 0$ when $z \int p(x)dx$ is sufficiently small, while $\theta(z, p; \mathbb{R}^d) > 0$ when $z \int p(x)dx$ is large enough.*

Sketch proof. The monotonicity follows from an obvious stochastic comparison argument. Since $z \int p(x)dx$ is the expected number of edges emanating from a given point, a branching argument shows that $\theta(z, p; \mathbb{R}^d) = 0$ when $z \int p(x)dx < 1$.

It remains to show that $\theta(z, p; \mathbb{R}^d) > 0$ when $z \int p(x)dx$ is large enough. For simplicity assume $\int p(x)dx = 1$ and $p(x - y) \geq \delta > 0$ whenever $|x - y| \leq 2r$. The following is taken from Georgii and Haggstrom (1996).

Divide $\mathbb{R}^d$ into cubic cells $\Delta(i), i \in Z^d$, with diameter at most r an pick a sufficiently large number n.

• Call a cell $\Delta(i)$ *good* if it contains at least n points which form a connected set relative to the edges of Γ in between them. This does not depend on the other cells and

$$P(\Delta(i) \text{ is good}) \geq \pi^z(N_i \geq n)[1 - (n - 1)(1 - \delta^2)^{n-2}] =: p_s \tag{133}$$

where N_i is the number of points in $\Delta(i)$, and the second term is an estimate for the probability that one of the n points is not connected to the first point by a sequence of two edges. **p_s is arbitrarily close to 1 when n and z are large enough.**

• Call two adjacent cells $\Delta(i), \Delta(j)$ *linked* if there exists an edge from some point in $\Delta(i)$ to some point in $\Delta(j)$. Conditionally on the event that $\Delta(i)$ and $\Delta(j)$ are good, this has probability at least $1 - (1 - \delta)^{n^2} =: p_b$, which is also close to 1 when n is large enough.

We have a site-bond percolation model in $\mathbb{Z}^d$ with p_s and p_b coupled to our original continuum percolation model with parameters z and p.

$$\begin{aligned}&\{\text{there exists an infinite cluster of linked good cells}\}\\ &\subset \{\text{there exists an infinite cluster in the Poisson random-edge model}\}.\end{aligned}$$

Hence

$$\theta(z, p; \mathbb{R}^d) \geq \frac{n}{|\Delta(0)|}\theta(p_s, p_b; \mathbb{Z}^d) \tag{134}$$

where $\theta(p_s, p_b; \mathbb{Z}^d)$ is the probability that the origin is connected to infinity in the discrete model. The right hand side dominates from below the average number of points of X in $\Delta(0)$ connected to infinity divided by the volume of $\Delta(0)$.

Hence $\theta(z, p; \mathbb{R}^d) > 0$ when z is large enough. $\square$

Stochastic comparison Refer to §6.1.1 for the definition of stochastic domination.

A simple point process P on a bounded Borel subset Λ of $\mathbb{R}^d$ (a probability measure on $\mathfrak{X}_\Lambda = \{X \in \mathfrak{X} : X \subset \Lambda\}$) has **Papangelou (conditional) intensity** $\gamma : \Lambda \times \mathfrak{X}_\Lambda \to [0, \infty[$ if P satisfies

$$\int P(dX) \sum_{x \in X} f(x, X \setminus \{x\}) = \int dx \int P(dX) \gamma(x|X) f(x, X)$$

for any measurable function $f : \Lambda \times \mathfrak{X}_\Lambda \to [0, \infty[$.

$\gamma(x|X)dx$ "conditional probability" for the existence of a particle in dx when the remaining configuration is X.

The Poisson process π^z_Λ on Λ has Papangelou intensity $\gamma(x|X) = z$.

Holley Preston inequality Consider the partial order induced by the inclusion relation on $\mathfrak{X}_\Lambda$.

Proposition 6.6 (Holley-Preston inequality)**.** *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Borel set and μ, μ' two probability measures on $\mathfrak{X}_\Lambda$ with Papangelou intensities γ resp. γ'. Suppose $\gamma(x|X) \le \gamma'(x|X')$ whenever $X \subset X'$ and $x \notin X' \setminus X$. Then $\mu \le \mu'$.*

Proof postponed to the next section.

6.3 The continuum Ising model

Point particles in $\mathbb{R}^d$ of two different types, plus and minus.

A configuration is a pair $\xi = (X^+, X^-)$.

Configuration space: $\Omega = \mathfrak{X}^2$.

Repulsive interspecies interaction pair potential of finite range, given by an even function $J : \mathbb{R}^d \to [0, \infty]$ with bounded support.

The Hamiltonian in a bounded Borel set $\Lambda \subset \mathbb{R}^d$ of a configuration $\xi = (X^+, X^-)$ is given by

$$H_\Lambda(\xi) := \sum_{x \in X^+, y \in X^- : \{x,y\} \cap \Lambda \neq \emptyset} J(x - y). \tag{135}$$

Example: classical Widom-Rowlinson model (1970) with a hard-core interspecies repulsion: $J(x-y) = \infty$ when $|x - y| \le 2r$ and $J(x - y) = 0$ otherwise.

Assumption:

$$\text{There exist } \delta, r > 0 \text{ such that } J(x - y) \ge \delta \text{ if } |x - y| \le 2r. \tag{136}$$

Gibbs distribution in Λ with activity $z > 0$ and boundary condition $\xi_{\Lambda^c} = (X^+_{\Lambda^c}, X^-_{\Lambda^c}) \in \mathfrak{X}^2_{\Lambda^c}$ is

$$G_\Lambda(d\xi_\Lambda|\xi_{\Lambda^c}) = Z^{-1}_{\Lambda|\xi_{\Lambda^c}} \exp[-H_\Lambda(\xi)]\, \pi^z_\Lambda(dX^+_\Lambda)\, \pi^z_\Lambda(dX^-_\Lambda)$$

Def. Gibbs measures $\mathcal{G} = \mathcal{G}(z)$ are the measures having G_Λ as conditional probabilities, for all Λ bounded Borel set in $\mathbb{R}^d$.

Existence of Gibbs measures. $G_\Lambda(\cdot|\xi_{\Lambda^c})$ has Papangelou intensity

$$\gamma(x|X^+, X^-) = \left\{ \begin{array}{ll} z \exp[-\sum_{y \in X^-} J(x - y)] & \text{if } x \in \Lambda^+ \\ z \exp[-\sum_{y \in X^+} J(x - y)] & \text{if } x \in \Lambda^- \end{array} \right\} \le z.$$

Holley-Preston implies that $G_\Lambda(\cdot|\xi_{\Lambda^c}) \le \pi^z_\Lambda \times \pi^z_\Lambda$. Compactness theorems for point processes show that for each $\xi \in \mathfrak{X}^2$, $G_\Lambda(\cdot|\xi_{\Lambda^c})$ has an accumulation point P as $\Lambda \nearrow \mathbb{R}^d$.

It is easy to see that $P \in \mathcal{G}$.

Uniqueness and phase transition We will show that the Gibbs measure is unique when z is small, whereas a phase transition (non-uniqueness) occurs when z is large.

It remains **open problem** whether there is a sharp activity threshold separating intervals of uniqueness and non-uniqueness. That is, a z_c such that for $z > z_c$ there are more than one extremal Gibbs measure and for $z < z_c$ there is only one Gibbs measure.

Proposition *For the continuum Ising model we have $\#\mathcal{G}(z) = 1$ when z is sufficiently small.*

Proof. Let P, $P' \in \mathcal{G}(z)$. We will show that $P = P'$ when z is small enough.

Let R be the range of J, i.e., $J(x) = 0$ when $|x| > R$.

Divide $\mathbb{R}^d$ into cubic cells $\Delta(i)$, $i \in \mathbb{Z}^d$, of linear size R.

Let p_c^* be the Bernoulli **site** percolation threshold of the graph with vertex set $\mathbb{Z}^d$ and edges between all points having distance 1 in the max-norm.

Consider the Poisson measure $Q^z = \pi^z \times \pi^z$ on $\Omega = \mathfrak{X}^2$.

Let ξ ,ξ' be two independent realizations of Q^z, and suppose z is so small that $Q^z \times Q^z(N_i + N_i' \geq 1) < p_c^*$, where N_i and N_i' are the numbers of particles (plus or minus) in $\xi \cap \Delta(i)$, respectively $\xi' \cap \Delta(i)$.

Then for any finite union Λ of cells we have $Q^z \times Q^z(\Lambda \overset{\geq 1}{\longleftrightarrow} \infty) = 0$, where $\{\Lambda \overset{\geq 1}{\longleftrightarrow} \infty\}$ denotes the event that a cell in Λ belongs to an infinite connected set of cells $\Delta(i)$ containing at least one particle in either ξ or ξ'.

Holley-Preston imply that $P \times P' \leq Q^z \times Q^z$. Hence $P \times P'(\Lambda \overset{\geq 1}{\longleftrightarrow} \infty) = 0$.

In other words, given two independent realizations ξ and ξ' of P and P' there exists a **random corridor** of width R around Λ which is completely **free of particles**.

Given $\Delta \supset \Lambda$, denote $K_\Lambda(\Delta)$ the set of (ξ, ξ') such that "there is an empty corridor of width R in Δ around Λ and there is no $\Delta' \subset \Delta$ with this property".

Now, suppose that for any local set B depending on Λ and boundary conditions ξ and ξ', we have

$$\begin{aligned} &G_\Delta \times G_\Delta(B \times \mathfrak{X} \cap K_\Lambda(\Delta)|(\xi, \xi')_{\Delta^c}) \\ &\qquad = G_\Delta \times G_\Delta(\mathfrak{X} \times B \cap K_\Lambda(\Delta)|(\xi, \xi')_{\Delta^c}) \end{aligned} \tag{137}$$

then, integrate with respect to $P \times P'$ to obtain

$$(P \times P')(B \times \mathfrak{X} \cap K_\Lambda(\Delta)) = (P \times P')(\mathfrak{X} \times B \cap K_\Lambda(\Delta)) \tag{138}$$

Summing over all choices of $K_\Lambda(\Delta)$, we get $P(B) = P'(B)$, for any bounded measurable B in Ω. This implies $P = P'$.

It remains to show (137).

If Δ does not contain a corridor of width R around Λ free of ξ and ξ' particles, then both sides of (137) are 0.

If Δ does contain such corridor, the boundary conditions $(\xi, \xi')_{\Delta^c}$ have the same effect as the empty boundary conditions. Hence, we can replace ξ and ξ' by empty configurations. Since the event $K_\Lambda(\Delta)$ and the specifications $(G_\Delta \times G_\Delta)(\cdot|\emptyset \times \emptyset)$are invariant under the map $(\xi, \xi') \to (\xi', \xi)$, the equation (137) holds. □

Existence of phase transition Follows from percolation in a suitable random-cluster model.

Gibbs distribution with + boundary conditions.

$$G_\Lambda^+ = \int \pi_{\Lambda^c}^z(dY_{\Lambda^c}^+)\, G_\Lambda(\cdot|Y_{\Lambda^c}^+, \emptyset) \tag{139}$$

This is the specificarion with a Poisson boundary condition of plus-particles and no minus-particle off Λ.

Random-cluster model: Fix the activity z. Define the probability measure χ_Λ^z on $\mathfrak{X} \times \mathcal{E}$ describing random graphs (Y, E) in $\mathbb{R}^d$:

$$\chi_\Lambda^z(dY, dE) = Z_{\Lambda|Y_{\Lambda^c}}^{-1}\, 2^{k(Y,E)}\, \pi^z(dY)\, \mu_Y^{p,\Lambda}(dE).$$

where $k(Y,E)$ is the number of clusters of the graph (Y,E);

$$Z_{\Lambda|Y_{\Lambda^c}} := \int 2^{k(Y,E)}\, \pi^z(dY_\Lambda)\, \mu_Y^{p,\Lambda}(dE).$$

is the normalization factor and $\mu_Y^{p,\Lambda}$ is the probability measure on E for which the edges $e=\{x,y\}\subset Y$ are drawn independently with probability

$$p(x-y) = \begin{cases} 1-e^{-J(x-y)} & \text{if } e=\{x,y\}\not\subset Y_{\Lambda^c} \\ 1 & \text{otherwise.} \end{cases} \tag{140}$$

χ_Λ is called the *continuum random-cluster* distribution in Λ with connection probability function p and **wired** boundary condition. This means that all points connected to Y_{Λ^c} belong to the same cluster.

If we eliminate the factor $2^{k(Y,E)}$ we have just the continuum percolation model with a unique cluster containing Y_{Λ^c}.

In the **Widom-Rowlinson** case of a hard-core interspecies repulsion the randomness of the edges disappears, indeed, if for $x,y\in Y$,we have $\|x-y\|<r$ then $\{x,y\}\in E$. In this case χ_Λ^z describes a **dependent Boolean percolation** model. The dependency comes from the factor $2^{k(Y,E)}$.

Coupling between the random-cluster model and the continuum Ising model Let P_Λ be the product distribution on $\mathfrak{X}\times\mathfrak{X}\times\mathcal{E}$ given by

$$P_\Lambda(dX^+,dX^-,dE) = \pi^z(dX^+)\,\pi_\Lambda^z(dX^-)\,\mu_{X^+\cup X^-}^{p,\Lambda}(dE)$$

with p given by (140) with $Y=X^+\cup X^-$.

Define the set of configurations with no connections between particles of different sign:

$$A_\Lambda := \{(X^+,X^-,E)\in\mathfrak{X}\times\mathfrak{X}\times\mathcal{E}\,:\, E\in\mathcal{E}(X^+\cup X^-)$$
$$\text{and } E\not\ni\{x,y\} \text{ for } x\in X^+,\ y\in X^-\}$$

Define Q_Λ as the conditioned measure

$$Q_\Lambda = P_\Lambda(\cdot|A_\Lambda)$$

Proposition 6.7. *For any bounded box Λ in $\mathbb{R}^d$, let (X^+,X^-,E) be distributed with Q_Λ. Then*

1. *The (X^+,X^-) marginal of Q_Λ has distribution G_Λ^+.*
2. *The random graph $(X^+\cup X^-,E)$ has distribution χ_Λ^z.*
3. *The conditioned law of $(X^+\cup X^-,E)$ given $\xi=(X^+,X^-)$ is the following. Independently for all $e=\{x,y\}\in E(Y)$ let $e\in E$ with probability*

$$p_{\Lambda,X^+,X^-}(e) = \begin{cases} 1-e^{-J(x-y)} & \text{if } e\subset X^+ \text{ or } e\subset X^- \text{ and } e\cap\Lambda\neq\emptyset \\ 1 & \text{if } e\subset\Lambda^c, \\ 0 & \text{otherwise.} \end{cases}$$

4. *The conditional distribution of (X^+,X^-) given (Y,E), denoted $Q_\Lambda(\mathrm{d}X^+,\mathrm{d}X^-|(Y,E))$, is the following. For each finite cluster C of (Y,E) let $C\subset X^+$ or $C\subset X^-$ according to independent flips of a fair coin; the unique infinite cluster of (Y,E) containing Y_{Λ^c} is included into X^+.*

Proof. **1.**

$$\begin{aligned} & Q_\Lambda(dX^+,dX^-) \\ = {} & \frac{1}{P_\Lambda(A_\Lambda)}\pi^z(dX^+)\pi_\Lambda^z(dX^-)\int_{E:(X^+,X^-,E)\in A_\Lambda}\mu_{X^+\cup X^-}^{p,\Lambda}(dE) \\ = {} & \frac{1}{P_\Lambda(A_\Lambda)}\pi^z(dX^+)\pi_\Lambda^z(dX^-)\prod_{x\in X^+,y\in X^-}(1-p_\Lambda(\{x,y\})) \\ = {} & \frac{1}{P_\Lambda(A_\Lambda)}\pi^z(dX^+)\pi_\Lambda^z(dX^-)\prod_{x\in X^+,y\in X^-}\exp(-J(x-y)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{P_\Lambda(A_\Lambda)} \pi^z(dX^+)\pi^z_\Lambda(dX^-)\exp(-H_\Lambda(X^+,X^-)) \\
&= G^+_\Lambda(X^+,X^-) && (141)
\end{aligned}
$$

2. Let $f : \mathfrak{X}\times\mathcal{E}$ be a test function. Then

$$
\begin{aligned}
&\int_{\mathfrak{X}\times\mathcal{E}} f(X^+\cup X^-,E)\, Q_\Lambda(dX^+,dX^-,dE) \\
&= \frac{1}{P_\Lambda(A_\Lambda)} \int_{A_\Lambda} f(X^+\cup X^-,E)\pi^z(dX^+)\pi^z_\Lambda(dX^-)\mu^{p,\Lambda}_{X^+\cup X^-}(dE) \\
&= \frac{1}{P_\Lambda(A_\Lambda)} \int_{\mathfrak{X}\times\mathcal{E}} f(Y,E) \\
&\quad \times \Big(\sum_{\substack{(X^+,X^-):X^+\cup X^-=Y\\ (X^+,X^-,E)\in A_\Lambda}} P_\Lambda((X^+,X^-,E)\in A_\Lambda|(Y,E))\Big)\, \pi^{z_\Lambda}(dY)\mu^{p,\Lambda}_Y(dE)
\end{aligned}
$$

where π^{z_Λ} is a PP in $\mathbb{R}^d$ with intensity $z_\Lambda(x) = z(1+\mathbf{1}\{x\in\Lambda\})$. Observe that $\pi^{z_\Lambda}(dY) = \frac{1}{Z}2^{\#Y_\Lambda}\pi^z(\mathrm{d}Y)$, where Z is the normalization.

$$
= \frac{1}{ZP_\Lambda(A_\Lambda)} \int_{\mathfrak{X}\times\mathcal{E}} f(Y,E)\, 2^{\#Y} \Big(\sum_{\substack{(X^+,X^-):X^+\cup X^-=Y\\ (X^+,X^-,E)\in A_\Lambda}} \left(\tfrac{1}{2}\right)^{\#Y} \Big)\, \pi^z(dY)\, \mu^{p,\Lambda}_Y(dE)
$$

where we used that the probability that a set of points Y gets partitioned into fixed X^+ and X^- is $2^{-\#Y}$, because each point has probability $1/2$ to belong X^+ (or to X^-).

Using that the cardinality of $\{(X^+,X^-,E)\in A_\Lambda : X^+\cup X^- = Y\}$ is $2^{k(Y,E)}$, we get:

$$
\begin{aligned}
&= \frac{1}{ZP_\Lambda(A_\Lambda)} \int_{\mathfrak{X}\times\mathcal{E}} f(Y,E)\, 2^{k(Y,E)}\, \pi^z(dY)\mu^{p,\Lambda}_Y(dE) \\
&= \int f(Y,E)\, \chi^z_\Lambda(dY,dE). && (142)
\end{aligned}
$$

The normalization comes for free because we started with a probability.

3. Fix E and denote $p(x,y) = Q_\Lambda(\{x,y\}\in E|X^+,X^-)$. Since $(X^+,X^-,E)\in A_\Lambda$, we have that $p(x,y)=0$ if $x\in X^+$ and $y\in X^-$, so that E does not contain vertices with different sign. The other edges intersecting Λ may take any value while those contained in Λ^c must be 1. Hence $p(x,y) = p_{\Lambda,X^+,X^-}(x,y)$ and

$$
Q_\Lambda((X^+\cup X^-,E)\,|\,X^+,X^-) = \prod_{\{x,y\}\in E} p(x,y) \qquad (143)
$$

4.

$$
Q_\Lambda(X^+,X^-\,|\,(Y,E)) \quad = \prod_{C \text{ cluster of } (Y,E)} \frac{1}{2} = \left(\frac{1}{2}\right)^{k(Y,E)}. \qquad \square
$$

Percolation and phase transition

The random-cluster representation gives the following key identity: For any finite box $\Delta\subset\Lambda$,

$$
\int [\#X^+_\Delta - \#X^-_\Delta] G^+_\Lambda(dX^+,dX^-) = \int \#\{x\in Y_\Delta : x\leftrightarrow Y_{\Lambda^c}\}\chi_\Lambda(dY,dE);
$$

where $x\leftrightarrow Y_{\Lambda^c}$ means that x is connected to a point of Y_{Λ^c} in the graph (Y,E). This is because the finite clusters have the same probability to belong to X^+ and to X^- and the points connected to the infinite cluster belong to X^+.

The **difference between the mean number of plus- and minus-particles** in Δ corresponds to the **percolation probability in** χ_Λ.

The percolation probability in χ_Λ is positive for large z.

Denote ν_Λ the point marginal distribution of χ_Λ. It satisfies

$$\nu_\Lambda f := \int f(Y)\, \chi_\Lambda(dY, dE), \tag{144}$$

for test functions $f : \mathfrak{X} \to \mathbb{R}$.

Then, item 4 of Proposition 6.7 implies

$$\chi_\Lambda(dY, dE) = \nu_\Lambda(dY)\varphi_{\Lambda,Y}(dE)$$

with

$$\varphi_{\Lambda,Y}(dE) = \frac{1}{Z_\Lambda} 2^{k(E)} \mu_Y^{p,\Lambda}(dE)$$

Proposition 6.8 (Edge domination)**.** *$\varphi_{\Lambda,Y}$ is stochastically larger than the Bernoulli edge measure $\mu_Y^{\tilde{p}}$ for which edges are drawn independently between points x, $y \in Y$ with probability*

$$\tilde{p}(x-y) = \frac{1-e^{-\delta}}{(1-e^{-\delta}) + 2e^{-\delta}}, \text{ for } |x-y| \le 2r, \tag{145}$$

and with probability 0 otherwise, where δ and r are as in assumption (136).

Proof. We fix $Y \sim \nu$ and consider edges distributed with $\mu(dE) := \varphi_{\Lambda,Y}(dE)$ on one hand and with $\mu'(dE)\mu_Y^{\tilde{p}}$ on the other hand. In order to apply Holley inequality compute

$$\begin{aligned} \mathrm{p}(e) &:= \mu(\eta(e) = 1 | \eta_{E(Y)\setminus\{e\}}) \\ &= \frac{p(x,y) 2^{k(Y,E(\eta^1))} \mu_Y^p(\eta_{E(Y)\setminus\{x,y\}})}{\left[p(x,y)2^{k(Y,E(\eta^1))} + (1-p(x,y))2^{k(Y,E(\eta^0))}\right] \mu_Y^p(\eta_{E(Y)\setminus\{x,y\}})} \end{aligned}$$

where $E(\eta) = \{e \in E(Y) : \eta(e) = 1\}$ and

$$\begin{aligned} &\eta^1(\{x,y\}) = 1 \text{ and } \eta^1(e) = \eta(e) \text{ for } e \neq \{x,y\}, \\ &\eta^0(\{x,y\}) = 0 \text{ and } \eta^0(e) = \eta(e) \text{ for } e \neq \{x,y\} \end{aligned} \tag{146}$$

The numerator is the probability under μ_Y^p of the configuration $\eta_{E(Y)\setminus\{e\}}$ off e and $\eta(e) = 1$. In the denominator we have the marginal distribution of $\eta_{E(Y)\setminus\{e\}}$ under the same measure.

We have

$$k(Y, E(\eta^0)) \le k(Y, E(\eta^1)) + 1, \tag{147}$$

because connecting x and y decreases at most by one the number of clusters. Hence,

$$\mathrm{p}(x,y) \ge \frac{p(x,y)}{p(x,y) + 2(1-p(x,y))} \ge \frac{1-e^{-\delta}}{(1-e^{-\delta}) + 2e^{-\delta}} = \tilde{\mathrm{p}}(x,y).$$

if $|x-y| > 2r$, by hypothesis. Apply Holley inequality to conclude. □

Proposition 6.9 (Point domination)**.** *There exists $\alpha > 0$ such that the point marginal ν_Λ of χ_Λ dominates a Poisson process of rate αz:*

$$\nu \ge \pi^{\alpha z}. \tag{148}$$

Proof. The point marginal ν_Λ has Papangelou intensity

$$\gamma(x|Y) = \frac{z \displaystyle\int 2^{k(Y\cup x, E)} \phi_{\Lambda, Y\cup x}(dE)}{\displaystyle\int 2^{k(Y,E)} \phi_{\Lambda,Y}(dE)}.$$

To get a lower estimate for $\gamma(x|Y)$ one has to compare the effect on the number of clusters in (Y, E) when a particle at x and corresponding edges are added. In principle, this procedure could connect a large number of distinct clusters lying close to x, so that $k(Y \cup x, \cdot)$ was much smaller than $k(Y, \cdot)$. However, one can show that this occurs only with small probability, so that

$$\gamma(x|Y) \ge \alpha z \text{ for some } \alpha > 0. \quad \square$$

Conclusion: The previous propositions imply that χ_Λ **is stochastically larger than the Poisson random-edge measure** $P_{\alpha z,\tilde{p}}$ defined in (11).

Hence,

$$\int \#\{x \in Y_\Delta : x \leftrightarrow Y_{\Lambda^c}\}\, \chi_\Lambda(dY, dE) \geq \ell(\Delta)\, \theta(\alpha z, \tilde{p}; \mathbb{R}^d)$$

Finally, since $G^+_\Lambda \leq \pi^z \times \pi^z$ by (17), the Gibbs distributions G^+_Λ have a cluster point $P^+ \in \mathcal{G}(z)$, a Gibbs measure satisfying

$$\int [\#X^+_\Delta - \#X^-_\Delta] P^+(dX^+, dX^-) \geq \theta(\alpha z, \tilde{p}; \mathbb{R}^d).$$

By spatial averaging one can achieve that P^+ is in addition translation invariant. Together with the continuum percolation Theorem, saying $\theta(z, p; \mathbb{R}^d) > 0$ for sufficiently large zp, this leads to the following theorem.

Theorem *For the continuum Ising model on $\mathbb{R}^d$, $d \geq 2$, with Hamiltonian (135) and sufficiently large activity z there exist two translation invariant Gibbs measures P^+ and P^- having a majority of plus- resp. minus-particles and related to each other by the plus-minus interchange.*

This result is due to Georgii and Haggstrom (1996). In the special case of the Widom-Rowlinson model it has been derived independently in the same way by Chayes, Chayes, and Kotecky (1995). The first proof of phase transition in the Widom-Rowlinson model was found by Ruelle in 1971, and for a soft but strong repulsion by Lebowitz and Lieb in 1972. Gruber and Griffiths (1986) used a direct comparison with the lattice Ising model in the case of a species-independent background hard core.

As a matter of fact, one can make further use of stochastic monotonicity. (In contrast to the preceding theorem, this only works in the present case of two particle types.) Introduce a partial order '$\leq$ ' on $\Omega = \mathfrak{X}^2$ by writing $(X^+, X^-) \leq (Y^+, Y^-)$ when $X^+ \subset Y^+$ and $X^- \supset Y^-$. (21). A straightforward extension of Holley-Preston then shows that the measures G^+_Λ decrease stochastically relative to this order when Λ increases. (This can be also deduced from the couplings obtained by perfect simulation, see the paper of Georgii, Section 4.4.) It follows that P^+ is in fact the limit of these measures, and is in particular translation invariant. Moreover, one can see that P^+ is stochastically maximal in $\mathcal{G}$ in this order.

Corollary *For the continuum Ising model with any activity $z > 0$, a phase transition occurs if and only if*

$$\int \hat{P}^+(dX^+, dX^-) \mu_{p,X^+}(0 \stackrel{+}{\longleftrightarrow} \infty) > 0;$$

here $\hat{P}^+$ is the Palm measure of P^+, and the relation $0 \stackrel{+}{\longleftrightarrow} \infty$ means that the origin belongs to an infinite cluster in the graph with vertex set X^+ and random edges drawn according to the probability function $p = 1 - e^{-J}$.

It is not known whether P^+ and P^- are the only extremal elements of $\mathcal{G}(z)$ when $d = 2$, as it is the case in the lattice Ising model. However, using a technique known in physics as the Mermin–Wagner theorem one can show the following.

Theorem *If J is twice continuously differentiable then each $P \in \mathcal{G}(z)$ is translation invariant.*

A proof can be found in Georgii (1999). The existence of non-translation invariant Gibbs measures in dimensions $d \geq 3$ is an open problem.

7 Perfect simulation of Markov jump processes

Cadenas de Markov a tiempo continuo

Vamos a considerar procesos Markovianos de salto en un conjunto finito S con tiempo $t \in \mathbb{R}^+$. Decimos que un proceso X_t es un proceso Markoviano de saltos si

$$P(X_{t+h} = y \,|\, X_t = x,\, X_s = x_s,\; 0 \leq s < t) = h\, q(x, y) + o(h). \tag{149}$$

$q(x, y)$ es la *tasa de salto* de x a $y \neq x$.

Ejemplo, proceso de Poisson S de intensidad λ induce un proceso Markoviano de salto. Defina $N(t) := S \cap [o, t]$.

$$\begin{aligned} P(N(t+h) = x+1) | N(t) = x) &= P(N(t, t+h] = 1) \\ &= \lambda\, h\, e^{-\lambda h} + P(\text{otras cosas})\,, \end{aligned}$$

Como el evento “otras cosas” está contenido en el evento “hay 2 o más puntos del proceso de Poisson en el intervalo de tiempo $[t, t+h)$” y la probabilidad de ese evento es $o(h)$, tenemos que $q(x, x+1) = \lambda$ y el resto de las tasas es 0.

Construcción usando procesos de Markov bi-dimensionales Queremos construir un proceso X_t con tasas $q(x, y)$, es decir, que satisfaga (149). La tasa de salida del estado x se denota

$$\lambda_x = \sum_y q(x, y),$$

Consideremos un proceso de Poisson bi-dimensional $M(\cdot)$. Para cada estado x consideramos una partición del intervalo $I_x = [0, \lambda_x]$ en Borelianos $B(x, y)$ de medida $q(x, y)$, $y \neq x$:

$$I_x = \dot\cup_{y \in S \setminus \{x\}} B(x, y); \quad B(x, y) \cap B(x, y') = \emptyset, \text{ for } y \neq y'.$$

$$|B(x, y)| = q(x, y), \quad x \neq y. \tag{150}$$

Fijemos $Y_0 = x_0$, un estado arbitrario y $T_0 = 0$. Sea T_1 la primera coordenada del primer evento del proceso $M(\cdot)$ en la banda $[0, \infty) \times I_{x_0}$:

$$T_1 = \inf\{t > 0 : M([0, t] \times I_{Y_0}) > 0\}.$$

Defina $(T_1, U_1) \in M$, el único punto que realiza el ínfimo, con $U_1 \in I_{Y_0}$. Como $(B(Y_0, y) : y \in S)$ es una partición de I_{Y_0}, hay un único $Y_1 \in S$ tal que $U_1 \in B(Y_0, Y_1)$.

Iterativamente, asumimos que T_{n-1} y Y_{n-1} están determinados y definimos

$$T_n = \inf\{t > T_{n-1} : M((I_{Y_{n-1}} \times T_{n-1}, t] > 0\}.$$

Defina $(T_n, U_n) \in M$, el único punto que realiza el ínfimo, con $U_n \in I_{Y_{n-1}}$. Como $(B(Y_{n-1}, y) : y \in S)$ es una partición de $I_{Y_{n-1}}$, hay un único $Y_n \in S$ tal que $U_n \in B(Y_{n-1}, Y_n)$.

Definition 7.1. Defina

$$X_t = Y_n, \text{ si } t \in [T_n, T_{n+1}), \quad \text{para } t \in [0, \infty) \tag{151}$$

Así, para cada realización del proceso de Poisson bidimensional M, construimos una realización del proceso $(X_t : t \in [0, \infty))$. T_n es el n-ésimo instante de salto; Y_n es el n-ésimo estado visitado por el proceso.

Proposition 7.2. *El proceso* $(X_t : t \in [0, T_\infty))$ *definido en* (151) *satisface* (149).

Proof. Por definición,

$$\mathbb{P}(X_{t+h} = y \,|\, X_t = x) \tag{152}$$

$$= \mathbb{P}\big\{M((t, t+h] \times B(x, y)) = 1\big\} + \mathbb{P}(\text{otras cosas}), \tag{153}$$

donde el evento {otras cosas} está contenido en el evento

$$\{M((t, t+h] \times [0, \lambda_x]) \geq 2\},$$

el proceso de Poisson M contiene dos o más puntos en el rectángulo $[0, \lambda_x] \times (t, t+h]$. Por definición de $M(\cdot)$, tenemos

$$\mathbb{P}(M((t, t+h] \times [0, \lambda_x]) \geq 2) = o(h) \quad \text{and} \tag{154}$$

$$\mathbb{P}(M((t, t+h] \times B(x, y) = 1) = hq(x, y) + o(h). \tag{155}$$

Esto demuestra la proposición. ◻

Example 7.3. Fila con un servidor y espacio limitado de espera. Espacio de estados $G = \{0, 1, 2\}$. X_t es el número de clientes en el sistema en el instante t. Los clientes llegan a tasa λ y los servicios son exponenciales a tasa μ. Las tasas son:

$$q(0,1) = q(1,2) \;=\; \lambda \tag{156}$$
$$q(1,0) = \mu; \quad q(2,1) \;=\; 2\mu \tag{157}$$
$$q(x,y) = 0, \text{ en los otros casos.} \tag{158}$$

Los intervalos usados en la construcción son los siguientes:

$$B(0,1) = B(1,2) \;=\; [0, \lambda] \tag{159}$$
$$B(1,0) = [\lambda, \lambda + \mu]; \; B(2,1) = [0, 2\mu]. \tag{160}$$

Todas las tasas están acotadas por $\max\{\lambda + \mu, 2\mu\}$.

Nacimiento puro X_t en $\{0, 1, 2, \dots\}$ con tasas

$$q(x, x+1) = \lambda x \tag{161}$$
$$q(x,y) = 0, \text{ en los otros casos.} \tag{162}$$

La tasa de llegadas en el instante t es proporcional al número de llegadas hasta ese instante. Los intervalos son

$$B(x, x+1) = [0, \lambda x) \tag{163}$$

Kolmogorov equations It is useful to use the following matrix notation. Let Q be the matrix with entries

$$q(x,y) \qquad \text{se } x \neq y \tag{164}$$
$$q(x,x) = -\lambda_x \;=\; -\sum_{y \neq x} q(x,y). \tag{165}$$

and P_t be the matrix with entries

$$p_t(x,y) \;=\; \mathbb{P}(X_t = y \mid X_0 = x).$$

Con esta notación, las ecuaciones de Chapman-Kolmogorov dicen

$$P_{t+s} \;=\; P_t P_s. \tag{166}$$

for all $s, t \geq 0$. To see it compute

$$\begin{aligned} p_{t+s}(x,y) &= \mathbb{P}(X_{t+s} = y \mid X_0 = x) \\ &= \sum_z \mathbb{P}(X_s = z \mid X_0 = x)\mathbb{P}(X_{t+s} = y \mid X_s = z) \\ &= \sum_z p_s(x,z) p_t(z,y). \end{aligned} \tag{167}$$

This is the (x, z) entry of $P_t P_s$.

Proposition 7.4 (Kolmogorov equations)**.** *The following identities hold*

$$P_t' = Q P_t \quad \textit{(Kolmogorov Backward equations)}$$

$$P_t' = P_t Q \quad \textit{(Kolmogorov Forward equations)}$$

for all $t \geq 0$, where P_t' is the matrix having as entries $p_t'(x,y)$ the derivatives of the entries of the matrix P_t.

Proof. Backward equations. Using Chapman-Kolmogorov,

$$p_{t+h}(x,y) - p_t(x,y) = \sum_z p_h(x,z) p_t(z,y) - p_t(x,y)$$

$$= (p_h(x,x) - 1)p_t(x,y)) + \sum_{z\neq x} p_h(x,z)p_t(z,y).$$

Dividing by h and taking h to zero we obtain $p_t'(x,y)$ in the left hand side. To compute the right hand side, observe that

$$p_h(x,x) = 1 - \lambda_x h + o(h).$$

Hence

$$\lim_h \frac{p_h(x,x) - 1}{h} = -\lambda_x = q(x,x).$$

Analogously, for $x \neq y$

$$p_h(x,y) = q(x,y)h + o(h)$$

and

$$\lim_h \frac{p_h(x,y)}{h} = q(x,y).$$

This shows the Kolmogorov Backward equations. The forward equations are proven analogously. To start, use Chapman-Komogorov to write

$$p_{t+h}(x,y) = \sum_z p_t(x,z)p_h(z,y). \quad \square$$

Las ecuaciones backward dicen $P_t' = QP_t$. Si P_t fuera un número, tendríamos

$$P_t = e^{Qt}$$

Formalmente podemos definir la matriz

$$e^{Qt} = \sum_{n\geq 0} \frac{(Qt)^n}{n!} = \sum_{n\geq 0} Q^n \frac{t^n}{n!}$$

y diferenciar para obtener

$$\frac{d}{dt} e^{Qt} = \sum_{n\geq 1} Q^n \frac{t^{n-1}}{(n-1)!} = Q \sum_{n\geq 1} Q^{n-1} \frac{t^{n-1}}{(n-1)!} = QP_t$$

A pesar que en general el producto no es conmutativo, tenemos que $QP_t = P_tQ$. Para verlo de otra manera,

$$QP_t = \sum_{n\geq 0} QQ^{n-1} \frac{t^{n-1}}{(n-1)!} = \sum_{n\geq 0} Q^{n-1} \frac{t^{n-1}}{(n-1)!} Q = P_tQ.$$

Invariant measures

Definition 7.5. We say that π is an *stationary distribution* for (X_t) if

$$\sum_x \pi(x)p_t(x,y) = \pi(y) \quad \text{Balance equations} \tag{168}$$

$$\sum_x \pi(x) = 1 \tag{169}$$

that is, if the distribution of the initial state is given by π, then the distribution of the process at time t is also given by π for any $t \geq 0$.

Theorem 7.6. *A distribution π is stationary for a process with rates $q(x,y)$ if and only if*

$$\sum_x \pi(x)q(x,y) = \pi(y) \sum_z q(y,z). \tag{170}$$

Condition (170) can be interpreted as a *flux condition*: the entrance rate under π to state y is the same as the exit rate from y. For this reason the equations (170) are called *balance equations.*

Proof. Assume π stationary, then (168) read

$$\pi P_t = \pi.$$

Differentiating we get

$$\begin{aligned} 0 &= \sum_x \pi(x) p_t'(x,y) = \sum_x \pi(x) \sum_z p_t(x,z) Q(z,y) \\ &= \sum_z \sum_x \pi(x) p_t(x,z) Q(z,y) = \sum_z \pi(z) Q(z,y) \end{aligned}$$

where we have applied the forward equations. This proves (170). Reciprocally, equations (170) say

$$\pi Q = 0.$$

Applying Kolmogorov backwards equations we get

$$(\pi P_t)' = \pi P_t' = \pi Q P_t = 0;$$

In other words, if the initial state of a process is chosen accordingly to the law π, the law of the process at any future time t is still π. This is because P_0 is the identity matrix and $\pi P_0 = \pi$. $\square$

Proceso clima Hay 3 estados: sol, nublado, lluvia. El tiempo que está en sol es exponencial 1/3 de donde pasa a nublado, queda un tiempo exponencial 1/4 cuando empieza a llover y llueve un tiempo exponencial 1, cuando vuelve a sol. La matriz de tasas es

$$\begin{pmatrix} -1/3 & 1/3 & 0 \\ 0 & -1/4 & 1/4 \\ 1 & 0 & -1 \end{pmatrix}$$

$\pi Q = \pi$ equivale a las ecuaciones

$$\begin{array}{llll} -\frac{1}{3}\pi(1) & & +\pi(3) & = 0 \\ \frac{1}{3}\pi(1) & -\frac{1}{4}\pi(2) & & = 0 \\ & \frac{1}{4}\pi(2) & -\pi(3) & = 0 \end{array}$$

La solución es $\pi = (\frac{3}{8}, \frac{4}{8}, \frac{1}{8})$. Podríamos haber obtenido esto con procesos de renovación con recompensa. La fracción de tiempo que estamos en cada estado es el conciente entre el tiempo medio en el estado y el tiempo medio del ciclo.

Recurrencia de Harris y convergencia exponencial al equilibrio En el siguiente teorema probamos simultaneamente la existencia y la convergencia a velocidad exponencial bajo una condición que es conocida como *recurrencia de Harris.* Defina

$$\gamma(z) := \min_x Q(x,z), \qquad \gamma(Q) := \sum_z \gamma(z) \tag{171}$$

Theorem 7.7. *Sea X_t un proceso de Markov con tasas Q. Si $\gamma > 0$, entonces (X_t) tiene una única distribución estacionaria π. Además, el proceso converge a π en variación total, a velocidad exponencial con coeficiente γ:*

$$\sup_x \frac{1}{2} \sum_z |\pi(z) - P_t(x,z)| < e^{-\gamma t}.$$

Proof. Primero vamos a demostrar que si hay una distribución estacionaria π, entonces vale la convergencia en variación total. Después demostraremos la existencia y unicidad de π.

Sin pérdida de generalidad asumimos que el espacio de estados es $\{1, \dots, K\} \subset \mathbb{N}$, para algún $K > 0$.

Convergencia. Construiremos dos cadenas con estados iniciales distintos con saltos governados por los puntos del mismo proceso de Poisson bidimensional M y la misma partición $B(x.y)$, elegida de manera de controlar el tiempo de encuentro de las marginales.

Construya una familia de intervalos sucesivos $J(z)$ disjuntos de tamaño

$$|J(z)| = \gamma(z).$$

La longitud del intervalo $\dot\cup_z J(z)$ es igual a γ.

A continuación, para cada estado x construya disjuntos sucesivos $I(x,z)$ de tamaño

$$|I(x,z)| = q(x,z) - \gamma(z);$$

localizados a la derecha de los intervalos $J(z)$.

Finalmente defina

$$B(x,z) = J(z) \cup I(x,z)$$

Como $B(x,z)$ tiene medida $q(x,z)$, la construcción de X_t usando el proceso de Poisson bidimensional M se puede hacer con estas particiones.

Vamos a realizar (X_t, X'_t) con estado inicial $(X_0, X'_0) = (x_0, x'_0)$ y cada marginal es governada por los puntos de M y la partición B recién construída.

Si el proceso en el instante $t-$ se encuentra en el estado (x,j) y M contiene al punto (t,u), con u en el intervalo

$$\cup_z J(z) = [0,\gamma),$$

entonces ambos procesos coalescen. Más precisamente, hay tres casos:

(a) $u \in J(z)$ para $z \notin \{x, x'\}$, en ese caso ambas marginales saltan a z;

(b) $u \in J(x)$, en ese caso la segunda marginal salta a x y la primera marginal se queda en x;

(c) $u \in J(x')$, en ese caso la primera marginal salta a x' y la segunda marginal se queda en x'.

Esto se puede hacer porque por debajo de γ la partición $B(x,z)$ no depende de x:

$$B(x,z) \cap [0,\gamma) = B(x',z) \cap [0,\gamma), \quad \text{para todo } z, x, x'.$$

El instante de coalescencia se denota τ:

$$\tau = \inf\{t > 0 : M([0,t] \times [0,\gamma)) > 0\}.$$

Por lo tanto,

$$t > \tau \quad \text{implica} \quad X_t = X'_t, \tag{172}$$

además

$$\tau \sim \text{ Exponencial}(\gamma) : \quad P(\tau > t) = e^{-\gamma t}. \tag{173}$$

Para concluir, escribimos

$$\begin{aligned}
\sum_z |P_t(y,z) - P_t(x,z)| &= \sum_z |P(X_t = z) - P(X'_t = z)| \\
&= \sum_z |E(\mathbf{1}\{X_t = z\} - \mathbf{1}\{X'_t = z\})| \\
&\le E\Big(\sum_z |\mathbf{1}\{X_t = z\} - \mathbf{1}\{X'_t = z\}|\Big) \\
&= 2E\mathbf{1}\{X_t \ne X'_t\} = 2P(X_t \ne X'_t) \\
&\le 2P(\tau > t) = 2e^{-\gamma t}, \text{ usando (172) y (173).}
\end{aligned}$$

Si el estado inicial X'_0 es aleatorio con distribución estacionaria π,

$$\begin{aligned}
\sum_z |\pi(z) - P_t(x,z)| &= \sum_z \Big| \sum_y \pi(y) P_t(y,z) - \sum_y \pi(y) P_t(x,z) \Big| \\
&\le \sum_y \pi(y) \sum_z |P_t(y,z) - P_t(x,z)| \\
&\le \sum_y \pi(y) 2e^{-\gamma t} = 2e^{-\gamma t}
\end{aligned}$$

Esto demuestra la convergencia a velocidad exponencial a la distribución estacionaria, cuando $t \to \infty$.

Existencia de la distribución estacionaria. Simulación perfecta

Denote $X^x_{[s,t]}$ el proceso que tiene una condición inicial $x_s = x$ en el instante s y utiliza los puntos de M en la banda $[s,t] \times [0,\infty)$. Esa construcción es invariante por traslaciones:

$$(X^x_{[s,s+t]}, t \geq 0) \text{ tiene la misma distribución que } (X^x_{[0,t]}, t \geq 0)$$

Vamos a sacar el límite cuando s se va a menos infinito. Sea

$$\tau(t) := \sup\{s < t : M([s,t] \times [0,\gamma)) > 0\}$$

En el instante $\tau(t)$, el proceso asume un valor aleatorio Z con distribución

$$P(Z = z) = \frac{\gamma(z)}{\gamma}$$

independiente del pasado y desde ahí evoluciona de acuerdo a los puntos de M en $[\tau(t), t] \times \mathbb{R}^+$:

$$X^x_{[s,t]} = X_{[\tau(t),t]} = \sum_z \mathbf{1}\{Z = z\} X^z_{[\tau(t),t]}, \quad \text{para todo } x,\ s \leq \tau(t).$$

Si definimos

$$Z_t := X_{[\tau(t),t]}, \quad t \in \mathbb{R} \tag{174}$$

tenemos que Z_t es Markov con tasas Q y es estacionaria: $P(Z_t = z)$ no depende de t. Por lo tanto, llamando π a la distribución de Z_t, vale que π es invariante para Q. $\square$

Ejemplo. Estados $\{1,2,3\}$.

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 4 & -5 & 1 \\ 2 & 3 & -5 \end{pmatrix} \tag{175}$$

$$(\gamma(z)) = (2,1,1) \qquad \gamma = 4.$$

$$(J(z)) = ([0,2),[2,3),[3,4))$$

$$(Q(i,z) - \gamma(z)) = \begin{pmatrix} * & 0 & 1 \\ 2 & * & 0 \\ 0 & 2 & * \end{pmatrix}$$

$$I(i,j) = \begin{pmatrix} * & \emptyset & [4,5) \\ [4,6) & * & \emptyset \\ \emptyset & [4,6) & * \end{pmatrix}$$

La existencia de una única distribución estacionaria y convergencia bajo la hipótesis de recurrencia positiva se demuestra como en el caso discreto. La diferencia es que no existe el problema de la periodicidad.

Theorem 7.8. *Si el proceso Markoviano de salto (X_t) es irreducible y tiene una distribución estacionaria π, entonces*

$$\lim_t p_t(x,y) = \pi(y).$$

Proof. Como $p_t(i,j) > 0$, tenemos que para todo $h > 0$ la cadena discreta con matriz de transición $p_h(x,y)$ es recurrente positiva y aperiódica y tiene distribución estacionaria π. Por lo tanto

$$\lim_n p_{nh}(i,j) = \pi(j). \quad \square$$

8 Perfect simulation of low density Gibbsian point processes

This section is based on [7]. Here we propose a perfect simulation algorithm and construction of measures locally absolutely continuous with respect to a Poisson process. They will be invariant measures of *interacting* birth-and-death processes. Perfect samples of finite windows of the *infinite-volume* measure

Background. Propp and Wilson's *Coupling from the Past* (CFTP). Fill's *Interruptible Algorithm* see `http://dimacs.rutgers.edu/~dbwilson/exact`. Spatial processes: Kendall and Moeller. All the above apply for finite state space or finite regions (*coupling with finite coalescence time*)

Basic example: seed + grain exclusion process

Consider a Poisson process $\tilde{S} \subset \mathbb{R}^d \times \mathbb{R}^+$ with intensity measure

$$\lambda \, dx \, h(r) \, dr; \; x \in \mathbb{R}^d, \; r \in \mathbb{R}^+,$$

with $\int h(r)dr = 1$. Call $\tilde{\mu}$ the law of tS.

If $\tilde{s} = (x, r) \in \tilde{S}$, we interpret that there is a *seed* at s with radious r.

Let $\gamma(x, r) = B(x, r)$: ball centered at x with radious r.

Let $\tilde{S}$ be the set of seeds plus grains and define the set of configuration without grain intersections

$$A_\Lambda = \{\tilde{S} \subset \mathbf{G} : \gamma(x, r) \cap \gamma(x', r') = \emptyset, \text{ for all } (x, r), (x', r') \in \tilde{S}, \, x, x' \in \Lambda\}.$$

Let

$$\mu_\Lambda := \tilde{\mu}(\cdot | \, A_\Lambda) \tag{176}$$

By Holley-Preston, this measure is dominated by $\tilde{\mu}$ (exercise), and hence there is a Gibbs measure for the specifications induced by μ_Λ, by taking limit along subsequences.

We want to construct (perfect simulate) a Gibbs measure with respect to the specifications induced by μ_Λ.

$$\mu := \text{“}\widetilde{\mu}(\cdot | \, A_{\mathbb{R}^d})\text{”}.$$

The right hand side is not really defined.

Usually one does this as a limit of finite-box measures (thermodynamic limit), as we did for the continuum Ising.

Here we will construct directly an infinite volume configuration coupled with configurations in Λ, simultaneously for all Λ, and then show that the infinite volume configuration is indeed the almost sure thermodynamic limit. This will work for low density.

The **idea** is the basis of MCMC (Markov chain Monte Carlo): find a Markov process in the set of point configurations having μ as invariant measure and then perfectly simulate this Markov process as we did with pure jump Markov processes in finite sets.

The Markov process will be a birth and death process of grains, with constant death rate and birth rate given by the Papangelou intensity induced by μ. This is a somehow *local* interaction.

We start defining a free birth and death process (that means no interactions).

8.1 Free birth and death spatial process:

Let $\mathbf{G}$ be a measurable set and consider an intensity measure w on $\mathbf{G}$.

In the case of seed-grain process, $\mathbf{G} = \{B(x, r) : x \in \mathbb{R}^d, r \in \mathbb{R}^+\}$ and for $\gamma = B(x, r)$, we have $w(d\gamma) = \lambda \, dx \, h(r) \, dr$.

We want a process $(\eta_t)_{t \in \mathbb{R}}$ with

birth rate $w(d\gamma)$ and

death rate of each $\gamma \in \eta_t$: 1.

Harris graphical construction of the free process

We construct a birth and death point process $(\eta_t)_{t\in\mathbb{R}}$, where $\eta_t \subset \mathbf{G}$ is a countable subset of $\mathbf{G}$ for each t.

Let $\mathbf{C}$ be a Poisson process on the space $\mathbf{G}\times\mathbb{R}\times\mathbb{R}^+$ with intensity

$$w(d\gamma)\, dt\, e^{-s} ds \tag{177}$$

An element of $\mathbf{C}$ is a triplet $C = (\gamma, T, S)$, with $\gamma \in \mathbf{G}$, $T \in \mathbb{R}$, $S \in \mathbb{R}^+$. We identify the point $C = (\gamma, T, S)$ with the space-time *cilinder*

$$C = \gamma \times [T, T+S) \qquad \text{(abusing notation)} \tag{178}$$

and interpret γ as the *basis*, T as the *birth-time*, S as the *life-time* and $T+S$ as the *death-time* of a the basis γ of the cylinder C.

We say that $C \in \mathbf{C}$ is *alive* at time t if $T \le t < T+S$. Define the process

$$\tilde\eta_t \;:=\; \Big\{\text{Basis}(C)\;:\;C\in\mathbf{C},\, C \text{ alive at } t\Big\} \tag{179}$$

(this maybe a multiset). Call $(\tilde\eta_t)_{t\in\mathbb{R}}$ the **stationary free birth death process**.

Semigroup and generator For a function $f:\mathbf{G}\to\mathbb{R}$, we define

$$\tilde S_t f(\eta) := E(f(\tilde\eta_t)|\tilde\eta_0 = \eta)). \tag{180}$$

$\tilde S_t$ is a semigroup. Define

Define the *generator* of the process by

$$\widetilde{L}f(\eta) = \sum_{\gamma\in\eta}[f(\eta\setminus\gamma) - f(\eta)] + \int_{\mathbf{G}} w(d\gamma)[f(\eta\cup\{\gamma\}) - f(\eta)]$$

It satisfies the Kolmogorov equations

$$\frac{d}{dt}\tilde S_t f(\eta) = \tilde L \tilde S_t f(\eta) \qquad \text{Backward equations}$$
$$\frac{d}{dt}\tilde S_t f(\eta) = \tilde S_t \tilde L f(\eta) \qquad \text{Forward equations}$$

Stationary measure: law of $\tilde\eta_t$ is $\widetilde{\mu}$ (Poisson).

The process in finite time intervals We define the free process for *positive* times in $[0,\infty)$ with initial configuration $\eta_0 = \eta \subset \mathbf{G}$ by including *initial cylinders*

$$\mathbf{C}_0^\eta = \Big\{\big(\gamma, 0, S_\gamma\big) : \gamma\in\eta\Big\}$$

where S_γ are independent exp(1) variables. The sets of cylinders of $\mathbf{C}$ born at positive times are denoted by

$$\mathbf{C}^+ := \{C\in\mathbf{C}\;:\;\text{Birth}(C)\ge 0\}$$

Define

$$\tilde\eta_t^\eta \;=:\; \Big\{\text{Basis}(C)\;:\;C\in\mathbf{C}^+\cup\mathbf{C}_0,\, C \text{ alive at time } t\Big\}$$

Then $(\tilde\eta_t^\eta)_{t\ge 0}$ has initial configuration η and generator $\tilde L$.

8.2 Birth and death spatial process with exclusions:

We now want to construct a birth and death process η_t with rate of birth of γ:

$$w(d\gamma)\times \mathbf{1}\{\gamma \text{ does not intersect present balls}\}, \tag{181}$$

is the Papangelou intensity for $\mu = \tilde{\mu}(\cdot | A_{\mathbb{R}^d})$.

Rate of death of γ: 1.

General principle If $\widetilde{\mu}$ is reversible for the (free) process without exclusions, then μ is reversible for the process with exclusions.

This is clear for finite systems: If we have a process with rate matrix Q and μ is reversible for Q, that is,

$$\mu(x)Q(x,y) = \mu(y)Q(y,x) \tag{182}$$

and we construct a new matrix Q_A where the jumps outside a given set of states A vanish, that is,

$$Q_A(x,y) := Q(x,y)\mathbf{1}\{y \notin A\}, \tag{183}$$

then the measure $\mu(\cdot|A)$ is reversible for Q_A.

Hence, to simulate a random variable with distribution $\mu(\cdot|A)$ one can use the Markov process with rate matrix Q_A.

Heat bath dynamics Want to construct $\eta_t \in \{0,1\}^{\mathbf{G}}$ with generator:

$$Lf(\eta) = \sum_{\gamma\in\eta}[f(\eta \setminus \gamma) - f(\eta)] + \int_{\mathbf{G}} w(d\gamma)M(\gamma \,|\, \eta)\,[f(\eta \cup \{\gamma\}) - f(\eta)]$$

where $M(\gamma \,|\, \eta) := \mathbf{1}\{\gamma$ does not intersects balls in $\eta\}$; this is the Papangelou intensity of μ.

The measure μ is reversible for this dynamics. The proof is based on the following heuristic microscopics:

$$\begin{aligned} &= (\text{weight of } \eta \cup \gamma) \times (\text{rate of } \eta \cup \gamma \to \eta) \\ (\text{weight of } \eta) \times (\text{rate of } \eta \to \eta \cup \gamma)& \end{aligned}$$

which gives

$$w(\eta \cup \gamma) \times 1 = w(\eta) \times M(\gamma|\eta)$$

assuming that η is a Poisson process with intensity $w(\gamma)$.

Graphical construction. Finite-volume, finite time Consider γ's contained in finite space-region Λ and a finite time interval $[0,t]$

1. Run free process starting from η_0.
2. Deaths as before.
3. Assume the first (attempted) birth is by the cylinder (γ, t, s):
   - If no intersections then
   $$\eta_t \leftarrow \eta_{t-} \cup \gamma$$
   - Otherwise
   $$\eta_{t_1} \leftarrow \eta_{t_1 -}$$
4. Iterate

In terms of cylinders we can see this as a two-sweep scheme:

first sweep: generate free cylinders born from 0 to t,

second sweep: iteratively erase cylinders (γ, t, s) with $M(\gamma|\eta_{t-}) = 0$.

$\mathbf{K}_{[0,t]}(\Lambda, \eta_0)$ the resultant set of kept cylinders

$$\eta_t := \Big\{\text{Basis}(C) \,:\, C \in \mathbf{K}_{[0,t]}(\Lambda, \eta_0),\, C \text{ alive at } t\Big\}\,.$$

Finite-volume, double infinite time (time-stationary construction)

Random times $\{\tau_j(\mathbf{C}) \in \mathbb{R} : j \in \mathbb{Z}\}$, such that

(a) $\tau_j \to \pm\infty$ for $j \to \pm\infty$ and

(b) $\big(\cup_i [T_i, T_i + S_i]\big) \cap \big(\cup_j \{\tau_j\}\big) = \emptyset$.

At each τ_j no cylinder is alive.

Kept cylinders obtained independently in each $[\tau_i, \tau_{i+1})$.

$\mathbf{K}(\Lambda)$:= the (time stationary random) set of kept cylinders.

Law of

$$\eta_t = \Big\{\text{Basis}(C) \,:\, C \in \mathbf{K}(\Lambda),\, C \text{ alive at } t\Big\}$$

is exactly the (unique) invariant measure μ_Λ.

Infinite-volume construction

The problem here is that there is no first mark in time and there are alive cylinders at all times.

The first generation of ***ancestors*** of a cylinder C is defined as the set of cylinders C' born before C and alive at the birth-time of C whose basis intersect the basis of C.

$\mathbf{A}_1^C$: First generation of ancestors of C.

$\mathbf{A}_n^C$: n-th generation of ancestors of C := union of first generation of ancestors of the cylinders belonging to the $(n-1)$-th generation of C:

$$\mathbf{A}_n^C = \cup_{C' \in \mathbf{A}_{n-1}^C} \mathbf{A}_1^{C'} \tag{184}$$

Notice that a cylinder C' may belong to more than a generation of ancestors of C.

Clan of ancestors of C: $\mathbf{A}^C := \cup_{n\geq 1} \mathbf{A}_n^C$

The construction of the set $\mathbf{K}$ of kept cylinders works if **the clan of ancestors of C is finite**. In this case we can apply the cleaning algorithm as follows. Pick a cylinder C and its clan of ancestors. Use the exclusion from younger to older cylinders in the clan which cylinders are kept or erased.

Result: $\mathbf{K} \subset \mathbf{C}$ set of *kept* cylinders; $\mathbf{C}_{[0,t]}$ cylinders born in $[0,t]$; $\mathbf{K}_{[0,t]}$ kept cylinders born in $[0,t]$

Theorem 8.1. *(i) If $\mathbf{A}^C \cap \mathbf{C}_{[0,t]}$ is almost surely finite for all C alive at time t, then*

$$\eta_t \;=\; \Big\{\mathit{Basis}(C) \,:\, C \in \mathbf{K}_{[0,t]}(\eta_0),\, C \text{ alive at } t\Big\}$$

has generator L.

(ii) If $\mathbf{A}^C$ is almost surely finite for all C, then

$$\eta_t = \Big\{\mathit{Basis}(C) \,:\, C \in \mathbf{K},\, C \text{ alive at } t\Big\}\,.$$

is stationary with generator L.

Furthermore, the measure μ given by the (Marginal) law of η_t is the unique stationary measure for the process.

Conditions allowing such a construction

Theorem 8.2 (Conditions guaranteeing the finiteness of the clan of branching ancestors)**.** *Define*

$$\alpha \,:=\, \sup_\gamma \frac{1}{|\gamma|} \int_{\mathbf{G}} |\theta| \mathbf{1}\{\theta \cap \gamma \neq \emptyset\} w(d\theta).$$

(i) If $\alpha < \infty$ then $\mathbf{A}^C \cap \mathbf{C}_{[0,t]}$ is almost surely finite for all C alive at time t.

(ii) If $\alpha < 1$, then $\mathbf{A}^C$ is almost surely finite for all C.

Proof. This works by dominating the n-th generation of a fixed cylinder C by a multitype branching process with mean "size type" α. $\square$

Oriented percolation

The **backwards oriented percolation model of cylinders** is defined by the oriented bonds $C \to C'$ if C' is an ancestor of C, that is, if the basis of C and C' intersect and C' is alive when C is born.

Let $m(\gamma, \theta)$ be the mean number of cylinders of basis θ in the first generation of a cylinder of basis γ. These are cylinders born at negative times $-t$ and have a lifetime at least t, so they survive to intersect the grain born at time zero. Its average number is, therefore,

$$m(\gamma, \theta) = w(\theta) I(\gamma, \theta) \Big[\int_{-\infty}^{0} dt \int_{t}^{\infty} ds\, e^{-s} \Big] \; = \; w(\theta) I(\gamma, \theta) \; \cdot \; 1 \, . \tag{185}$$

Define $m^n(\gamma, \theta)$ as the mean number of cylinders of basis θ incompatible with a cylinder of basis γ in the n-th generation of ancestors, m^n is the matrix-product of m by itself n times.

We will show that a sufficient condition for absence of oriented percolation is

$$\sum_{n \geq 1} \sum_{\theta} m^n(\gamma, \theta) < \infty, \qquad \text{for all } \gamma. \tag{186}$$

Let $q : \mathbf{G} \to \mathbb{R}^+$ be the area of the grain, and assume $\inf_\gamma q(\gamma) \geq 1$, (see Lemma 5.15 of Fernández, Ferrari and Garcia (2001)). Denote

$$\alpha_q \, := \, \sup_{\gamma} \frac{1}{q(\gamma)} \sum_{\theta} q(\theta)\, m(\gamma, \theta), \tag{187}$$

we have

$$\sum_{\theta} m^n(\gamma, \theta) \; \leq \; \alpha_q^n \, q(\gamma) \, . \tag{188}$$

In our case $q(\gamma)$ is the area of the circle centered at γ.

Domination by branching process We introduce a multitype branching process $\mathbf{B}_n$, in the set of cylinders, which will dominate $\mathbf{A}_n$. We look "backwards in time" and let "ancestors" play the role of "branches". Births in the original marked Poisson process correspond to dissapearance of branches. We reserve the words "birth" and "death" for the original forward-time Poisson process.

Proposition 8.3 (Domination by branching)**.** *For any given set of cylinders $\{C_1, \dots, C_k\}$, there exist* independent *random sets $\mathbf{B}_1^{C_i}$ with the same marginal distribution as $\mathbf{A}_1^{C_i}$, such that*

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{C_i} \subset \bigcup_{i=1}^{k} \mathbf{B}_1^{C_i}. \tag{189}$$

Proof. Denote $C \prec C'$ if and only if $\text{Birth}(C) \leq \text{Birth}(C')$.

For a finite set of cylinders $\{C_1, \dots, C_k\}$ such that $C_i \prec C_{i+1}$, $i = 1, \dots, k-1$, define

$$\widetilde{\mathbf{A}}_1^{C_j} = \mathbf{A}_1^{C_j} \setminus \left(\bigcup_{\ell=1}^{j-1} \mathbf{A}_1^{C_\ell} \right). \tag{190}$$

This ensures that $\widetilde{\mathbf{A}}_1^{C_j}$ are independent sets and

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{C_i} = \bigcup_{i=1}^{k} \widetilde{\mathbf{A}}_1^{C_i}. \tag{191}$$

On the other hand, for any C, $\widetilde{\mathbf{A}}_1^{C}$ is stochastically dominated by $\mathbf{A}_1^{C}$ [that is, there exists a joint realization $(\widetilde{\mathbf{A}}_1^{C}, \mathbf{A}_1^{C})$ such that $\mathbb{P}(\widetilde{\mathbf{A}}_1^{C} \subset \mathbf{A}_1^{C}) = 1$]. From this observation and (191) we get (189). $\square$

The procedure defined by $\mathbf{B}_1$ naturally induces a multitype branching process in the space of cylinders. We define the n-th generation of the branching process by

$$\mathbf{B}_n^C = \{ \mathbf{B}_1^{C'} : C' \in \mathbf{B}_{n-1}^C \} \tag{192}$$

where for all C', $\mathbf{B}_1^{C'}$ has the same distribution as $\mathbf{A}_1^{C'}$ and are independent random sets depending only on C'. Inductively,

$$\mathbf{A}_n^C \subset \mathbf{B}_n^C. \tag{193}$$

Indeed,

$$\mathbf{A}_n^C = \bigcup_{C'\in\mathbf{A}_{n-1}^C} \mathbf{A}_1^{C'} = \bigcup_{C'\in\mathbf{A}_{n-1}^C} \widetilde{\mathbf{A}}_1^{C'} \tag{194}$$

where in the definition of $\widetilde{\mathbf{A}}_1^{C'}$ we use $\{C_1,\dots,C_k\} = \cup_{i=0}^{n-1}\mathbf{A}_i^C$. Hence, the inductive hypothesis $\mathbf{A}_i^C \subset \mathbf{B}_i^C$, for $i = 1,\dots,n-1$, yields (193).

In consistence with our previous notation, we denote

$$\mathbf{B}^C = \bigcup_{n\geq 0}\mathbf{B}_n^C \;\; ; \quad \mathbf{B}^{x,t} = \bigcup_{n\geq 0}\mathbf{B}_n^{x,t} \;\; ; \quad \mathbf{B}^{\Upsilon} = \bigcup_{x\in\Upsilon}\mathbf{B}^{x,0}, \tag{195}$$

the branching clans of C, (x,t) and Υ (at time 0) respectively. By (193),

$$\mathbf{A}^C \subset \mathbf{B}^C;\;\; \mathbf{A}^{x,t} \subset \mathbf{B}^{x,t};\;\; \mathbf{A}^{\Upsilon} \subset \mathbf{B}^{\Upsilon}. \tag{196}$$

Definition: the **space-witdth** is the volume occupied by the projection of the bases of the cylinders in the family. The **time-lenght** is the lenght of the time interval between t and the first birth in the family of ancestors of (x,t).

We have the following dominations, by (196),

$$\mathrm{TL}\,(\mathbf{A}^C) \leq \mathrm{TL}\,(\mathbf{B}^C)\;;\; \mathrm{TL}\,(\mathbf{A}^{x,t}) \leq \mathrm{TL}\,(\mathbf{B}^{x,t})\;;\; \mathrm{TL}\,(\mathbf{A}^{\Upsilon}) \leq \mathrm{TL}\,(\mathbf{B}^{\Upsilon})\;, \tag{197}$$

and similarly for the respective space widths.

Multitype branching of γ's The (multitype) branching process $\mathbf{B}_n$ induces naturally a multitype branching process in the set of contours. For a cylinder C with basis γ and birth-time 0, define $\mathbf{b}_n^\gamma \in \mathbb{N}^{\mathbf{G}}$ as the number of cylinders in the nth generation of ancestors of C with basis θ:

$$\mathbf{b}_n^\gamma(\theta) = \Big|\Big\{C' \in \mathbf{B}_n^C : \mathrm{Basis}(C') = \theta\Big\}\Big|\,. \tag{198}$$

This process will be useful in estimating the space properties of the clans of ancestors. We have the following relationship:

$$\sum_\theta \mathbf{b}_n^\gamma(\theta) = |\mathbf{B}_n^C|. \tag{199}$$

The process $\mathbf{b}_n$ is a multitype branching process whose offspring distributions are Poisson with means

$$\begin{aligned} m(\gamma,\theta) &= \mathbf{1}\{\gamma \not\sim \theta\}\; w(\theta)\int_0^\infty e^{-t}dt \\ &= \mathbf{1}\{\gamma \not\sim \theta\}\int_0^\infty e^{-t}dt \\ &= \mathbf{1}\{\gamma \not\sim \theta\} w(\theta). \end{aligned} \tag{200}$$

To see this, notice that the cylinders C' with basis θ that are potential ancestors of C (with basis γ) form a Poisson process of rate $\mathbf{1}\{\gamma \not\sim \theta\}\, w(\theta)$. Each of those cylinders is an ancestor of C if its lifetime is bigger than the difference between the birth-time of C and C'. The lifetimes of different cylinders are independent exponentially distributed random variables of rate 1. The probability that the lifetime of any given cylinder is bigger than t is given by e^{-t}. Hence, the birth-times of the ancestors of C with basis θ form a (non homogeneous) Poisson process of rate depending on t given by $\mathbf{1}\{\gamma \not\sim \theta\}\, w(\theta)\, e^{-t}$. The mean number of births is therefore given by (200).

Lemma 8.4. *The means* (200) *satisfy. Here* $q(\gamma) = |\gamma|$.

$$\sum_\theta m^n(\gamma,\theta) \;\leq\; \sum_\theta |\theta|\, m^n(\gamma,\theta) \;\leq\; |\gamma|\,\alpha^n, \tag{201}$$

where α *is defined in* (187).

Proof.

$$\begin{aligned}
&\sum_\theta |\theta|\, m^n(\gamma,\theta) \\
&= \sum_{\gamma_1:\gamma_1\not\sim\gamma} w(\gamma_1) \sum_{\gamma_2:\gamma_2\not\sim\gamma_1} w(\gamma_2) ... \sum_{\theta:\theta\not\sim\gamma_{n-1}} |\theta|\, w(\theta) \\
&= |\gamma| \sum_{\gamma_1:\gamma_1\not\sim\gamma} \frac{|\gamma_1|}{|\gamma|} w(\gamma_1) \sum_{\gamma_2:\gamma_2\not\sim\gamma_1} \frac{|\gamma_2|}{|\gamma_1|} w(\gamma_2) ... \sum_{\theta:\theta\not\sim\gamma_{n-1}} \frac{|\theta|}{|\gamma_{n-1}|} w\theta \\
&\le |\gamma| \Big(\sup_\gamma \sum_{\theta:\theta\not\sim\gamma} \frac{|\theta|}{|\gamma|} w(\theta) \Big)^n . \ \square
\end{aligned} \tag{202}$$

This Lemma shows, in particular, that the branching process $\mathbf{b}_n$ is subcritical if $\alpha < 1$.

8.3 Continuous-time branching process

Let C be a cylinder with basis γ and birth-time 0. Combing backwards continuously in time the branching clan $\mathbf{B}^C$ we define a continuous-time multitype branching process $\psi_t^\gamma(\theta) =$ number of contours of type θ present at time t (of this process) whose initial configuration is δ_γ. Each $C' \in \mathbf{B}^C$ is a branch, that is, belongs to the first generation of ancestors of a unique cylinder $U(C')$ in $\mathbf{B}^C$. In the branching process ψ_t all the branches (ancestors) of $U(C')$ appear simultaneously at the birth of $U(C')$, that is when $U(C')$ dissapears if we look backwards in time. Therefore the part of C' in the interval $[\mathrm{Birth}(U(C')), \mathrm{Death}(C')]$ is ignored. Formally,

$$\psi_t^\gamma(\theta) = \Big|\Big\{ C' \in \bar{\mathbf{B}}^C : \mathrm{Basis}(C') = \theta, \mathrm{Birth}(C') < -t < \mathrm{Birth}(U(C')), \mathrm{Life}(C') \ni t \Big\}\Big|. \tag{203}$$

In the process ψ_t, each contour γ lives a mean-one exponential time after which it dies and gives birth to k_θ contours θ, $\theta \in \mathbf{G}$, with probability

$$\prod_\theta \frac{e^{m(\gamma,\theta)}\, m(\gamma,\theta)^{k_\theta}}{k_\theta!} \tag{204}$$

for $k_\theta \ge 0$. These are independent Poisson distributions of mean $m(\gamma,\theta)$. The infinitesimal generator of the process is given by

$$Lf(\psi) = \sum_{\gamma\in\mathbf{G}} \psi(\gamma) \sum_{\eta\in\mathcal{Y}_0(\gamma)} \prod_{\theta:\eta(\theta)\ge 1} \frac{e^{m(\gamma,\theta)}\, m(\gamma,\theta)^{\eta(\theta)}}{\eta(\theta)!} \ [f(\psi+\eta-\delta_\gamma) - f(\psi)] \tag{205}$$

where $\psi, \eta \in \mathcal{Y}_0 = \{\psi \in \mathbb{N}^{\mathbf{G}}; \sum_\theta \psi(\theta) < \infty\}$ and $\mathcal{Y}_0(\gamma) = \{\psi \in \mathcal{Y}_0; \psi(\theta) \ge 1 \text{ implies } \theta \not\sim \gamma\}$ and $f : \mathcal{Y}_0 \to \mathbb{N}$.

The branching process ψ_t^γ allows us to estimate the time-length of a clan, due to the obvious fact:

$$\sum_\theta \psi_t^\gamma(\theta) = 0 \ \text{ implies } \ \mathrm{TL}\,(\mathbf{B}^C) < t\ . \tag{206}$$

Let $M_t(\gamma,\theta)$ be the mean number of contours of type θ in ψ_t and $R_t(\gamma)$ its sum over θ:

$$M_t(\gamma,\theta) = \mathbb{E}\psi_t^\gamma(\theta)\ ; \qquad R_t(\gamma) = \sum_\theta M_t(\gamma,\theta)\ . \tag{207}$$

The bound we need is given in the next Lemma.

Lemma 8.5. *The mean number of branches, $R_t(\gamma)$ satisfies*

$$\mathbb{P}\Big(\sum_\theta \psi_t^\gamma(\theta) > 0\Big) \ \le\ R_t(\gamma) \ \le\ |\gamma|\, e^{(\alpha-1)t}. \tag{208}$$

Proof. The first inequality is immediate because $\sum_\theta \psi_t^\gamma(\theta)$ assumes non-negative integer values and $R_t(\gamma)$ is its mean value.

To show the second inequality we first use the generator given by (205) to get the Kolmogorov backwards equations for $R_t(\gamma)$:

$$\frac{d}{dt}R_t(\gamma) = \sum_{\gamma'} m(\gamma,\gamma')R_t(\gamma') - R_t(\gamma). \tag{209}$$

Since $R_0(\gamma') \equiv 1$, the solution is

$$R_t(\gamma) = \sum_{\gamma'} \Big[\exp[t(m-I)]\Big](\gamma,\gamma') \tag{210}$$

where m is the matrix with entries $m(\gamma,\gamma')$ and I is the identity matrix. This can be rewritten as

$$R_t(\gamma) \;=\; e^{-t}\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\gamma'} m^n(\gamma,\gamma') \tag{211}$$

$$\leq\; e^{-t}\sum_{n\geq 0}\frac{t^n}{n!}\,|\gamma|\,\alpha^n\;, \tag{212}$$

where the last bound is just the leftmost inequality in (201). □

8.4 Time length and space width

We are now ready to provide bounds for the time length and space width of the percolation clan.

Theorem 8.6. *If $\beta > \beta^*$ (i.e. $\alpha(\beta) < 1$), then*

(i) The probability of backward oriented percolation is zero.

(ii) For any positive b,

$$\mathbb{P}\Big(TL\,(\mathbf{A}^{x,t}) > bt\Big) \;\leq\; \alpha_0\, e^{-(1-\alpha)bt} \tag{213}$$

(iii)

$$\mathbb{E}\Big(SW(\mathbf{A}^{x,t})\Big) \;\leq\; \frac{\alpha_0(\beta)}{1-\alpha(\beta)} \tag{214}$$

(iv)

$$\mathbb{E}\Big(\exp[a\,SW(\mathbf{A}^{x,t})]\Big) \;\leq\; \frac{\alpha_0(\beta-a)}{1-\alpha(\beta-a)} \tag{215}$$

(v)

$$\mathbb{P}\Big(SW(\mathbf{A}^{x,t}) \geq \ell\Big) \;\leq\; \frac{\alpha_0(\widetilde\beta)}{1-\alpha(\widetilde\beta)}\; e^{-(\beta-\widetilde\beta)\,\ell} \tag{216}$$

for any $\widetilde\beta \in (\beta^*,\beta)$.

Proof (i) We follow an idea of Hall (1985). For each $C \in \mathbf{C}$ we use the domination (193) and the identity (199). Therefore, to prove that there is no backward oriented percolation it is enough to prove that, for fixed γ

$$\mathbb{P}\Bigg(\sum_\theta \mathbf{b}_n^\gamma(\theta) \neq 0 \text{ for infinitely many } n\Bigg) = 0. \tag{217}$$

Since $\mathbf{b}_n^\gamma(\theta)$ assumes non negative integer values, by Borel-Cantelli Lemma, a sufficient condition for (217) is

$$\sum_n\sum_\theta m^n(\gamma,\theta) < \infty. \tag{218}$$

But this follows from Lemma 8.4.

(ii) By (206) and (197) for each $x \in \gamma$ and $s \le t$

$$\sum_\theta \psi_t^\gamma(\theta) = 0 \ \text{ implies } \ \mathrm{TL}\,(\mathbf{A}^{x,0}) \le t. \tag{219}$$

Hence,

$$\begin{aligned}\mathbb{P}(\mathrm{TL}\,(\mathbf{A}^{x,0}) > t) \ &\le \ \sum_{\gamma \ni x} \mathbb{P}(\eta_0(\gamma) = 1)\, R_t(\gamma) \\ &\le \ \sum_{\gamma \ni x} w(\gamma)\, R_t(\gamma) \ \le \ \alpha_0\, e^{-(1-\alpha)t} \end{aligned}\tag{220}$$

by the rightmost inequality in (208).

(iii) We find upperbounds for the space diameter of the backwards percolation clan through upperbounds for the total number of occupied points by the multitype branching process $\mathbf{b}_n$ defined by (198). In fact,

$$\mathrm{SW}\,(\mathbf{A}^{x,0}) \le \sum_{\gamma \ni x} \eta_0(\gamma) \sum_n \sum_\theta |\theta|\, \mathbf{b}_n^\gamma(\theta). \tag{221}$$

By (**??**), $\mathbb{E}\eta_0(\gamma) \le w(\gamma)$, hence by (201)

$$\begin{aligned}\mathbb{E}\Big(\sum_{\gamma \ni x} \eta_0(\gamma) \sum_n \sum_\theta |\theta|\, \mathbf{b}_n^\gamma(\theta)\Big) \ &\le \ \sum_{\gamma \ni x} w(\gamma) \sum_n \sum_\theta |\theta|\, m^n(\gamma, \theta) \\ &\le \ \alpha_0 \sum_n \alpha^n. \end{aligned}\tag{222}$$

(iv) Write

$$\begin{aligned}\mathbb{E}(e^{a\mathrm{SW}}) &= \sum_\ell e^{a\ell}\, \mathbb{P}(\mathrm{SW} = \ell) \\ &\le \sum_\ell e^{a\ell} \sum_k \sum_{\gamma_1, \ldots, \gamma_k} \mathbf{1}\Big\{|\gamma_1 \cup \cdots \cup \gamma_k| = \ell\Big\} \\ &\qquad \times \mathbb{P}\Big(\gamma_1 \ni 0, \mathbf{b}_1^{\gamma_1}(\gamma_2) \ge 1, \ldots, \mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \ge 1\Big)\,. \end{aligned}\tag{223}$$

By the Markovian property of $\mathbf{b}_n$ we get

$$\begin{aligned}&\mathbb{P}\Big(\gamma_1 \ni 0, \mathbf{b}_1^{\gamma_1}(\gamma_2) \ge 1, \ldots, \mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \ge 1\Big) \\ &\quad = \ \mathbb{P}(\gamma_1 \ni 0)\, \mathbb{P}(\mathbf{b}_1^{\gamma_1}(\gamma_2) \ge 1) \cdots \mathbb{P}(\mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \ge 1) \\ &\quad \le \ \mathbf{1}\Big\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \ldots, \gamma_{k-1} \not\sim \gamma_k\Big\} \prod_{i=1}^k w(\gamma)_i \end{aligned}\tag{224}$$

Substituting this in (223) and using that

$$e^{a\ell}\, \mathbf{1}\Big\{|\gamma_1 \cup \cdots \cup \gamma_k| = \ell\Big\} \ \le \ \mathbf{1}\Big\{|\gamma_1 \cup \cdots \cup \gamma_k| = \ell\Big\} \exp\Big(a \sum_{i=1}^k |\gamma_i|\Big) \tag{225}$$

we get

$$\begin{aligned}\mathbb{E}(e^{a\mathrm{SW}}) \ &\le \ \sum_k \sum_{\gamma_1, \ldots, \gamma_k} \mathbf{1}\Big\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \ldots, \gamma_{k-1} \not\sim \gamma_k\Big\} \exp\Big(-(\beta - a) \sum_{i=1}^k |\gamma_i|\Big) \\ &= \ \sum_k \sum_{\gamma_1 \ni 0} |\gamma_1|\, e^{-(\beta-a)|\gamma_1|} \frac{1}{|\gamma_1|} \sum_{\gamma_2 \not\sim \gamma_1} |\gamma_2| e^{-(\beta-a)|\gamma_2|} \cdots \frac{1}{|\gamma_{k-1}|} \sum_{\gamma_k \not\sim \gamma_{k-1}} e^{-(\beta-a)|\gamma_k|} \\ &\le \ \alpha_0(\beta - a) \sum_{k \ge 0} \alpha(\beta - a)^k. \end{aligned}\tag{226}$$

(v) It suffices to use (iv) and the exponential Chevichev inequality and to notice that a must be less than $\beta - \beta^*$ to avoid a zero in the denominator of (215). $\square$

Remarks. **1.** Part (ii) can in fact be proven by a more elementary argument not requiring the continuous-time construction of Section 8.3. The argument gives the same rate of decay as in (213) but a worse leading constant. Let us sketch it.

$$\begin{aligned} &\mathbb{P}\Big(\mathrm{TL}\,(\mathbf{A}^{x,0}) > t\Big) \\ &\quad \le \sum_k \sum_{\gamma_1,\dots,\gamma_k} \mathbf{1}\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \dots, \gamma_{k-1} \not\sim \gamma_k\}\, \mathbb{P}(S_1 + \cdots + S_k > t) \end{aligned} \tag{227}$$

where S_i are independent mean one exponentially distributed random variables and independent of γ_i. The time S_i represents the period between the birth of γ_i and γ_{i+1}. As the sum of independent exponentials is a gamma distribution with parameters k and 1,

$$\mathbb{P}(S_1 + \cdots + S_k > t) \;=\; e^{-t} \sum_{i=0}^{k} \frac{t^i}{i!}\,. \tag{228}$$

Therefore (227) is bounded by

$$\begin{aligned} &e^{-t} \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{k\ge i} \sum_{\gamma_1 \ni 0} |\gamma_1|\, w(\gamma_1) \frac{1}{|\gamma_1|} \sum_{\gamma_2 \not\sim \gamma_1} |\gamma_2| w(\gamma_2) \dots \frac{1}{|\gamma_{k-1}|} \sum_{\gamma_k \not\sim \gamma_{k-1}} w(\gamma_k) \\ &\le e^{-t} \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{k\ge i} \alpha_0\, \alpha^{k-1} \\ &= \frac{\alpha_0}{\alpha(1-\alpha)}\, e^{-(1-\alpha)t}. \end{aligned} \tag{229}$$

2. In Fernández, Ferrari and Garcia (1998) we offered an alternative proof of part (iii), based on a the computation of the exponential moment of the total population of a subcritical single-type branching process, which dominates the space width. However this proof works in a smaller range of β.

Perfect simulation of invariant measures of infinite volume exclusion birth-and-death processes in finite window Λ:

Two steps

1. Construction of $\mathbf{A}^{\Lambda,0}$: cylinders alive at time 0 and their clans of ancestors. (Contact process of cylinders.)

   Under our conditions, it contains only a finite number of cylinders.

2. Cleaning algorithm: Decide, using the exclusion rule, from younger to older cylinders which cylinders are kept or erased.

   Result: $\mathbf{K}^{\Lambda,0}$ set of *kept* cylinders.

Define

$$\eta \cap \Lambda := \{\mathrm{Basis}(C) \,:\, C \in \mathbf{K}^{\Lambda,0},\, C \text{ alive at time } 0\}$$

for γ intersecting Λ.

This configuration has the marginal distribution of the infinite-volume measure μ on the set $\mathbf{G}_\Lambda = \{\theta \in \mathbf{G} : \theta \cap \Lambda \ne \emptyset\}$.

8.5 Construction of Gibbs measures

If the measure is locally absolutely continuous with respect to $\widetilde{\mu}$ when restricted to a finite region:

$$\mu(d\eta) = \frac{1}{Z} e^{-H_\Lambda(\eta)} \widetilde{\mu}(d\eta)$$

that is, the Hamiltonian acts only in the points inside some region Λ. We are interested in Gibbs measures with respect to these specifications, which are absolutely continuous with respect to a Poisson process $\tilde{\mu}$.

We propose a dynamics with rate of birth:

$$w(d\gamma) \times \frac{e^{-H(\eta\cup\gamma)}}{e^{-H(\eta)}}$$

which is the Papangelou intensity measure for μ, and rate of death 1.

The generator is now

$$Lf(\eta) = \sum_{\gamma\in\eta}[f(\eta\setminus\gamma) - f(\eta)] + \int_{\mathbf{G}} w(d\gamma)\, \frac{e^{-H(\eta\cup\gamma)}}{e^{-H(\eta)}}\, [f(\eta\cup\{\gamma\}) - f(\eta)]$$

and the semigroup

$$S_t f(\eta) := E(f(\eta_t)|\eta_0 = \eta)). \tag{230}$$

We say that μ is reversible for (η_t) if for any test function f, g, we have

$$\int d\mu\, g\, S_t f = \int d\mu\, f\, S_t g$$

Under suitable conditions, the next identity implies μ reversible.

$$\int d\mu\, g\, Lf = \int d\mu\, f\, Lg$$

These equations are consequence of the following heuristic microscopics:

$$\begin{aligned} &= (\text{weight of } \eta\cup\gamma) \times (\text{rate of } \eta\cup\gamma \to \eta). \\ (\text{weight of } \eta) &\times (\text{rate of } \eta \to \eta\cup\gamma) \end{aligned}$$

which gives:

$$e^{-H(\eta\cup\gamma)} \times 1 = e^{-H(\eta)} \times \frac{e^{-H(\eta\cup\gamma)}}{e^{-H(\eta)}}$$

We construct the dynamics in function of a Poisson process $\mathbf{C}$ on $\mathbf{G}\times\mathbb{R}\times\mathbb{R}^+\times[0,1]$ with intensity

$$w(d\gamma)\, dt\, e^{-s}\, ds\, du. \tag{231}$$

A point of this Poisson process is a *flagged cylinder*

$$C = (\gamma, T, S, U) \tag{232}$$

where $T = \text{birth}(C) \in \mathbb{R}$
$S = \text{life}(C) \sim \text{exponential}(1)$
$U = \text{flag}(C) \sim \text{Uniform}[0,1]$

We perform the same construction as before but now we include $C(\gamma, T, S, U)$ in $\mathbf{K}$ if

$$U \le M(\gamma|\eta_{T-}) \tag{233}$$

where

$$M(\gamma|\eta) = \frac{e^{-H(\eta\cup\gamma)}}{e^{-H(\eta)}}. \tag{234}$$

9 The free Bose gas

This section is based on Armendáriz, Ferrari and Yuhjtman [1].

Feynman 1953: Partition function of the free Bose Gas:

$$Z_{\Lambda,N} = \frac{1}{N!} \sum_{\sigma\in S_N} \int_{\Lambda^N} e^{-\alpha\sum_i \|x_i - x_{\sigma(i)}\|^2}\, dx_1 \dots dx_N, \tag{235}$$

$\alpha > 0$ temperature. $S_N :=$ set of permutations of $\{1, \dots, N\}$. $\Lambda :=$ bounded subset of $\mathbb{R}^d$.

Finite-volume spatial random permutation:

Configuration space:

$$(\underline{x},\sigma)\in\Lambda^N\times S_N$$

Canonical measure $G_{\Lambda,N}$:

$$G_{\Lambda,N}g := \frac{1}{Z_{\Lambda,N}}\frac{1}{N!}\sum_{\sigma\in S_N}\int_{\Lambda^N} g(\underline{x},\sigma)\, e^{-\alpha\sum_{i=1}^N \|\sigma(x_i)-x_i\|^2}dx_1\dots dx_N. \tag{236}$$

$Z_{\Lambda,N}$ = normalization (partition function), $\underline{x}=(x_1,\dots,x_N)$.

g test function

Infinite-volume spatial random permutations

We want to to construct a random spatial permutation

$$(\chi,\sigma) \text{ with law denoted } \mu_\rho$$

where χ is a point process and $\sigma:\chi\to\chi$ is a permutation, that is, a bijection.

We ask μ_ρ to be translation-invariant, with *point density* ρ and to be Gibbs for the specifications induced by $G_{\Lambda,N}$.

Loops and spatial permutations We say that $\gamma=[x_1,\dots,x_k]$ meaning $[x_2,\dots,x_k,x_1]=[x_1,\dots,x_k]$ is an **unrooted loop** of size k. The support of γ is denoted by $\{\gamma\}:=\{x_1,\dots,x_k\}$. We consider γ as a permutation of $\{\gamma\}$, by setting $\gamma(x_i)=x_{i+1}$. In particular, γ is a spatial permutation.

Observe that there is a bijection **spatial permutation** $\leftrightarrow$ **loop configuration** given by

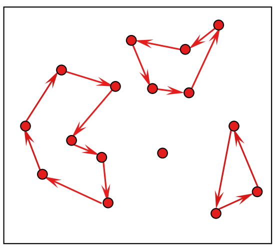
$$(\chi,\sigma)\mapsto\Gamma,$$

where $\gamma\in\Gamma$ if γ is an unrooted root satisfying $\{\gamma\}\subset\chi$ and $\gamma(x)=\sigma(x)$ for all $x\in\{\gamma\}$. Reciprocally, $\Gamma\mapsto(\chi,\sigma)$, the spatial permutation defined by $\chi=\cup_{\gamma\in\Gamma}\{\gamma\}$; $\sigma(x)=\gamma(x)$ for $x\in\{\gamma\}$, $\gamma\in\Gamma$.

Loop factorization of the spatial random permutation density. Take χ finite and observe that the weight of a spatial random permutation (χ,σ) is the product of the weights of the corresponding loops of length 1:

$$e^{-\alpha\sum_{x\in\chi}\|\sigma(x)-x\|^2} = \prod_{\gamma\in\sigma} e^{-\alpha\sum_{x\in\{\gamma\}}\|\gamma(x)-x\|^2}$$

This has been observed by Sütő [19] and suggest that a spatial random permutation can be seen as a point process, where the points are loops. Our approach is based on this somehow obvious observation.



Loops induced by a spatial random permutation in a box. An arrow from x to y means $y=\sigma(x)$. An isolated dot at x means $x=\sigma(x)$, that is, a loop of length 1.

We will *propose* a Poisson point process of loops and then will show that its distribution is a Gibbs measure for the specifications induced by $G_{\Lambda,N}$.

Gaussian loop soup in $\mathbb{R}^d$.

Space of **unrooted loops**: $D:=\Big(\cup_{k\geq1}(\mathbb{R}^d)^k\Big)/\sim$,
where the equivalence relation $\sim$ is defined by $(x_1,\dots,x_k)\sim(x_2,\dots,x_k,x_1)$; the class of equivalence of a vector $(x_1,\dots,x_k)$ is denoted $[x_1,\dots,x_k]$.

Fix an activity parameter $\lambda \in (0,1]$ and define the **loop soup intensity measure** on D:

$$Q_\lambda^{\mathrm{ls}}(d[x_1,\dots,x_k]) := \tfrac{\lambda^k}{k}\left(\tfrac{\alpha}{\pi}\right)^{kd/2} e^{-\alpha\sum_{i=0}^{k-1}\|x_i-\gamma(x_i)\|^2}\,dx_1\dots dx_k,$$

The denominator k compensates the fact that each rooted k-loop is counted k times.

Given a Borel set $A\subset\mathbb{R}^d$ and an unrooted loop $\gamma\in D_k$ with support $\{\gamma\}=\{x_0,\dots,x_{k-1}\}$, consider the cardinality of the set $A\cap\{\gamma\}$

$$n_A(\gamma) := \sum_{j=0}^{k-1}\mathbf{1}_A(x_j). \tag{237}$$

If the set has finite Lebesgue measure $|A|<\infty$, the mean density of points belonging to k-loops in A is defined as

$$\rho_{k,\lambda}(A) := \frac{1}{|A|}\int_{D_k} n_A(\gamma)\,Q_{k,\lambda}(d\gamma)\,. \tag{238}$$

where $Q_k(d\gamma)=\mathbf{1}\{\gamma\in D_k\}\,Q(d\gamma)$.

Proposition 9.1. *For any compact set $A\subset\mathbb{R}^d$,*

$$\rho_{k,\lambda}=\rho_{k,\lambda}(A)=\Big(\frac{\alpha}{\pi}\Big)^{d/2}\frac{\lambda^k}{k^{d/2}}\,. \tag{239}$$

Proof. Denote

$$p(x,y) := \Big(\frac{\alpha}{\pi}\Big)^{d/2}\exp\big(-\alpha\|x-y\|^2\big).$$

the density of a Gaussian random variable in $\mathbb{R}^d$ with mean x and covariance matrix $(2/\alpha)\mathrm{Id}$, where Id here is the identity matrix.

We have

$$\begin{aligned}
\rho_{k,\lambda}(A) &= \frac{\lambda^k}{k|A|}\int_{(\mathbb{R}^d)^k}\Big(\sum_{j=0}^{k-1}\mathbf{1}_A(x_j)\Big)\prod_{i=0}^{k-1}p(x_i,x_{i+1})\,dx_0\dots dx_{k-1}\\
&= \frac{\lambda^k}{k|A|}\sum_{j=0}^{k-1}\int_{(\mathbb{R}^d)^k}\mathbf{1}_A(x_j)\prod_{i=0}^{k-1}p(x_i,x_{i+1})\,dx_0\dots dx_{k-1}\\
&= \frac{\lambda^k}{k|A|}\sum_{j=0}^{k-1}\int_{\mathbb{R}^d}\mathbf{1}_A(x_j)\Big(\frac{\alpha}{\pi k}\Big)^{d/2}dx_j\\
&= \frac{\lambda^k}{k|A|}\Big(\frac{\alpha}{\pi k}\Big)^{d/2}k|A| = \Big(\frac{\alpha}{\pi}\Big)^{d/2}\frac{\lambda^k}{k^{d/2}}\,.
\end{aligned}$$

□

We call $\rho_{k,\lambda}$ the point density of the measure $Q_{k,\lambda}$. The density of Q_λ is defined by

$$\rho(\lambda) := \sum_{k\ge1}\rho_{k,\lambda}\,. \tag{240}$$

We have

$$\rho(\lambda) = \Big(\frac{\alpha}{\pi}\Big)^{d/2}\sum_{k\ge1}\frac{\lambda^k}{k^{d/2}}<\infty \iff \begin{cases} d\le 2 \text{ and } 0\le\lambda<1, \text{ or}\\ d\ge 3 \text{ and } 0\le\lambda\le 1\,. \end{cases} \tag{241}$$

Define the critical density ρ_c by

$$\rho_c := \Big(\frac{\alpha}{\pi}\Big)^{d/2}\sum_{k\ge1}\frac{1}{k^{d/2}}. \tag{242}$$

The function $\rho:[0,1]\to[0,\rho_c]$ is invertible; let $\lambda(\rho)$ be its inverse.

If λ and d satisfy (241), so that $\rho(\lambda)<\infty$, define the *Gaussian loop soup intensity measure* Q_λ^{ls} on D as

$$Q_\lambda^{\mathrm{ls}} := \sum_{k\ge1}Q_{k,\lambda}\,. \tag{243}$$

This measure has point density $\rho(\lambda)$.

Proposition 9.2. *Let d and λ be as in (241). Then $Q^{\rm ls}_\lambda$ is σ-finite.*

Proof. Need to show that there is a countable partition of D in sets with finite $Q^{\rm ls}_\lambda$ measure.

We start by showing that $Q_{k,\lambda}$ is σ-finite. Let $B_j \subset \mathbb{R}^d$ be the ball of radius j in $\mathbb{R}^d$, and define

$$\Upsilon_{k,j} := \big\{\gamma \in D_k : \{\gamma\} \cap B_j \neq \emptyset\big\} \subset D_k\,.$$

Then $\bigcup_j \Upsilon_{k,j} = D_k$, and

$$Q_{k,\lambda}(\Upsilon_{k,j}) \le |B_j|\,\rho_{k,\lambda}(B_j) = |B_j|\rho_{k,\lambda} < \infty,$$

by (238) and Proposition 9.1. Letting $\Upsilon_j := \bigcup_{k\ge 1}\Upsilon_{k,j}$, we have $\bigcup_{j\ge1}\Upsilon_j = D$ and

$$Q^{\rm ls}_\lambda(\Upsilon_j) \le \sum_{k\ge1} Q_{k,\lambda}(\Upsilon_{k,j}) \le \sum |B_j|\rho_{k,\lambda} = \rho|B_j| < \infty, \quad \text{by (241).} \qquad \square$$

Definition 9.3. *Let d and λ satisfy (241). Define the* ***loop-soup*** *by*

$$\begin{aligned} \Gamma^{\rm ls}_\lambda &:= \textit{ Poisson process on } D \textit{ with intensity } Q^{\rm ls}_\lambda,\\ \mu^{\rm ls}_\lambda &:= \textit{ Law of } \Gamma^{\rm ls}_\lambda. \end{aligned}$$

Analogous to Brownian loop soup. Lawler and Werner 2004, Lawler and Trujillo Ferreras 2007, Le Jan 2017.

A sample Γ of a Gaussian loop soup is a countable collection of unrooted Gaussian loops in $\mathbb{R}^d$, with the property that any compact set contains finitely many points in the supports of these loops.

The density of the loop soup Let $\mathfrak{X}$ be the set of locally finite loop soup configurations,

$$\mathfrak{X} := \{\Gamma \subset D : \textstyle\sum_{\gamma\in\Gamma} |\{\gamma\}\cap A| < \infty, \text{ for all compact } A \subset \mathbb{R}^d\}. \tag{244}$$

Given $\Upsilon \subset D$ and $\Gamma \in \mathfrak{X}$ let

$$\mathfrak{X}_\Upsilon := \big\{\Gamma \in \mathfrak{X} : \Gamma \subseteq \Upsilon\big\}. \tag{245}$$

Assume $Q^{\rm ls}_\lambda(\Upsilon) < \infty$, then, by definition of Poisson process, for measurable, bounded $g : \mathfrak{X}_\Upsilon \to \mathbb{R}$,

$$\mu^{\rm ls}_\lambda g = e^{-Q^{\rm ls}_\lambda(\Upsilon)} \sum_{\ell\ge0} \frac{1}{\ell!} \int_\Upsilon \cdots \int_\Upsilon g\big(\{\gamma_1,\dots,\gamma_\ell\}\big)\, Q^{\rm ls}_\lambda(d\gamma_1)\dots Q^{\rm ls}_\lambda(d\gamma_\ell) \tag{246}$$

$$= \sum_{\ell\ge0} \frac{e^{-Q^{\rm ls}_\lambda(\Upsilon)}}{\ell!} \int_\Upsilon \cdots \int_\Upsilon g\big(\{\gamma_1,\dots,\gamma_\ell\}\big)\, f^{\rm ls}_\lambda\big(\{\gamma_1,\dots,\gamma_\ell\}\big)\, d\gamma_1\dots d\gamma_\ell, \tag{247}$$

where

$$f^{\rm ls}_\lambda\big(\{\gamma_1,\dots,\gamma_\ell\}\big) := \textstyle\prod_{i=1}^{\ell} \omega_\lambda(\gamma_i), \tag{248}$$

$$\omega_\lambda\big([x_0,\dots,x_{k-1}]\big) := \lambda^k \textstyle\prod_{i=0}^{k-1} p(x_i,x_{i+1}) \quad \text{with } x_k = x_0, \tag{249}$$

and where for any bounded measurable $h : \Upsilon \to \mathbb{R}$,

$$\int_D h(\gamma)\, d\gamma := \sum_{k\ge1} \frac{1}{k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h\big([x_0,\dots,x_{k-1}]\big)\, dx_0\dots dx_{k-1}, \tag{250}$$

if the right hand side is well defined. Taking $h(\gamma) = h_1(\gamma)\,\omega_\lambda(\gamma)\mathbf{1}_{\{\gamma\in\Upsilon\}}$ for some bounded $h_1 : \Upsilon \to \mathbb{R}$ and recalling the assumption $Q^{\rm ls}_\lambda(\Upsilon) < \infty$, we conclude that (250) is well defined for this h and it is bounded by $\|h_1\|_\infty Q^{\rm ls}_\lambda(\Upsilon)$.

The function $e^{-Q^{\rm ls}_\lambda(\Upsilon)} f^{\rm ls}_\lambda$ is the *density* of the Gaussian loop soup at fugacity λ in the set Υ; in particular, if $g \equiv 1$ we have $\mu^{\rm ls}_\lambda g = 1$.

9.1 Finite-volume loop soup and spatial permutations

We show that the Gaussian loop soup in a compact set $A \subset \mathbb{R}^d$ conditioned to have n points has the same law as the spatial random permutation with density (236).

Recall that there is a bijection between $\mathfrak{X}$ and the set

$$\{(\chi, \sigma) : \chi \text{ locally finite and } \sigma \text{ contains only finite cycles}\} \tag{251}$$

the space of finite cycle permutations with locally finite supports.

The set of loops with supports contained in A and the space of loop-soup configurations contained in A with exactly n points are denoted by

$$D_A := \{\gamma \in D : \{\gamma\} \subset A\}, \tag{252}$$

$$\mathfrak{X}_A := \mathfrak{X}_{D_A}, \tag{253}$$

$$\mathfrak{X}_{A,n} := \big\{\Gamma \in \mathfrak{X}_A : \textstyle\sum_{\gamma\in\Gamma} |\gamma| = n\big\}. \tag{254}$$

Canonical measures The Gaussian loop soup restricted to loops contained in A is defined by

$$\mu^{\mathrm{ls}}_{A,\lambda} := \text{Poisson process on } \mathfrak{X}_A \text{ with intensity } \mathbf{1}_{D_A}(\gamma)\, Q^{\mathrm{ls}}_\lambda(d\gamma). \tag{255}$$

We now show that $\mu^{\mathrm{ls}}_{A,\lambda}$ conditioned to have n points in A equals the spatial random permutation with density $G_{A,n}$ defined in (236).

Proposition 9.4. *For any measurable, bounded test function $g : \mathfrak{X}_{A,n} \to \mathbb{R}$, we have*

$$\frac{1}{\mu^{\mathrm{ls}}_{A,\lambda}(\mathfrak{X}_{A,n})} \int_{\mathfrak{X}_{A,n}} g(\Gamma)\, \mu^{\mathrm{ls}}_{A,\lambda}(d\Gamma) = G_{A,n} g \tag{256}$$

Proof. Since there is a bijection between the supports of these probability measures, it suffices to verify that the weights assigned by their densities to any given configuration satisfy a fixed ratio. Let (χ, σ) be a spatial permutation such that $\chi \subset A$ and $|\chi| = n$. Let Γ be the cycle decomposition of (χ, σ); clearly $\Gamma \in \mathfrak{X}_{A,n}$. Then, by the definition of Poisson process, the loop soup conditioned density of $\Gamma \in \mathfrak{X}_{A,n}$ is

$$f^{\mathrm{ls}}_\lambda(\Gamma|\mathfrak{X}_{A,n}) = \frac{e^{-Q^{\mathrm{ls}}_\lambda(D_A)}}{\mu^{\mathrm{ls}}_\lambda(\mathfrak{X}_{A,n})} \prod_{\gamma\in\Gamma} \omega_\lambda(\gamma) = \frac{\lambda^n e^{-Q^{\mathrm{ls}}_\lambda(D_A)}}{\mu^{\mathrm{ls}}_\lambda(\mathfrak{X}_{A,n})} \prod_{\gamma\in\Gamma} \omega_1(\gamma), \tag{257}$$

where ω_λ was defined in (249).

On the other hand, the density of the canonical measure $G_{A,n}$ can be written as a function of the cycle decomposition of σ by

$$\frac{1}{Z_{A,n}} f_{A,n}(\chi, \sigma) = \frac{1}{Z_{A,n}}\, e^{-\alpha H(\chi,\sigma)} = \frac{1}{Z_{A,n}} \prod_{\gamma\in(\chi,\sigma)} \omega_1(\gamma). \qquad \square$$

Grand-canonical measures The grand-canonical spatial random permutation at fugacity $\lambda \le 1$ associated to the canonical density (236) is defined by

$$\mu_{A,\lambda}\, g := \frac{1}{Z_{A,\lambda}} \sum_{n\ge 0} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{A^n} g(\underline{x}, \sigma) e^{-\alpha H(\underline{x},\sigma)} d\underline{x}, \tag{258}$$

where $\underline{x} = (x_1, \dots, x_n)$, $H(\underline{x}, \sigma) := \sum_{i=1}^n \|x_i - x_{\sigma(i)}\|^2$ and

$$Z_{A,\lambda} := \sum_{n\ge 0} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{A^n} e^{-\alpha H(\underline{x},\sigma)} d\underline{x}.$$

Proposition 9.5. *Let $\lambda \le 1$. The Gaussian loop soup at fugacity λ restricted to A defined in (255) and the grand-canonical measure (258) at the same fugacity are equivalent:*

$$\mu^{\mathrm{ls}}_{A,\lambda} = \mu_{A,\lambda}. \tag{259}$$

Proof. We look at both measures as point processes in D_A. To prove the proposition it suffices to show that the measures have the same Laplace functionals. Given $\psi : D_A \to \mathbb{R}_+$, define $g : \mathfrak{X}_A \to \mathbb{R}$ as

$$g(\Gamma) := \exp\Big(-\textstyle\sum_{\gamma\in\Gamma}\psi(\gamma)\Big).$$

By Campbell's theorem,

$$\begin{aligned}
\mu^{\mathrm{ls}}_{A,\lambda} g &= \int_{\mathfrak{X}_A} \mu^{\mathrm{ls}}_\lambda(d\Gamma) e^{-\sum_{\gamma\in\Gamma}\psi(\gamma)} = \exp\Big(\int_{D_A} (e^{-\psi(\gamma)}-1) Q^{\mathrm{ls}}_\lambda(d\gamma)\Big) \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \exp\Big(\sum_{k\geq 1}\frac{1}{k!}a_k\Big),
\end{aligned}$$

where, using the definition of $Q_{k,\lambda}$ and denoting $\tilde\lambda := (\alpha/\pi)^{d/2}\lambda$,

$$\begin{aligned}
a_k &:= \tilde\lambda^k (k-1)! \int_{A^k} dx_1\dots dx_k\, e^{-\alpha H([x_1,\dots,x_k])}\, e^{-\psi([x_1,\dots,x_k])} \\
&= \tilde\lambda^k \sum_{\gamma\in\mathcal{C}_k}\int_{A^k} dx_1\dots dx_k\, e^{-\alpha H(\underline{x},\gamma)}\, e^{-\psi(\underline{x},\gamma)}
\end{aligned}$$

where $\mathcal{C}_k$ is the set of cycles of size k with elements $\{1,\dots,k\}$, $\underline{x} = (x_1,\dots,x_n)$, and $(\underline{x},\gamma) := [x_1, x_{\gamma(1)},\dots,x_{\gamma^{k-1}(1)}]$; notice that $\mathcal{C}_k$ has cardinality $(k-1)!$. By Lemma 9.6 below we have

$$\begin{aligned}
\mu^{\mathrm{ls}}_{A,\lambda} g &= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0}\frac{1}{n!}\sum_{P\in\mathcal{P}_n}\prod_{I\in P} a_{|I|} \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0}\frac{\tilde\lambda^n}{n!}\sum_{P\in\mathcal{P}_n}\prod_{I\in P}\sum_{\gamma\in\mathcal{C}_{|I|}}\int_{A^{|I|}} dx_1\dots dx_{|I|} e^{-\alpha H([x_1,\dots,x_{|I|}])}\, e^{-\psi([x_1,\dots,x_{|I|}])} \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0}\frac{\tilde\lambda^n}{n!}\sum_{\sigma\in\mathcal{S}_n}\prod_{\gamma\in\sigma}\int_{A^{|\gamma|}} dx_1\dots dx_{|\gamma|}\, e^{-\alpha H(\underline{x},\gamma)} e^{-\psi(\underline{x},\gamma)} \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0}\frac{\tilde\lambda^n}{n!}\sum_{\sigma\in\mathcal{S}_n}\int_{A^n} dx_1\dots dx_n\, e^{-\alpha H(\underline{x},\sigma)}\prod_{\gamma\in\sigma} e^{-\psi((x_i : i\in\{\gamma\}),\gamma)} \\
&= \mu_{A,\lambda} g,
\end{aligned}$$

where $(\underline{x},\sigma)$ is the spatial permutation that maps x_i to $x_{\sigma(i)}$ and $\{\gamma\}$ is the set of indices that appear in the cycle γ. ☐

Lemma 9.6 (Combinatorial lemma). *Let $(a_n)_{n\geq 1}\in\mathbb{C}^{\mathbb{N}}$ be such that $\sum_{n\geq 1}\frac{1}{n!}|a_n| < \infty$. Then*

$$\exp\Big(\sum_{n\geq 1}\frac{1}{n!}\,a_n\Big) = 1 + \sum_{n\geq 1}\frac{1}{n!}\sum_{P\in\mathcal{P}_n}\prod_{I\in P} a_{|I|}, \tag{260}$$

where $\mathcal{P}_n$ is the set of partitions of $\{1,\dots,n\}$ into non-empty sets,

$$\mathcal{P}_n = \Big\{P \text{ partition of } \{1,\dots,n\} : \emptyset\notin P\Big\}, \tag{261}$$

and $|I|$ stands for the cardinality of the set I.

Proof. By the series expansion of the exponential function

$$\begin{aligned}
\exp\Big(\sum_{n\geq 1}\frac{1}{n!}\,a_n\Big) &= \sum_{j\geq 0}\frac{1}{j!}\Big(\sum_{\ell\geq 1}\frac{1}{\ell!}a_l\Big)^j \\
&= 1 + \sum_{j\geq 1}\frac{1}{j!}\sum_{\substack{i_1,\dots,i_j \\ i_\ell\geq 1}}\frac{a_{i_1}}{i_1!}\cdots\frac{a_{i_j}}{i_j!} \\
&= 1 + \sum_{j\geq 1}\frac{1}{j!}\sum_{n\geq 1}\sum_{\substack{i_1+\dots+i_j=n \\ i_\ell\geq 1}}\frac{a_{i_1}}{i_1!}\cdots\frac{a_{i_j}}{i_j!}
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{j\geq 1} \frac{1}{j!} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{i_1+\dots+i_j=n\\ i_\ell\geq 1}} \binom{n}{i_1 \dots i_j} a_{i_1}\dots a_{i_j} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{j\geq 1} \frac{1}{j!} \sum_{\substack{i_1+\dots+i_j=n\\ i_\ell\geq 1}} \binom{n}{i_1 \dots i_j} a_{i_1}\dots a_{i_j} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{j\geq 1} \sum_{P=\{I_1,\dots,I_j\}\in\mathcal{P}_n} a_{|I_1|}\dots a_{|I_j|} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} a_{|I|}.
\end{aligned}
$$

The change of the order of summation in the fifth line above is justified by the absolute convergence of $\sum_{n\geq 1} \frac{1}{n!} a_n$. □

Point correlations of the loop soop Define

$$
K_\lambda(x,y) := \sum_{k\geq 1} \left(\frac{\alpha}{\pi k}\right)^{d/2} \lambda^k\, e^{-\frac{\alpha}{k}\|x-y\|^2}. \tag{262}
$$

Proposition 9.7 (Point correlations)**.** *The n-point correlation density of $\nu_\lambda^{\mathrm{ls}}$ is given by*

$$
\varphi_\lambda^{\mathrm{ls}}(x_1,\dots,x_n) = perm\big(K_\lambda(x_i,x_j)\big)_{i,j=1}^n, \tag{263}
$$

where $perm(A)$ is the permanent of the matrix $A\in\mathbb{R}^{n\times n}$.

Sketch of the proof. We compute the 3-point correlation density. To simplify notation, in this proof we will denote $\mu = \mu_\lambda^{\mathrm{ls}}$, $Q = Q_\lambda^{\mathrm{ls}}$ and $K_{xy} = K_\lambda(x,y)$. Given pairwise disjoint bounded Borel sets $A,B,C\subset\mathbb{R}^d$, the third moment measure for the point marginal $\nu_\lambda^{\mathrm{ls}}$ over $A\times B\times C$ is given by

$$
\int n_A(\Gamma)\, n_B(\Gamma)\, n_C(\Gamma)\, \mu(d\Gamma) \tag{264}
$$

$$
\begin{aligned}
&= \int \textstyle\sum_{\gamma\in\Gamma} n_A(\gamma) \sum_{\gamma'\in\Gamma} n_B(\gamma') \sum_{\gamma''\in\Gamma} n_C(\gamma'')\, \mu(d\Gamma) \\
&= \int \Big(\textstyle\sum_{\gamma\in\Gamma} n_A(\gamma)\, n_B(\gamma)\, n_C(\gamma) \\
&\quad + \textstyle\sum_{\gamma\in\Gamma} n_A(\gamma) \sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma} n_B(\gamma')\, n_C(\gamma') \\
&\quad + \textstyle\sum_{\gamma\in\Gamma} n_B(\gamma) \sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma} n_A(\gamma')\, n_C(\gamma') \\
&\quad + \textstyle\sum_{\gamma\in\Gamma} n_C(\gamma) \sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma} n_A(\gamma')\, n_B(\gamma') \\
&\quad + \textstyle\sum_{\gamma\in\Gamma} n_A(\gamma) \sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma} n_B(\gamma') \sum_{\gamma''\in\Gamma\setminus\{\gamma,\gamma'\}} n_C(\gamma'') \Big)\mu(d\Gamma)
\end{aligned} \tag{265}
$$

$$
\begin{aligned}
&= Q(n_A\, n_B\, n_C) + Q(n_A)\, Q(n_B\, n_C) \\
&\quad + Q(n_B)\, Q(n_A\, n_B) + Q(n_C)\, Q(n_A\, n_B) + Q(n_A)\, Q(n_B)\, Q(n_C).
\end{aligned} \tag{266}
$$

To go from (265) to (266) we use that μ is a Poisson process of loops, then

(a) the expectation of the product of functions of different loops factorize (Theorem 3.2 in [17]), and

(b) $\int \sum_{\gamma\in\Gamma} g(\gamma)\mu(d\Gamma) = \int_D g(\gamma) Q(d\gamma)$, denoted $Q(g)$, by Campbell's theorem.

Define

$$
\langle a_1 \dots a_k\rangle := \big\{\gamma\in D : \gamma \text{ goes through } a_1,\dots,a_k \text{ in this order}\big\} \tag{267}
$$

and compute

$$
Q(n_A\, n_B\, n_C) = \int \textstyle\sum_{a\in\gamma} \mathbf{1}_A(a) \sum_{b\in\gamma} \mathbf{1}_B(b) \sum_{c\in\gamma} \mathbf{1}_C(c)\, Q(d\gamma) \tag{268}
$$

$$
= \int \sum_{\substack{\{a,b,c\}\subset\{\gamma\}\\ a\in A, b\in B, c\in C}} \big(\mathbf{1}_{\langle abc\rangle}(\gamma) + \mathbf{1}_{\langle acb\rangle}(\gamma)\big)\, Q(d\gamma) \tag{269}
$$

$$= \int_A \int_B \int_C (K_{ab}K_{bc}K_{ca} + K_{ac}K_{cb}K_{ba})\, dc\, db\, da, \tag{270}$$

where (269) follows from partitioning the set of cycles that go through a, b, c according to the order in which they visit the points, and (270) can be proved using the argument applied to compute the density ρ.

Using the same argument to compute the other terms in (266), we conclude that the third moment measure (264) is absolutely continuous with respect to Lebesgue measure in $(\mathbb{R}^d)^3$ with Radon-Nikodym derivative

$$\begin{aligned}\varphi_\lambda^{\mathrm{ls}}(x,y,z) = {} & K_{xx}K_{yy}K_{zz} + K_{xx}K_{yz}K_{zy} + K_{xy}K_{yx}K_{zz} \\ & + K_{xy}K_{yz}K_{zx} + K_{xz}K_{yx}K_{zy} + K_{xz}K_{yy}K_{zx},\end{aligned}$$

which proves (263) for $n = 3$; see Fig. 2. We leave the proof of the general case to the reader. $\square$

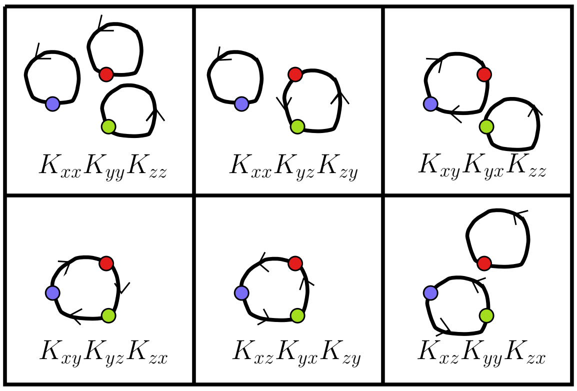


Figure 2: Gaussian loop soup 3-point correlations. A directed lace between two points means that the loop goes through the points in the indicated order. Point x is blue, y is red and z is green.

9.2 Gaussian interlacements

We work now in $d \geq 3$.

Space of Doubly infinite trajectories:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{R}^d,\ \lim_{n\to\pm\infty} \|w(n)\| = \infty \right\}$$

Define $X_n(w) := w(n)$, the position of the walk at time n.

Denote $P^x :=$ distribution of a double infinite random walk with Normal$(0, \frac{1}{2\alpha}\mathrm{Id})$ increments, starting at x.

Intensity via capacity (Sznitman): Define the entrance time of w in a compact set A by:

$$T_A(w) := \inf \left\{ n \in \mathbb{Z},\ X_n(w) \in A \right\} \in\ (-\infty, \infty].$$

For a test function $g : W \to \mathbb{R}$, define

$$Q_A^{\mathrm{cap}} g := \int_A E^x \left[g\, \mathbf{1}_{\{T_A = 0\}} \right] dx.$$

The measure $e_A(x) := P^x[T_A = 0]$ is called *equilibrium measure* and its integral $\int_A e_A(x)\, dx$ is called *Capacity* of A.

Intensity via visit debiasing: Denote the number of visits to A by

$$n_A(w) := \sum_{n\in\mathbb{Z}} \mathbf{1}_A(X_n(w))$$

Define:

$$Q_A^{\text{unif}} g := \int_A E^x \left[\frac{g}{n_A} \right] dx.$$

Under this measure, the weight of a trajectory intersectig A is inversely proportional to the number of visits to A.

Define the time shift θ on W by $[\theta w](k) := w(k+1)$; the time shift acts on functions by $(\theta g)(w) := g(\theta w)$. Since the Lebesgue measure is reversible for the random walk, we have

$$\int_{\mathbb{R}^d} E^x[g]\, dx = \int_{\mathbb{R}^d} E^x[\theta^i g]\, dx, \quad \text{for any } i \in \mathbb{Z}, \tag{271}$$

for any bounded measurable test function $g : W \to \mathbb{R}$.

Proposition 9.8. *For any bounded set $A \subset \mathbb{R}^d$ and measurable bounded function $g : W \to \mathbb{R}$ invariant under time shifts, $g = \theta g$, we have*

$$Q_A^{\text{unif}} g = Q_A^{\text{cap}} g. \tag{272}$$

Proof. Write

$$\begin{aligned}
Q_A^{\text{unif}} g &= \int_A dx\, E^x \left[\frac{g}{n_A} \textstyle\sum_{i \le 0} \mathbf{1}_{\{T_A = i\}} \right] \\
&= \sum_{i \le 0} \int_{\mathbb{R}^d} dx\, E^x \left[\mathbf{1}_A(X_0) \mathbf{1}_{\{T_A = i\}} \frac{g}{n_A} \right] && \text{by Fubini} \\
&= \sum_{i \ge 0} \int_{\mathbb{R}^d} dx\, E^x \left[\mathbf{1}_A(X_i) \mathbf{1}_{\{T_A = 0\}}\, \theta^i \Big(\frac{g}{n_A} \Big) \right] && \text{by (271)} \\
&= \int_A dx\, E^x \left[\mathbf{1}_{\{T_A = 0\}} \frac{g}{n_A} \textstyle\sum_{i \ge 0} \mathbf{1}_A(X_i) \right] && \text{since } \theta^i \Big(\frac{g}{n_A} \Big) = \frac{g}{n_A} \\
&= \int_A dx\, E^x \left[\mathbf{1}_{\{T_A = 0\}}\, g \right] = Q_A^{\text{cap}} g.
\end{aligned}$$

□

We also have that the measures Q_A^{cap} and Q_A^{unif} are finite:

$$Q_A^{\text{unif}}(W) = Q_A^{\text{cap}}(W) = \text{cap}(A) \le |A|. \tag{273}$$

Lemma 9.9. *Let $A \subset B$ be bounded sets of $\mathbb{R}^d$, and let g be a test function that is invariant under time shifts, $g = \theta g$. Then*

$$Q_B^{\text{cap}} g \mathbf{1}_{\{T_A < \infty\}} = Q_A^{\text{cap}} g \qquad \textit{(compatibility)} \tag{274}$$

$$Q_B^{\text{cap}} g = Q_A^{\text{cap}} g + Q_{B \setminus A}^{\text{cap}} g \mathbf{1}_{\{T_A = \infty\}} \qquad \textit{(additivity)}, \tag{275}$$

The same holds for Q^{unif}.

Proof. Writing $\mathbf{1}_{\{T_A < \infty\}} = \sum_{i \in \mathbb{Z}} \mathbf{1}_{\{T_A = i\}}$ we have

$$\begin{aligned}
Q_B^{\text{cap}} g \mathbf{1}_{\{T_A < \infty\}} &= \sum_{i \ge 0} \int dx\, \mathbb{E}^x \left[\mathbf{1}_{\{T_B = 0\}} \mathbf{1}_{\{T_A = i\}}\, g \right] \\
&= \sum_{i \ge 0} \int dx\, \mathbb{E}^x \left[\theta^i \big(\mathbf{1}_{\{T_B = -i\}} \mathbf{1}_{\{T_A = 0\}}\, g \big) \right] \\
&= \sum_{j \le 0} \int dx\, \mathbb{E}^x \left[\mathbf{1}_{\{T_B = j\}} \mathbf{1}_{\{T_A = 0\}}\, g \right] \\
&= \int dx\, \mathbb{E}^x \left[\mathbf{1}_{\{T_A = 0\}}\, g \textstyle\sum_{i \le 0} \mathbf{1}_{\{T_B = i\}} \right] = Q_A^{\text{cap}} g,
\end{aligned}$$

since $\mathbf{1}_{\{T_A = 0\}} \sum_{i \le 0} \mathbf{1}_{\{T_B = i\}} = \mathbf{1}_{\{T_A = 0\}}$. This proves (274). To get (275) write

$$\begin{aligned}
Q_B^{\text{cap}} g &= Q_B^{\text{cap}} g (\mathbf{1}_{\{T_A < \infty\}} + \mathbf{1}_{\{T_A = \infty\}}) \\
&= Q_A^{\text{cap}} g + Q_B^{\text{cap}} g \mathbf{1}_{\{T_A = \infty\}} = Q_A^{\text{cap}} g + Q_{B \setminus A}^{\text{cap}} g \mathbf{1}_{\{T_A = \infty\}}.
\end{aligned}$$

Since $g \mathbf{1}_{\{T_A < \infty\}} = \theta(g \mathbf{1}_{\{T_A < \infty\}})$, $g = \theta g$ and $g \mathbf{1}_{\{T_A = \infty\}} = \theta(g \mathbf{1}_{\{T_A = \infty\}})$, Proposition 9.8 implies that (274) and (275) hold for Q^{unif} as well. □

Now let us identify trajectories that differ by time shift: given two doubly-infinite trajectories $w, w' \in W$, we say that $w \sim w'$ if there exists $k \in \mathbb{Z}$ such that $w' = \theta^k w$. Let

$$\widetilde{W} := W/\sim \tag{276}$$

be the space of trajectories modulo time shift, $\pi : W \to \widetilde{W}$ the projection, and $\widetilde{\mathcal{W}}$ the push-forward σ-algebra on $\widetilde{W}$.

Given $A \subset \mathbb{R}^d$ let

$$\begin{aligned} W_A &:= \{w \in W : T_A(w) < \infty\}, \quad \text{trajectories intersecting } A \\ \widetilde{W}_A &:= \pi(W_A), \quad \text{classes of trajectories intersecting } A \end{aligned}$$

If $g : W \to \mathbb{R}$ is shift invariant then it can be extended to $\tilde{g} : \widetilde{W} \to \mathbb{R}$ by $g(\tilde{w}) = g(w)$, for any choice of representative $w \in \pi^{-1}(\tilde{w})$.

Proposition 9.10. *There exists a unique σ-finite measure Q^{ri} on $(\widetilde{W}, \widetilde{\mathcal{W}})$ such that for each bounded set $A \subset \mathbb{R}^d$*

$$\mathbf{1}_{\widetilde{W}_A} Q^{\mathrm{ri}} = \pi_* Q_A^{\mathrm{cap}} = \pi_* Q_A^{\mathrm{unif}}, \tag{277}$$

where $\pi_* Q_A^{\mathrm{cap}}$ and $\pi_* Q_A^{\mathrm{unif}}$ denote the push-forward measures defined by

$$(\pi_* Q_A^{\mathrm{cap}})\tilde{g} := Q_A^{\mathrm{cap}}(\tilde{g} \circ \pi) \tag{278}$$

Proof. Let $\tilde{g} : \widetilde{W} \to \mathbb{R}$ and define $g : W \to \mathbb{R}$ by $g = \tilde{g} \circ \pi$ (composition). Then $\theta g = g$ and

$$\pi_* Q_A^{\mathrm{cap}} \tilde{g} = Q_A^{\mathrm{cap}} \tilde{g} \circ \pi = Q_A^{\mathrm{cap}} g = Q_A^{\mathrm{unif}} g = Q_A^{\mathrm{unif}} \tilde{g} \circ \pi = \pi_* Q_A^{\mathrm{unif}} \tilde{g},$$

by Proposition 9.8. This proves the second equality in (277).

Let $\{A_n\}_{n \geq 1}$ be an increasing sequence of bounded Borel sets in $\mathbb{R}^d$ such that $A_n \nearrow_{n \to \infty} \mathbb{R}^d$. Then $\widetilde{W} = \bigcup_{n \geq 1} \widetilde{W}_{A_n}$ and uniqueness of the measure satisfying (277) follows. Define Q^{ri} on $\widetilde{W}_{A_n}$ by

$$\mathbf{1}_{\widetilde{W}_{A_n}} Q^{\mathrm{ri}} := \pi_* Q_{A_n}^{\mathrm{cap}}. \tag{279}$$

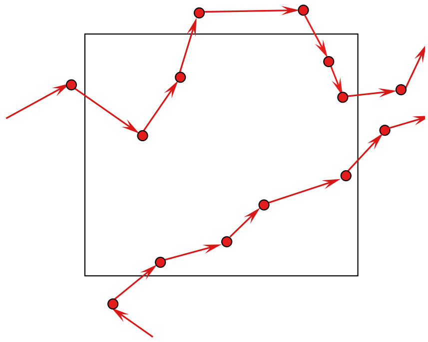
Let A be a bounded set and take n sufficiently large such that $A_n \supset A$. Then

$$\mathbf{1}_{\widetilde{W}_A} \mathbf{1}_{\widetilde{W}_{A_n}} \left(\pi_* Q_{A_n}^{\mathrm{cap}}\right) = \mathbf{1}_{\widetilde{W}_A} \left(\pi_* Q_{A_n}^{\mathrm{cap}}\right) = \pi_* \mathbf{1}_{W_A} Q_{A_n}^{\mathrm{cap}} = \pi_* Q_A^{\mathrm{cap}}, \tag{280}$$

where the last identity follows from (274). When $A = A_m$ for some $m < n$, (280) proves that the definition (279) is consistent and that the measure Q^{ri} defined in (279) satisfies (277). By (277), $Q^{\mathrm{ri}}(\widetilde{W}_{A_n}) = Q_{A_n}^{\mathrm{cap}}(W) = \mathrm{cap}(A_n) < \infty$, which proves the σ-finite property. $\square$

Gaussian interlacements Let $d \geq 3$ and $\beta > 0$. The Gaussian random interlacements process at point density ρ is

$$\begin{aligned} \Gamma_\rho^{\mathrm{ri}} &:= \text{ Poisson process on } \widetilde{W} \text{ with intensity } \rho Q^{\mathrm{ri}}, \\ \mu_\rho^{\mathrm{ri}} &:= \text{ Law of } \Gamma_\rho^{\mathrm{ri}}. \end{aligned}$$



Like Sznitman 2010 Brownian interlacements.

Construction of a random interlacement at density ρ

1. Sample a Poisson process χ_0 of parameter ρ.
2. Take a bounded box Λ
3. To each point in $\chi_0 \cap \Lambda$ sample a double-infinity Gaussian walk.
4. Accept the walk with probability 1 over number of visits to Λ.

4' (Alternative to item (4).Accept the walk if $T_\Lambda(w) = 0$.

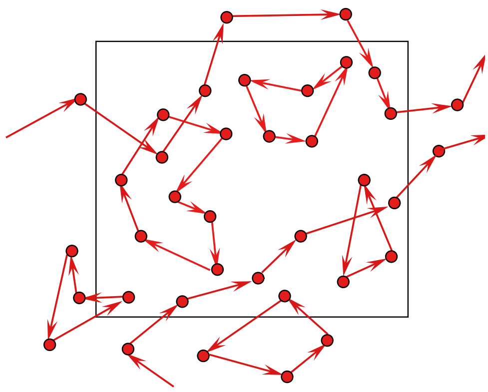
5. The accepted walks will be a sample of the random interlacement intersecting Λ.
6. To sample in $\mathbb{R}^d$, consider a partition $(\Lambda_j)_{j\geq 1}$ of $\mathbb{R}^d$ with Λ_j bounded.
7. Perform the procedure (1) to (5) in each $\Lambda_1, \Lambda_2, \dots$ successively.
8. Reject walks with starting point in Λ_j that have points in previous visited boxes.

Gaussian permutation at density $\rho > 0$ Define

$$(\chi, \sigma)_\rho := \Gamma^{\mathrm{ls}}_{\lambda(\rho \wedge \rho_c)} \cup \Gamma^{\mathrm{ri}}_{(\rho - \rho_c)^+}$$

This is a superposition of independent realizations of:

- Gaussian loup soup at density $\min\{\rho, \rho_c\}$
- Gaussian interlacement at density $(\rho - \rho_c)^+$.



The Gaussian random permutation is Markov and Gibbs $\Lambda \subset \mathbb{R}^d$ and a spatial permutation $\Gamma = (\zeta, \kappa) \in \mathfrak{X}$,

$$\begin{aligned}
I_\Lambda \zeta &:= \zeta \cap \Lambda, && \text{red points}\\
O_\Lambda \zeta &:= \zeta \cap \Lambda^c, && \text{purple and yellow points}\\
U_\Lambda \zeta &:= \{u \in \zeta \cap \Lambda^c : \kappa(u) \in \Lambda\}, && \text{yellow}\\
V_\Lambda \zeta &:= \{v \in \zeta \cap \Lambda^c : \kappa^{-1}(v) \in \Lambda\}, && \text{yellow}
\end{aligned}$$

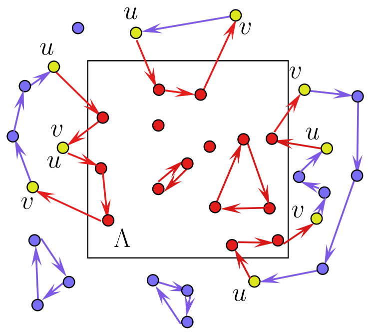
and the maps

$$\begin{aligned}
I_\Lambda \kappa : I_\Lambda \zeta \cup U_\Lambda \zeta \to I_\Lambda \zeta \cup V_\Lambda \zeta, &\quad I_\Lambda \kappa(x) = \kappa(x) \quad \text{red arrows,}\\
O_\Lambda \kappa : O_\Lambda \zeta \setminus U_\Lambda \zeta \to O_\Lambda \zeta \setminus V_\Lambda \zeta, &\quad O_\Lambda \kappa(x) = \kappa(x) \quad \text{purple arrows.}
\end{aligned}$$

Define the inside and outside projections (with respect to Λ) by

$$I_\Lambda(\zeta, \kappa) := (I_\Lambda \zeta, I_\Lambda \kappa), \qquad O_\Lambda(\zeta, \kappa) := (O_\Lambda \zeta, O_\Lambda \kappa).$$

Decomposition of a loop soup intersecting Λ



Inside = red; outside = purple + yellow

Markov property

Proposition 9.11. *The Gaussian random permutation μ_ρ is Markov:*

$$\begin{aligned}&\mu_\rho\big(dI_\Lambda(\Gamma)\big|\, O_\Lambda(\Gamma) \text{ occurs outside } \Lambda\big)\\ &\qquad = \mu_\rho\big(dI_\Lambda(\Gamma)\big|\, (U_\Lambda, V_\Lambda) \text{ occur outside } \Lambda\big).\end{aligned}$$

Proof in the subcritical case (for the Loop soup) Conditioning on purple and yellow, the law of red points and arrows depends only on the labeled yellow points. Conditioned on labeled yellow points, purple points and arrows are independent of red points and arrows.

Recall the notation $D_A := \{\gamma \in D : \{\gamma\} \subset A\}$, and denote $\partial D_A := \big(D_A \cup D_{A^c}\big)^c$. Note that the restricted Gaussian loop soups

$$\begin{aligned}&\Gamma_\lambda^{\rm ls} \cap D_A \quad \text{(loops contained in } A),\\ &\Gamma_\lambda^{\rm ls} \cap D_{A^c} \quad \text{(loops contained in } A^c),\\ &\Gamma_\lambda^{\rm ls} \cap \partial D_A \quad \text{(loops intersecting } A \text{ and } A^c)\end{aligned} \tag{281}$$

are independent Poisson processes with intensity measures $Q_\lambda^{\rm ls}\mathbf{1}_{D_A}$, $Q_\lambda^{\rm ls}\mathbf{1}_{D_{A^c}}$, $Q_\lambda^{\rm ls}\mathbf{1}_{\partial D_A}$, respectively, and form a partition of $\Gamma_\lambda^{\rm ls}$.

Due to the independence of the partition (281), the inside and outside components of $\Gamma_\lambda^{\rm ls}$ are partitioned into independent pieces as follows,

$$I_A(\Gamma_\lambda^{\rm ls}) = \big(\Gamma_\lambda^{\rm ls} \cap D_A\big) \,\dot\cup\, \partial I_A(\Gamma_\lambda^{\rm ls}), \tag{282}$$

$$O_A(\Gamma_\lambda^{\rm ls}) = \big(\Gamma_\lambda^{\rm ls} \cap D_{A^c}\big) \,\dot\cup\, \partial O_A(\Gamma_\lambda^{\rm ls}), \tag{283}$$

where

$$\begin{aligned}\partial I_A(\Gamma) := \big\{&\eta = (u, x_1, \dots, x_{\ell(\eta)}, v) :\\ &u \in U_A(\Gamma),\, x_i = \kappa^i(u) \in A,\, v = \kappa^{\ell(\eta)+1}(u) \in V_A(\Gamma)\big\},\end{aligned} \tag{284}$$

$$\begin{aligned}\partial O_A(\Gamma) := \big\{&\eta' = (v, y_1, \dots, y_{\ell(\eta')}, u) :\\ &v \in V_A(\Gamma),\, y_i = \kappa^i(v) \in A^c,\, u = \kappa^{\ell(\eta')+1}(v) \in U_A(\Gamma)\big\},\end{aligned} \tag{285}$$

where $\ell(\eta) = \min\{\ell \geq 1 : \kappa^{\ell+1}(u) \in V_A(\Gamma)\}$ and $\ell(\eta') = \min\{\ell \geq 0 : \kappa^{\ell+1}(v) \in U_A(\Gamma)\}$. These numbers count the number of points visited by the associated path, excluding the endpoints u and v.

In the figure, an element of $\partial I_A(\Gamma)$ is given by a red path linking two yellow points with labels u and v respectively, while an element of $\partial O_A(\Gamma)$ is a purple path that links two yellow points v and u. Each path in the inside boundary $\partial I_A(\Gamma)$ contains at least one point in A, so that $\ell(\eta) \geq 1$, while the outside boundary $\partial O_A(\Gamma)$ might contain a path (v, u) with $v \in V_A$, $u \in U_A$; $\ell(\eta') = 0$ in this case. There is no path when $u = v \in U_A \cap V_A$.

Given $\eta = (u, x_1, \dots, x_\ell, v) \in \partial I_A(\Gamma)$ and $\eta' = (v, y_1, \dots, y_\ell, u) \in \partial O_A(\Gamma)$, consider the weights

$$\omega(\eta) := p(u, x_1)\, p(x_\ell, v) \prod_{i=1}^{\ell-1} p(x_i, x_{i+1}) \qquad \ell \geq 1, \tag{286}$$

$$\omega(\eta') := p(v, y_1)\, p(y_\ell, u) \prod_{i=1}^{\ell-1} p(y_i, y_{i+1}) \qquad \ell \geq 1, \tag{287}$$

$$\omega(v, u) := p(v, u) \qquad \ell = 0, \tag{288}$$

where p the Brownian transition density at time $\frac{2}{\alpha}$. By (249), the weight of a cycle $\gamma = [x_0, \ldots, x_{n-1}]$ is given by

$$\omega_\lambda(\gamma) = \lambda^n \prod_{i=0}^{n-1} p(x_i, x_{i+1}), \quad \text{with } x_n = x_0. \tag{289}$$

If γ intersects both A and A^c, this weight factorizes as

$$\omega_\lambda(\gamma) = \lambda^n \prod_{\eta \in \partial I_A(\gamma)} \omega(\eta) \prod_{\eta' \in \partial O_A(\gamma)} \omega(\eta'). \tag{290}$$

Replacing (290) in (248) we obtain

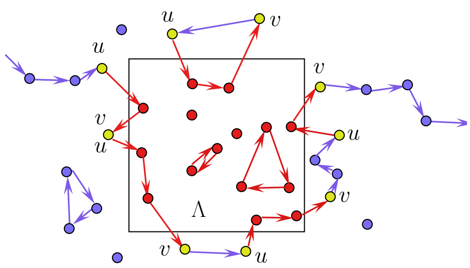
$$\begin{aligned} f_\lambda^{\text{ls}}(\Gamma \cap \partial D_A) &= e^{-Q_\lambda^{\text{ls}}(\partial D_A)}\, \lambda^{|U_A(\Gamma) \cup V_A(\Gamma)|} \\ &\qquad \prod_{\gamma \in \Gamma \cap \partial D_A} \Big[\prod_{\eta \in \partial I_A(\gamma)} \omega(\eta)\, \lambda^{\ell(\eta)} \prod_{\eta' \in \partial O_A(\gamma)} \omega(\eta') \lambda^{\ell(\eta')} \Big] \\ &= e^{-Q_\lambda^{\text{ls}}(\partial D_A)}\, \lambda^{|U_A(\Gamma) \cup V_A(\Gamma)|} \\ &\qquad \prod_{\eta \in \partial I_A(\Gamma)} \lambda^{\ell(\eta)} \omega(\eta) \prod_{\eta' \in \partial O_A(\Gamma)} \lambda^{\ell(\eta')} \omega(\eta')\,. \end{aligned} \tag{291}$$

In view of the partition into independent processes (281) and the representation (291) above, we conclude that

$$\begin{aligned} \mu_\rho\big(dI_A(\Gamma) \big| O_A(\Gamma)\big) = \frac{1}{Z}\, & f_\lambda^{ls}(\Gamma \cap D_A)\, dx_1 \ldots dx_{|\zeta \cap A|} \\ & \prod_{\eta \in \partial I_A(\Gamma)} \lambda^{\ell(\eta)} \omega(\eta)\, dx_1^\eta \ldots dx_{\ell(\eta)}^\eta, \end{aligned} \tag{292}$$

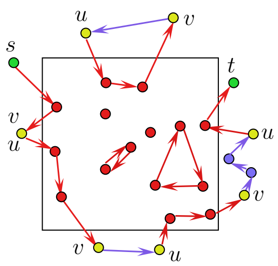
where Z is a normalizing constant $Z\big(\alpha, \lambda, A, U_A(\Gamma), V_A(\Gamma)\big)$. Identity (292) above implies that the conditioned measure on the left only depends on the sets $U_A(\Gamma)$ and $V_A(\Gamma)$, proving the Markov property in the critical and subcritical cases $\rho \leq \rho_c$.

Supercritical case



In this case one cuts the part of the infinite trajectories that are outside Λ and not directly connected to Λ and proceeds in the same way as in the loop-soup.

An important remark is that the intensity $\lambda = 1$ in this case makes equally distributed the pieces η in the inside of Λ for both loops and infinite trajectories.

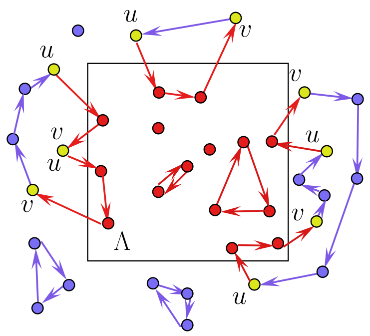


Gaussian random permutation is Gibbs

Λ-compatibility between infinite volume random permutations:

$$(\chi, \sigma) \sim_\Lambda (\zeta, \kappa)$$

if they have same yellow points and purple points and arrows.



Conditioned Hamiltonian: For $(\chi, \sigma) \sim_\Lambda (\zeta, \kappa)$

$$H_\Lambda((\chi, \sigma)|(\zeta, \kappa)) := \sum_{x \in [\chi \cap \Lambda] \cup [\kappa^{-1}(\zeta \cap \Lambda) \setminus \Lambda]} \|x - \sigma(x)\|^2.$$

Fix yellow and purple and sum over red points and arrows.

Specifications $G_{\Lambda,\lambda}(\cdot|(\zeta, \kappa)) :=$ law of red points and arrows.

Gaussian random permutation on $\mathbb{R}^d$ is Gibbs

Theorem 9.12 (AFY 2019). *For $d \geq 3$ and $\lambda \leq 1$ the loop soup measure*

$$\mu_\lambda^{\mathrm{ls}} \text{ is Gibbs for the specifications } (G_{\Lambda,\lambda} : \Lambda \text{ compact}):$$

$$\mu_\lambda^{\mathrm{ls}} g = \int d\mu_\lambda^{\mathrm{ls}}(\zeta, \kappa)\, G_{\Lambda,\lambda}(g|(\zeta, \kappa)) \qquad DLR$$

For all $\rho \geq \rho_c$ the measure

$$\mu_1^{\mathrm{ls}} * \mu_{\rho-\rho_c}^{\mathrm{ri}} \text{ is Gibbs for the specifications } (G_{\Lambda,1} : \Lambda \text{ compact}).$$

Corollary 9.13 (AFY 2019). *Point and permutation marginals can be computed explicitely.*

9.3 Point marginal of Gaussian interlacements

Correlations

$$\nu_\rho^{\mathrm{ri}} := \text{Point marginal of Gaussian interlacements } \mu_\rho^{\mathrm{ri}}$$

Correlations:

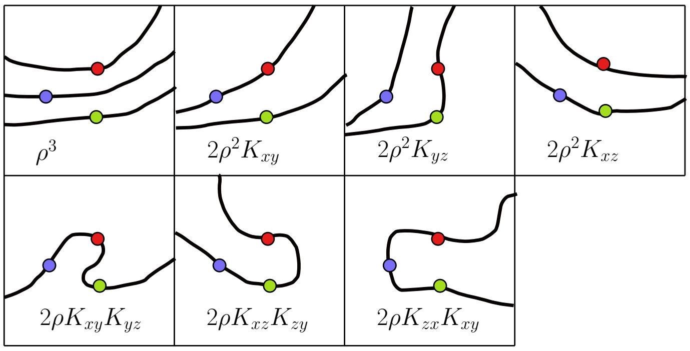
$$\varphi_\rho^{\mathrm{ri}}(x_1, \dots, x_n) = \sum_{P \in \mathcal{P}_n} \prod_{I \in P} \sum_{\sigma \in \mathcal{S}_I} V_\rho(x_{\sigma(i_1)}, \dots, x_{\sigma(i_{|I|})}). \tag{293}$$

$\mathcal{P}_n :=$ partitions of $\{1, \dots, n\}$ with nonempty sets,
$\mathcal{S}_I :=$ permutations of I,
$(i_1, \dots, i_{|I|})$ arbitrary order of I and

$$V_\rho(x_1, \dots, x_\ell) := \rho K_{x_1,x_2} \dots K_{x_{\ell-1},x_\ell}$$

(Here $\lambda = 1$ and $K_{xy} = K_1(x, y)$)

$$\begin{aligned}\varphi_\rho^{\mathrm{ri}}(x, y, z) = \rho^3 &+ 2\rho^2 K_{xy} + 2\rho^2 K_{yz} + 2\rho^2 K_{xz} \\ &+ 2\rho K_{xy} K_{yz} + 2\rho K_{xz} K_{zy} + 2\rho K_{zx} K_{xy}.\end{aligned}$$



Thermodynamic limit of Point marginal

Theorem 9.14 (Shirai-Takahashi, Tamura-Ito)**.** *Fix density* $\rho > 0$*.*

$G^{\text{point}}_{\Lambda,|\Lambda|\rho}$:= *law of point-marginal with* $|\Lambda|\rho$ *points.*

Subcritical $\rho \le \rho_c$ *or* $d \le 2$ *Fichtner 1991; Tamura-Ito 2006.*

$$G^{\text{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{\text{TI}}_{\rho} \text{ as } \Lambda \nearrow \mathbb{R}^d.$$

Supercritical $\rho > \rho_c$ *and* $d \ge 3$ *Tamura-Ito 2007.*

$$G^{\text{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{point}_{\rho} = \nu^{TI}_{\rho_c} * \nu^{\infty}_{\rho-\rho_c}.$$

Theorem 9.15 (AFY)**.** *Point marginal of Gaussian random permutation coincide with thermodynamic limit above:*

$\nu^{\text{TI}}_{\rho} = \nu^{\text{ls}}_{\lambda(\rho)}$*, point marginal of loop soup at fugacity* $\lambda(\rho)$*.*

$\nu^{\infty}_{\rho} = \nu^{\text{ri}}_{\rho}$*, point marginal of Gaussian interlacements at* ρ*.*

Partial "Thermodynamic limit" of permutation marginal

$G^{\text{permut}}_{\Lambda,|\Lambda|\rho} := \sigma$-marginal of $G_{\Lambda,|\Lambda|\rho}$

$$G^{\text{permut}}_{\Lambda,\rho} \Rightarrow \nu^{\text{permut}}_{\rho} \text{ for } \textit{cycle-size distribution}.$$

Macroscopic cycles: cycles with size bigger than $\varepsilon|\Lambda|$.

Subcritical case. $\rho \le \rho_c$ or $d = 1, 2$

The expected fraction of points in macroscopic cycles is zero. BU 2011

Supercritical case. $d \ge 3$ and $\rho > \rho_c$

(a) expected fraction of points in macroscopic cycles is $\frac{\rho-\rho_c}{\rho}$.

(b) Rescaled macroscopic cycles have random lenght:
Benfatto, Cassandro, Merola Presutti 2005.

Poisson-Dirichlet distribution (as uniform permut): Betz-Ueltschi 2011.

Current problems Thermodynamic limit of canonical measure. That is, the Gaussian random permutation in a box Λ should converge to the infinite volume GRP constructed here.

Extensions: Poisson process on $\mathbb{Z}^d$

Other interactions besides Gaussian.

Quantum case, when the Brownian trajectories from x to y interact. (BCMP 2005 treated the mean field case).

10 Branching process

Proceso de ramificación Z_n = tamaño de una población en el instante n. Cada individuo tiene un número aleatorio de hijos distribuidos como una variable aleatoria $\xi \geq 0$ con media $E\xi = \mu < \infty$ y distribución $P(\xi = j) = p_j$. Sean $\xi_{n,k}, n, k \geq 1$ iid con la misma distribución de ξ. Aquí $\xi_{n,k}$ es el número de hijos que tiene el k-ésimo individuo vivo en el instante n. Definimos $Z_0 = 1$ y para $n \geq 1$,

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k}$$

Condicionando a Z_{n-1},

$$EZ_n = E(E(Z_n|Z_{n-1})) = \mu EZ_{n-1} = \mu^2 EZ_{n-2} = \mu^n.$$

Teorema subcrítico *Si $\mu < 1$ entonces*

$$P(Z_n \geq 1, \text{ para todo } n \geq 0) = 0$$

Proof. Por la desigualdad de Markov

$$P(Z_n \geq 1) \leq EZ_n = \mu^n \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{si } \mu < 1.$$

Como $\{Z_n \geq 1\} \nearrow \{Z_n > 0, \text{ para todo } n \geq 0\}$, podemos concluir.

Teorema crítico *Si $\mu = 1$ y $P(\xi = 1) < 1$, entonces*

$$P(Z_n \geq 1, \text{ para todo } n \geq 0) = 0$$

Teorema supercrítico *Si $\mu > 1$, entonces*

$$P(Z_n \geq 1, \text{ para todo } n \geq 0) > 0$$

Defina la función $\phi : [0,1] \to [0,1]$ por $\phi(0) = p_0$ y para $s \in (0,1]$,

$$\phi(s) = \sum_{j\geq 0} s^j p_j$$

ϕ es continua en $[0,1]$ y si $p_0 + p_1 < 1$ (que asumimos, si no, se trata de un paseo aleatorio que ya vimos), para $s \in (0,1)$,

$$\phi'(s) = \sum_{j\geq 1} j s^{j-1} p_j > 0$$
$$\phi''(s) = \sum_{j\geq 2} j(j-1) s^{j-2} p_j > 0$$

O sea que ϕ es estrictamente creciente y estrictamente convexa en el intervalo $(0,1)$. Además $\lim_{s\nearrow 1} \phi'(s) = \mu$.

Defina $\theta_n = P(Z_n = 0|Z_0 = 1)$. Como $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$, tenemos $\theta_n \leq \theta_{n+1}$. Además $\theta_0 = 0$, $\theta_n \leq 1$. Por lo tanto $\theta_n \nearrow \theta_\infty \leq 1$.

Condicionando a la primera generación,

$$\begin{aligned}\theta_n &= \sum_{j\geq 0} P(Z_n = 0|Z_1 = j) P(Z_1 = j|Z_0 = 1) \\ &= \sum_{j\geq 0} \left(P(Z_{n-1} = 0|Z_0 = 1)\right)^j p_j \\ &= \phi(\theta_{n-1}).\end{aligned}$$

La segunda igualdad se explica así: la probabilidad que el proceso se extinga en el instante n dado que hay j individuos en el instante 1 es igual a la probabilidad que cada una de las familias de los j individuos vivos en el instante 1 se haya extinguido en el instante n. Para concluir observe que las j familias evolucionan independientemente, y hay $n-1$ generaciones entre el instante 1 y el n.

Sacando límites, vemos que $\theta_\infty = \phi(\theta_\infty)$, un punto fijo de ϕ.

Si ρ es un punto fijo,

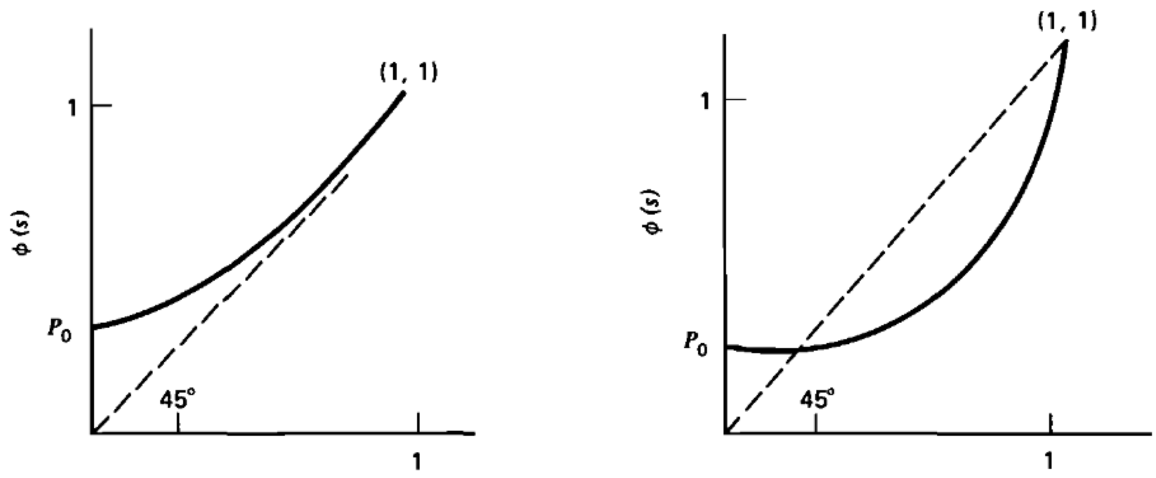
$$\rho = \sum_{k\geq 0} \rho^k p_k \geq \rho^0 p_0 = P(X_1 = 0) = \theta_1.$$

Como $\theta_1 \leq \rho$ y si $\theta_n \leq \rho$ entonces $\theta_{n+1} = \phi(\theta_n) \leq \phi(\rho) = \rho$, tenemos que $\theta_n \leq \rho$ para todo n. Es decir que θ_n converge al menor de los puntos fijos.

Dem del teorema crítico Si $\phi'(1) = \mu = 1$ y $p_1 < 1$, como ϕ es estrictamente convexa $\phi(s) > s$ para $s \in (0,1)$ y ϕ tiene 1 como único punto fijo. Por lo tanto $\theta_n \to 1$. $\square$

Dem del teorema supercrítico. Si $\phi'(1) = \mu > 1$, entonces hay un único $\rho < 1$ tal que $\phi(\rho) = \rho$. Para ver esto, observe que $\phi(0) = p_0 \geq 0$, $\phi(1) = 1$ y $\phi'(1) = \mu > 0$, lo que implica que hay un único punto fijo ρ menor que 1. Unicidad es consecuencia de la estricta convexidad de ϕ. Por lo tanto $\theta_n \nearrow \rho < 1$. $\square$

Distinguimos dos casos: A la izquierda $\phi(s) > s$ para todo $s \in (0,1)$ y a la derecha $\phi(s) = s$ para



algún $s \in (0,1)$. En la figura de la izquierda $\phi'(1) \leq 1$ y en la de la derecha $\phi'(1) > 1$.

Ejemplo. Considere que la distribución del número de hijos es Poisson con parámetro λ. Es decir

$$p_j = \frac{e^{-\lambda}\lambda^j}{j!}$$

La generadora de momentos es

$$\phi(s) = \exp(\lambda(s-1))$$

Por lo que la ecuación para el punto fijo es

$$\rho = \exp(\lambda(\rho - 1))$$

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