Algebraic winding numbers

Daniel Perrucci[♭*] Marie-Françoise Roy[♯]

♭ Departamento de Matemática, FCEN, Universidad de Buenos Aires and IMAS UBA-CONICET,
Buenos Aires, Argentina,
♯ IRMAR (UMR CNRS 6625), Université de Rennes,
Campus de Beaulieu, 35042 Rennes Cedex, France.

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Abstract

In this paper, we propose a new algebraic winding number and prove that it computes the number of complex roots of a polynomial in a rectangle, including roots on edges or vertices with appropriate counting. The definition makes sense for the algebraic closure $\mathbf{C} = \mathbf{R}[i]$ of a real closed field $\mathbf{R}$, and the root counting result also holds in this case. We study in detail the properties of the algebraic winding number defined in [3] with respect to complex root counting in rectangles. We extend both winding numbers to rational functions, obtaining then algebraic versions of the argument principle for rectangles.

Keywords: Root counting, Cauchy index, Winding number, Argument principle.

MSC2020: 12D10, 13J30, 14Q20.

1 Introduction

The classical argument principle applied to a rational function $F/G \in \mathbb{C}(Z) \setminus \{0\}$ on a rectangle Γ, states that, as long as F/G has no zeros or poles on the boundary $\partial\Gamma$ of Γ, the winding number of the curve $(F/G) \circ \partial\Gamma$, which can be computed analytically as

$$\frac{1}{2\pi i} \int_{\partial\Gamma} \frac{(F/G)'(z)}{(F/G)(z)} dz,$$

counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G inside Γ (see [1, Chapter 4, Section 5.2]).

In this paper we give a new algebraic definition of the winding number, proving an algebraic version of the argument principle on a rectangle Γ in full generality: the algebraic winding number counts

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the number of zeros (with multiplicity) minus the number of poles (with order) even when there are zeroes of poles on the boundary of Γ. As can be expected, zeroes (resp. poles) on the edges of Γ count for 1/2 (resp. -1/2), while zeroes or poles at the vertices count for 1/4 (resp. -1/4). Moreover the algebraic version of the argument principle is valid for the algebraic closure $\mathbf{C} = \mathbf{R}[i]$ of any real closed field $\mathbf{R}$, even in situations where the integration is not available. A first algebraic definition of the winding number already appears in [3] but is not adapted to count roots at the vertices.

The strategy of the proof is to check that the count of zeroes is correct for the basic cases of constants and degree one complex polynomials and to use additivity properties of the algebraic winding numbers with respect multiplication of rational functions to prove the general case. These additivity properties are proved through a univariate auxiliary product formula, since the algebraic winding numbers are defined by univariate Cauchy indices associated to the real and imaginary part of the rational function on the edges of Γ. This strategy was already used in [3] but the additivity of the previous algebraic number was not universally true, and moreover the auxiliary product formula was also not valid in all cases.

Finally, studying in all needed details the delicate situations involved, we obtain a complete algebraic proof of the argument principle, with no restriction on the edges and vertices. We also generalize the results in [3], clarify some of its statements and complete its proofs.

After this informal presentation, the introduction proceeds now with the definition of the Cauchy index, the definition of the two algebraic winding numbers, for polynomials and for rational functions, and finally states the main results of the paper and explains the organization of the following sections.

1.1 Definition of the Cauchy index

We define the Cauchy index of a pair of univariate polynomials (P, Q) on an interval. This definition coincides with the classical Cauchy index of the rational function P/Q when both P, Q do not vanish at the endpoints of the interval and is needed in full generality here.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbf{R}$ be a real closed field and $x \in \mathbf{R}$. We consider the sign of x as usual as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Given $P, Q \in \mathbf{R}[X] \setminus \{0\}$ they can be written uniquely as

$$P = (X - x)^{\operatorname{mult}_x(P)} P_x, \qquad Q = (X - x)^{\operatorname{mult}_x(Q)} Q_x,$$

with $\operatorname{mult}_x(P), \operatorname{mult}_x(Q) \in \mathbb{N}$, $P_x, Q_x \in \mathbf{R}[X]$ and $P_x(x) \neq 0, Q_x(x) \neq 0$. We consider the valuation defined by x on $\mathbf{R}(X)$ by

$$\operatorname{val}_x(P/Q) = \begin{cases} \operatorname{mult}_x(P) - \operatorname{mult}_x(Q) & \text{if } P \neq 0, \\ +\infty & \text{if } P = 0. \end{cases}$$

Let $P, Q \in \mathbf{R}[X]$. We define the Sign of (P, Q) at x, which is the sign of the rational function P/Q at x whenever this makes sense, and 0 otherwise. The reason why we consider pairs of polynomials instead of rational functions is because in the following sections, the case $Q = 0$ will also be of use.

Definition 1 *For $P, Q \in \mathbf{R}[X]$ and $x \in \mathbf{R}$,*

$$\operatorname{Sign}(P,Q,x) := \begin{cases} \operatorname{sign}(P_x(x)Q_x(x)) & \text{if } P \neq 0, Q \neq 0 \text{ and } \operatorname{val}_x(P/Q) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily checked that whenever $P(x)$ and $Q(x)$ are not simultaneously 0, $\operatorname{Sign}(P,Q,x) = \operatorname{sign}(P(x)Q(x))$.

We also consider the sign variation defined as follows.

Definition 2 *For $P, Q \in \mathbf{R}[X]$ and $x \in \mathbf{R}$,*

$$\operatorname{Var}_x(P,Q) := \frac{1}{2} - \frac{1}{2}\operatorname{Sign}(P,Q,x).$$

Also, for $a, b \in \mathbf{R}$,

$$\operatorname{Var}_a^b(P,Q) := \operatorname{Var}_a(P,Q) - \operatorname{Var}_b(P,Q) = -\frac{1}{2}\operatorname{Sign}(P,Q,a) + \frac{1}{2}\operatorname{Sign}(P,Q,b).$$

Note that $\operatorname{Var}_x(P,Q)$ coincides with the usual notion of sign variation whenever $P(x)$ and $Q(x)$ are not simultaneously 0, since in this case:

$$\operatorname{Var}_x(P,Q) = \begin{cases} 0 & \text{if } P(x)Q(x) > 0, \\ \frac{1}{2} & \text{if } P(x)Q(x) = 0, \\ 1 & \text{if } P(x)Q(x) < 0. \end{cases}$$

We define the Cauchy index of (P,Q) first at a point and then on an interval, following [3]. As before, we need to consider pairs of polynomials instead of rational functions in order to not to exclude the case $Q = 0$.

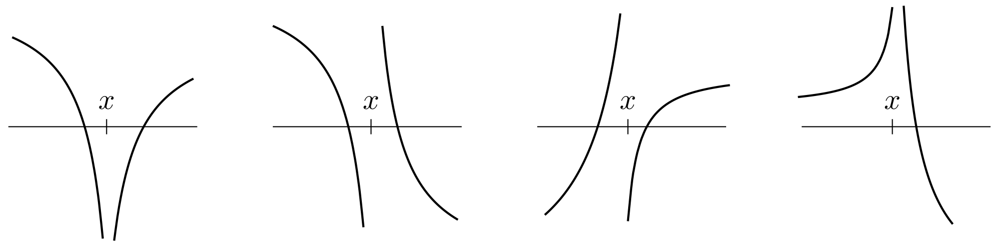
Definition 3 *For $P, Q \in \mathbf{R}[X]$, $x \in \mathbf{R}$ and $\varepsilon \in \{+,-\}$,*

$$\operatorname{Ind}_x^{\varepsilon}(P,Q) := \begin{cases} \frac{1}{2}\operatorname{sign}(P_x(x)Q_x(x)) & \text{if } P \neq 0, Q \neq 0, \varepsilon = + \text{ and } \operatorname{val}_x(P/Q) < 0, \\ \frac{1}{2}(-1)^{\operatorname{val}_x(P/Q)}\operatorname{sign}(P_x(x)Q_x(x)) & \text{if } P \neq 0, Q \neq 0, \varepsilon = - \text{ and } \operatorname{val}_x(P/Q) < 0, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\operatorname{Ind}_x(P,Q) := \operatorname{Ind}_x^+(P,Q) - \operatorname{Ind}_x^-(P,Q).$$

It is easy to see that, if $\operatorname{val}_x(P/Q) < 0$, $\operatorname{Ind}_x^+(P,Q)$ is half the sign of P/Q at a sufficiently small interval to the right of x and $\operatorname{Ind}_x^-(P,Q)$ is half the sign of P/Q at a sufficiently small interval to the left of x. We illustrate the definition of Cauchy index at a point considering the graph of the rational function P/Q around x in different cases.



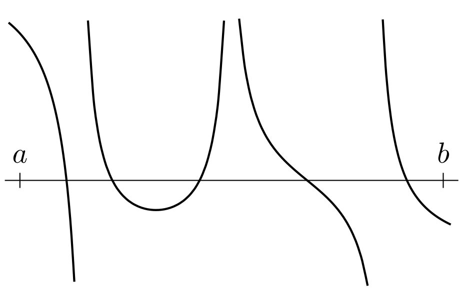
$\mathrm{Ind}_x(P,Q)=0$ $\qquad$ $\mathrm{Ind}_x(P,Q)=1$ $\qquad$ $\mathrm{Ind}_x(P,Q)=-1$ $\qquad$ $\mathrm{Ind}_x(P,Q)=0$

Definition 4 *For $P, Q \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$,*

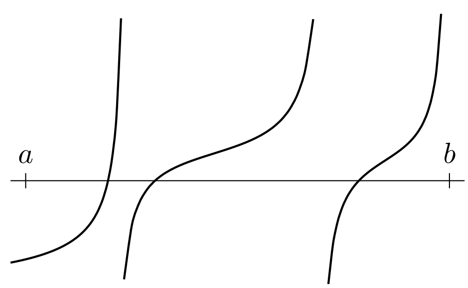
$$\mathrm{Ind}_a^b(P,Q) := \begin{cases} \mathrm{Ind}_a^+(P,Q) + \sum_{x\in(a,b)} \mathrm{Ind}_x(P,Q) - \mathrm{Ind}_b^-(P,Q) & \text{if } a<b, \\ -\mathrm{Ind}_b^a(P,Q) & \text{if } b<a, \\ 0 & \text{if } a=b. \end{cases}$$

Note that the sum is well-defined since for any P and Q we have $\mathrm{Ind}_x(P,Q) \neq 0$ only for a finite number of $x \in \mathbf{R}$.

In the following picture we consider again the graph of the function P/Q, this time in $[a,b]$.



$\mathrm{Ind}_a^b(P,Q) = 1+0+1 = 2$



$\mathrm{Ind}_a^b(P,Q) = -1-1-\frac{1}{2} = -\frac{5}{2}$

If P and Q are not simultaneously 0, then any common factor of P and Q can be simplified in both P and Q without changing the value of $\mathrm{Sign}(P,Q,x)$, $\mathrm{Var}_a^b(P,Q)$ or $\mathrm{Ind}_a^b(P,Q)$ for any x, a, b in $\mathbf{R}$.

The algorithmic symbolic computation of the Cauchy index of (P,Q) can be done using Sturm sequences as in [3, Section 3] or subresultant polynomials as in [5]. Classically, the Cauchy index of the rational function P/Q is defined on intervals under the assumption that P and Q do not vanish at the endpoints and can be computed by various symbolic methods (see [2, Chapter 9]).

1.2 Two different algebraic winding numbers

First, we recall the algebraic winding number w which was introduced in [3] with the aim of providing a real algebraic proof of the fundamental theorem of algebra.

Let $\mathbf{C} = \mathbf{R}[i]$ be the algebraic closure of the real closed field $\mathbf{R}$.

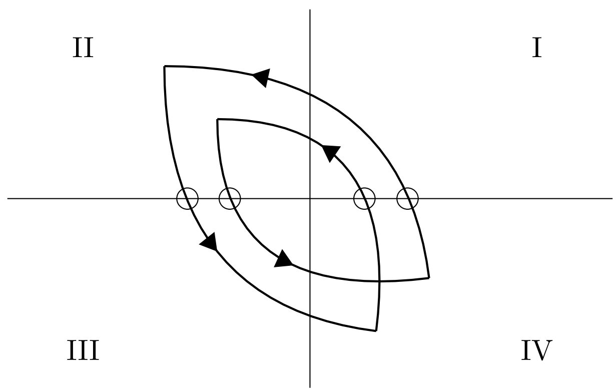
Notation 5 *For $F \in \mathbf{C}[X,Y]$, we denote F_{re} and F_{im} the real and imaginary parts of F, i.e. the unique polynomials in $\mathbf{R}[X,Y]$ such that the identity*

$$F(X,Y) = F_{\mathrm{re}}(X,Y) + iF_{\mathrm{im}}(X,Y)$$

in $\mathbf{C}[X,Y]$ holds.

Definition 6 *([3, Definition 4.2]) Let $F \in \mathbf{C}[X,Y]$, $x_0, x_1, y_0, y_1 \in \mathbf{R}$ with $x_0 < x_1$ and $y_0 < y_1$, and $\Gamma := [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. We consider the following* winding number w *of F on $\partial\Gamma$:*

$$\begin{aligned} \mathrm{w}(F \mid \partial\Gamma) \;\; := \;\; & \tfrac{1}{2}\Big(\mathrm{Ind}_{x_0}^{x_1}(F_{\mathrm{re}}(T,y_0), F_{\mathrm{im}}(T,y_0)) + \mathrm{Ind}_{y_0}^{y_1}(F_{\mathrm{re}}(x_1,T), F_{\mathrm{im}}(x_1,T)) \\ & + \mathrm{Ind}_{x_1}^{x_0}(F_{\mathrm{re}}(T,y_1), F_{\mathrm{im}}(T,y_1)) + \mathrm{Ind}_{y_1}^{y_0}(F_{\mathrm{re}}(x_0,T), F_{\mathrm{im}}(x_0,T))\Big). \end{aligned}$$



$\mathrm{w}(F \mid \partial\Gamma) = 2$

The idea behind this definition is, if we go through the curve $F \circ \partial\Gamma$ following the counterclock sense, to count one half of a turn each time this curve crosses the X-axis from quadrant IV to I or from quadrant II to III, and minus one half of a turn each time it crosses the X-axis from quadrant I to IV or from quadrant III to II. Since these crossings coincide with jumps of the rational function $F_{\mathrm{re}}/F_{\mathrm{im}}$ from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ respectively, the Cauchy index is an appropriate algebraic tool to count the number of turns counterclockwise, which is (when F does not vanish on $\partial\Gamma$) the classical definition of the winding number.

We consider a new variable Z together with the inclusion $\mathbf{C}[Z] \subset \mathbf{C}[X,Y]$ through the identity $Z = X + iY$.

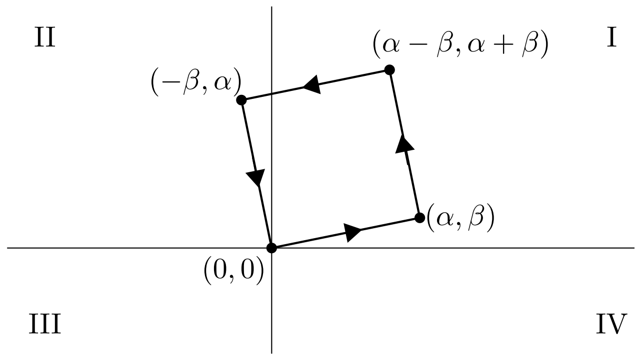
Example 7 *Let $F = Z$ and $\Gamma = [0,1] \times [0,1]$, then*

$$\mathrm{w}(F \mid \partial\Gamma) = \frac{1}{2}\Big(\mathrm{Ind}_0^1(T,0) + \mathrm{Ind}_0^1(1,T) + \mathrm{Ind}_1^0(T,1) + \mathrm{Ind}_1^0(0,T)\Big) = \frac{1}{4}.$$

This basic example illustrates why we consider the Cauchy index of a pair of polynomials (P,Q) rather than the Cauchy index of a rational function P/Q: otherwise the Cauchy index $\mathrm{Ind}_0^1(T,0)$ would not be well defined.

However the algebraic winding number w is not additive with respect multiplication by a constant as we see now.

Example 8 *Let* $\alpha, \beta \in \mathbf{R}$ *with* $0 < \beta < \alpha$, $\gamma = \alpha + i\beta \in \mathbf{C}$ *and* $\Gamma = [0,1] \times [0,1] \subset \mathbf{R}^2$. *Then* $\mathrm{w}(\gamma \,|\, \partial\Gamma) = 0$, $\mathrm{w}(Z \,|\, \partial\Gamma) = \frac{1}{4}$ *but* $\mathrm{w}(\gamma Z \,|\, \partial\Gamma) = 0$ *instead of* $\frac{1}{4}$.



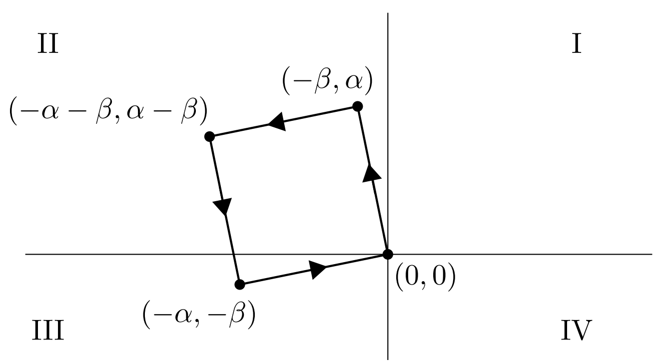
One of the main contributions of this paper is the definition of a new algebraic winding number W, more symmetric than w, which works in general for computing the number of roots with multiplicity on rectangles, counting one half for roots on edges and one quarter for roots on vertices (Theorem 12).

Definition 9 *Let* $F \in \mathbf{C}[X,Y]$, $x_0, x_1, y_0, y_1 \in \mathbf{R}$ *with* $x_0 < x_1$ *and* $y_0 < y_1$, *and* $\Gamma := [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. *We define the following* winding number W *of* F *on* $\partial\Gamma$*:*

$$\mathrm{W}(F \,|\, \partial\Gamma) \;:=\; \frac{1}{2}\Big(\mathrm{w}(F \,|\, \partial\Gamma) + \mathrm{w}(iF \,|\, \partial\Gamma)\Big).$$

In order to motivate the definition of W, let us go back to Example 8.

Example 10 *Continuing Example 8, we know already that* $\mathrm{w}(\gamma Z \,|\, \partial\Gamma) = 0$. *Note that even though the curve* $(\gamma Z) \circ \partial\Gamma$ *does not cross the* X*-axis, it crosses the* Y*-axis. In other words, the curve* $(i\gamma Z) \circ \partial\Gamma$ *crosses the* X*-axis and we have* $\mathrm{w}(i\gamma Z \,|\, \partial\Gamma) = 1/2$ *as we see in the following picture.*



Finally $\mathrm{W}(\gamma Z \,|\, \partial\Gamma) = 1/4$. *It can be checked that* $\mathrm{W}(\gamma \,|\, \partial\Gamma) = 0$ *and* $\mathrm{W}(Z \,|\, \partial\Gamma) = 1/4$, *so the additivy of* W *with respect to multiplication holds in this special case.*

We shall indeed see later that W is always additive with respect to multiplication in $\mathbf{C}[Z] \setminus \{0\}$ (Proposition 23).

1.3 Extension of algebraic winding numbers to rational functions

For $F \in \mathbf{C}[X,Y]$, we denote $\overline{F} \in \mathbf{C}[X,Y]$ the conjugate polynomial

$$\overline{F}(X,Y) = F_{\text{re}}(X,Y) - iF_{\text{im}}(X,Y).$$

Now consider $F, G \in \mathbf{C}[X,Y]$ with $G \neq 0$. Then $G_{\text{re}}^2 + G_{\text{im}}^2 \neq 0 \in \mathbf{R}[X,Y]$ and in $\mathbf{C}(X,Y)$ we have the identity

$$\frac{F}{G} = \frac{F\overline{G}}{G\overline{G}} = \underbrace{\frac{(F\overline{G})_{\text{re}}}{G_{\text{re}}^2 + G_{\text{im}}^2}}_{\in \mathbf{R}(X,Y)} + i \underbrace{\frac{(F\overline{G})_{\text{im}}}{G_{\text{re}}^2 + G_{\text{im}}^2}}_{\in \mathbf{R}(X,Y)},$$

which defines the real and imaginary part of F/G. Note that the quotient in $\mathbf{R}(X,Y)$ between the real and imaginary parts of $F/G \in \mathbf{C}(X,Y)$ is the same as the quotient in $\mathbf{R}(X,Y)$ of the real and imaginary part of $F\overline{G} \in \mathbf{C}[X,Y]$. This motivates the following extension of the definition of the algebraic winding numbers to the field of rational functions.

Definition 11 *Let $F/G \in \mathbf{C}(X,Y)$, $x_0, x_1, y_0, y_1 \in \mathbf{R}$ with $x_0 < x_1$ and $y_0 < y_1$, and $\Gamma := [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. We define the following* winding numbers *of F/G on $\partial\Gamma$:*

$$\begin{aligned} \mathrm{w}(F/G \,|\, \partial\Gamma) &:= \mathrm{w}(F\overline{G} \,|\, \partial\Gamma), \\ \mathrm{W}(F/G \,|\, \partial\Gamma) &:= \mathrm{W}(F\overline{G} \,|\, \partial\Gamma). \end{aligned}$$

It can be easily checked that w and W are well defined for $F/G \in \mathbf{C}(X,Y)$, in the sense that they do not depend on the election of the numerator F and the denominator G in $\mathbf{C}[X,Y]$.

1.4 Goals of the paper

The main results of the paper are two algebraic versions of the argument principle on rectangles.

Our main goal is to prove Theorem 12 which shows that the new algebraic winding number W counts complex zeroes and poles with full generality.

As before, let $\mathbf{R}$ be a real closed field and $\mathbf{C} = \mathbf{R}[i]$ its algebraic closure.

Theorem 12 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle. Then $\mathrm{W}(F/G \,|\, \partial\Gamma)$ counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G in Γ. Zeros and poles on the edges count for one half. Zeros and poles on the vertices count for one quarter.*

The key tool to prove Theorem 12 is to prove that, in opposition to w, W is additive in general with respect to multiplication in $\mathbf{C}(Z) \setminus \{0\}$ (Proposition 23).

Note that Theorem 12 provides immediately a computer algebra subdivision method to isolate the complex roots of a polynomial in a rectangle Γ by cutting Γ into four rectangles, computing W for each of them and iterating the construction. This method relies only on computing the univariate Cauchy index which is a basic algorithm in real algebraic geometry (see for example [2]). However such an algorithm is not optimal in terms of computational complexity. The complexity of the isolation of complex roots remains a very active domain of research. See [4] for a classical reference in the topic.

The second goal of this paper (Theorem 13) is to clarify the root counting properties of w. We show that w also counts complex zeroes and poles under mild hypothesis. Before stating this result, we extend the valuation we considered before to $\mathbf{C}(Z)$.

Given $F, G \in \mathbf{C}[Z] \setminus \{0\}$ they can be written uniquely as

$$F = (Z - z)^{\mathrm{mult}_z(F)}\, F_z, \qquad G = (Z - z)^{\mathrm{mult}_z(F)}\, G_z,$$

with $\mathrm{mult}_z(F), \mathrm{mult}_z(G) \in \mathbb{N}$, $F_z, G_z \in \mathbf{C}[Z]$ and $F_z(z) \neq 0, G_z(z) \neq 0$. We consider the valuation defined by z on $\mathbf{C}(Z)$ by

$$\mathrm{val}_z(F/G) = \begin{cases} \mathrm{mult}_z(F) - \mathrm{mult}_z(G) & \text{if } F \neq 0, \\ +\infty & \text{if } F = 0. \end{cases}$$

Theorem 13 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F/G has even valuation at the vertices of Γ. Then $\mathrm{w}(F/G \,|\, \partial\Gamma)$ counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G in Γ. Zeros and poles on the edges count for one half. Zeros and poles on the vertices count for one quarter.*

The key tool to prove Theorem 12 is to prove that w is additive with respect to multiplication in $\mathbf{C}(Z) \setminus \{0\}$ when restricted to rational functions with even valuation at the vertices of Γ (Proposition 22).

Note that in the particular case of a polynomial $F \in \mathbf{C}[Z] \setminus \{0\}$ which does not vanish at the vertices of Γ, Theorem 13 gives as a corollary the following result.

Theorem 14 *([3, Theorem 5.1]) Let $F \in \mathbf{C}[Z] \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F does not vanish at the vertices of Γ. Then $\mathrm{w}(F \,|\, \partial\Gamma)$ counts the number of zeros (with multiplicity) of F in Γ. Zeros on the edges count for one half.*

The proof in [3] of Theorem 14 ([3, Theorem 5.1]) has a flaw related to the fact that the auxiliary product formula ([3, Theorem 4.5]) does not hold in full generality (we will discuss the topic of the auxiliary product formula in Section 2). For a polynomial F which does not vanish on the boundary of Γ, the proof of Theorem 14 can be fixed easily (see details in [6, Lemma 20 and Theorem 39]); but to the best of our knowledge, Theorem 14 for a polynomial F with roots on the edges of Γ, was not proved yet.

1.5 Organization of the paper

The rest of the paper is organized as follows. In Section 2 we discuss the auxiliary product formula on univariate polynomials coming from [3, Theorem 4.5]. In Section 3 we study product formulas for complex rational functions, more precisely the additivity properties of the algebraic winding numbers w and W with respect to multiplication in $\mathbf{C}(Z) \setminus \{0\}$, using the auxiliary product formula. Finally, in Section 4, we use the additivity properties of the algebraic winding numbers W and w to prove Theorem 12 and Theorem 13.

2 Auxiliary product formula for univariate polynomials

Let $F, H \in \mathbf{C}[Z]\setminus\{0\}$ and let $\Gamma \subset \mathbf{R}^2$ be a rectangle. The strategy developed in [3] to relate $\mathrm{w}(FH \mid \partial\Gamma)$ with $\mathrm{w}(F \mid \partial\Gamma) + \mathrm{w}(H \mid \partial\Gamma)$ is to consider first the polynomials P, Q, R, S obtained when restricting F_{re}, F_{im}, H_{re}, H_{im} to an edge of Γ, then the polynomials $PR-QS, PS+QR$ which are the real and imaginary part of the product FH restricted to this edge of Γ, and then relating $\mathrm{Ind}_a^b(PR-QS, PS+QR)$, with $\mathrm{Ind}_a^b(P,Q) + \mathrm{Ind}_a^b(R,S)$, where $a < b$ are the endpoints of the interval parametrizing the edge of Γ under consideration.

For any $P, Q, R, S \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$ with $a < b$, we consider the following auxiliary product formula coming from [3, Theorem 4.5]:

$$\mathrm{Ind}_a^b(PR-QS, PS+QR) = \mathrm{Ind}_a^b(P,Q) + \mathrm{Ind}_a^b(R,S) - \mathrm{Var}_a^b(PS+QR, QS). \tag{1}$$

The auxiliary product formula (1) does not hold with full generality, as can be seen in the following example appearing already in [6].

Example 15 *Let $a = 0$, $b = 1$, $P = 1$, $Q = X$, $R = X-1$, $S = X$. Then $PR - QS = -X^2 + X - 1$, $PS + QR = X^2$ and*

$$\mathrm{Ind}_0^1(PR-QS, PS+QR) = -\frac{1}{2}, \quad \mathrm{Ind}_0^1(P,Q) = \frac{1}{2}, \quad \mathrm{Ind}_0^1(R,S) = -\frac{1}{2}, \quad \mathrm{Var}_0^1(PS+QR, QS) = 0,$$

so the auxiliary product formula (1) does not hold.

In the rest of the section, we characterize exactly the cases where the auxiliary product formula (1) holds, and we provide an alternative formula for the cases in which (1) does not hold. To this aim, we introduce first the following definition.

Definition 16 *Let $P, Q, R, S \in \mathbf{R}[X]$ and $c \in \mathbf{R}$. We say that $c \in \mathbf{R}$ is a* bad number *for P, Q, R, S, if $Q, S \neq 0$ and c satisfies the following two conditions:*

- $\mathrm{val}_c(P/Q) = \mathrm{val}_c(R/S) < 0$,
- $\mathrm{val}_c((PS+QR)/QS) = 0$.

If P, Q, R, S are clear from the context, we simply say that c is a bad number.

To illustrate this definition, 0 is a bad number and 1 is not a bad number in Example 15.

Note that if c is a bad number, then $Q, S \neq 0$ by definition but also necessarily $P, R, PS + QR \neq 0$. Note also that since bad numbers are roots of Q and S, there is at most a finite number of bad numbers in $\mathbf{R}$.

Next result proves that the auxiliary product formula is valid when the endpoints of the interval are not bad numbers.

Proposition 17 *Let $P, Q, R, S \in \mathbf{R}[X]$ with P and Q not simultaneously 0 and R and S not simultaneously 0, and $a, b \in \mathbf{R}$ with $a < b$. If a and b are not bad numbers, the auxiliary product formula (1)*

$$\mathrm{Ind}_a^b(PR-QS, PS+QR) = \mathrm{Ind}_a^b(P,Q) + \mathrm{Ind}_a^b(R,S) - \mathrm{Var}_a^b(PS+QR, QS)$$

holds.

The second result studies the situation when the endpoints of the interval are bad numbers. This result will not be needed in the rest of the paper, but we include it for the sake of completeness.

Proposition 18 *Let $P, Q, R, S \in \mathbf{R}[X]$ with P and Q not simultaneously 0 and R and S not simultaneously 0, and $a, b \in \mathbf{R}$ with $a < b$.*

i) If a is a bad number and b is not a bad number,

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) \ = \ \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \frac{1}{2}\mathrm{Sign}(PS + QR, QS, b).$$

ii) If b is a bad number and a is not a bad number,

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) \ = \ \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) + \frac{1}{2}\mathrm{Sign}(PS + QR, QS, a).$$

iii) If a and b are both bad numbers,

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) \ = \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S).$$

It can be easily checked that in Example 15, the equation of item i) holds instead of equation (1).

In order to move faster to the study of the properties of the algebraic winding numbers, we postpone the proofs of Propositions 17 and 18, which are rather lengthy and technical, to an annex at the end of the paper.

3 Product formulas for complex rational functions

The strategy to study the additivity of w and W with respect to product in $\mathbf{C}(Z) \setminus \{0\}$ is based on the auxiliary product formula (1) proved in Proposition 17.

We consider $F/G, H/I \in \mathbf{C}(Z) \setminus \{0\}$ and a rectangle $\Gamma = [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. Without loss of generality, throughout this section we assume that F and G are coprime and H and I are coprime.

We first discuss the relation between $\mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma)$ with $\mathrm{w}(F/G \,|\, \partial\Gamma) + \mathrm{w}(H/I \,|\, \partial\Gamma)$.

Notation 19 *We take the parametrizations (T, y_0), (x_1, T), (T, y_1) and (x_0, T) of the lines containing the bottom, right, top and left edges of Γ and consider the following polynomials in $\mathbf{R}[T]$:*

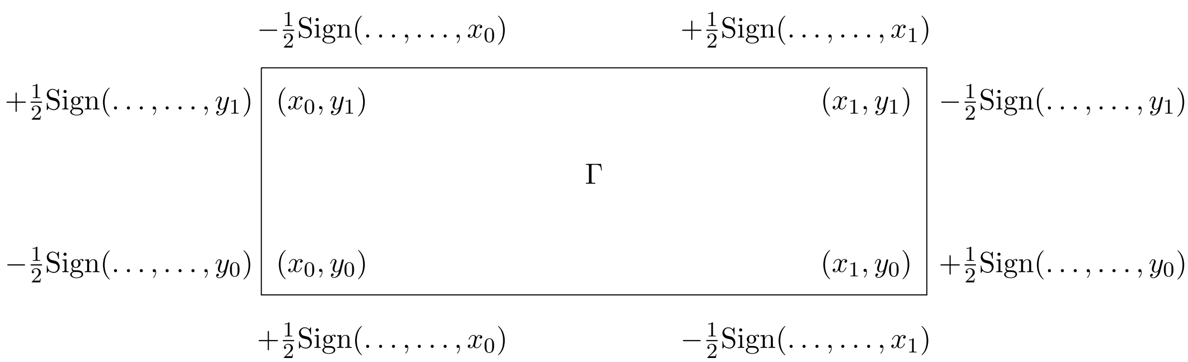
$$\begin{array}{llll}
P_1(T) = (F\overline{G})_{\mathrm{re}}(T, y_0), & Q_1(T) = (F\overline{G})_{\mathrm{im}}(T, y_0), & R_1(T) = (H\overline{I})_{\mathrm{re}}(T, y_0), & S_1(T) = (H\overline{I})_{\mathrm{im}}(T, y_0), \\
P_2(T) = (F\overline{G})_{\mathrm{re}}(x_1, T), & Q_2(T) = (F\overline{G})_{\mathrm{im}}(x_1, T), & R_2(T) = (H\overline{I})_{\mathrm{re}}(x_1, T), & S_2(T) = (H\overline{I})_{\mathrm{im}}(x_1, T), \\
P_3(T) = (F\overline{G})_{\mathrm{re}}(T, y_1), & Q_3(T) = (F\overline{G})_{\mathrm{im}}(T, y_1), & R_3(T) = (H\overline{I})_{\mathrm{re}}(T, y_1), & S_3(T) = (H\overline{I})_{\mathrm{im}}(T, y_1), \\
P_4(T) = (F\overline{G})_{\mathrm{re}}(x_0, T), & Q_4(T) = (F\overline{G})_{\mathrm{im}}(x_0, T), & R_4(T) = (H\overline{I})_{\mathrm{re}}(x_0, T), & S_4(T) = (H\overline{I})_{\mathrm{im}}(x_0, T).
\end{array}$$

Using Proposition 17 (four times), if x_0, x_1 are not bad numbers for P_1, Q_1, R_1, S_1 or P_3, Q_3, R_3, S_3

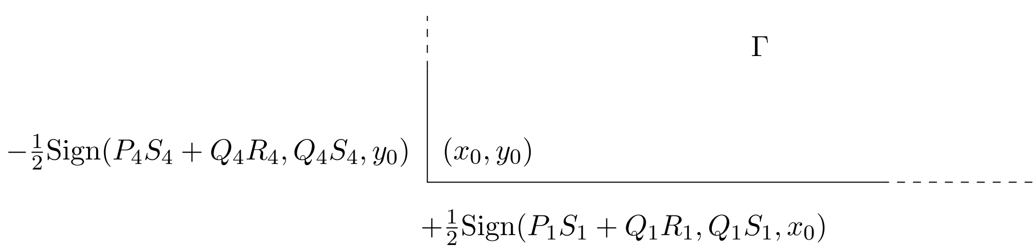
and y_0, y_1 are not bad numbers for P_2, Q_2, R_2, S_2 or P_4, Q_4, R_4, S_4,

$$
\begin{aligned}
& 2\left(\mathrm{w}(F/G\cdot H/I \,|\, \partial\Gamma) - \mathrm{w}(F/G \,|\, \partial\Gamma) \;-\; \mathrm{w}(H/I \,|\, \partial\Gamma)\right) \\
= \;& 2\left(\mathrm{w}(F\overline{G}H\overline{I} \,|\, \partial\Gamma) - \mathrm{w}(F\overline{G} \,|\, \partial\Gamma) \;-\; \mathrm{w}(H\overline{I} \,|\, \partial\Gamma)\right) \\
= \;& -\operatorname{Var}_{x_0}^{x_1}(P_1S_1+Q_1R_1, Q_1S_1) \;-\; \operatorname{Var}_{y_0}^{y_1}(P_2S_2+Q_2R_2, Q_2S_2) \\
& -\operatorname{Var}_{x_1}^{x_0}(P_3S_3+Q_3R_3, Q_3S_3) \;-\; \operatorname{Var}_{y_1}^{y_0}(P_4S_4+Q_4R_4, Q_4S_4).
\end{aligned}
$$

Therefore, $\mathrm{w}(F/G\cdot H/I \,|\, \partial\Gamma)$ coincides with $\mathrm{w}(F/G \,|\, \partial\Gamma)+\mathrm{w}(H/I \,|\, \partial\Gamma)$ if and only if the four Var add up to 0.



For instance, zooming around vertex (x_0, y_0) we have:



It would be then enough to prove that at each vertex, the adding Sign cancels with the substracting Sign. Unfortunately, this is not the case in general, as shown by the following example.

Example 20 *(Continuation of Example 8) Let* $F = Z, G = 1,\ H = \alpha + i\beta$ *with* $0 < \beta < \alpha,\ I = 1$ *and* $\Gamma = [0,1]\times[0,1]$*. Then* $S_1(T) = S_2(T) = S_3(T) = S_4(T) = \beta \neq 0$*, so there are no bad numbers for* $P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3$ *or* P_4, Q_4, R_4, S_4*. In addition,*

$$
\begin{aligned}
P_1(T) = T, \quad Q_1(T) = 0, \quad R_1(T) = \alpha, \quad S_1(T) = \beta, \\
P_4(T) = 0, \quad Q_4(T) = T, \quad R_4(T) = \alpha, \quad S_4(T) = \beta
\end{aligned}
$$

and at vertex (x_0, y_0) *we have*

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \mathrm{Sign}(\beta T, 0, 0) = 0$$

but

$$\mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0) = \mathrm{Sign}(\alpha T, \beta T, 0) = 1.$$

It can be checked that the adding Sign *cancels with the substracting* Sign *at the remaining three vertices* $(x_1, y_0), (x_1, y_1), (x_0, y_1)$. *Actually, we have already observed in Example 8 that* w *is not additive in this case.*

Another limitation to the method proposed above is the existence of bad numbers for P_i, Q_i, R_i, S_i for some $i = 1, \dots, 4$. The following example shows that the assumption that the rational functions (or even polynomials) do not vanish at the vertices of the rectangle (as made in Theorem 14) is not enough to avoid them.

Example 21 *(Continuation of Example 15) Let* $F = iZ + 1$, $G = 1$, $H = (1+i)Z - 1$, $I = 1$ *and* $\Gamma = [0, 1] \times [0, 2]$. *Then* $F/G = F$ *and* $H/I = H$ *do not vanish at any of the vertices of* Γ*; yet the polynomials* P_1, Q_1, R_1, S_1 *are exactly the polynomials* P, Q, R, S *in Example 15, and we have seen already that* 0 *is a bad number and therefore the auxiliary product formula (1) does not hold.*

Nevertheless, next proposition shows that w is indeed additive with respect to product in $\mathbf{C}(Z) \setminus \{0\}$ under mild hypothesis.

Proposition 22 *Let* $F/G, H/I \in \mathbf{C}(Z) \setminus \{0\}$ *and* $\Gamma \subset \mathbf{R}^2$ *a rectangle such that* F/G *and* H/I *have even valuation at the vertices of* Γ. *Then*

$$\mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma) + \mathrm{w}(H/I \,|\, \partial\Gamma).$$

On the other hand, with respect to the new algebraic winding number W, next proposition shows that it is additive with respect to product in $\mathbf{C}(Z) \setminus \{0\}$ with full generality.

Proposition 23 *Let* $F/G, H/I \in \mathbf{C}(Z) \setminus \{0\}$ *and* $\Gamma \subset \mathbf{R}^2$ *a rectangle. Then*

$$\mathrm{W}(F/G \cdot H/I \,|\, \partial\Gamma) = \mathrm{W}(F/G \,|\, \partial\Gamma) + \mathrm{W}(H/I \,|\, \partial\Gamma).$$

Before the proofs of these propositions, we perform some preliminary computations and introduce some notation we will use repeatedly in the rest of the section.

We consider a new variable $\overline{Z}$, together with the inclusion $\mathbf{C}[Z, \overline{Z}] \subset \mathbf{C}[X, Y]$ through the identities $Z = X + iY, \overline{Z} = X - iY$. For $G(Z) = \sum_j \gamma_j Z^j \in \mathbf{C}[Z]$, let $\overline{G}(\overline{Z}) = \sum_j \overline{\gamma_j}\,\overline{Z}^j \in \mathbf{C}[\overline{Z}]$. In this way, note that

$$\overline{G}(\overline{Z}) = \overline{G}(X, Y) \in \mathbf{C}[X, Y].$$

Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$. We define $z_0 = x_0 + iy_0 \in \mathbf{C}$ and

$$\begin{aligned} e &= \mathrm{val}_{z_0}(F/G), \qquad & p + iq &= F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0, \\ f &= \mathrm{val}_{z_0}(H/I), \qquad & r + is &= H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0. \end{aligned}$$

If $e \geq 0$, since F and G are coprime, $F(Z)\overline{G}(\overline{Z}) = (Z - z_0)^e F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z})$ with

$$F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z}) = p + iq + (Z - z_0)A(Z, \overline{Z}) + (\overline{Z} - \overline{z_0})B(Z, \overline{Z}) \in \mathbf{C}[Z, \overline{Z}] \subset \mathbf{C}[X, Y] \tag{2}$$

for some $A, B \in \mathbf{C}[Z, \overline{Z}]$. Therefore,

$$\begin{aligned} P_1(T) &= (T - x_0)^e \left(p + (T - x_0)A_1(T)\right), \\ Q_1(T) &= (T - x_0)^e \left(q + (T - x_0)B_1(T)\right), \end{aligned} \tag{3}$$

for some $A_1, B_1 \in \mathbf{R}[T]$. If e is even,

$$\begin{aligned} P_4(T) &= (-1)^{\frac{e}{2}}(T-y_0)^e\left(p+(T-y_0)A_{4,\mathrm{e}}(T)\right), \\ Q_4(T) &= (-1)^{\frac{e}{2}}(T-y_0)^e\left(q+(T-y_0)B_{4,\mathrm{e}}(T)\right), \end{aligned} \tag{4}$$

for some $A_{4,\mathrm{e}}, B_{4,\mathrm{e}} \in \mathbf{R}[T]$. If e is odd,

$$\begin{aligned} P_4(T) &= (-1)^{\frac{e-1}{2}}(T-y_0)^e\left(-q+(T-y_0)A_{4,\mathrm{o}}(T)\right), \\ Q_4(T) &= (-1)^{\frac{e-1}{2}}(T-y_0)^e\left(p+(T-y_0)B_{4,\mathrm{o}}(T)\right), \end{aligned} \tag{5}$$

for some $A_{4,\mathrm{o}}, B_{4,\mathrm{o}} \in \mathbf{R}[T]$.

Similarly, if $e < 0$, since F and G are coprime, $F(Z)\overline{G}(\overline{Z}) = (\overline{Z} - \overline{z_0})^{-e}F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z})$ with $F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z})$ as in (2). Therefore, we obtain formulas for P_1, Q_1, P_4, Q_4 as in (3), (4) and (5) replacing e by $-e$, and additionally multiplying P_4 and Q_4 by (-1) in (5) (e odd).

If $f \geq 0$, since H and I are coprime, $H(Z)\overline{I}(\overline{Z}) = (Z - z_0)^e H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z})$ with

$$H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z}) = r + is + (Z - z_0)C(Z, \overline{Z}) + (\overline{Z} - \overline{z_0})D(Z, \overline{Z}) \in \mathbf{C}[Z, \overline{Z}] \subset \mathbf{C}[X, Y] \tag{6}$$

for some $C, D \in \mathbf{C}[Z, \overline{Z}]$. Therefore,

$$\begin{aligned} R_1(T) &= (T-x_0)^f\left(r+(T-x_0)C_1(T)\right), \\ S_1(T) &= (T-x_0)^f\left(s+(T-x_0)D_1(T)\right), \end{aligned} \tag{7}$$

for some $C_1, D_1 \in \mathbf{R}[T]$. If f is even,

$$\begin{aligned} R_4(T) &= (-1)^{\frac{f}{2}}(T-y_0)^f\left(r+(T-y_0)C_{4,\mathrm{e}}(T)\right), \\ S_4(T) &= (-1)^{\frac{f}{2}}(T-y_0)^f\left(s+(T-y_0)D_{4,\mathrm{e}}(T)\right), \end{aligned} \tag{8}$$

for some $C_{4,\mathrm{e}}, D_{4,\mathrm{e}} \in \mathbf{R}[T]$. If f is odd,

$$\begin{aligned} R_4(T) &= (-1)^{\frac{f-1}{2}}(T-y_0)^f\left(-s+(T-y_0)C_{4,\mathrm{o}}(T)\right), \\ S_4(T) &= (-1)^{\frac{f-1}{2}}(T-y_0)^f\left(r+(T-y_0)D_{4,\mathrm{o}}(T)\right), \end{aligned} \tag{9}$$

for some $C_{4,\mathrm{o}}, D_{4,\mathrm{o}} \in \mathbf{R}[T]$.

Finally, if $f < 0$, since H and I are coprime, $H(Z)\overline{I}(\overline{Z}) = (\overline{Z} - \overline{z_0})^{-f}H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z})$ with $H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z})$ as in (6). Therefore, we obtain formulas for R_1, S_1, R_4, S_4 as in (7), (8) and (9) replacing f by $-f$, and additionally multiplying R_4 and S_4 by (-1) in (9) (f odd).

Now, we prove a lemma, which is a particular case of Proposition 22 where one of the rational functions is a constant.

Lemma 24 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$, $\gamma \in \mathbf{C} \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F/G has even valuation at the vertices of Γ. Then*

$$\mathrm{w}(\gamma F/G \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma).$$

Using Lemma 24 and the definition of W (Definition 9), we obtain immediately the following result.

Lemma 25 *If F/G has even valuation at the vertices of* Γ*,*

$$\mathrm{W}(F/G \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma).$$

Proof of Lemma 24: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$ and $\gamma = \alpha + i\beta$. The statement is clear if $\beta = 0$, so we suppose $\beta \neq 0$. We take $H = \alpha + i\beta$ and $I = 1$, and using Notation 19 we have

$$R_1(T) = R_2(T) = R_3(T) = R_4(T) = \alpha,$$

$$S_1(T) = S_2(T) = S_3(T) = S_4(T) = \beta.$$

Since S_1, S_2, S_3, S_4 are constant, there are no bad numbers for P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3 or P_4, Q_4, R_4, S_4. Using Proposition 17 (four times) as explained at the beginning of the section, and the fact that $\mathrm{w}(\alpha + i\beta \,|\, \partial\Gamma) = 0$, we have

$$\begin{aligned} & 2\Big(\mathrm{w}(\gamma F/G \,|\, \partial\Gamma) - \mathrm{w}(F/G \,|\, \partial\Gamma)\Big) \\ = \;& 2\Big(\mathrm{w}((\alpha+i\beta)F\overline{G} \,|\, \partial\Gamma) - \mathrm{w}(F\overline{G} \,|\, \partial\Gamma) - \mathrm{w}(\alpha+i\beta \,|\, \partial\Gamma)\Big) \\ = \;& -\mathrm{Var}_{x_0}^{x_1}(\beta P_1 + \alpha Q_1, \beta Q_1) - \mathrm{Var}_{y_0}^{y_1}(\beta P_2 + \alpha Q_2, \beta Q_2) \\ & -\mathrm{Var}_{x_1}^{x_0}(\beta P_3 + \alpha Q_3, \beta Q_3) - \mathrm{Var}_{y_1}^{y_0}(\beta P_4 + \alpha Q_4, \beta Q_4). \end{aligned}$$

Therefore, it is enough to prove that the four Var add up to 0. Concentrating at vertex (x_0, y_0), we will prove that

$$\mathrm{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) = \mathrm{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0).$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$, we have that $\beta p + \alpha q$, βq are not simultaneously 0. If $e \geq 0$, using (3) and (4) we obtain

$$\mathrm{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) = \mathrm{sign}((\beta p + \alpha q)\beta q) = \mathrm{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0).$$

If $e < 0$ the same identity holds with a similar proof.

Finally, the analysis for the three remaining vertices (x_1, y_0), (x_1, y_1) and (x_0, y_1) is identical. $\square$

Proof of Proposition 22: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$ and take $\gamma = \alpha + i\beta \in \mathbf{C}$. Multiplying F/G by γ and considering for instance the parametrization of the bottom edge of Γ, we define

$$P_{\gamma,1}(T) = (\gamma F\overline{G})_{\mathrm{re}}(T, y_0), \quad Q_{\gamma,1}(T) = (\gamma F\overline{G})_{\mathrm{im}}(T, y_0).$$

Using Notation 19 we have

$$\begin{aligned} P_{\gamma,1}(T) &= \alpha P_1(T) - \beta Q_1(T), \\ Q_{\gamma,1}(T) &= \alpha Q_1(T) + \beta P_1(T). \end{aligned}$$

Let $c \in \mathbf{R}$. If $\mathrm{val}_c(P_1/Q_1) < 0$, then for $\beta \neq 0$ we have $\mathrm{val}_c(P_{\gamma,1}/Q_{\gamma,1}) \geq 0$. If $\mathrm{val}_c(P_1/Q_1) \geq 0$, then for generic $\alpha, \beta \in \mathbf{R}$ we still have $\mathrm{val}_c(P_{\gamma,1}/Q_{\gamma,1}) \geq 0$.

This implies that, multiplying F/G and H/I by suitable constants if necessary, we can assume without loss of generality that x_0, x_1 are not bad numbers for P_1, Q_1, R_1, S_1 or P_3, Q_3, S_3, R_3 and y_0, y_1 are not bad numbers for P_2, Q_2, R_2, S_2 or P_4, Q_4, S_4, R_4. In addition we may assume that $F_z(z)\overline{G_z(z)}$ and $H_z(z)\overline{I_z(z)}$ are not real for each of the four vertices z of Γ. By Lemma 24, this does not change $w(F/G \,|\, \partial\Gamma)$, $w(H/I \,|\, \partial\Gamma)$ or $w(F/G \cdot H/I \,|\, \partial\Gamma)$.

Again, using Proposition 17 (four times) we have

$$\begin{aligned}
& 2\left(\mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma) - \mathrm{w}(F/G \,|\, \partial\Gamma) - \mathrm{w}(H/I \,|\, \partial\Gamma)\right) \\
= \;& 2\left(\mathrm{w}(F\overline{G}H\overline{I} \,|\, \partial\Gamma) - \mathrm{w}(F\overline{G} \,|\, \partial\Gamma) - \mathrm{w}(H\overline{I} \,|\, \partial\Gamma)\right) \\
= \;& -\operatorname{Var}_{x_0}^{x_1}(P_1S_1 + Q_1R_1, Q_1S_1) \;-\; \operatorname{Var}_{y_0}^{y_1}(P_2S_2 + Q_2R_2, Q_2S_2) \\
& -\operatorname{Var}_{x_1}^{x_0}(P_3S_3 + Q_3R_3, Q_3S_3) \;-\; \operatorname{Var}_{y_1}^{y_0}(P_4S_4 + Q_4R_4, Q_4S_4).
\end{aligned}$$

Therefore, it is enough to prove that the four Var add up to 0. Concentrating at vertex (x_0, y_0), we will prove that

$$\operatorname{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \operatorname{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0).$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ and $r + is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0$ are such that $q \neq 0$ and $s \neq 0$, we have that $qs \neq 0$. If $e \geq 0$ and $f \geq 0$, using (3), (4), (7) and (8) we obtain

$$\operatorname{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \operatorname{sign}((ps + qr)qs) = \operatorname{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0).$$

If $e < 0$ or $f < 0$ the same identity holds with a similar proof.

Finally, the analysis for the three remaining vertices (x_1, y_0), (x_1, y_1) and (x_0, y_1) is identical. □

We focus now on Proposition 23. As before, we prove first a lemma, which is a particular case of it where one of the rational functions is a constant.

Lemma 26 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$, $\gamma \in \mathbf{C} \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle. Then*

$$\mathrm{W}(\gamma F/G \,|\, \partial\Gamma) = \mathrm{W}(F/G \,|\, \partial\Gamma).$$

Proof: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$ and $\gamma = \alpha + i\beta$. The statement is clear if $\beta = 0$, so we suppose $\beta \neq 0$. We take $H = \alpha + i\beta$ and $I = 1$, and using Notation 19 we have

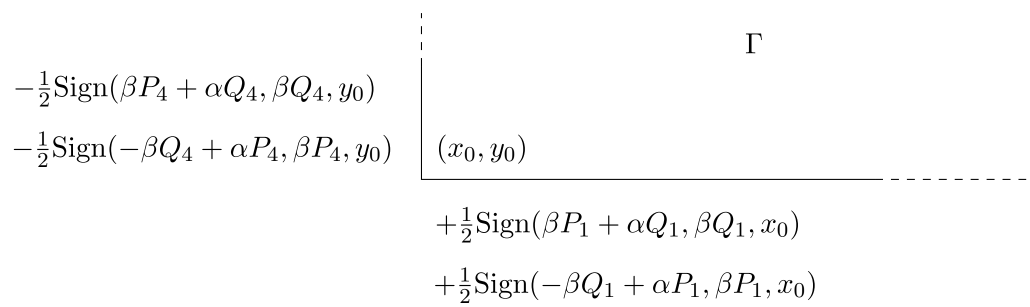
$$R_1(T) = R_2(T) = R_3(T) = R_4(T) = \alpha,$$

$$S_1(T) = S_2(T) = S_3(T) = S_4(T) = \beta.$$

Since S_1, S_2, S_3, S_4 are constant, there are no bad numbers for $P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2$; P_3, Q_3, R_3, S_3; P_4, Q_4, R_4, S_4; $-Q_1, P_1, R_1, S_1; -Q_2, P_2, R_2, S_2; -Q_3, P_3, R_3, S_3$ or $-Q_4, P_4, R_4, S_4$. Using Proposition 17 (eight times) and the fact that $\mathrm{w}(\alpha+i\beta\,|\,\partial\Gamma)=0$, we have

$$
\begin{aligned}
& 4\Big(\mathrm{W}(\gamma F/G\,|\,\partial\Gamma) - \mathrm{W}(F/G\,|\,\partial\Gamma)\Big) \\
= \;& 4\Big(\mathrm{W}((\alpha+i\beta)F\overline{G}\,|\,\partial\Gamma) - \mathrm{W}(F\overline{G}\,|\,\partial\Gamma)\Big) \\
= \;& 2\Big(\mathrm{w}((\alpha+i\beta)F\overline{G}\,|\,\partial\Gamma) - \mathrm{w}(F\overline{G}\,|\,\partial\Gamma) - \mathrm{w}(\alpha+i\beta\,|\,\partial\Gamma)\Big) \\
& + 2\Big(\mathrm{w}((\alpha+i\beta)iF\overline{G}\,|\,\partial\Gamma) - \mathrm{w}(iF\overline{G}\,|\,\partial\Gamma) - \mathrm{w}(\alpha+i\beta\,|\,\partial\Gamma)\Big) \\
= \;& -\operatorname{Var}_{x_0}^{x_1}(\beta P_1+\alpha Q_1,\beta Q_1) - \operatorname{Var}_{y_0}^{y_1}(\beta P_2+\alpha Q_2,\beta Q_2) \\
& -\operatorname{Var}_{x_1}^{x_0}(\beta P_3+\alpha Q_3,\beta Q_3) - \operatorname{Var}_{y_1}^{y_0}(\beta P_4+\alpha Q_4,\beta Q_4) \\
& -\operatorname{Var}_{x_0}^{x_1}(-\beta Q_1+\alpha P_1,\beta P_1) - \operatorname{Var}_{y_0}^{y_1}(-\beta Q_2+\alpha P_2,\beta P_2) \\
& -\operatorname{Var}_{x_1}^{x_0}(-\beta Q_3+\alpha P_3,\beta P_3) - \operatorname{Var}_{y_1}^{y_0}(-\beta Q_4+\alpha P_4,\beta P_4).
\end{aligned}
$$

Therefore, it is enough to prove that the eight Var add up to 0. Zooming around vertex (x_0,y_0) we have



We will prove that

$$
\begin{aligned}
& \operatorname{Sign}(\beta P_1+\alpha Q_1,\beta Q_1,x_0)+\operatorname{Sign}(-\beta Q_1+\alpha P_1,\beta P_1,x_0) \;= \\
= \;& \operatorname{Sign}(\beta P_4+\alpha Q_4,\beta Q_4,y_0)+\operatorname{Sign}(-\beta Q_4+\alpha P_4,\beta P_4,y_0).
\end{aligned}
\tag{10}
$$

Since $p+iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ we have that $\beta p+\alpha q$ and βq are not simultaneously 0 and $-\beta q+\alpha p$, βp are not simultaneously 0. Suppose $e \geq 0$.

If e is even, using (3) and (4) we have

$$\operatorname{Sign}(\beta P_1+\alpha Q_1,\beta Q_1,x_0) = \operatorname{sign}((\beta p+\alpha q)\beta q) = \operatorname{Sign}(\beta P_4+\alpha Q_4,\beta Q_4,y_0),$$

$$\operatorname{Sign}(-\beta Q_1+\alpha P_1,\beta P_1,x_0) = \operatorname{sign}((-\beta q+\alpha p)\beta p) = \operatorname{Sign}(-\beta Q_4+\alpha P_4,\beta P_4,y_0).$$

If e is odd, using (3) and (5) we have

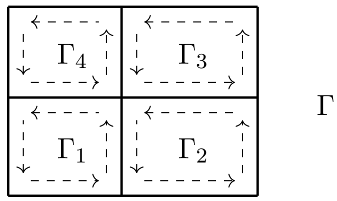
$$\operatorname{Sign}(\beta P_1+\alpha Q_1,\beta Q_1,x_0) = \operatorname{sign}((\beta p+\alpha q),\beta q) = \operatorname{Sign}(-\beta Q_4+\alpha P_4,\beta P_4,y_0),$$

$$\operatorname{Sign}(-\beta Q_1+\alpha P_1,\beta P_1,x_0) = \operatorname{sign}((-\beta q+\alpha p)\beta p) = \operatorname{Sign}(\beta P_4+\alpha Q_4,\beta Q_4,y_0).$$

Then, in both cases we have that identity (10) holds. If $e < 0$ the same identity holds with a similar proof.

Finally, the analysis for the three remaining vertices (x_1, y_0), (x_1, y_1) and (x_0, y_1) is identical. $\square$

Proof of Proposition 23: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$. From the definition of w and W, it follows that for any function in $\mathbf{C}[X, Y]$, if we subdivide Γ in four subrectangles $\Gamma_1, \dots, \Gamma_4$ as in the picture below, then the winding number w or W on $\partial\Gamma$ equals the sum of the respective winding numbers on $\partial\Gamma_i$ for $1 \leq i \leq 4$.



If F, G, H and I vanish at more than one vertex of Γ, we choose a point $z = (x, y) \in \Gamma$ such that (x, y), (x_0, y), (x_1, y), (x, y_0) and (x, y_1) are not roots of any of them. Using z to subdivide Γ in four rectangles, F, G, H and I vanish at most at one vertex of each of these four rectangles. So without loss of generality we replace Γ by Γ_1 and we make the assumption that F, G, H and I do not vanish at $(x_0, y_1), (x_1, y_1), (x_1, y_0)$. In particular this implies that F/G and H/I have even valuation at these three vertices of Γ.

As in the proof of Proposition 22, multiplying F/G and H/I by suitable constants if necessary, we can suppose without loss of generality that x_0, x_1 and y_0, y_1 are not bad numbers for all the finitely many 4-uples of polynomials we will use along the proof. In addition we may assume that $F_z(z)\overline{G_z(z)}$ and $H_z(z)\overline{I_z(z)}$ are not real or purely imaginary for each of the four vertices z of Γ. By Lemma 26, this does not change $\mathrm{W}(F/G \,|\, \partial\Gamma)$, $\mathrm{W}(H/I \,|\, \partial\Gamma)$ or $\mathrm{W}(F/G \cdot H/I \,|\, \partial\Gamma)$.

The proof is done in several cases according to the parity of the valuations e and f of F/G and H/I at $z_0 = x_0 + iy_0$.

If e and f are both even, then F/G and H/I have even valuation at the four vertices of Γ and the statement follows from Proposition 22 since, by Lemma 25,

$$\mathrm{W}(F/G \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma),\ \mathrm{W}(H/I \,|\, \partial\Gamma) = \mathrm{w}(H/I \,|\, \partial\Gamma),\ \text{and}\ \mathrm{W}(F/G{\cdot}H/I \,|\, \partial\Gamma) = \mathrm{w}(F/G{\cdot}H/I \,|\, \partial\Gamma).$$

If e is odd and f is even, suppose first $e \geq 0$ and $f \geq 0$. Since H/I has even valuation at the four vertices of Γ, by Lemma 25, $\mathrm{W}(H/I \,|\, \partial\Gamma) = \mathrm{w}(H/I \,|\, \partial\Gamma)$. Then, using Proposition 17 (eight times) for P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3; P_4, Q_4, R_4, S_4; $-Q_1, P_1, R_1, S_1$; $-Q_2, P_2, R_2, S_2$;

$-Q_3, P_3, R_3, S_3$; and $-Q_4, P_4, R_4, S_4$, we have

$$
\begin{aligned}
& 4\Big(\mathrm{W}(F/G\cdot H/I\,|\,\partial\Gamma) -\ \mathrm{W}(F/G\,|\,\partial\Gamma) -\ \mathrm{W}(H/I\,|\,\partial\Gamma)\Big) \\
& 4\Big(\mathrm{W}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) -\ \mathrm{W}(F\overline{G}\,|\,\partial\Gamma) -\ \mathrm{w}(H\overline{I}\,|\,\partial\Gamma)\Big) \\
= \quad & 2\Big(\mathrm{w}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) - \mathrm{w}(F\overline{G}\,|\,\partial\Gamma) - \mathrm{w}(H\overline{I}\,|\,\partial\Gamma)\Big) \\
& + 2\Big(\mathrm{w}(iF\overline{G}H\overline{I}\,|\,\partial\Gamma) - \mathrm{w}(iF\overline{G}\,|\,\partial\Gamma) - \mathrm{w}(H\overline{I}\,|\,\partial\Gamma)\Big) \\
= \quad & -\ \mathrm{Var}_{x_0}^{x_1}(P_1S_1+Q_1R_1, Q_1S_1)\ -\ \mathrm{Var}_{y_0}^{y_1}(P_2S_2+Q_2R_2, Q_2S_2) \\
& -\ \mathrm{Var}_{x_1}^{x_0}(P_3S_3+Q_3R_3, Q_3S_3)\ -\ \mathrm{Var}_{y_1}^{y_0}(P_4S_4+Q_4R_4, Q_4S_4) \\
& -\ \mathrm{Var}_{x_0}^{x_1}(-Q_1S_1+P_1R_1, P_1S_1)\ -\mathrm{Var}_{y_0}^{y_1}(-Q_2S_2+P_2R_2, P_2S_2) \\
& -\ \mathrm{Var}_{x_1}^{x_0}(-Q_3S_3+P_3R_3, P_3S_3)\ -\ \mathrm{Var}_{y_1}^{y_0}(-Q_4S_4+P_4R_4, P_4S_4).
\end{aligned}
$$

Therefore, it is enough to prove that the eight Var add up to 0. Concentrating around vertex (x_0, y_0) we will prove that

$$
\begin{aligned}
& \mathrm{Sign}(P_1S_1+Q_1R_1, Q_1S_1, x_0) + \mathrm{Sign}(-Q_1S_1+P_1R_1, P_1S_1, x_0) \quad = \\
= \quad & \mathrm{Sign}(P_4S_4+Q_4R_4, Q_4S_4, y_0) + \mathrm{Sign}(-Q_4S_4+P_4R_4, P_4S_4, y_0).
\end{aligned}
$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ and $r + is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0$ are such that $p, q, s \neq 0$, we have that $qs \neq 0$ and $ps \neq 0$. Then, using (3), (5), (7) and (8) we have

$$
\mathrm{Sign}(P_1S_1+Q_1R_1, Q_1S_1, x_0) = \mathrm{sign}((ps+qr)qs) = \mathrm{Sign}(-Q_4S_4+P_4R_4, P_4S_4, y_0)
$$

$$
\mathrm{Sign}(-Q_1S_1+P_1R_1, P_1S_1, x_0) = \mathrm{sign}((-qs+pr)ps) = \mathrm{Sign}((P_4S_4+Q_4R_4, Q_4S_4, y_0).
$$

If $e < 0$ or $f < 0$ the same identity holds with a similar proof.

For the remaining three vertices, since $(F\overline{G})_{\mathrm{re}}$ $(F\overline{G})_{\mathrm{im}}$, $(H\overline{I})_{\mathrm{re}}$ and $(H\overline{I})_{\mathrm{im}}$ do not vanish at them, a simple evaluation gives

$$
\begin{aligned}
\mathrm{Sign}(P_1S_1+Q_1R_1, Q_1S_1, x_1) &= \mathrm{Sign}(P_2S_2+Q_2R_2, Q_2S_2, y_0), \\
\mathrm{Sign}(-Q_1S_1+P_1R_1, P_1S_1, x_1) &= \mathrm{Sign}(-Q_2S_2+P_2R_2, P_2S_2, y_0), \\
\mathrm{Sign}(P_2S_2+Q_2R_2, Q_2S_2, y_1) &= \mathrm{Sign}(P_3S_3+Q_3R_3, Q_3S_3, x_1), \\
\mathrm{Sign}(-Q_2S_2+P_2R_2, P_2S_2, y_1) &= \mathrm{Sign}(-Q_3S_3+P_3R_3, P_3S_3, x_1), \\
\mathrm{Sign}(P_3S_3+Q_3R_3, Q_3S_3, x_0) &= \mathrm{Sign}(P_4S_4+Q_4R_4, Q_4S_4, y_1), \\
\mathrm{Sign}(-Q_3S_3+P_3R_3, P_3S_3, x_0) &= \mathrm{Sign}(-Q_4S_4+P_4R_4, P_4S_4, y_1).
\end{aligned}
$$

If e is even and f is odd we interchange F/G with H/I and proceed exactly as before.

If e and f are both odd, suppose first $e \geq 0$ and $f \geq 0$. Since $F/G \cdot H/I$ has even valuation at the four vertices of Γ, by Lemma 25, $\mathrm{W}(F/G \cdot H/I \,|\, \partial\Gamma) = \mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma)$. Then using Proposition 17 (eight times) for P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3; P_4, Q_4, R_4, S_4; $-Q_1, P_1, -S_1, R_1$; $-Q_2, P_2, -S_2, R_2$; $-Q_3, P_3, -S_3, R_3$; and $-Q_4, P_4, -S_4, R_4$, we have

$$
\begin{aligned}
& 4\Big(\mathrm{W}(F/G \cdot H/I \,|\, \partial\Gamma) - \mathrm{W}(F/G \,|\, \partial\Gamma) - \mathrm{W}(H/I \,|\, \partial\Gamma)\Big) \\
& 4\Big(\mathrm{w}(F\overline{G}H\overline{I} \,|\, \partial\Gamma) - \mathrm{W}(F\overline{G} \,|\, \partial\Gamma) - \mathrm{W}(H\overline{I} \,|\, \partial\Gamma)\Big) \\
= \quad & 2\Big(\mathrm{w}(F\overline{G}H\overline{I} \,|\, \partial\Gamma) - \mathrm{w}(F\overline{G} \,|\, \partial\Gamma) - \mathrm{w}(H\overline{I} \,|\, \partial\Gamma)\Big) \\
& + 2\Big(\mathrm{w}(iF\overline{G}iH\overline{I} \,|\, \partial\Gamma) - \mathrm{w}(iF\overline{G} \,|\, \partial\Gamma) - \mathrm{w}(iH\overline{I} \,|\, \partial\Gamma)\Big) \\
= \quad & -\mathrm{Var}_{x_0}^{x_1}(P_1S_1 + Q_1R_1, Q_1S_1) \; - \; \mathrm{Var}_{y_0}^{y_1}(P_2S_2 + Q_2R_2, Q_2S_2) \\
& -\mathrm{Var}_{x_1}^{x_0}(P_3S_3 + Q_3R_3, Q_3S_3) \; - \; \mathrm{Var}_{y_1}^{y_0}(P_4S_4 + Q_4R_4, Q_4S_4) \\
& -\mathrm{Var}_{x_0}^{x_1}(-Q_1R_1 - P_1S_1, P_1R_1) \; - \; \mathrm{Var}_{y_0}^{y_1}(Q_2R_2 - P_2S_2 -, P_2R_2) \\
& -\mathrm{Var}_{x_1}^{x_0}(-Q_3R_3 - P_3S_3, P_3R_3) \; - \; \mathrm{Var}_{y_1}^{y_0}(-Q_4R_4 - P_4S_4, P_4R_4).
\end{aligned}
$$

Therefore, it is enough to prove that the eight Var add up to 0. Concentrating around vertex (x_0, y_0), we will prove that

$$
\begin{aligned}
& \mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) + \mathrm{Sign}(-Q_1R_1 - P_1S_1, P_1R_1, x_0) \quad = \\
= \quad & \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0) + \mathrm{Sign}(-Q_4R_4 - P_4S_4, P_4R_4, y_0).
\end{aligned}
$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ and $r + is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0$ are such that $p, q, r, s \neq 0$, we have that $qs \neq 0$ and $pr \neq 0$. Then, using (3), (5), (7) and (9) we have

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \mathrm{sign}((ps + qr)qs) = \mathrm{Sign}(-Q_4R_4 - P_4S_4, P_4R_4, y_0),$$

$$\mathrm{Sign}(-Q_1R_1 - P_1S_1, P_1R_1, x_0) = \mathrm{sign}(-(ps + qr)pr) = \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0).$$

If $e < 0$ or $f < 0$ the same identity holds with a similar proof.

For the remaining three vertices, since $(F\overline{G})_{\mathrm{re}}$ $(F\overline{G})_{\mathrm{im}}$, $(H\overline{I})_{\mathrm{re}}$ and $(H\overline{I})_{\mathrm{im}}$ do not vanish at them, a simple evaluation gives

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_1) = \mathrm{Sign}(P_2S_2 + Q_2R_2, Q_2S_2, y_0),$$

$$\mathrm{Sign}(-Q_1R_1 - P_1S_1, P_1R_1, x_1) = \mathrm{Sign}(-Q_2R_2 - P_2S_2, P_2R_2, y_0),$$

$$\mathrm{Sign}(P_2S_2 + Q_2R_2, Q_2S_2, y_1) = \mathrm{Sign}(P_3S_3 + Q_3R_3, Q_3S_3, x_1),$$

$$\mathrm{Sign}(-Q_2R_2 - P_2S_2, P_2R_2, y_1) = \mathrm{Sign}(-Q_3R_3 - P_3S_3, P_3R_3, x_1),$$

$$\mathrm{Sign}(P_3S_3 + Q_3R_3, Q_3S_3, x_0) = \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_1),$$

$$\mathrm{Sign}(-Q_3R_3 - P_3S_3, P_3R_3, x_0) = \mathrm{Sign}(-Q_4R_4 - P_4S_4, P_4R_4, y_1).$$

□

4 Proofs of the main results

In this final section, we focus on the proof of our main results, using the additivity properties of w and W.

In [3, Proposition 4.4], it is shown that w does the right counting in the case of monic linear polynomials. Similarly, the following lemma shows that W does the right counting in the basic cases we will need.

Lemma 27 *Let $\Gamma \subset \mathbf{R}^2$ be a rectangle.*

For $\gamma \in \mathbf{C}$, $\mathrm{W}(\gamma \,|\, \partial\Gamma) = 0$.

For $F(Z) = Z - z_0$ with $z_0 \in \mathbf{C}$,

$$\mathrm{W}(F \,|\, \partial\Gamma) = \begin{cases} 1 & \text{if } z_0 \text{ is in the interior of } \Gamma, \\ \frac{1}{2} & \text{if } z_0 \text{ is in one of the edges of } \Gamma, \\ \frac{1}{4} & \text{if } z_0 \text{ is in one of the vertices of } \Gamma, \\ 0 & \text{if } z_0 \text{ is in the exterior of } \Gamma. \end{cases}$$

and

$$\mathrm{W}(1/F \,|\, \partial\Gamma) = \begin{cases} -1 & \text{if } z_0 \text{ is in the interior of } \Gamma, \\ -\frac{1}{2} & \text{if } z_0 \text{ is in one of the edges of } \Gamma, \\ -\frac{1}{4} & \text{if } z_0 \text{ is in one of the vertices of } \Gamma, \\ 0 & \text{if } z_0 \text{ is in the exterior of } \Gamma. \end{cases}$$

We omit the proof of Lemma 27 since it follows from a straightforward computation.

Proof of Theorem 12: By Lemma 27, W does the right counting for non-zero constants, linear monic polynomials and their inverses, and by Proposition 23, W is additive with respect to multiplication in $\mathbf{C}(Z) \setminus \{0\}$. This proves the theorem. □

Proof of Theorem 13: Theorem 13 is a corollary of Theorem 12, since under the assumption that F/G has even valuation at the vertices of Γ, using Lemma 25, we have $\mathrm{W}(F/G \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma)$. □

An alternative proof for Theorem 13 follows from first proving that w does the right counting for non-zero constants, monic linear polynomials ([3, Proposition 4.4]), and its inverses; then proving that if $F(Z) = (Z - z_0)^2$ with z_0 a vertex of Γ, w does the right counting for F and $1/F$, and finally, using the additivity property for w proven in Proposition 22. This proof avoids completely the definition of W.

Annex: Proof from Section 2

In this annex we prove Propositions 17 and 18 from Section 2.

We start recalling the inversion formula (see [3, Theorem 3.9]), which relates the Cauchy index with the sign variation on an interval.

Theorem 28 *Let $P, Q \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$. Then*

$$\mathrm{Ind}_a^b(P,Q) + \mathrm{Ind}_a^b(Q,P) = \mathrm{Var}_a^b(P,Q).$$

Next, we prove two auxiliary lemmas. The proof of these lemmas appears already in [3] as part of the proof of [3, Theorem 4.5] (also as part of the proof of [6, Lemma 20]), but we also include these proofs here for completeness.

Lemma 29 *Let $P, Q, R, S \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$ with $a < b$. If $P = 0$ or $Q = 0$ but P and Q are not simultaneously 0, the auxiliary product formula (1) holds. Similarly, if $R = 0$ or $S = 0$ but R and S are not simultaneously 0 the auxiliary product formula (1) holds.*

Proof: If $P = 0$ and $Q \neq 0$, using the inversion formula (Theorem (28)) with R and S we have

$$\begin{aligned}
\mathrm{Ind}_a^b(PR - QS, PS + QR) &= \mathrm{Ind}_a^b(-S, R) \\
&= \mathrm{Ind}_a^b(R, S) - \mathrm{Var}_a^b(R, S) \\
&= \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \mathrm{Var}_a^b(PS + QR, QS).
\end{aligned}$$

On the other hand, if $P \neq 0$ and $Q = 0$,

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) = \mathrm{Ind}_a^b(R, S) = \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \mathrm{Var}_a^b(PS + QR, QS).$$

□

Lemma 30 *Let $P, Q, R, S \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$ with $a < b$. If $Q, S \neq 0$ and $PS + QR = 0$, the auxiliary product formula (1) holds.*

Proof: Since $Q \neq 0$, $S \neq 0$ and $PS + QR = 0$, we have $P/Q = -R/S \in \mathbf{R}(X)$ and then

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) = 0 = \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \mathrm{Var}_a^b(PS + QR, QS).$$

□

We are ready for the proof of Proposition 17. Again, part of this proof appears already in the proof of [3, Theorem 4.5] (also in the proof of [6, Lemma 20]). The new part is essentially Case 3, where bad numbers in the interior of the interval are considered.

Proof of Proposition 17 : If at least one of the polynomials P, Q, R, S or $PQ + RS$ is 0, identity (1) holds by Lemmas 29 and 30. So in the rest of the proof we assume that none of these polynomials is 0. We divide P and Q by $\gcd(P, Q)$ and R and S by $\gcd(R, S)$, so without loss of generality we also assume that P and Q are coprime and R and S are coprime.

We divide the interval $[a, b]$ in finitely many subintervals $[a', b']$ and it is enough to prove that identity (1) holds in each of these subintervals. We consider all the roots of P, Q, R, S or $PS + QR$ in $[a, b]$ (possibly none), this includes all bad numbers in $[a, b]$ (again, possibly none). We divide $[a, b]$ in as many subintervals as needed in such a way that each subinterval contains at most one of these roots and additionally:

- if the root is not a bad number, then it is an endpoint of the subinterval,
- if the root is a bad number, then it is an interior point of the subinterval.

This is possible because a and b are not bad numbers. We consider then several cases as follows.

Case 1: There is no root of Q, S or $PS+QR$ in $[a', b']$, then

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = \mathrm{Ind}_{a'}^{b'}(P,Q) = \mathrm{Ind}_{a'}^{b'}(R,S) = 0$$

and

$$\mathrm{Sign}(PS+QR, QS, a') = \mathrm{Sign}(PS+QR, QS, b')$$

so identity (1) holds in $[a', b']$.

Case 2: There is one root of Q, S or $PS+QR$ in $[a', b']$ which is not a bad number, and therefore is an endpoint of $[a', b']$. In this case, by composing with the linear change $X \mapsto a' + b' - X$ (which interchanges a' and b') we can always suppose that the root is a'. We split this case in many cases:

Case 2a: If $Q(a') \neq 0, S(a') \neq 0$ and $(PS+QR)(a') = 0$, then

$$\mathrm{Ind}_{a'}^{b'}(P,Q) = \mathrm{Ind}_{a'}^{b'}(R,S) = \mathrm{Sign}(PS+QR, QS, a') = 0.$$

On the other hand

$$\frac{P(a')}{Q(a')} = -\frac{R(a')}{S(a')},$$

so

$$(PR-QS)(a') = Q(a')S(a')\underbrace{\left(\frac{P(a')}{Q(a')}\frac{R(a')}{S(a')} - 1\right)}_{<0} \neq 0.$$

Write $PS+QR = (X-a')^{\mu}T$ with $\mu = \mathrm{mult}_{a'}(PS+QR) > 0$. Note that $\mathrm{sign}(T(a')) = \mathrm{sign}(T(b')) = \mathrm{sign}((PS+QR)(b'))$. Then

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = -\frac{1}{2}\mathrm{sign}\big(Q(a')S(a')T(a')\big) = -\frac{1}{2}\mathrm{Sign}(PS+QR, QS, b')$$

so identity (1) holds in $[a', b']$.

Case 2b: If $Q(a') = 0$ and $S(a') \neq 0$, since P and Q have no common roots, then $(PS+QR)(a') \neq 0$ and we have that

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = \mathrm{Ind}_{a'}^{b'}(R,S) = \mathrm{Sign}(PS+QR, QS, a') = 0.$$

Write $Q = (X-a')^{\mu}Q_{a'}$ with $\mu = \mathrm{mult}_{a'}(Q) > 0$. Note that $\mathrm{sign}(Q_{a'}(a')) = \mathrm{sign}(Q(b'))$. Then

$$\mathrm{Ind}_{a'}^{b'}(P,Q) = \frac{1}{2}\mathrm{sign}\big(P(a')Q_{a'}(a')\big) = \frac{1}{2}\mathrm{sign}\Big(\big((PS{+}QR)Q_{a'}S\big)(a')\Big) = \frac{1}{2}\mathrm{Sign}(PS{+}QR, QS, b')$$

so identity (1) holds in $[a', b']$.

Case 2c: If $Q(a') \neq 0$ and $S(a') = 0$ we proceed in a similar way to the previous case.

Case 2d: If $Q(a') = 0$ and $S(a') = 0$, then $(PS + QR)(a') = 0$, and since P and Q have no common roots and R and S have no common roots, $P(a') \neq 0$, $R(a') \neq 0$.

Write $PS + QR = (X - a')^{\mu_0} T$ with $\mu_0 = \operatorname{mult}_{a'}(PS + QR) > 0$, $Q = (X - a')^{\mu_1} Q_{a'}$ with $\mu_1 = \operatorname{mult}_{a'}(Q) > 0$ and $S = (X - a')^{\mu_2} S_{a'}$ with $\mu_2 = \operatorname{mult}_{a'}(S) > 0$. Note that $\operatorname{val}_{a'}(P/Q) = -\mu_1$, $\operatorname{val}_{a'}(R/S) = -\mu_2$ and $\operatorname{val}_{a'}((PS + QR)/QS) = \mu_0 - \mu_1 - \mu_2$. We denote

$$\begin{aligned} \sigma_1 &:= \operatorname{sign}(P(a')) \in \{-1, 1\}, \\ \sigma_2 &:= \operatorname{sign}(R(a')) \in \{-1, 1\}, \\ \sigma_3 &:= \operatorname{sign}(T(a')) \in \{-1, 1\}, \\ \sigma_4 &:= \operatorname{sign}(Q_{a'}(a')) \in \{-1, 1\}, \\ \sigma_5 &:= \operatorname{sign}(S_{a'}(a')) \in \{-1, 1\}. \end{aligned}$$

Since a' is not a bad number, either $\mu_1 \neq \mu_2$ or $\mu_1 = \mu_2$ but $\mu_0 \neq \mu_1 + \mu_2$. Note that if $\mu_1 \neq \mu_2$, then again $\mu_0 = \min\{\mu_1, \mu_2\} \neq \mu_1 + \mu_2$. So, in any case we have

$$\operatorname{Sign}(PS + QR, QS, a') = 0.$$

On the other hand, we have

$$\begin{aligned} \operatorname{Ind}_{a'}^{b'}(PR - QS, PS + QR) &= \tfrac{1}{2}\sigma_1\sigma_2\sigma_3, \\ \operatorname{Ind}_{a'}^{b'}(P, Q) &= \tfrac{1}{2}\sigma_1\sigma_4, \\ \operatorname{Ind}_{a'}^{b'}(R, S) &= \tfrac{1}{2}\sigma_2\sigma_5, \\ \tfrac{1}{2}\operatorname{Sign}(PS + QR, QS, b') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5. \end{aligned}$$

We need to prove that

$$\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_4 + \sigma_2\sigma_5 - \sigma_3\sigma_4\sigma_5$$

or, equivalently,

$$\big(\sigma_1\sigma_2 + \sigma_4\sigma_5\big)\sigma_3 = \sigma_1\sigma_4 + \sigma_2\sigma_5. \tag{11}$$

We take into account that $\sigma_1 = \operatorname{sign}(P(b'))$, $\sigma_2 = \operatorname{sign}(R(b'))$, $\sigma_3 = \operatorname{sign}((PS + QR)(b'))$, $\sigma_4 = \operatorname{sign}(Q(b'))$ and $\sigma_5 = \operatorname{sign}(S(b'))$ and one final time we split in cases as follows.

- If $\sigma_1 = \sigma_5$ and $\sigma_2 = \sigma_4$, then $\sigma_3 = 1$ and equation (11) holds.
- If $\sigma_1 = -\sigma_5$ and $\sigma_2 = -\sigma_4$ then $\sigma_3 = -1$ and equation (11) holds.
- In every other case, exactly three elements in the set $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5\}$ are equal and the remaining one is different. Then

  $$\sigma_1\sigma_2 + \sigma_4\sigma_5 = \sigma_1\sigma_4 + \sigma_2\sigma_5 = 0$$

  and equation (11) holds.

So, identity (1) holds in $[a', b']$.

Case 3: There is one root c of Q, S or $PS + QR$ in $[a', b']$ which is a bad number, and therefore $c \neq a'$ and $c \neq b'$. Since c is a bad number, c is indeed a root of Q, S and $PS + QR$. Also, since P and Q have no common root and R and S have no common root, $P(c) \neq 0$, $R(c) \neq 0$.

Write $PS+QR = (X-c)^{2\mu}T$, $Q = (X-c)^{\mu}Q_c$, $S = (X-c)^{\mu}S_c$ with $\mu = \mathrm{mult}_c(Q) = \mathrm{mult}_c(S) = -\mathrm{val}_c(P/Q) = -\mathrm{val}_c(R/S) > 0$. We denote

$$\begin{aligned}\sigma_1 &:= \mathrm{sign}(P(c)) \in \{-1,1\},\\ \sigma_2 &:= \mathrm{sign}(R(c)) \in \{-1,1\},\\ \sigma_3 &:= \mathrm{sign}(T(c)) \in \{-1,1\},\\ \sigma_4 &:= \mathrm{sign}(Q_c(c)) \in \{-1,1\},\\ \sigma_5 &:= \mathrm{sign}(S_c(c)) \in \{-1,1\}.\end{aligned}$$

Since $\mathrm{val}_c((PR-QS)/(PS+QR)) = -2\mu$ is even, we have

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = 0.$$

On the other hand, we have

$$\begin{aligned}\mathrm{Ind}_{a'}^{b'}(P,Q) &= \tfrac{1}{2}(1-(-1)^{\mu})\sigma_1\sigma_4,\\ \mathrm{Ind}_{a'}^{b'}(R,S) &= \tfrac{1}{2}(1-(-1)^{\mu})\sigma_2\sigma_5,\\ \tfrac{1}{2}\mathrm{Sign}(PS+QR, QS, a') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5,\\ \tfrac{1}{2}\mathrm{Sign}(PS+QR, QS, b') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5.\end{aligned}$$

We need to prove that $\sigma_1\sigma_4 + \sigma_2\sigma_5 = 0$. Since $\mu > 0$ and

$$(X-c)^{2\mu}T = PS+QR = (X-c)^{\mu}(PS_c + Q_cR),$$

we conclude that

$$P(c)S_c(c) + Q_c(c)R(c) = 0$$

and therefore

$$\sigma_1\sigma_5 = \mathrm{sign}(P(c)S_c(c)) = -\mathrm{sign}(Q_c(c)R(c)) = -\sigma_2\sigma_4,$$

but then

$$\sigma_1\sigma_4 = \sigma_1\sigma_5^2\sigma_4 = -\sigma_2\sigma_4^2\sigma_5 = -\sigma_2\sigma_5,$$

and identity (1) holds in $[a', b']$.

□

We conclude this annex with the proof of Proposition 18.

Proof of Proposition 18: We start with item i). Since there are bad numbers, we have $P, Q, R, S, PS+QR \neq 0$. As in the proof of Proposition 17, we divide P and Q by $\gcd(P,Q)$ and R and S by $\gcd(R,S)$, so without loss of generality, we assume that P and Q are coprime and R and S are coprime.

Now, we consider $a' \in (a,b)$ such that there is no root of P, Q, R, S or $PS+QR$ in $(a, a']$. Then a' is not a bad number and therefore by Lemma 17, identity (1) holds in $[a', b]$. It is enough then to prove that

$$\mathrm{Ind}_a^{a'}(PR-QS, PS+QR) \;=\; \mathrm{Ind}_a^{a'}(P,Q) + \mathrm{Ind}_a^{a'}(R,S) - \frac{1}{2}\mathrm{Sign}(PS+QR, QS, a').$$

Since $Q(a) = 0$ and $S(a) = 0$, then $(PS + QR)(a) = 0$, and since P and Q have no common roots and R and S have no common roots, $P(a) \neq 0$, $R(a) \neq 0$.

Write $PS + QR = (X - a)^{2\mu} T$, $Q = (X - a)^{\mu} Q_a$ $S = (X - a)^{\mu} S_a$ with $\mu = \mathrm{mult}_a(Q) = \mathrm{mult}_a(S) = -\mathrm{val}_a(P/Q) = -\mathrm{val}_a(R/S) > 0$. We denote

$$
\begin{aligned}
\sigma_1 &:= \mathrm{sign}(P(a)) \in \{-1, 1\}, \\
\sigma_2 &:= \mathrm{sign}(R(a)) \in \{-1, 1\}, \\
\sigma_3 &:= \mathrm{sign}(T(a)) \in \{-1, 1\}, \\
\sigma_4 &:= \mathrm{sign}(Q_a(a)) \in \{-1, 1\}, \\
\sigma_5 &:= \mathrm{sign}(S_a(a)) \in \{-1, 1\}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{Ind}_a^{a'}(PR - QS, PS + QR) &= \tfrac{1}{2}\sigma_1\sigma_2\sigma_3, \\
\mathrm{Ind}_a^{a'}(P, Q) &= \tfrac{1}{2}\sigma_1\sigma_4, \\
\mathrm{Ind}_a^{a'}(R, S) &= \tfrac{1}{2}\sigma_2\sigma_5, \\
\tfrac{1}{2}\mathrm{Sign}(PS + QR, QS, a') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5.
\end{aligned}
$$

So we need to prove that

$$\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_4 + \sigma_2\sigma_5 - \sigma_3\sigma_4\sigma_5.$$

The rest of the proof is exactly as in Case 2d of the proof of Proposition 17.

The proof of item ii) is similar to the proof of item i). The proof of item iii) follows easily by introducing an intermediate point between a and b which is not a bad number and applying items i) and ii) to the new two subintervals. □

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