Discrete Percolation

1. Prove Kolmogorov's zero-one law. Suppose that $(X_n : n \in \mathbb{N})$ is a sequence of independent random variables. Then the tail sigma-algebra $\mathcal{T}$ of $(X_n : n \in \mathbb{N})$ contains only events of probability 0 or 1. Moreover, any $\mathcal{T}$-measurable random variable is almost surely constant. See Theorem 2.6.1 of the book of Norris *Probability and measure.*

2. Show that the event $E :=$ *there is an infinite cluster* is a tail event. Use the obvious extension of Kolmogorov 0-1 law to independent random variables indexed by d-dimensional integersw to show that E has probability 0 or 1.

3. Show that $\theta > 0$ if and only if $P(\text{there is an infinite cluster}) = 1$. See Theorem 1.11, pag. 19 in the book of Grimmett, *Percolation.*

4. Show that the number of sites in the cluster of the origin can be dominated by the total population of a branching process. Use this to show that if the branching process is subcritical, there is no percolation.

Continuum Percolation

5. Show that $\theta(z, p, \mathbb{R}^d)$ is increasing as function of z and $p(\cdot)$.

6. Describe the branching argument in the continuum percolation case. Use that to show that $\theta(z, p; \mathbb{R}^d) = 0$ when $z \int p(x)dx < 1$.

7. Show that there is percolation in $\mathbb{R}^d$ as a corollary of percolation in $\mathbb{R}^2$ (proven in the notes).

Domination

8. Let P, P' be two probability measures on $\mathcal{X}$. We say that P is stochastically smaller than P', denoted $P \le P'$, if $\int f dP \le \int f dP'$ for all measurable bounded increasing f on Ω. Prove that this is equivalent to have a coupling $\hat{P}$ on Ω^2 with marginals P and P' such that $\hat{P}(\xi \le \xi') = 1$.

9. Show that if P_λ is the distribution of a Poisson process of density $\lambda(\cdot)$, then $\lambda(x) \le \lambda_1(x)$ for all x implies $P_\lambda \le P_{\lambda_1}$, stochastically.

Continuum Ising model

10. (a) Give an algorithm to construct a sample of the measure P_Λ. (b) Give an algorithm to construct a sample of the measure Q_Λ.

11. Take a configuration $(X^+, X^-, E) \sim Q_\Lambda$. Consider the marginals (X^+, X^-) and $(X^+ \cup X^-, E)$. Show that for $\Delta \subset \Lambda$,

$$\int [\#X^+_\Delta - \#X^-_\Delta] \, Q_\Lambda(dX^+, dX^-, dE) = \int \#\{x \in (X^+_\Delta \cup X^-_\Delta : x \leftrightarrow X^+_{\Lambda^c}\} \, Q_\Lambda(dX^+, dX^-, dE).$$

12. Let π^{z_Λ} be a PP in $\mathbb{R}^d$ with intensity $z_\Lambda(x) = z(1 + \mathbf{1}\{x \in \Lambda\})$. Prove that $\pi^{z_\Lambda}(dY) = \frac{1}{Z} 2^{\# Y_\Lambda} \pi^z(dY)$, where Z is the normalization. Compute Z.

13. Let

$$\varphi_{\Lambda,Y}(dE) = \frac{1}{Z_\Lambda} 2^{k(E)} \mu_Y^{p,\Lambda}(dE) \tag{1}$$

Prove that $\varphi_{\Lambda,Y} \geq \mu_Y^{\tilde{p}}$, for some positive $\tilde{p}$.

14. Denote ν_Λ the point marginal distribution of χ_Λ, that is, for test functions $f : \mathfrak{X} \to \mathbb{R}$,

$$\nu_\Lambda f := \int f(Y)\, \chi_\Lambda(dY, dE). \tag{2}$$

Give a sketch proof of the domination $\nu_\Lambda \geq \pi^{\alpha z}$ for some $\alpha > 0$.

15. Introduce a partial order $\leq$ on $\mathfrak{X}^2$ by writing $(X^+, X^-) \leq (Y^+, Y^-)$ when $X^+ \subset Y^+$ and $X^- \supset Y^-$. Show that the measures G_Λ^+ decrease stochastically relative to this order when Λ increases.

16. Use the previous exercise to show that as $\Lambda \nearrow \mathbb{R}^d$, G_Λ^+ converges to $P^+ \in \mathcal{G}$.