hhh

1. Denote
$$p(x,y) := \Big(\frac{\alpha}{\pi}\Big)^{d/2} \exp\big(-\alpha\|x-y\|^2\big).$$
and prove that for any $j \in \{0,\dots,k-1\}$ we have
$$\int_{(\mathbb{R}^d)^k} \mathbf{1}_A(x_j) \prod_{i=0}^{k-1} p(x_i,x_{i+1})\, dx_0 \dots dx_{k-1} = \int_{\mathbb{R}^d} \mathbf{1}_A(x_j) \Big(\frac{\alpha}{\pi k}\Big)^{d/2} dx_j,$$
with the convention $x_k = x_0$.

2. Let $(X_i)_{i\geq 1}$ be independent random variables with distribution Normal$(0, \frac{2}{\alpha}\,\mathrm{Id})$ in $\mathbb{R}^d$. Let $Y_i = X_1 + \cdots + X_i$ and for fixed $k \geq 1$ define
$$Z_i = Y_i - \frac{i}{k} Y_k \tag{1}$$
Show that the distribution of $(Z_0,\dots,Z_{k-1})$ is absolutely continuous with respect to Lebesgue measure in $(\mathbb{R}^d)^{k-1}$ with density
$$f(z_1,\dots,z_{k-1}) := \big(\tfrac{\alpha}{\pi}\big)^{kd/2} e^{-\alpha \sum_{i=0}^{k-1} \|z_i - z_{i+1}\|^2}.$$
with the convention $z_k = z_0 = 0$.

3. Let $\lambda \leq 1$ and $A \subset \mathbb{R}^d$ be a bounded Borel set. Define the grand canonical measure by
$$\mu_{A,\lambda}\, g := \frac{1}{Z_{A,\lambda}} \sum_{n\geq 0} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{A^n} g(\underline{x},\sigma) e^{-\alpha H(\underline{x},\sigma)} d\underline{x}, \tag{2}$$
where $g : \mathfrak{X}_A \to \mathbb{R}$ is a bounded test function. Show that the Gaussian loop soup $\mu_{A,\lambda}$ restricted to A (that is, with intensity $1\{\{\gamma\} \subset A\}\, Q_\lambda(d\gamma)$) and the grand-canonical measure $\mu_{A,\lambda}$ are equivalent.

Random interlacements

4. Let $A \subset B$ be bounded sets of $\mathbb{R}^d$, and let g be a test function that is invariant under time shifts, $g = \theta g$. Prove that
$$Q_B^{\mathrm{cap}} g \mathbf{1}_{\{T_A<\infty\}} = Q_A^{\mathrm{cap}} g \qquad \text{(compatibility)} \tag{3}$$
$$Q_B^{\mathrm{cap}} g = Q_A^{\mathrm{cap}} g + Q_{B\setminus A}^{\mathrm{cap}} g \mathbf{1}_{\{T_A=\infty\}} \qquad \text{(additivity)}, \tag{4}$$
The same holds for Q^{unif}.

5. Compute the point correlations of the Gaussian random interlacements.