Pure jump processes

1. Prove that the process X_t in Definition 7.1 in the notes satisfy
$$\mathbb{P}(X_{t+h} = y | X_t = x) = hq(x, y) + o(h).$$
and it is Markov.

2. Define
$$P_t(x, y) \;=\; \mathbb{P}(X_t = y \,|\, X_0 = x).$$
and show the Kolmogorov equations
$$P_t' = QP_t \quad \text{(Kolmogorov Backward equations)}$$
$$P_t' = P_tQ \quad \text{(Kolmogorov Forward equations)}$$
for all $t \geq 0$, where P_t' is the matrix having as entries $p_t'(x, y)$ the derivatives of the entries of the matrix P_t.

3. We say that π is an *stationary distribution* for (X_t) if $\sum_x \pi(x) = 1$ and it satisfies the *balance equations*:
$$\sum_x \pi(x)p_t(x, y) = \pi(y)$$
Prove that a distribution π is stationary for a process with rates $q(x, y)$ if and only if
$$\sum_x \pi(x)q(x, y) = \pi(y) \sum_z q(y, z). \tag{1}$$

4. Prove that if $\pi(x)q(x, y) = \pi(y)q(y, x)$ then π is reversible for the process and π is invariant.

5. Consider a queuing system with 1 server and bounded waiting place. The state space is $\{0, 1, 2\}$ and X_t is the number of customers in the system at time t. Customers arrive at rate λ and services are exponential at rate μ:
$$q(0, 1) = q(1, 2) \;=\; \lambda \tag{2}$$
$$q(1, 0) = \mu; \quad q(2, 1) \;=\; 2\mu \tag{3}$$
$$q(x, y) = 0, \text{ otherwise.} \tag{4}$$
Compute the invariant measure for this process and decide if it is reversible.

Perfect simulation

6. Let
$$Z_t := X_{[\tau(t),t]}, \quad t \in \mathbb{R}$$
Prove that Z_t is jump Markov with rates Q and it is stationary: $P(Z_t = z)$ does not depend on t.

7. Construct a stationary version of $(X_t)_{t\in\mathbb{R}}$ for the process of exercise 5 using a Poisson process in $\mathbb{R}^2$.