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Harness processes and harmonic crystals

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Abstract

In the Hammersley harness processes the $\mathbb{R}$-valued height at each site $i \in \mathbb{Z}^d$ is updated at rate 1 to an average of the neighboring heights plus a centered random variable (the noise). We construct the process "a la Harris" simultaneously for all times and boxes contained in $\mathbb{Z}^d$. With this representation we compute covariances and show L^2 and almost sure time and space convergence of the process. In particular, the process started from the flat configuration and viewed from the height at the origin converges to an invariant measure. In dimension three and higher, the process itself converges to an invariant measure in L^2 at speed $t^{1-d/2}$ (this extends the convergence established by Hsiao). When the noise is Gaussian the limiting measures are Gaussian fields (harmonic crystals) and are also reversible for the process.

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1. Introduction

The harness process

The *harness process* is a continuous-time version of the serial harness introduced by Hammersley [9]. Let $P = (p(i, j), i, j \in \mathbb{Z}^d)$ be a translation invariant finite-range stochastic matrix (that is, $p(i, j) \geq 0$, $\sum_j p(i, j) = 1$ for all i, $p(i, i + j) = 0$ if $|j| > v$ for some v and $p(i, j) = p(0, j - i)$ for all i, j). Let the *noise* $G(\mathrm{d}x)$ be a centered distribution with variance 1.

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The state space is $\mathcal{X} = \mathbb{R}^{\mathbb{Z}^d}$. We consider a family of processes in subsets $\Lambda \subset \mathbb{Z}^d$ with boundary conditions $\gamma \in \mathcal{X}$. For configurations $\eta \in \mathcal{X}$ and bounded cylinder functions $f : \mathcal{X} \to \mathbb{R}$ define the generator

$$L^{\Lambda,\gamma} f(\eta) = \sum_{i\in\Lambda} \int G(\mathrm{d}\varepsilon)[f(P_i(\eta_\Lambda\gamma_{\Lambda^c}) + \sigma\varepsilon e_i) - f(\eta)] \tag{1}$$

where the standard deviation of the noise $\sigma > 0$ is a parameter, $e_i(j)$ is the indicator function of $i = j$, $P_i\eta$ is the configuration

$$(P_i\eta)(i) = \sum_{j\in\mathbb{Z}^d} p(i,j)\eta(j); \quad (P_i\eta)(j) = \eta(j) \text{ for } j \neq i; \tag{2}$$

and the juxtaposition $\eta_\Lambda\gamma_{\Lambda^c} \in \mathcal{X}$ is defined by

$$(\eta_A\gamma_{A^c})(i) = \begin{cases} \eta(i), & \text{if } i \in A, \\ \gamma(i), & \text{if } i \in A^c. \end{cases} \tag{3}$$

In other words, at all times the sites outside Λ have fixed configuration γ and those inside are updated at rate 1 with a P-weighted mean of the neighbors plus an independent centered random variable. When the boundary configuration γ is the *flat configuration* $\gamma(i) \equiv 0$ we write L^Λ.

Basis [1,2] proves that there exist Markov processes (η_t) in $\mathbb{R}^{\mathbb{Z}^d}$ with generators $L^{\Lambda,\gamma}$, that is, processes satisfying

$$\lim_{h\to 0} \frac{1}{h}\mathbb{E}[f(\eta_{t+h}) - f(\eta_t) \mid \mathcal{F}_t] = L^{\Lambda,\gamma} f(\eta_t) \tag{4}$$

for bounded cylinder functions f, where $\mathcal{F}_t$ is the σ-algebra generated by $\{\eta_s, s \le t\}$. His proof works in a more general context of metric spaces. The existence is immediate if Λ is finite but for infinite Λ it is necessary to impose that the boundary conditions γ do not grow too fast (see (46) later). Hsiao [11,12] shows the existence of invariant measures in dimensions $d \ge 3$ and gives conditions for the convergence of the process to the invariant measures. The discrete-time version is called "serial harness" by Hammersley and its tail behavior has been studied by Toom [15].

The Gaussian Gibbs fields

For each finite $\Lambda \subset \mathbb{Z}^d$ let $H^\Lambda : \mathcal{X} \to \mathbb{R}$ be the Hamiltonian

$$H^\Lambda(\eta) = \frac{\beta}{2} \sum_{i\in\Lambda} \sum_{j\in\mathbb{Z}^d} p(i,j)(\eta(i) - \eta(j))^2. \tag{5}$$

For finite $\Lambda \subset \mathbb{Z}^d$ and $\gamma \in \mathcal{X}$ define the measure $\mu^{\Lambda,\gamma}$ on $\mathbb{R}^\Lambda$ by

$$\mu^{\Lambda,\gamma}(f) = \frac{1}{Z^{\Lambda,\gamma}} \int_{\mathbb{R}^\Lambda} f(\eta) e^{-H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c})} \prod_{i\in\Lambda} \mathrm{d}\eta(i) \tag{6}$$

where $\mathrm{d}\eta(i)$ is the Lebesgue measure in the ith coordinate of Λ. The elements of the family

$$\{\mu^{\Lambda,\gamma} : \Lambda \subset \mathbb{Z}^d \text{ finite}, \gamma \in \mathcal{X}\} \tag{7}$$

are called *local specifications*. When γ is the flat configuration we write μ^Λ. One of the main problems in Statistical Mechanics is to find a measure on $\mathcal{X}$ whose conditional probabilities are given by the specifications (6) (DLR equations, see the book of Georgii [8] or the monograph of Bovier [3]; for the Gaussian fields this has been solved by Spitzer [14] and Dobrushin [5]). More precisely, we say that a measure μ is a Gibbs measure with specifications $\mu_{\Lambda,\gamma}$ if for all finite Λ and continuous $f : \mathbb{R}^\Lambda \to \mathbb{R}$, the conditional probabilities exist μ almost surely and satisfy

$$\mu(\cdot \mid \mathcal{F}^c_\Lambda)(\gamma_{\Lambda^c}) = \mu^{\Lambda,\gamma} \qquad \mu \text{ a.s.} \tag{8}$$

where $\mathcal{F}^c_\Lambda$ is the σ-algebra generated by γ_{Λ^c}.

Harnesses

The motivation of Hammersley [9] was the construction of probability measures μ on $\mathbb{R}^{\mathbb{Z}^d}$ with the property

$$\mu(\eta(x) \mid \eta(y), y \neq x) = \sum_y p(x, y)\eta(y) \tag{9}$$

that is, the expected value under μ of the height at x conditioned on the heights at the other sites is a convex combination (taken with the matrix p) of the heights at the other sites. Measures μ satisfying (9) are called *harnesses*. Williams [16] constructs Gaussian measures that are harnesses when p is a nearest neighbor symmetric random walk in $\mathbb{Z}^d$. Kingman [13] proposes the construction of harnesses in L^1. The Gaussian Gibbs fields satisfying (8) are harnesses.

Results

The point of this paper is a simultaneous construction (coupling) of versions of the processes $(\eta_t^{\Lambda,\gamma})$ and configurations $\eta^{\Lambda,\gamma}$ with law $\mu^{\Lambda,\gamma}$ for all Λ and γ, in the same probability space. Then we show L_2 and almost sure time and space convergence. This is based on a Harris graphical construction of the harness process on a probability space generated by a family of one-dimensional marked stationary Poisson processes indexed by $\mathbb{Z}^d$. Epochs of the Poisson process correspond to updating times of the Harness process; the marks are independent and identically distributed random variables with distribution G. This construction allows one to represent the process starting at time s with the flat configuration as

$$\eta_{[s,t]}(i) := \sum_{j\in\Lambda} \sum_{n:T_n(j)\in[s,t]} \varepsilon_n(j) b_n(i, j) \tag{10}$$

for $t \geq s$. Here $\varepsilon_n(j)$ is the noise associated with $T_n(j)$, the nth Poisson epoch of site j, and $b_n(i, j)$ is the probability that *given the Poisson epochs*, a random walk starting at time t at site i jumping at the Poisson epochs backwards in time is at site j at time $T_n(j)$. The jumps of the walk have law p. Since $b_n(i, j)$ are functions of the Poisson epochs, $\eta_{[s,t]}(i)$ is a function of the Poisson epochs in the interval $[s, t]$ and the noises associated with them. This representation is the continuous analogue of equation (8.2) in [9]. It is reminiscent of what is called *duality* in interacting particle systems and goes in parallel with the backwards representation of the *random average process* in [6].

We show that for each fixed t the process $(\eta_{[t-s,t]}(i), s \geq 0)$ is a martingale with uniformly bounded second moments for $d \geq 3$ and hence for each fixed t it converges almost surely to a limit denoted $\eta_t(i)$. We also show that the rate of L^2 convergence is bounded by a constant times $s^{1-d/2}$, improving the weakly convergence established by Hsiao [11,12]. The limiting process

$(\eta_t, t \in \mathbb{R})$ is a stationary harness process. For $d \leq 2$ we study the process pinned at zero at the origin (for which the site at the origin is not updated and remains zero) and the process as seen from the height at the origin. We prove similar results in those cases. To our knowledge these results are new for $d = 1, 2$. The graphical construction and the martingale property are shown in Section 2.

The process can be defined in subsets of $\Lambda \subset \mathbb{Z}^d$ by assuming that the heights outside Λ are fixed. Using the superlabel Λ for the process restricted to Λ with the heights outside Λ equal to zero we get a family of stationary processes $((\eta_t^\Lambda, t \in \mathbb{R}), \Lambda \subset \mathbb{Z}^d)$. We show that under suitable conditions, for each t, the one-time marginal family $(\eta_t^\Lambda, \Lambda \subset \mathbb{R}^d)$ converges coordinatewise in L^2 to an infinite volume configuration $\eta_t^{\mathbb{Z}^d}$ as $\Lambda \nearrow \mathbb{Z}^d$. The time and space convergence results are proven in Theorem 9 in Section 4.

The one- and two-point correlations are computed in Section 3 using the following random walk representation of the second moments of the differences: For $i \in \mathbb{Z}^d$,

$$\mathbb{E}(\eta_{[s,t]}(i) - \eta_{[s,t]}(0))^2 = 2\int_0^{t-s} (\mathbb{P}(D_u^0 = 0) - \mathbb{P}(D_u^i = 0))\mathrm{d}u \tag{11}$$

where D_u^i is the position at time u of a symmetric random walk starting at i at time 0. The transition probabilities of this walk are homogeneous except at the origin; they are given in (22). This walk also appears in [11] for computing the correlations of the stationary law of η_t.

The law of η_t^Λ is the unique invariant measure for the harness process when Λ is finite; recall that the boundary conditions we are taking "pin" the process to the external configuration. This is proven in Theorem 9 using the representation (10). In the infinite case there are infinitely many invariant measures. In particular, if h is a harmonic function for p, for $d \geq 3$ the law of $\eta_t^{\mathbb{Z}^d} + h$ is invariant for the harness process. We conjecture that for $d \geq 3$ the law of $\eta_t^{\mathbb{Z}^d}$ is the unique ergodic invariant measure with mean zero. Hsiao [11] proved that this is the only ergodic invariant measure with mean zero and finite variance. To eliminate the restriction of finite variance it would be sufficient to show the following random version of the ergodic theorem: Let η be a configuration chosen from an ergodic measure μ with mean zero and $\mathbb{P}$ the probability induced by the Poisson processes, then

$$\lim_{s\to\infty} \sum_j b_{[0,-s]}(i,j)\eta(j) = 0 \qquad \mathbb{P}\text{-a.s. } \mu\text{-a.s.} \tag{12}$$

where $b_{[0,-s]}(i,j)$ is the probability conditioned on the Poisson epochs that the backwards walk starting at i at time 0 is at j at time $-s$. The ergodicity of μ implies that (12) holds μ-a.s. if we replace $b_{[0,-s]}(i,j)$ by its averages. The limit (12) is related to the asymptotic behavior of the *no-noise* harness process $\underline{\eta}_t$ defined in (47), a harness process with zero noise (that is, $G(\mathrm{d}x) = \delta_0(x)$). In this process the heights are updated at the Poisson times to the p-average of the other heights. The problem is to characterize the set of initial configurations for which this process converges to the "all-zero" configuration.

Under the assumptions that the noise G is Gaussian (that is $G(\mathrm{d}x) = (2\pi)^{-1/2}e^{-x^2/2}\mathrm{d}x$) and that $p(0,0) = 0$, Hsiao [11] proved that the Gaussian Gibbs field μ^Λ is reversible for the harness process in any Λ. Indeed, since the conditional distribution under μ^Λ of $\eta(i)$ given $(\eta(j), j \neq i)$ has a Gaussian law centered at $\sum_j p(i,j)\eta(j)$, the harness process is just the so-called *heat bath* dynamics at continuous time. The weak convergence of μ^Λ to $\mu^{\mathbb{Z}^d}$ for $d \geq 3$ has been proven by Spitzer [14]; we provide here convergence in L^2 and a simultaneous construction of $(\xi^\Lambda)_\Lambda$ for an increasing sequence of finite sets $\Lambda \nearrow \mathbb{Z}^d$ satisfying that ξ^Λ has law μ^Λ and converges almost surely to a configuration $\xi^{\mathbb{Z}^d}$ with law $\mu^{\mathbb{Z}^d}$, the infinite volume Gibbs measure with specifications (6). This is done in Proposition 10. The almost sure convergence of η_t^Λ as $\Lambda \nearrow \mathbb{Z}^d$ for $d \geq 3$ remains open. We prove similar results for the process pinned at the origin and the process as seen from the height at the origin.

Compared with the work of Hsiao who considered $d \geq 3$, our constructive approach permits us (a) to treat (bounded or unbounded) regions Λ contained in $\mathbb{Z}^d$ and the difference process in dimensions $d = 1$ and 2 and (b) to compute non-equilibrium correlation functions. Hsiao also considered the case when p is sub-stochastic; we discuss this with Pechersky [7].

2. Harris graphical construction

Let $(\mathcal{T}, \mathcal{E}, \mathcal{U})$ be a collection of independent marked rate-1 Poisson processes on $\mathbb{R}$:

$$(\mathcal{T}, \mathcal{E}, \mathcal{U}) := ((T_n(i), \varepsilon_n(i), U_n(i)); i \in \mathbb{Z}^d, n \in \mathbb{Z}) \tag{13}$$

where $T_n(i)$ is the nth epoch of a stationary Poisson process of rate 1 (that is, $T_0(i) < 0 \leq T_1(i)$, $T_1(i)$, $-T_0(i)$ and $T_n(i) - T_{n-1}(i)$ for $n \neq 1$ are i.i.d. exponential with mean 1); $\varepsilon_n(i)$ are i.i.d. centered random variables with variance 1 and $U_n(i)$ are i.i.d. in $\mathbb{Z}^d$ with law $p(i, \cdot)$. Furthermore $T_n(i) - T_{n-1}(i)$, $\varepsilon_{n'}(i')$, $U_{n''}(i'')$, $n, n', n'', i, i', i'' \in \mathbb{Z}$ are mutually independent random variables. Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability and expectation induced by these processes.

Fix $t \in \mathbb{R}$ and let $(B^{i,\Lambda}_{[t,u]}, u \leq t)$ be a *backward* random walk starting at site i at time t and jumping at the Poisson epochs backwards in time according to the $U_n(j)$ variables and *absorbed* at Λ^c. That is, $B^{i,\Lambda}_{[t,t]} = i$ and if at time $u+$ the walk is at $j \in \Lambda$, $T_n(j) = u$ and $U_n(j) = j'$, then at this time the walk jumps to j'. If $j \notin \Lambda$ then it stays at j for ever.

For $s \leq t$ define $\eta^\Lambda_{[s,t]}(i)$ as the expectation of the sum of the noise variables $\varepsilon_n(i)$ encountered by $B^{i,\Lambda}_{[t,\cdot]}$ in the (backwards) interval $[t, s]$ *conditioned on the jump times*. More precisely, define $\eta^\Lambda_{[s,s]}(i) \equiv 0$ and for $t \geq s$,

$$\eta^\Lambda_{[s,t]}(i) := \sum_{j \in \Lambda} \sum_{n: T_n(j) \in [s,t]} \varepsilon_n(j) b^\Lambda_{[t,T_n(j)]}(i, j) \tag{14}$$

where, abusing notation by calling $\mathcal{T}$ the σ-algebra generated by $\mathcal{T}$,

$$b^\Lambda_{[t,u]}(i, j) = b^\Lambda_{[t,u]}(i, j|\mathcal{T}) := \mathbb{P}\left(B^{i,\Lambda}_{[t,u]} = j \mid \mathcal{T}\right) \tag{15}$$

for $u \leq t$; that is, $b^\Lambda_{[t,u]}(i, j)$ is a function of the Poisson epochs in the interval $[u, t]$ and it is independent of $\mathcal{E}$ and $\mathcal{U}$.

For each $s \in \mathbb{R}$, expressions (14) and (15) define a random process $(\eta^\Lambda_{[s,t]}, t \geq s)$ as a (deterministic) function of $((T_n(j), \varepsilon_n(j)) : T_n(j) \in [s, \infty))$. The sums (14) are almost surely finite as a consequence of the finite range of p and the fact that there are only a finite number of Poisson epochs in bounded time intervals.

We also define the process starting with a configuration $\zeta \in \mathcal{X}$ at time s by $\eta^{\Lambda,\zeta}_{[s,s]}(i) \equiv \zeta$ and for $t \geq s$,

$$\eta^{\Lambda,\zeta}_{[s,t]}(i) := \sum_{j \in \Lambda} \sum_{n: T_n(j) \in [s,t]} \varepsilon_n(j) b^\Lambda_{[t,T_n(j)]}(i, j) + \sum_{j \in \Lambda} b^\Lambda_{[t,s]}(i, j)\zeta(j). \tag{16}$$

This is defined for configurations ζ that do not increase too fast for guaranteeing that the sum in (16) is almost surely finite. A sufficient condition is that ζ belongs to Ξ_Λ, where

$$\Xi_\Lambda := \left\{ \zeta : \sum_{j\in\Lambda} p_t^\Lambda(i,j)\zeta(j) < \infty \text{ for all } i \in \Lambda, t > 0 \right\}, \tag{17}$$

where p_t^Λ is the probability that a continuous random walk with rates p, absorbed at sites in Λ^c starting at i at time zero, is at j at time t. Notice that $p_{t-s}^\Lambda = \mathbb{E}(b_{[t,s]}^\Lambda(i,j))$.

Proposition 1. *For any $d \ge 1$, $\Lambda \subset \mathbb{Z}^d$ and $s \in \mathbb{R}$, the process $(\eta_{[s,t]}^{\Lambda,\zeta}, t \ge s)$ defined in (14) has generator L^Λ (in the sense of (4)) and initial condition ζ at time s.*

Proof. For any $s \in \mathbb{R}$ the process $(\eta_{[s,t]}^{\Lambda,\zeta}, t \ge s)$ as defined by (16) satisfies the following infinitesimal evolution:

$$\eta_{[s,t]}^{\Lambda,\zeta}(i) = \begin{cases} \eta_{[s,t-]}^{\Lambda,\zeta}(i), & \text{if } t \text{ is not a epoch of } \mathcal{T}(i) \\ \sum_{j\in\mathbb{Z}^d} p(i,j)\eta_{[s,t-]}^{\Lambda,\zeta}(j) + \varepsilon_n(i), & \text{if } t = T_n(i) \end{cases} \tag{18}$$

from where it follows that $\eta_{[s,t]}^{\Lambda,\zeta}$ has generator L. □

In the following we use the notation:

$$b_n^\Lambda(i,j) := b_{[t,T_n(j)]}^\Lambda(i,j). \tag{19}$$

Proposition 2. *For each $i \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ the process $(\eta_{[t-s,t]}^\Lambda(i), s \ge 0)$ is a martingale with respect to the filtration $(\mathcal{F}_s)_{s\ge 0}$, where $\mathcal{F}_s$ is the sigma algebra generated by $(\eta_{[t-u,t]}^\Lambda)_{u\le s}$.*

Proof. For $r > s$ the expectation of $\eta_{[t-r,t]}^\Lambda - \eta_{[t-s,t]}^\Lambda$ given $\mathcal{F}_s$ vanishes because it is the mean of a (random) finite sum of randomly weighted centered variables $\varepsilon_n(j)$ independent of the weights and of the past. Indeed, for $0 \le s \le r$,

$$\begin{aligned} &\mathbb{E}\left(\eta_{[t-r,t]}^\Lambda - \eta_{[t-s,t]}^\Lambda \mid \mathcal{F}_s\right) \\ &\quad = \mathbb{E}\left[\sum_{j\in\mathbb{Z}^d} \sum_{n:T_n(j)\in[t-r,t-s]} \varepsilon_n(j) b_n^\Lambda(i,j) \middle| \mathcal{F}_s \right] \\ &\quad = \mathbb{E}\left[\mathbb{E}\left[\sum_{j\in\mathbb{Z}^d} \sum_{n:T_n(j)\in[t-r,t-s]} \varepsilon_n(j) b_n^\Lambda(i,j) \middle| \mathcal{T}, \mathcal{F}_s \right] \middle| \mathcal{F}_s \right] \\ &\quad = \mathbb{E}\left[\sum_{j\in\mathbb{Z}^d} \sum_n \mathbb{E}[\varepsilon_n(j) \mid \mathcal{T}, \mathcal{F}_s] b_n^\Lambda(i,j) \mathbb{I}\{n : T_n(j) \in [t-r, t-s]\} \middle| \mathcal{F}_s \right] = 0 \end{aligned}$$

where the third identity follows from Fubini and the fact that both $q_n(i,j)$ and $T_n(j)$ are $\mathcal{T}$-measurable; the fourth identity follows because (a) for $T_n(j) \in [t-u, t-s]$, $\varepsilon_n(j)$ is independent of $\mathcal{F}_s$, (b) $\varepsilon_n(j)$ is independent of $\mathcal{T}$ for all n and j and (c) $\varepsilon_n(j)$ are centered random variables. □

3. Covariances

This section collects bounds for the relevant covariances. The main tool is an expression of the covariances of the process in $\mathbb{Z}^d$ as a function of the potential kernel of a symmetric random walk. These covariances are bounds for the covariances in the box $\Lambda \subset \mathbb{Z}^d$; this works for $d \geq 3$. As a consequence the relevant variances are uniformly bounded in time and space. When $p(0,0) = 0$, we use results from the Gaussian case to bound the variances when $d = 1, 2$ for the process "pinned" at the origin and for the process "as seen from the height at the origin". The results are summarized in Corollary 8 later.

We start with an elementary computation.

Lemma 3. *Let $\Lambda' \subset \Lambda \subset \mathbb{Z}^d$ and $s' \leq s \leq t$. For all $i \in \Lambda$, $j \in \Lambda'$*

$$\mathbb{E}\left[\eta^{\Lambda}_{[s,t]}(i)\eta^{\Lambda'}_{[s',t]}(j)\right] = \mathbb{E}\left[\sum_{k\in\Lambda}\ \sum_{n:T_n(k)\in[s,t]} b^{\Lambda}_n(i,k)b^{\Lambda'}_n(j,k)\right]. \tag{20}$$

Proof. Using the definition, conditioning on the Poisson marks and integrating with respect to the disorder variables, the left hand side of (20) equals

$$\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}^d}\ \sum_{n:T_n(k)\in[s,t]} \varepsilon_n(k)b^{\Lambda}_n(i,k)\right)\left(\sum_{k\in\mathbb{Z}^d}\ \sum_{n:T_n(k)\in[s',t]} \varepsilon_n(k)b^{\Lambda'}_n(j,k)\right)\middle|\mathcal{T}\right]\right]$$

$$= \mathbb{E}\left[\sum_{k\in\mathbb{Z}^d}\sum_{k'\in\mathbb{Z}^d}\ \sum_{n:T_n(k)\in[s,t]}\ \sum_{n':T_{n'}(k')\in[s',t]} \mathbb{E}(\varepsilon_n(k)\varepsilon_{n'}(k')|\mathcal{T})b^{\Lambda}_n(i,k)b^{\Lambda'}_{n'}(j,k')\right] \tag{21}$$

where we can interchange sums and conditional expectations as the sums are $\mathcal{T}$ almost surely finite. From (21) we get the right hand side of (20) because $\varepsilon_n(k)$ are i.i.d. independent of $\mathcal{T}$ with variance 1. □

Covariances in $\mathbb{Z}^d$

Let D^i_t be a continuous time random walk on $\mathbb{Z}^d$ starting at i with the following (symmetric) transition rates:

$$p_D(i,j) = \begin{cases} p(0, j-i) + p(0, i-j), & \text{if } i \neq 0;\\ \displaystyle\sum_{k\in\mathbb{Z}^d} p(0,k)p(0,k+j), & \text{if } i = 0.\end{cases} \tag{22}$$

Lemma 4. *Let $d \geq 1$ and $-\infty < s \leq t$. For $i, j \in \mathbb{Z}^d$, $\mathbb{E}\eta^{\mathbb{Z}^d}_{[s,t]}(i) = 0$ and*

$$\mathbb{E}\left(\eta^{\mathbb{Z}^d}_{[s,t]}(i)\right)^2 = \mathbb{E}\left[\sum_{k\in\mathbb{Z}^d}\ \sum_{n:T_n(k)\in[s,t]} b^{\mathbb{Z}^d}_n(i,k)^2\right] = \int_0^{t-s} \mathbb{P}(D^0_u = 0)\mathrm{d}u \tag{23}$$

$$\mathbb{E}\left(\eta^{\mathbb{Z}^d}_{[s,t]}(j) - \eta^{\mathbb{Z}^d}_{[s,t]}(i)\right)^2 = 2\int_0^{t-s} (\mathbb{P}(D^0_u = 0) - \mathbb{P}(D^{i-j}_u = 0))\mathrm{d}u. \tag{24}$$

Proof. Taking $\Lambda = \Lambda' = \mathbb{Z}^d$, $s = s'$ in (20) gives the first identity in (23). The middle expression in (23) is the average number of Poisson epochs used simultaneously by $B^{i,\Lambda}_{[t,s]}$ and $\bar{B}^{i,\Lambda}_{[t,s]}$, where

$\bar{B}^{j,\Lambda}_{[t,s]}$ is a random walk that uses the same Poisson epochs as $B^{i,\Lambda}_{[t,s]}$ but independent jump variables $\bar{U}_n(\cdot)$. Noting that $\bar{B}^{j,\Lambda}_{[t,s]} - B^{i,\Lambda}_{[t,s]}$ has the same law as D^{i-j}_{t-s}, the second identity in (23) follows. In this computation the expected number of Poisson marks at the origin seen by D^{i-j}_u, for $u \in [t,s]$, equals the right hand side of (23) because the jump rate at the origin is 1. The same considerations show (24). □

We now get bounds for the time integrals.

Lemma 5. *There exist constants C and $C(i)$ such that for $s > 1$,*

$$\int_s^\infty \mathbb{P}(D^0_u = 0)\mathrm{d}u < Cs^{1-d/2}, \qquad \text{for } d \geq 3 \tag{25}$$

$$\int_s^\infty (\mathbb{P}(D^0_u = 0) - \mathbb{P}(D^i_u = 0))\mathrm{d}u < C(i)s^{-d/2}. \tag{26}$$

Proof. Since D is a local perturbation of a symmetric finite range random walk, we have $P(D^0_u = 0) < Cs^{-d/2}$, from where one gets (25) with another constant. Differentiating with respect to u and using the Kolmogorov Backwards equation we get

$$P(D^0_s = 0) = \int_s^\infty \sum_i p_D(0,i)(\mathbb{P}(D^0_u = 0) - \mathbb{P}(D^i_u = 0))\mathrm{d}u.$$

Since the differences are positive, we get (26) with $C(i) = C/p_D(0,i)$ when $p_D(0,i) > 0$. An inductive step shows (26) for all i. □

Next we show that if $p(0,0) = 0$, the variances of the process pinned at zero are uniformly bounded. The property holds for all centered noises of variance 1, but the proof uses the fact that the Gibbs measure with specifications (6) is reversible for the process with Gaussian noise. This is the case only when $p(0,0) = 0$.

Lemma 6. *Assume $p(0,0) = 0$. Then for all $d \geq 1$, $i \in \Lambda$, $\Lambda \subset \mathbb{Z}^d$ there exist constants $V^{\Lambda\setminus\{0\}}(i) < \infty$ such that*

$$\mathbb{E}\left[\eta^{\Lambda\setminus\{0\}}_{[s,t]}(i)\right]^2 \leq V^{\Lambda\setminus\{0\}}(i) < \infty. \tag{27}$$

Proof. From (20) we see that the variances do not depend on the particular distribution G provided its variance is 1. Hence we can assume without loss of generality that the noise is Gaussian. Theorem 12 later says that under $p(0,0) = 0$ and Gaussian noise there exists a Gibbs measure $\mu^{\Lambda\setminus\{0\}}$ reversible (and hence invariant) for the process. That is,

$$\int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)\mathbb{E}f\left(\eta^{\Lambda\setminus\{0\},\xi}_{[s,t]}\right) = \int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)f(\xi) \tag{28}$$

for cylinder continuous $f : \mathcal{X} \to \mathbb{R}$. The variances $V^{\Lambda\setminus\{0\}}(i) =: \int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)\xi(i)^2$ are finite for all $\Lambda \subset \mathbb{Z}^d$ (see (59) later). Then, using (16) and the invariance property (28),

$$\begin{aligned} V^{\Lambda\setminus\{0\}}(i) &= \int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)\mathbb{E}\left[\eta^{\Lambda\setminus\{0\},\xi}_{[s,t]}(i)\right]^2 \\ &= \mathbb{E}\left[\eta^{\Lambda\setminus\{0\}}_{[s,t]}(i)\right]^2 + \int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)\mathbb{E}\left(\sum_{k\in\Lambda} b^{\Lambda\setminus\{0\}}_{[t,s]}(i,k)\xi(k)\right)^2. \end{aligned} \tag{29}$$

(The crossed terms cancel because $\varepsilon_n(k)$ are centered and independent of ξ and b.) This shows (27). $\square$

Variances are monotone in time and Λ:

Lemma 7. *For $i \in \mathbb{Z}^d$, $\Lambda \subset \bar{\Lambda}$ and $t \geq s \geq \bar{s}$,*

$$\mathbb{E}\left[\eta^{\Lambda}_{[s,t]}(i)\right]^2 \leq \mathbb{E}\left[\eta^{\bar{\Lambda}}_{[\bar{s},t]}(i)\right]^2 \tag{30}$$

$$\mathbb{E}\left[\eta^{\mathbb{Z}^d}_{[s,t]}(i) - \eta^{\mathbb{Z}^d}_{[s,t]}(0)\right]^2 \leq \mathbb{E}\left[\eta^{\mathbb{Z}^d}_{[\bar{s},t]}(i) - \eta^{\mathbb{Z}^d}_{[\bar{s},t]}(0)\right]^2. \tag{31}$$

Proof. Using (20) with $\Lambda = \Lambda'$ and $s = s'$:

$$\begin{aligned}\mathbb{E}\left[\eta^{\Lambda}_{[s,t]}(i)\right]^2 &= \mathbb{E}\left[\sum_{k\in\Lambda}\sum_{n:T_n(k)\in[s,t]} b^{\Lambda}_n(i,k)^2\right] \\ &\leq \mathbb{E}\left[\sum_{k\in\bar{\Lambda}}\sum_{n:T_n(k)\in[\bar{s},t]} b^{\bar{\Lambda}}_n(i,k)^2\right] = \mathbb{E}\left[\eta^{\bar{\Lambda}}_{[\bar{s},t]}(i)\right]^2 \end{aligned}\tag{32}$$

where the inequality follows from the fact that the probabilities absorbed at Λ are dominated by the ones absorbed at $\bar{\Lambda}$: if $\Lambda \subset \bar{\Lambda}$, then $b^{\Lambda}_n(i,k) \leq b^{\bar{\Lambda}}_n(i,k)$. This shows monotonicity in Λ for (30). Variances of martingales are non-decreasing in time, showing time monotonicity in (30) and (31). $\square$

Corollary 8. *There exist constants $C(i)$ such that for all Λ and $s \leq t$*

(a) *For $d \geq 3$, $\mathbb{E}[\eta^{\Lambda}_{[s,t]}(i)]^2 < C(i)$.*

(b) *For $d \geq 1$, $\mathbb{E}[\eta^{\mathbb{Z}^d}_{[s,t]}(i) - \eta^{\mathbb{Z}^d}_{[s,t]}(0)]^2 < C(i)$.*

(c) *Assuming $p(0,0) = 0$, for $d \geq 1$, $\mathbb{E}[\eta^{\Lambda\setminus\{0\}}_{[s,t]}(i)]^2 < C(i)$.*

Proof. (a) follows from (30), (23) and (25). Obtain (b) from (31), (24) and (26) and (c) from (30) and (27). $\square$

4. Time and space convergence

The process $(\eta^{\Lambda}_{[s,t]} : t \geq s)$ has "flat boundary conditions" outside Λ and "flat initial condition" at time s. We state the results for this case and later comment on general boundary and initial conditions. We first show that under suitable conditions the process (14) is well defined when $s = -\infty$ and it is in fact a stationary version of the harness process. In particular, when the noise is Gaussian, the marginal law of this process at any time t has a Gibbs distribution with specifications (6) which are also reversible for the harness processes with Gaussian noise. In one and two dimensions there is no Gibbs measure with specifications (6) (see [8], Chapter 13). The harness process should not converge to a probability measure for $d = 1, 2$ (delocalization); see [15] for the discrete-time version. However both the harness process pinned at the origin and the process "as seen from the height at the origin" converge to the pinned Gibbs measure $\mu^{\mathbb{Z}^d\setminus\{0\}}$. The L^2 time convergence for $d \geq 3$ was proven by Hsiao [11]; we obtain the convergence bounds (35).

Theorem 9. *The following hold*
A.s. time convergence. *Assume either* (a) $d \geq 3$ *or* (b) $\Lambda \neq \mathbb{Z}^d$ *and* $p(0,0) = 0$. *For each* $t \in \mathbb{R}$, $i \in \mathbb{R}^d$, *as* $s \to \infty$, $\eta^{\Lambda}_{[t-s,t]}(i)$ *converges almost surely to a random variable* $\eta^{\Lambda}_t(i)$*:*

$$\lim_{s\to\infty} \eta^{\Lambda}_{[t-s,t]}(i) = \eta^{\Lambda}_t(i) \quad a.s. \tag{33}$$

For $d \geq 1$, $\eta^{\mathbb{Z}^d}_{[t-s,t]}(i) - \eta^{\mathbb{Z}^d}_{[t-s,t]}(0)$ *converges almost surely to a random variable* $\Delta^{\mathbb{Z}^d}_t(i)$*:*

$$\lim_{s\to\infty} \left[\eta^{\mathbb{Z}^d}_{[t-s,t]}(i) - \eta^{\mathbb{Z}^d}_{[t-s,t]}(0)\right] = \Delta^{\mathbb{Z}^d}_t(i) \quad a.s. \tag{34}$$

L_2 **time convergence**. *There exist positive constants* $C, C(i) < \infty$ *such that for* $d \geq 1$, $\Lambda \subset \mathbb{Z}^d$ *and* $s \geq 0$,

$$\mathbb{E}\left(\eta^{\Lambda}_{[t-s,t]}(i) - \eta^{\Lambda}_t(i)\right)^2 \leq C s^{1-d/2}. \tag{35}$$

(These bounds are relevant only for $d \geq 3$.*)*

$$\lim_{s\to\infty} \mathbb{E}\left(\eta^{\Lambda\setminus\{0\}}_{[t-s,t]}(i) - \eta^{\Lambda\setminus\{0\}}_t(i)\right)^2 = 0 \tag{36}$$

$$\mathbb{E}\left(\eta^{\mathbb{Z}^d}_{[t-s,t]}(i) - \eta^{\mathbb{Z}^d}_{[t-s,t]}(0) - \Delta^{\mathbb{Z}^d}_t(i)\right)^2 \leq C(i) s^{-d/2}. \tag{37}$$

Stationarity. *The processes* $(\eta^{\Lambda}_t,\ t \in \mathbb{R})$ *and* $(\Delta^{\mathbb{Z}^d}_t,\ t \in \mathbb{R})$ *are stationary Markov with generators* L^{Λ} *and* $\widetilde{L}$ *respectively, where* $\widetilde{L}$ *is given later in* (45).
Uniqueness for finite Λ. *If* Λ *has a finite number of points, then the law of* η^{Λ}_t *is the unique invariant measure for the process with generator* L^{Λ}.
L_2 **space convergence.** *For either* $d \geq 3$ *or* $\Lambda \neq \mathbb{Z}^d$,

$$\lim_{\Lambda'\nearrow\Lambda} \mathbb{E}\left(\eta^{\Lambda'}_t(i) - \eta^{\Lambda}_t(i)\right)^2 = 0. \tag{38}$$

Proof.
A.s. time convergence. Fix $t \in \mathbb{R}$. By Proposition 2, the process $(\eta^{\Lambda}_{[t-s,t]}(i), s \geq t)$ is a martingale. By Corollary 8 its variances are uniformly bounded under the given conditions — since the origin plays no special role, it is not a loss of generality to assume that $0 \notin \Lambda$. Analogously, the process as seen from the height at the origin $(\eta^{\Lambda}_{[t-s,t]}(i) - \eta^{\Lambda}_{[t-s,t]}(0), s \geq 0)$ is a martingale with uniformly bounded variances under the given conditions. Martingales with uniformly bounded variances converge almost surely [10].
L_2 **time convergence**.

$$\mathbb{E}\left(\eta^{\Lambda}_{[t-s,t]}(i) - \eta^{\Lambda}_t(i)\right)^2 \tag{39}$$

$$= \mathbb{E}\left(\eta^{\Lambda}_{[t-s,t]}(i)\right)^2 + \mathbb{E}\left(\eta^{\Lambda}_t(i)\right)^2 - 2\mathbb{E}\left(\eta^{\Lambda}_{[t-s,t]}(i)\eta^{\Lambda}_t(i)\right) \tag{40}$$

$$= \mathbb{E}\sum_{k\in\mathbb{Z}^d}\left(\sum_{n:T_n(k)\in[t-s,t]} + \sum_{n:T_n(k)\in(-\infty,t]} - 2\sum_{n:T_n(k)\in[t-s,t]}\right) b^{\Lambda}_{[t,T_n]}(i,k)^2$$

$$= \mathbb{E}\sum_{k\in\mathbb{Z}^d}\sum_{n:T_n(k)\in(-\infty,t-s)} b^{\Lambda}_{[t,T_n]}(i,k)^2 \leq \mathbb{E}\sum_{k\in\mathbb{Z}^d}\sum_{n:T_n(k)\in(-\infty,t-s)} b^{\mathbb{Z}^d}_{[t,T_n]}(i,k)^2$$

$$= \int_s^{\infty} \mathbb{P}(D^0_u = 0)\mathrm{d}u < C s^{1-d/2} \tag{41}$$

where the second identity comes from (20), the inequality from (30) and the final identity can be shown as (23). This shows the inequality in (35).

By the martingale property,

$$\mathbb{E}\left(\eta^{\Lambda\setminus\{0\}}_{[t-s,t]}(i)-\eta^{\Lambda\setminus\{0\}}_{t}(i)\right)^2=\mathbb{E}\left(\eta^{\Lambda\setminus\{0\}}_{[t-s,t]}(i)\right)^2-\mathbb{E}\left(\eta^{\Lambda\setminus\{0\}}_{t}(i)\right)^2$$

which converges to 0 as $s\to\infty$ because it is an increasing bounded sequence by Lemmas 6 and 7. This shows (36). Analogously, using (24) and (26),

$$\begin{aligned}&\mathbb{E}\left(\eta^{\mathbb{Z}^d}_{[t-s,t]}(i)-\eta^{\mathbb{Z}^d}_{[t-s,t]}(0)-\Delta^{\mathbb{Z}^d}_t(i)\right)^2\\&\quad=2\int_s^\infty(\mathbb{P}(D^0_u=0)-\mathbb{P}(D^i_u=0))\mathrm{d}u<C(i)s^{-d/2}.\end{aligned}\tag{42}$$

Stationarity. The construction of η^Λ_t commutes with the time-translation operator: $\eta^\Lambda_t(\omega+u)=\eta^\Lambda_{t+u}(\omega)$, where $\omega=((T_n(i),\varepsilon_n(i),U_n(i)):i\in\mathbb{Z}^d,n\in\mathbb{Z})$ and $\omega+u:=((T_n(i)+u,\varepsilon_n(i),U_n(i)):i\in\mathbb{Z}^d,n\in\mathbb{Z})$ are identically distributed. The Markov property follows as in (18).

Uniqueness. Let ξ be a random configuration in $\mathbb{R}^\Lambda$. with invariant distribution for the process, then ξ has the same law as the random configuration

$$\xi^\Lambda_{[s,t]}(i):=\sum_{j\in\Lambda}\sum_{n:T_n(j)\in[s,t)}\varepsilon_n(j)\,b^\Lambda_n(i,j)+\sum_{j\in\Lambda}b^\Lambda_{[t,s]}(i,j)\xi(j)\tag{43}$$

(recall $b^\Lambda_{[t,s]}(i,j)=\mathbb{P}(B^{\Lambda,i}_{[t,s]}=j|\mathcal{T})$). Since Λ is finite and the walk $B^{\Lambda,i}_{[t,s]}$ is absorbed at Λ^c, $b^\Lambda_{[t,s]}(i,j)$ goes to zero a.s. as $s\to-\infty$ and so does the second sum in (43). This implies that ξ and η^Λ_t (which is the limit of the first sum) have the same law.

L_2 **space convergence**. Fix $i\in\Lambda$. Using (20) we get for $\Lambda\supset\Lambda'\ni i$,

$$\mathbb{E}(\eta^{\Lambda'}_t(i)-\eta^\Lambda_t(i))^2=\mathbb{E}\sum_{j\in\Lambda'}\sum_{n:T_n(j)\le t}[b^\Lambda_n(i,j)-b^{\Lambda'}_n(i,j)]^2.\tag{44}$$

The summand in (44) is bounded by $(b^\Lambda_n(i,j))^2+(b^{\Lambda'}_n(i,j))^2\le 2(b^{\mathbb{Z}^d}_n(i,j))^2$ which is integrable for $d\ge3$ by (25) or if $\Lambda\ne\mathbb{Z}^d$ for $d=1,2$ by (27). Then, since $\lim_{\Lambda'\nearrow\mathbb{Z}^d}b^{\Lambda'}_n(i,j)=b^\Lambda_n(i,j)$ a.s., (44) goes to zero as $\Lambda'\nearrow\Lambda$. □

The pinned process and the processes as seen from the height at the origin

The height at the origin of the process $\eta^{\Lambda\setminus\{0\}}_t(0)$ remains always equal to zero. For this reason, we call it the process *pinned at zero*.

For fixed s, the process $(\eta^{\mathbb{Z}^d}_{[s,t]}-\eta^{\mathbb{Z}^d}_{[s,t]}(0),t\ge s)$ is called the *process as seen from the height at the origin*. Its generator is

$$\begin{aligned}\widetilde{L}f(\eta)&=\sum_{i\ne0}\int G(\mathrm{d}\varepsilon)[f(P_i(\eta)+\sigma\varepsilon e_i)-f(\eta)]\\&\quad+\int G(\mathrm{d}\varepsilon)\left[f\left(\eta-\left(\sum_{\ell\ne0}p(0,\ell)\eta(\ell)+\sigma\varepsilon\right)\sum_{j\ne0}e_j\right)-f(\eta)\right].\end{aligned}\tag{45}$$

The first term corresponds to updatings of sites other than the origin while the second one corresponds to the shift all sites suffer when the origin is updated.

Convergence to the invariant measure

Due to the time stationarity of the marked Poisson processes, the law of $\eta^{\Lambda}_{[s,t]}$ depends only on $t-s$, and in particular for *each* $t \geq 0$, $\eta^{\Lambda}_{[-t,0]}$ has the same law as $\eta^{\Lambda}_{[0,t]}$. Hence, for cylinder Lipschitz functions f for which there exists a finite positive α satisfying $|f(\eta) - f(\eta')| \leq \alpha(\sum_k (\eta(k) - \eta'(k))^2)^{1/2}$ depending on the coordinates in the finite set $\mathrm{Supp}(f) \subset \mathbb{Z}^d$,

$$\begin{aligned}
|\mathbb{E} f(\eta^{\Lambda}_{[0,t]}) - \mu^{\Lambda} f| &= |\mathbb{E}(f(\eta^{\Lambda}_{[-t,0]}) - f(\eta^{\Lambda}_{[-\infty,0]}))| \\
&\leq \mathbb{E}\left(\alpha \sum_{i \in \mathrm{Supp}(f)} (\eta^{\Lambda}_{[-t,0]}(i) - \eta^{\Lambda}_{[-\infty,0]}(i))^2\right)^{1/2} \\
&\leq \left(\alpha \sum_{i \in \mathrm{Supp}(f)} \mathbb{E}(\eta^{\Lambda}_{[-t,0]}(i) - \eta^{\Lambda}_{[-\infty,0]}(i))^2\right)^{1/2} \\
&\leq (|\mathrm{Supp}(f)| \alpha C t^{-1+d/2})^{1/2}
\end{aligned}$$

by (35). The last bound is relevant only for $d \geq 3$. Analogously, using (37),

$$\left| \mathbb{E} f\left(\eta^{\mathbb{Z}^d}_{[0,t]} - \eta^{\mathbb{Z}^d}_{[0,t]}(0)\right) - \mu^{\mathbb{Z}^d \setminus \{0\}} f \right| \leq (|\mathrm{Supp}(f)| \alpha C t^{-d/2})^{1/2}.$$

Other initial and boundary conditions

Let $\Lambda \subset \mathbb{Z}^d$ and

$$\Gamma_{\Lambda} := \left\{ \gamma : \sum \bar{b}_{\Lambda}(i,j)\gamma(j) < \infty, \text{ for all } i \in \Lambda \right\} \tag{46}$$

where $b_{\Lambda}(i,j)$ is the probability that a continuous time random walk, with rates p, absorbed at the sites of Λ^c, starting at $i \in \Lambda$ is absorbed at site $j \in \Lambda^c$.

Let $\gamma \in \Gamma_{\Lambda}$ and $\zeta \in \Xi_{\Lambda}$ given in (17). Due to the linear property of the dynamics, the process $\eta^{\Lambda,\gamma,\zeta}_{[s,t]}$ with initial configuration $\eta^{\Lambda,\gamma,\zeta}_{[s,s]} = \zeta$ at time s and boundary conditions γ can be seen as the sum of a process with flat boundary and initial conditions plus a "no-noise" harness process.

The process $(\overline{\eta}^{\Lambda,\gamma,\zeta}_{[s,t]} : t \geq s)$ with initial configuration $\overline{\eta}^{\Lambda,\gamma,\zeta}_{[s,s]} = \zeta$ at time s and generator

$$\overline{L}^{\Lambda,\gamma} f(\eta) = \sum_{i \in \Lambda} [f(P_i(\eta_{\Lambda}\gamma_{\Lambda^c})) - f(\eta)] \tag{47}$$

is called the *no-noise harness process*; it has γ boundary conditions outside Λ. This is just a harness process with noise distribution concentrating mass on the point 0 so that the updating of site i is done using only the P_i average of the other heights. It is still a stochastic process because the updating times are governed by the Poisson processes $\mathcal{T}$. Let $\mathcal{H}^{\Lambda,\gamma}$ be the set of harmonic functions for p on Λ with γ boundary conditions:

$$\mathcal{H}^{\Lambda,\gamma} := \left\{ h \in \mathbb{R}^{\mathbb{Z}^d} : \sum_j p(i,j)h(j) = h(i), i \in \Lambda; h(j) = \gamma(j), i \in \Lambda^c \right\}.$$

Measures concentrating mass on $\mathcal{H}^{\Lambda,\gamma}$ are invariant for the no-noise process $\overline{\eta}^{\Lambda,\gamma,\cdot}_{[s,t]}$. Some questions naturally arise here: Do the invariant measures for the no-noise process concentrate mass on $\mathcal{H}^{\Lambda,\gamma}$? Does this process converge to one of the invariant measures? If yes, what is the speed of convergence?

We have the following decomposition

$$\eta^{\Lambda,\gamma,\zeta}_{[s,t]} = \eta^{\Lambda}_{[s,t]} + \overline{\eta}^{\Lambda,\gamma,\zeta}_{[s,t]}. \tag{48}$$

Notice however that the two processes use the same Poisson epochs. Measures in the set

$$\mathcal{I}^{\Lambda,\gamma} = \{\text{law of } \eta^{\Lambda}_t + h : h \in \mathcal{H}^{\Lambda,\gamma}\} \tag{49}$$

are invariant for the process $\eta^{\Lambda,\gamma}_{[0,t]}$. Are all invariant measures convex combinations of the measures in $\mathcal{I}^{\Lambda,\gamma}$? What are the domains of attraction of the measures in $\mathcal{I}^{\Lambda,\gamma}$?

Uniqueness

For $d \geq 3$, we conjecture that the law of $\eta^{\mathbb{Z}^d}_t$ is the unique ergodic invariant measure with zero mean (that is, such that $\mathbb{E}\eta_t(i) = 0$ for all i) for the process with generator $L^{\mathbb{Z}^d}$. Hsiao [11] has proven that the law of $\eta^{\mathbb{Z}^d}_t$ is the unique invariant measure with zero mean and uniformly bounded second moment. For $d = 1, 2$, we conjecture that the law of $\eta^{\mathbb{Z}^d\setminus\{0\}}_t$ is the unique ergodic (here we mean for the height differences) invariant measure with zero mean for the process with generator $L^{\mathbb{Z}^d\setminus\{0\}}$ and the unique ergodic measure with mean zero invariant for the pinned process $\eta^{\mathbb{Z}^d}_t - \eta^{\mathbb{Z}^d}_t(0)$.

A.s. space convergence

Let $(\Lambda_m : m \geq 0)$ be an increasing family of sets such that $\Lambda_m \nearrow \Lambda$. Assuming as extra condition that G is Gaussian, we exhibit a family of random configurations $(\xi^{\Lambda_m}_t : m \geq 0)$ with marginal laws μ^{Λ_m} converging almost surely as Λ_m increases to Λ. As noted by the referee, the *existence* of such a sequence is guaranteed by the Skorohod representation theorem; our aim here is to explicitly construct it.

Fix Λ_m and the Poisson configuration $\mathcal{T}$ and define $b^m_n(i, j) := b^{\Lambda_m}_n(i, j)$ (this is a function of $\mathcal{T}$). By (14), $\eta^m_{[s,t]}(i) := \eta^{\Lambda_m}_{[s,t]}(i)$ is a sum of the independent Gaussian random variables $\varepsilon_n(j)b^m_n(i, j)$, for n such that $T_n(j) \leq t$ and $j \in \mathbb{Z}^d$.

Since $b^m_n(i, j)$ is non-decreasing in m we can define $a^0_n(i, j) = 0$ and for $m \geq 1$,

$$a^m_n(i, j) := (b^m_n(i, j)^2 - b^{m-1}_n(i, j)^2)^{1/2} \tag{50}$$

(so that $\sum_{\ell=1}^m (a^\ell)^2 = (b^m)^2$). Let $Z^\ell_n(j)$ be a sequence of independent and identically distributed centered Gaussian random variables of variance 1 and let

$$W^m_n(i, j) := \sum_{\ell=1}^{m} a^\ell_n(i, j) Z^\ell_n(j). \tag{51}$$

Hence $W^m_n(i, j)$ are independent Gaussian random variables,

$$W^m_n(i, j) \stackrel{d}{=} \varepsilon_n(j) b^m_n(i, j) \tag{52}$$

and the random configuration ξ_t^m defined by

$$\xi_t^m(i) := \sum_j \sum_n W_n^m(i, j) \tag{53}$$

has the same law as η_t^m.

Proposition 10. *Assume $G(\mathrm{d}x) = (2\pi)^{-1/2} e^{-x^2/2} \mathrm{d}x$ (Gaussian noise). Then for either $d \geq 3$ or $\Lambda \neq \mathbb{Z}^d$,*

$$\lim_{m\to\infty} \xi^m(i) = \xi_t^\Lambda(i) \quad a.s. \tag{54}$$

and for $d \geq 1$, for any Λ,

$$\lim_{m\to\infty} (\xi_t^m(i) - \xi_t^m(0)) = \xi_t^\Lambda(i) - \xi_t^\Lambda(0) \quad a.s. \tag{55}$$

Proof. By Lemma 11 below, $(\xi_t^m(i), m \geq 1)$ is a martingale. Since it has uniformly bounded second moments by Corollary 8, it converges almost surely. □

Lemma 11. *For each $i \in \mathbb{Z}^d$, the family $(\xi_t^m(i), m \geq 1)$ is a martingale for the filtration $\mathcal{F}_m$ generated by the family of variables $\{T_n(j), (Z_n^\ell(j), \ell \leq m); j \in \Lambda_m, T_n(j) \leq t\}$.*

Proof. Take $m' \geq m$. Then

$$\xi_t^{m'}(i) - \xi_t^m(i) := \sum_{j\in\Lambda_{m'}} \sum_{\ell=m+1}^{m'} \sum_n a_n^\ell(i, j) Z_n^\ell(j) \tag{56}$$

which conditioned to $\mathcal{F}_m$ has mean zero because it is a weighted sum of $Z_n^\ell(j)$'s that are independent of the weights and of those $Z_n^\ell(j)$'s generating $\mathcal{F}_m$. □

5. Reversibility and Gibbs measures

Most results of the previous sections hold for any variance-1 noise and for any finite range matrix p. With this generality the properties of the law of η_t^Λ (which is an invariant measure for the process) are not well understood besides the knowledge of the covariances. However, if we assume

$$G(\mathrm{d}x) = (2\pi)^{-1/2} e^{-x^2/2} \mathrm{d}x \text{ (Gaussian noise) and } p(0, 0) = 0, \tag{57}$$

then for finite Λ the law of η_t^Λ is the finite volume Gibbs measure μ^Λ given by (6) and it is reversible for L^Λ. These properties extend to infinite Λ as well. This is the content of our next result.

Theorem 12. *Assume (57). Then,*

(1) *For either $d \geq 3$ or $\Lambda \neq \mathbb{Z}^d$, the distribution of η_t^Λ is the Gibbs measure μ^Λ with specifications (6) and boundary conditions $\gamma \equiv 0$ and the process $(\eta_t^\Lambda, t \in \mathbb{R})$ is reversible.*
(2) *For $d \geq 1$, the marginal (invariant) distribution of $\eta_t^{\mathbb{Z}^d} - \eta_t^{\mathbb{Z}^d}(0)$ is the Gibbs measure $\mu^{\mathbb{Z}^d\setminus\{0\}}$ with specifications (6) and $\gamma \equiv 0$ and the process $(\eta_t^{\mathbb{Z}^d} - \eta_t^{\mathbb{Z}^d}(0), t \in \mathbb{R})$ is reversible.*

The case $d \geq 3$ and $\Lambda = \mathbb{Z}^d$ is already contained in [11].

Proof. (1) For finite Λ the statements are proven in Lemma 13 below. For infinite Λ the existence of the infinite volume measure μ^{Λ} with specifications (6) is proven by Spitzer [14]; alternatively it follows either from the L^2 space convergence (38) in Theorem 9 or the a.s. space convergence of Proposition 10. The reversibility of the limiting measure μ^{Λ} follows then as in Lemma 13.

(2) The existence of the infinite volume Gibbs measure $\mu^{\mathbb{Z}^d \setminus \{0\}}$ is proven by Spitzer [14], see also [4]. We do not have an alternative proof in this case. The reversibility follows as in Lemmas 13 and 14 later. $\square$

Spitzer [14] (see [4] for the non-nearest neighbor case) proved that the covariances of $\mu^{\Lambda\setminus\{0\}}$ are given by

$$\int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)\xi(i)\xi(j) = \sum_{n\geq 0} \mathbb{P}(X_n^i = j, \tau^i > n) \tag{58}$$

where X_n^i is a random walk with probability transition matrix p and τ^i is the first time the walk hits the origin or Λ^c. This is the expected number of visits to j for the walk X_n starting at i before being absorbed at 0 or Λ^c. These covariances are finite in any dimension: the number of visits to j of the walk starting at j is a geometric random variable because after each visit the walk can be absorbed at 0 or (in dimensions $d \geq 3$) never visit j again. In particular there exist constants $C(i)$

$$V^{\Lambda\setminus\{0\}}(i) =: \int \mu^{\Lambda\setminus\{0\}}(\mathrm{d}\xi)\xi(i)^2 < C(i) < \infty \quad \text{for all } \Lambda \subset \mathbb{Z}^d. \tag{59}$$

The next lemma is essentially contained in Theorem 3.3 of Hsiao [11].

Lemma 13. *Assume (57) and Λ finite. Then the Gibbs measure $\mu^{\Lambda,\gamma}$ with Hamiltonian $H^{\Lambda}(\eta) = \frac{1}{2}\sum_{i,j} p(i,j)(\eta(i)-\eta(j))^2$ is reversible for each of the generators*

$$L_k^{\Lambda,\gamma} f(\eta) = \int G(\mathrm{d}\varepsilon)[f(P_k(\eta_\Lambda\gamma_{\Lambda^c}) + \varepsilon e_k) - f(\eta)], \quad k \in \Lambda. \tag{60}$$

(For definitions of P_k and $\eta_\Lambda\gamma_{\Lambda^c}$ see (1).)

Proof. Define $\mu = \mu^{\Lambda,\gamma}$, $L_k = L_k^{\Lambda,\gamma}$ and $\eta = \eta_\Lambda\gamma_{\Lambda^c}$. We need to show that $\mu(gL_kf) = \mu(fL_kg)$ for any continuous bounded functions f and g. By definition,

$$\begin{aligned}\int \mu(\mathrm{d}\eta)g(\eta)L_kf(\eta) &= \int \mu(\mathrm{d}\eta)g(\eta)\int \frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x[f(P_k\eta + e_kx) - f(\eta)] \\ &= \int \mu(\mathrm{d}\eta)\int \frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x\, g(\eta)f(P_k\eta + e_kx) - \mu(gf).\end{aligned} \tag{61}$$

Let $\bar\eta(k) := \sum_{i\neq k} p(k,i)\eta(i)$ (this does not depend on $\eta(k)$). Then,

$$\sum_{i\neq k} p(k,i)(\eta(k)-\eta(i))^2 = \sum_{i\neq k} p(k,i)(\eta(i)-\bar\eta(k))^2 + (\eta(k)-\bar\eta(k))^2.$$

Hence,

$$\int \mu(\mathrm{d}\eta)g(\eta)\int G(\mathrm{d}x)f(P_k\eta + e_kx) \tag{62}$$

$$
\begin{aligned}
&= \int \prod_{\ell \neq k} \mathrm{d}\eta(\ell) \\
&\quad \times \exp\left(-\frac{1}{2} \sum_{i,j \neq k} p(i,j)(\eta(j) - \eta(i))^2 - \frac{1}{2} \sum_{i \neq k} p(k,i)(\eta(i) - \bar{\eta}(k))^2 \right) \\
&\quad \times \int e^{-(\eta(k)-\bar{\eta}(k))^2/2} \mathrm{d}\eta(k) \int \frac{e^{-x^2/2}}{\sqrt{2\pi}} \mathrm{d}x\, g(\eta) f(P_k\eta + e_k x).
\end{aligned}
\tag{63}
$$

Change variables: $\eta' = P_k\eta + e_k x$ and $z = \eta(k) - \bar{\eta}(k)$. Since $\eta'(i) = \eta(i)$ for $i \neq k$, the second line in (62) remains unchanged on substituting η for η'. Noticing that $x = \eta'(k) - \overline{\eta'}(k)$ and $\eta = P_k\eta' - e_k z$, (63) reads

$$
\begin{aligned}
&\int \prod_{\ell \neq k} \mathrm{d}\eta'(\ell) \\
&\quad \times \exp\left(-\frac{1}{2} \sum_{i,j \neq k} p(i,j)(\eta'(j) - \eta'(i))^2 - \frac{1}{2} \sum_{i \neq k} p(k,i)(\eta'(i) - \overline{\eta'}(k))^2 \right) \\
&\quad \times \int \frac{e^{-z^2/2}}{\sqrt{2\pi}} \mathrm{d}z \int e^{-(\eta'(k)-\overline{\eta'}(k))^2/2} \mathrm{d}x\, g(P_k\eta' - e_k z) f(\eta') \\
&= \int \mu(\mathrm{d}\eta) f(\eta) \int G(\mathrm{d}z) g(P_k\eta + e_k z).
\end{aligned}
\tag{64}
$$

Subtracting $\mu(fg)$ in (62) and (64) we obtain $\mu(gLf) = \mu(fLg)$. $\square$

Free boundary conditions

To find the infinite volume measure $\mu^{\mathbb{Z}^d \setminus \{0\}}$ we need to introduce a family of processes and measures with free boundary conditions. Let $\Lambda \subset \mathbb{Z}^d$ and define

$$
\widetilde{p}^{\Lambda}(i,j) := \frac{p(i,j)}{\sum_{k \in \Lambda} p(i,k)}, \quad i, j \in \Lambda \tag{65}
$$

that is, a transition matrix for a walk that remains in Λ. Let $\widetilde{L}^{\Lambda}$, $\widetilde{H}^{\Lambda}$, $\widetilde{\mu}^{\Lambda}$ be the generator, Hamiltonian and Gibbs measure defined with $\widetilde{p}^{\Lambda}$. In the dynamics defined by $\widetilde{L}^{\Lambda}$ the mean is taken only inside Λ (no boundary conditions matter). Let also

$$
\widetilde{L}_k^{\Lambda} f(\eta) := \int G(\mathrm{d}\varepsilon)[f(\widetilde{P}_k(\eta_\Lambda) + \sigma\varepsilon e_k) - f(\eta_\Lambda)], \quad k \in \Lambda \setminus \{0\} \tag{66}
$$

be the one-site generator of site $k \in \Lambda \setminus \{0\}$. We are interested in two processes: the process with free boundary conditions pinned at zero and the process with free boundary conditions as seen from the height at the origin. The former one has generator $\sum_{k \in \Lambda \setminus \{0\}} \widetilde{L}_k^{\Lambda}$, while the second has generator $\sum_{k \in \Lambda \setminus \{0\}} \widetilde{L}_k^{\Lambda} + \widetilde{L}^{\Lambda,0} f(\eta)$ where the shift generator $\widetilde{L}^{\Lambda,0}$ is defined by

$$
\widetilde{L}_0^{\Lambda,0} f(\eta) := \int G(\mathrm{d}\varepsilon)[f(\eta_\Lambda - (\widetilde{P}_0(\eta_\Lambda)(0) + \varepsilon)\mathbf{1}) - f(\eta_\Lambda)] \tag{67}
$$

for f not depending on $\eta(0)$, where $\mathbf{1}$ is the configuration $\mathbf{1}(i) \equiv 1$.

Lemma 14. *Assume (57) and Λ finite. Then the Gibbs measure $\widetilde{\mu}^{\Lambda,0}$ is reversible for each of the generators $\widetilde{L}_k^{\Lambda}$ (and hence for the process pinned at zero with free boundary conditions) and for the shift generator $\widetilde{L}^{\Lambda,0}$ (and hence for the free process as seen from the height at the origin).*

Proof. The proof that the measure $\widetilde{\mu}^{\Lambda,0}$ is reversible for $\widetilde{L}_k^{\Lambda}$ for $k \neq 0$ goes as the proof of Lemma 13.

To show that $\widetilde{\mu}^{\Lambda,0}$ is reversible for $\widetilde{L}^{\Lambda,0}$ take g and f not depending on the height at the origin and compute

$$\int \widetilde{\mu}^{\Lambda,0}(\mathrm{d}\eta)g(\eta)\int G(\mathrm{d}x)f(\eta-(\bar{\eta}(0)+x)\mathbf{1}) \tag{68}$$

$$=\int\prod_{\ell\neq 0}d\eta(\ell) \tag{69}$$

$$\times\ \exp\left(-\frac{1}{2}\sum_{i,j\neq 0}\widetilde{p}(i,j)(\eta(j)-\eta(i))^2-\frac{1}{2}\sum_{k\neq 0}\widetilde{p}(0,k)\eta(k)^2\right)$$

$$\times\int\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x\,g(\eta)\,f(\eta-(\bar{\eta}(0)+x)\mathbf{1}). \tag{70}$$

Change variables: $z=\bar{\eta}(0)$ and $\eta'=\eta-(\bar{\eta}(0)+x)\mathbf{1}$. Then $\overline{\eta'}(0)=-x$ and

$$\sum_{k\neq 0}\widetilde{p}(0,k)\eta(k)^2+x^2=\sum_{k\neq 0}\widetilde{p}(0,k)\eta'(k)^2+z^2. \tag{71}$$

So (68) equals

$$=\int\prod_{\ell\neq 0}\mathrm{d}\eta'(\ell)\exp\left(-\frac{1}{2}\sum_{i,j\neq 0}\widetilde{p}(i,j)(\eta'(j)-\eta'(i))^2-\frac{1}{2}\sum_{k\neq 0}\widetilde{p}(0,k)\eta'(k)^2\right)$$

$$\times\int\frac{e^{-z^2/2}}{\sqrt{2\pi}}\mathrm{d}z\,g(\eta'-(\overline{\eta'}(0)+z)\mathbf{1})\,f(\eta')$$

$$=\int\widetilde{\mu}^{\Lambda,0}(\mathrm{d}\eta)f(\eta)\int G(\mathrm{d}z)g(\eta-(\bar{\eta}(0)+z)\mathbf{1}). \tag{72}$$

Subtracting $\widetilde{\mu}^{\Lambda,0}(fg)$ in (68) and (72) we obtain $\widetilde{\mu}^{\Lambda,0}(g\widetilde{L}^{\Lambda,0}f)=\widetilde{\mu}^{\Lambda,0}(f\widetilde{L}^{\Lambda,0}g)$. □

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