Rational certificates of non-negativity on semialgebraic subsets of cylinders

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Abstract

Let $g_1, \dots, g_s \in \mathbb{R}[X_1, \dots, X_n, Y]$ and $S = \{(\bar{x}, y) \in \mathbb{R}^{n+1} \mid g_1(\bar{x}, y) \geq 0, \dots, g_s(\bar{x}, y) \geq 0\}$ be a non-empty, possibly unbounded, subset of a cylinder in $\mathbb{R}^{n+1}$. Let $f \in \mathbb{R}[X_1, \dots, X_n, Y]$ be a polynomial which is positive on S. We prove that, under certain additional assumptions, for any non-constant polynomial $q \in \mathbb{R}[Y]$ which is positive on $\mathbb{R}$, there is a certificate of the non-negativity of f on S given by a rational function having as numerator a polynomial in the quadratic module generated by $g_1, \dots, g_s$ and as denominator a power of q.

Keywords: Positivstellensatz, Positive polynomials, Sums of squares, Quadratic modules.

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1 Introduction

Certificates of positivity and non-negativity by means of sums of squares is a topic whose roots go back to Hilbert's 17-th problem and its celebrated solution by Artin ([1]). Another milestone in the development of this theory is the Positivstellensatz by Krivine ([4]) and Stengle ([22]) which provides rational certificates for a multivariate polynomial f that is positive or non-negative on a basic closed semialgebraic set $S \subset \mathbb{R}^n$.

More recently, the famous works by Schmüdgen ([17]) and Putinar ([12]) led to a renewed interest in these certificates. Schmüdgen's Positivstellensatz ensures the existence of a polynomial certificate of non-negativity for a polynomial f which is positive on a compact set S. Under a stronger assumption which implies compactness of S, Putinar's Positivstellensatz establishes the existence of a simpler polynomial certificate. Several subsequent works extended the previous theorems in different directions including non-compact situations. We refer the reader to the survey by Scheiderer ([16]) and to the books by Marshall ([7]) and by Powers ([10]) for a comprehensive treatment of the subject; see also [20] for a specific reference on the moment problem and its connections with certificates of non-negativity.

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In this paper, we address the problem of the existence of certificates of non-negativity in the following setting. Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}, Y] = \mathbb{R}[X_1, \dots, X_n, Y]$ and

$$S = \{(\bar{x}, y) \in \mathbb{R}^{n+1} \mid g_1(\bar{x}, y) \geq 0, \dots, g_s(\bar{x}, y) \geq 0\}$$

be a non-empty, possibly unbounded, subset of a cylinder in $\mathbb{R}^{n+1}$. Let $f \in \mathbb{R}[\bar{X}, Y]$ be a polynomial which is positive on S. Under certain additional assumptions, we prove that, for any non-constant polynomial $q \in \mathbb{R}[Y]$ which is positive on $\mathbb{R}$, there is a certificate of the non-negativity of f on S given by a rational function having as numerator a polynomial in the quadratic module generated by $g_1, \dots, g_s$ and as denominator a power of q (see Theorem 1).

Variants of this problem have been considered previously. In [9] and in [3], Schmüdgen's Positivstellensatz and Putinar's Positivstellensatz are extended to cylinders with compact cross-section, under some additional assumptions on the polynomial f. In [5] and [6], among other results concerning non-negativity of polynomials on non-compact sets, the authors analyze the more general case of S being a subset of a cylinder and they prove the existence of a polynomial certificate for a small suitable perturbation of f provided that the polynomials defining S satisfy certain assumptions.

On the other hand, the existence of rational certificates having as a denominator a power of a fixed particular polynomial has been studied before in different frameworks. In [15], it is proved that a polynomial f which is positive on $\mathbb{R}^n$ is a sum of squares of rational functions having as denominators powers of $1 + \sum X_j^2$. Then in [13] and [14] this result is generalized to basic closed semialgebraic sets, under additional assumptions to control the behavior of the polynomial f at infinity.

Going back to Schmüdgen's and Putinar's Positivstellensatz, in [21] and [8] the authors develop a constructive approach in order to obtain bounds for the degrees of every term involved in these certificates. In this work, the main idea to prove Theorem 1 is, as in [9] and [3], to produce for each $y \in \mathbb{R}$ a certificate on the slice of S cut by the equation $Y = y$, in a parametric way such that all these certificates can be glued together in a single one. The procedure we follow on each slice is indeed an adaptation of the one in [21] and [8] using [11].

This slicing method is somehow complementary to the one in [6] where, instead, in order to deal with subsets of cylinders the fibres with respect to the projection on the variables $\bar{X}$ are considered to reduce the problem to the univariate case, and finally a clever glueing process is designed. Furthermore, this approach is related to the fibre theorem proved by Schmüdgen in [18] (see also its generalization in [19]), which reduces the moment problem for a closed (possibly unbounded) basic semialgebraic set to the moment problem for its fibres with respect to a polynomial map with bounded image.

The rest of the paper is organized in two sections. In Section 2 we introduce our assumptions and notation and state the main result, and then in Section 3 we prove it.

2 Assumptions and main result

We introduce the notation we will use throughout the paper.

Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}, Y]$. We consider the quadratic module generated by $\mathbf{g} := (g_1, \dots, g_s)$:

$$M(\mathbf{g}) = \left\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y]^2 \right\},$$

that is, the smallest quadratic module in $\mathbb{R}[\bar{X}, Y]$ that contains $g_1, \dots, g_s$. As in [6, Section 5] (also [3]), we make an assumption on $M(\mathbf{g})$ which is weaker than Archimedianity but captures a similar idea on the variables $\bar{X}$.

Assumption 1 *There exists $N \in \mathbb{R}_{>0}$ such that*

$$N - \sum_{1 \le j \le n} X_j^2 \in M(\mathbf{g}).$$

Note that under this assumption, the set S is included in the cylinder with compact cross section $\mathbf{B} \times \mathbb{R}$, where

$$\mathbf{B} := \{\bar{x} \in \mathbb{R}^n \mid \sum_{1 \le j \le n} x_j^2 \le N\}.$$

For $i = 1, \dots, s$, if

$$g_i(\bar{X}, Y) = \sum_{0 \le k \le m_i} g_{ik}(\bar{X}) Y^k \in \mathbb{R}[\bar{X}, Y]$$

with $g_{im_i}(\bar{X}) \neq 0$, we write

$$\widetilde{g}_i(\bar{X}, Y, Z) := Z^{m_i} g_i(\bar{X}, Y/Z) = \sum_{0 \le k \le m_i} g_{ik}(\bar{X}) Y^k Z^{m_i - k} \in \mathbb{R}[\bar{X}, Y, Z]$$

for the homogenization of g_i with respect to the variable Y. We make the following further assumptions on the polynomials $g_1, \dots, g_s$ and the set S they describe.

Assumption 2

1. *$S \neq \emptyset$.*
2. *For $i = 1, \dots, s$, $m_i = \deg_Y(g_i)$ is even.*
3. *$S_\infty := \{\bar{x} \in \mathbb{R}^n \mid g_{1m_1}(\bar{x}) \ge 0, \dots, g_{sm_s}(\bar{x}) \ge 0\} \subset \mathbf{B}$.*

Indeed, once Assumption 1 is made, the third condition in Assumption 2 could be replaced by the condition that S_∞ is bounded, since it is always possible to increase N if necessary. Nevertheless, for simplicity we assume that N is big enough. On the other hand, the following example shows that the third condition in Assumption 2 does not follow from Assumption 1 and the first two conditions in Assumption 2.

Example 1 *For $n = 2$, consider $g_1 = X_1 Y^2 + (1 - X_1^2 - X_2^2)$ and $g_2 = -X_1 Y^2 + 1$. Then*

$$2 - X_1^2 - X_2^2 = g_1 + g_1 \in M(\mathbf{g}),$$

so Assumption 1 is satisfied. In addition $S \neq \emptyset$ (moreover, it is not bounded) since $(0, 0, y) \in S$ for every $y \in \mathbb{R}$. However

$$S_\infty = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$$

is not a bounded set.

Let $q \in \mathbb{R}[Y]$ be a non-constant polynomial which is positive on $\mathbb{R}$ (and therefore, a sum of squares in $\mathbb{R}[Y]$),

$$q(Y) = \sum_{0 \le k \le m_0} q_k Y^k$$

with $m_0 > 0$ and $q_{m_0} \neq 0$. The assumption of q being positive on $\mathbb{R}$ implies m_0 is even and $q_{m_0} > 0$. We write

$$\widetilde{q}(Y, Z) := Z^{m_0} q(Y/Z) = \sum_{0 \le k \le m_0} q_k Y^k Z^{m_0 - k}.$$

Note that $\widetilde{q}(Y,Z)$ is a sum of squares in $\mathbb{R}[Y,Z]$, $\widetilde{q}$ is non-negative in $\mathbb{R}^2$ and it only vanishes at the origin.

Let

$$\mathbf{C} := \{(y,z) \in \mathbb{R}^2 \mid \widetilde{q}(y,z) = 1, z \geq 0\}$$

and

$$\widetilde{S} := \{(\bar{x}, y, z) \in \mathbb{R}^{n+2} \mid \ \widetilde{g}_1(\bar{x}, y, z) \geq 0, \dots, \widetilde{g}_s(\bar{x}, y, z) \geq 0, \ (y,z) \in \mathbf{C}\}.$$

For $\theta \in [0,\pi]$ and $\rho \in \mathbb{R}$, we have that

$$\widetilde{q}(\rho\cos(\theta), \rho\sin(\theta)) = \rho^{m_0}\widetilde{q}(\cos(\theta), \sin(\theta)).$$

Therefore, for any such θ, there exists a unique $\rho(\theta) \in [0,+\infty)$ such that $(\rho(\theta)\cos(\theta), \rho(\theta)\sin(\theta)) \in \mathbf{C}$, which is

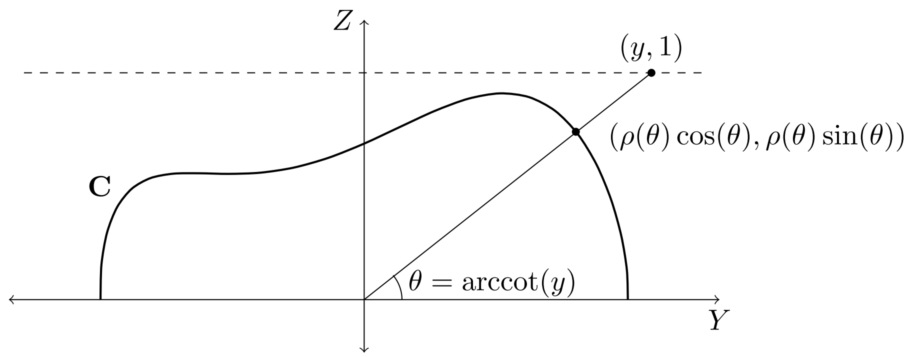
$$\rho(\theta) = \widetilde{q}(\cos(\theta), \sin(\theta))^{-1/m_0}$$

and satisfies $\rho(\theta) > 0$. Since the function $\rho : [0,\pi] \to \mathbb{R}$ is continuous,

$$\mathbf{C} = \{(\rho(\theta)\cos(\theta), \rho(\theta)\sin(\theta)) \mid \theta \in [0,\pi]\}$$

is a compact set. Moreover, the set $\widetilde{S} \cap \{z \neq 0\}$ is in bijection with S. This bijection is given by

$$\begin{array}{ccc} \widetilde{S} \cap \{z \neq 0\} & & S \\ (\bar{x}, y, z) & \mapsto & (\bar{x}, y/z) \\ \left(\bar{x}, \rho(\operatorname{arccot}(y))\frac{y}{\sqrt{y^2+1}}, \rho(\operatorname{arccot}(y))\frac{1}{\sqrt{y^2+1}}\right) & \leftarrow\!\shortmid & (\bar{x}, y) \end{array}$$



This implies

$$\widetilde{S} \cap \{z \neq 0\} \subset \mathbf{B} \times \mathbf{C}.$$

On the other hand, Assumption 2 implies

$$\widetilde{S} \cap \{z = 0\} = (S_\infty \times \{(-\rho(\pi), 0)\}) \cup (S_\infty \times \{(\rho(0), 0)\}) \subset \mathbf{B} \times \mathbf{C}.$$

We conclude that $\widetilde{S} \subset \mathbf{B} \times \mathbf{C}$ and therefore $\widetilde{S}$ is compact.

Similarly, for a polynomial

$$f(\bar{X}, Y) = \sum_{0 \leq k \leq m} f_k(\bar{X})Y^k \in \mathbb{R}[\bar{X}, Y]$$

with $f_m(\bar{X}) \neq 0$, we write

$$\widetilde{f}(\bar{X}, Y, Z) := Z^m f(\bar{X}, Y/Z) = \sum_{0 \leq k \leq m} f_k(\bar{X}) Y^k Z^{m-k} \in \mathbb{R}[\bar{X}, Y, Z]$$

for its homogenization with respect to the variable Y.

It is easy to see that f is positive on S if and only if $\widetilde{f}$ is positive on $\widetilde{S} \cap \{z \neq 0\}$. We make the following assumptions on the polynomial f (cf. [13, Theorem 4.2], [9, Definition 3], [3, Definition 3]).

Assumption 3

1. *$m = \deg_Y(f)$ is even.*
2. *$f_m(\bar{x}) > 0$ on S_∞.*

Under Assumptions 2 and 3, we have that $\widetilde{f} > 0$ on $\widetilde{S} \cap \{z = 0\}$.

We are ready now to state our main result, using the notation we introduced above.

Theorem 1 *Let $\mathbf{g} := g_1, \dots, g_s$ and f be polynomials in $\mathbb{R}[\bar{X}, Y]$ such that $f > 0$ on S and Assumptions 1, 2 and 3 hold. Let $q \in \mathbb{R}[Y]$ be a non-constant polynomial which is positive on $\mathbb{R}$. Then, there exists $M \in \mathbb{Z}_{\geq 0}$ such that $q^M f \in M(\mathbf{g})$.*

Note that, since $q \in \mathbb{R}[Y]$ is a sum of squares, multiplying on both sides by q if necessary, we may assume that M is even. If $q^M f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s$ with $\sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y]^2$, then the identity

$$f = \frac{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s}{q^M}$$

is a rational certificate of non-negativity for f on S. For each $y \in \mathbb{R}$, this identity can be evaluated to express $f(\bar{X}, y)$ as en explicit element of the quadratic module generated by $g_1(\bar{X}, y), \dots, g_s(\bar{X}, y)$ in $\mathbb{R}[\bar{X}]$, thus obtaining a certificate of non negativity on the slices of S cut by the equations $Y = y$ in a parametric way.

The following example ([3, Example 8]) shows that the second condition in Assumption 3 is necessary for the result to hold.

Example 2 *For $n = 1$ consider $g_1 = (1 - X^2)^3 \in \mathbb{R}[X, Y]$. Then $S = [-1, 1] \times \mathbb{R} \subset \mathbb{R}^2$ and*

$$\frac{4}{3} - X^2 = \frac{4}{3} X^2 \Big(X^2 - \frac{3}{2} \Big)^2 + \frac{4}{3} \Big(1 - X^2 \Big)^3 \in M(g_1)$$

(see also [7, Theorem 7.1.2]). Take $f(X, Y) = (1 - X^2) Y^2 + 1 \in \mathbb{R}[X, Y]$. It is clear that $f > 0$, however it is not the case that $f_2 = 1 - X^2$ is positive on $S_\infty = [-1, 1]$.

For any $q(Y) \in \mathbb{R}[Y]$ which is positive on $\mathbb{R}$, if we have an identity

$$q(Y)^M ((1 - X^2) Y^2 + 1) = \sum_j \Big(\sum_i p_{ji}(X) Y^i \Big)^2 + \sum_j \Big(\sum_i q_{ji}(X) Y^i \Big)^2 \big(1 - X^2 \big)^3,$$

every term on the right hand side has degree in Y bounded by $2m' = m_0 M + 2$ (where $m_0 = \deg(q)$), and then looking at the terms of degree $2m'$ in Y, we have

$$q_{m_0}^M (1 - X^2) = \sum_j p_{jm'}(X)^2 + \sum_j q_{jm'}(X)^2 \big(1 - X^2 \big)^3.$$

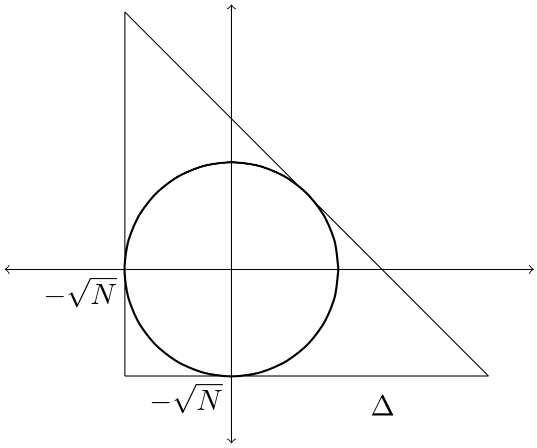
This implies that $1 - X^2$ belongs to the quadratic module generated by $(1 - X^2)^3$ in $\mathbb{R}[X]$, which is false since it is the well-known example from [23, Example].

3 Proof of the main result

As mentioned in the Introduction, the main idea to prove Theorem 1 is to produce in a parametric way, for each $y \in \mathbb{R}$ a certificate on the slice of S cut by the equation $Y = y$. To this end, we adapt the techniques in [21] and [8] using [11]. We keep the notation from the previous section.

Let $\Delta \subset \mathbb{R}^n$ be the following simplex containing $\mathbf{B}$:

$$\Delta = \{\bar{x} \in \mathbb{R}^n \mid x_j \geq -\sqrt{N} \text{ for } j = 1, \dots, n, \sum_{1 \leq j \leq n} x_j \leq \sqrt{nN}\}. \quad (1)$$



Since $\mathbf{C}$ is compact and $0 \notin \mathbf{C}$, there exist positive $\rho_1, \rho_2 \in \mathbb{R}$ which are respectively the minimum and maximum value of $\|(y, z)\|$ for $(y, z) \in \mathbf{C}$. Taking this into account, the following lemma can be proved similarly as [8, Lemma 11].

Lemma 2 *Let $f \in \mathbb{R}[\bar{X}, Y]$. There is a constant $K > 0$ such that, for every $\xi_1, \xi_2 \in \Delta \times \mathbf{C}$,*

$$|\widetilde{f}(\xi_1) - \widetilde{f}(\xi_2)| \leq K \, \|\xi_1 - \xi_2\|.$$

Moreover, the constant K can be computed in terms of n, the degrees in $\bar{X}$ and Y of f, the size of the coefficients of f, N and ρ_2.

As explained in the previous section, if f is positive on S and Assumptions 1, 2 and 3 are satisfied, then $\widetilde{f}$ is positive on

$$\widetilde{S} \ \subset \ \mathbf{B} \times \mathbf{C} \ \subset \ \Delta \times \mathbf{C}.$$

We denote

$$f^\bullet := \min\{\widetilde{f}(\bar{x}, y, z) \mid (\bar{x}, y, z) \in \widetilde{S}\} > 0.$$

Our first aim is to construct an auxiliary polynomial $h \in \mathbb{R}[\bar{X}, Y, Z]$ such that

- $h(\bar{x}, y, z)$ is positive on $\Delta \times \mathbf{C}$,
- $h(\bar{x}, y, 1) = q(y)^M f(\bar{x}, y) - p(\bar{x}, y)$, for a polynomial $p \in M(\mathbf{g})$.

Let r be the remainder of $m = \deg_Y(f)$ in the division by $m_0 = \deg(q) > 0$ and, for $i = 1, \dots, s$, let $e_i \in \mathbb{Z}$ be the minimum non-negative number such that m_0 divides $m_i + e_i$. We have then that $r, e_1, \dots, e_s$ are even and $0 \leq r, e_1, \dots, e_s \leq m_0 - 2$.

Proposition 3 *With our previous notation and assumptions, there exist* $\lambda, \alpha_1, \dots, \alpha_s \in \mathbb{R}_{>0}$, $k \in \mathbb{Z}_{\geq 0}$ *and* $M, M_1, \dots, M_s \in \mathbb{Z}_{\geq 0}$ *such that the polynomial*

$$h(\bar{X}, Y, Z) \;=\; \widetilde{q}(Y,Z)^M\, \widetilde{f}(\bar{X}, Y, Z) \;-$$

$$-\lambda(Y^2+Z^2)^{\frac{r}{2}} \sum_{1\leq i\leq s} \alpha_i (Y^2+Z^2)^{\frac{e_i}{2}} \widetilde{g}_i(\bar{X},Y,Z)(\alpha_i(Y^2+Z^2)^{\frac{e_i}{2}}\widetilde{g}_i(\bar{X},Y,Z) - \widetilde{q}(Y,Z)^{\frac{m_i+e_i}{m_0}})^{2k}\widetilde{q}(Y,Z)^{M_i},$$

is homogeneous in (Y,Z) *of degree* $\max\{m; r+(2k+1)(m_i+e_i), i=1,\dots,s\}$, *and satisfies*

$$h(\bar{x}, y, z) > \frac{f^{\bullet}}{2}$$

for all $(\bar{x}, y, z) \in \Delta \times \mathbf{C}$.

Note that for $i = 1, \dots, s$, the degree in (Y,Z) of

$$(Y^2+Z^2)^{\frac{r+e_i}{2}} \widetilde{g}_i(\bar{X},Y,Z)(\alpha_i(Y^2+Z^2)^{\frac{e_i}{2}}\widetilde{g}_i(\bar{X},Y,Z) - \widetilde{q}(Y,Z)^{\frac{m_i+e_i}{m_0}})^{2k}$$

has remainder r in the division by m_0. This ensures that, once k is fixed, there exist unique $M, M_1, \dots, M_s$ so that the degree and homogeneity conditions on (Y,Z) are satisfied.

Proof of Proposition 3: Since $\Delta \times \mathbf{C}$ is compact, for $i = 1, \dots, s$, we define

$$\beta_i = \sup_{\Delta\times\mathbf{C}} \left|(y^2+z^2)^{\frac{e_i}{2}}\widetilde{g}_i(\bar{x},y,z)\right|, \qquad \alpha_i = \frac{1}{\beta_i+1}$$

and

$$G_i(\bar{X},Y,Z) = \alpha_i(Y^2+Z^2)^{\frac{e_i}{2}}\widetilde{g}_i(\bar{X},Y,Z).$$

Then, for $i = 1, \dots, s$ we have

$$\sup_{\Delta\times\mathbf{C}} \left|G_i(\bar{x},y,z)\right| < 1.$$

The polynomial h in the statement of the proposition can be rewritten as

$$h(\bar{X}, Y, Z) \;=\; \widetilde{q}(Y,Z)^M\, \widetilde{f}(\bar{X}, Y, Z) \;-$$

$$-\lambda(Y^2+Z^2)^{\frac{r}{2}} \sum_{1\leq i\leq s} G_i(\bar{X},Y,Z)(G_i(\bar{X},Y,Z) - \widetilde{q}(Y,Z)^{\frac{m_i+e_i}{m_0}})^{2k}\widetilde{q}(Y,Z)^{M_i}.$$

For every $(\bar{x}, y, z) \in \Delta \times \mathbf{C}$, we have that $\widetilde{q}(y,z) = 1$; then,

$$h(\bar{x},y,z) = \widetilde{f}(\bar{x},y,z) - \lambda(y^2+z^2)^{\frac{r}{2}} \sum_{1\leq i\leq s} G_i(\bar{x},y,z)\left(G_i(\bar{x},y,z)-1\right)^{2k}.$$

For $i = 1, \dots, s$, if $G_i(\bar{x},y,z) \geq 0$, then $0 \leq G_i(\bar{x},y,z) < 1$. Now, it is not difficult to see that, for every $t \in [0,1]$, the inequality $t(t-1)^{2k} < \dfrac{1}{2ek}$ holds; therefore,

$$G_i(\bar{x},y,z)(G_i(\bar{x},y,z)-1)^{2k} < \frac{1}{2ek}. \tag{2}$$

Consider the set

$$A = \left\{(\bar{x},y,z) \in \Delta\times\mathbf{C} \;\mid\; \widetilde{f}(\bar{x},y,z) \leq \frac{3}{4}f^{\bullet}\right\}.$$

Note that $A \cap \widetilde{S} = \emptyset$, since $\widetilde{f}(\bar{x}, y, z) \geq f^\bullet$ for every $(\bar{x}, y, z) \in \widetilde{S}$.

For $(\bar{x}, y, z) \in (\Delta \times \mathbf{C}) - A$,

$$h(\bar{x}, y, z) \geq \widetilde{f}(\bar{x}, y, z) - \lambda\rho_2^r \sum_{\substack{1 \leq i \leq s, \\ G_i(\bar{x},y,z) \geq 0}} G_i(\bar{x}, y, z) \left(G_i(\bar{x}, y, z) - 1\right)^{2k}$$

and, as a consequence,

$$h(\bar{x}, y, z) > \frac{3}{4} f^\bullet - \frac{\lambda\rho_2^r s}{2ek}.$$

Therefore, if we take $k \geq \dfrac{2\lambda\rho_2^r s}{e f^\bullet}$, we have

$$h(\bar{x}, y, z) > \frac{f^\bullet}{2}.$$

For $\bar{\xi} = (\bar{x}, y, z) \in \Delta \times \mathbf{C}$, we define $H(\bar{\xi}) = \mathrm{dist}(\bar{\xi}, \widetilde{S})$ and $F(\bar{\xi}) = -\min\{0, G_1(\bar{\xi}), \ldots, G_s(\bar{\xi})\}$. Note that

$$F^{-1}(0) = \{\bar{\xi} \in \Delta \times \mathbf{C} \mid G_1(\bar{\xi}) \geq 0, \ldots, G_s(\bar{\xi}) \geq 0\} =$$
$$= \{\bar{\xi} \in \Delta \times \mathbf{C} \mid \widetilde{g}_1(\bar{\xi}) \geq 0, \ldots, \widetilde{g}_s(\bar{\xi}) \geq 0\} = \widetilde{S} = H^{-1}(0).$$

By the Łojasiewicz inequality (see [2, Corollary 2.6.7]), there exist positive constants L and c such that, for all $\bar{\xi} \in \Delta \times \mathbf{C}$,

$$\mathrm{dist}(\bar{\xi}, \widetilde{S})^L \leq c\, F(\bar{\xi}).$$

Since $A \cap \widetilde{S} = \emptyset$, for $\bar{\xi} \in A$, $F(\bar{\xi}) > 0$. Let i_0, with $1 \leq i_0 \leq s$, be such that $F(\bar{\xi}) = -G_{i_0}(\bar{\xi})$; then,

$$G_{i_0}(\bar{\xi}) \leq -\frac{1}{c}\, \mathrm{dist}(\bar{\xi}, \widetilde{S})^L.$$

Let $\bar{\xi}_0 \in \widetilde{S}$ be a point where the distance from $\bar{\xi}$ to $\widetilde{S}$ is attained, namely, $\mathrm{dist}(\bar{\xi}, \widetilde{S}) = \|\bar{\xi} - \bar{\xi}_0\|$. As $\frac{f^\bullet}{4} \leq \widetilde{f}(\bar{\xi}_0) - \widetilde{f}(\bar{\xi}) \leq K\|\bar{\xi}_0 - \bar{\xi}\|$, where K is the positive constant from Lemma 2, we deduce that

$$\mathrm{dist}(\bar{\xi}, \widetilde{S}) = \|\bar{\xi}_0 - \bar{\xi}\| \geq \frac{f^\bullet}{4K}$$

and, as a consequence,

$$G_{i_0}(\bar{\xi}) \leq -\frac{1}{c}\left(\frac{f^\bullet}{4K}\right)^L.$$

Together with inequality (2), this implies that

$$h(\bar{\xi}) \geq \widetilde{f}(\bar{\xi}) + \frac{\lambda\rho_1^r}{c}\left(\frac{f^\bullet}{4K}\right)^L - \frac{\lambda\rho_2^r(s-1)}{2ek}$$
$$= \left(\widetilde{f}(\bar{\xi}) - f^\bullet + \frac{\lambda\rho_1^r}{c}\left(\frac{f^\bullet}{4K}\right)^L\right) + \left(f^\bullet - \frac{\lambda\rho_2^r(s-1)}{2ek}\right)$$

Let $\bar{\xi}^\bullet \in \widetilde{S}$ be a point where the minimum $f^\bullet$ of $\widetilde{f}$ is attained in $\widetilde{S}$, that is, $\widetilde{f}(\bar{\xi}^\bullet) = f^\bullet$. By Lemma 2, we have

$$|\widetilde{f}(\bar{\xi}) - f^\bullet| = |\widetilde{f}(\bar{\xi}) - \widetilde{f}(\bar{\xi}^\bullet)| \leq K\|\bar{\xi} - \bar{\xi}^\bullet\|$$

and so, if $D := \mathrm{diam}(\Delta \times \mathbf{C})$,

$$|\widetilde{f}(\bar{\xi}) - f^\bullet| \leq KD.$$

Therefore, for $\lambda \geq KD\frac{c}{\rho_1^r}\left(\frac{4K}{f^\bullet}\right)^L$, we have that

$$\widetilde{f}(\bar{\xi}) - f^\bullet + \frac{\lambda \rho_1^r}{c}\left(\frac{f^\bullet}{4K}\right)^L \geq 0. \quad (3)$$

On the other hand, for $k \geq \frac{2\lambda\rho_2^r s}{ef^\bullet}$, the inequalities

$$\frac{\lambda\rho_2^r(s-1)}{2ek} \leq \frac{f^\bullet}{4}\frac{(s-1)}{s} < \frac{f^\bullet}{4}$$

hold and so,

$$f^\bullet - \frac{\lambda\rho_2^r(s-1)}{2ek} > \frac{3}{4}f^\bullet. \quad (4)$$

From (3) and (4), we conclude that

$$h(\bar{\xi}) > \frac{3}{4}f^\bullet.$$

Summarizing, for $\lambda \geq KD\frac{c}{\rho_1^r}\left(\frac{4K}{f^\bullet}\right)^L$ and $k \geq \frac{2\lambda\rho_2^r s}{ef^\bullet}$, we have that $h(\bar{x}, y, z) > \frac{f^\bullet}{2}$ for every $(\bar{x}, y, z) \in \Delta \times \mathbf{C}$. □

Remark 4 *From the proof of Proposition 3, it follows that, once the polynomials $g_1, \dots, g_s$ and q are fixed, for every f positive on S satisfying Assumptions 1, 2 and 3, an explicit bound for M in terms of $\deg(f)$, the size of the coefficients of f and $f^\bullet$ can be computed similarly as in [8] or [3].*

In order to prove our main result, we will apply the following effective version of Polya's theorem for a simplex (see [11, Theorem 3]).

Lemma 5 *Let $P \subset \mathbb{R}^n$ be an n-dimensional simplex with vertices $v_0, \dots, v_n$, and let $\bar{\ell} = \{\ell_0, \dots, \ell_n\}$ be the set of barycentric coordinates on P, i.e., $\ell_i \in \mathbb{R}[\bar{X}]$ is linear (affine) for $i = 0, \dots, n$,*

$$\bar{X} = \sum_{0\leq i\leq n} \ell_i(\bar{X})v_i, \quad 1 = \sum_{0\leq i\leq n} \ell_i(\bar{X}), \quad \text{and} \quad \ell_i(v_j) = \delta_{ij} \text{ for } 0 \leq i, j \leq n.$$

Let $h \in \mathbb{R}[\bar{X}]$ be a polynomial that is strictly positive on P. Then, for $\kappa \gg 0$, h has a representation of the form

$$h = \sum_{|\beta|\leq\kappa} b_\beta \bar{\ell}^\beta \qquad \text{with} \quad b_\beta > 0.$$

Moreover, for each β, $b_\beta \in \mathbb{R}$ is a linear combination of the coefficients of h, and an explicit bound for κ can be given in terms of the degree of h, the size of the coefficients of h, the minimum value of h in P and the vertices $v_0, \dots, v_n$.

Lemma 6 *For $N \in \mathbb{R}_{>0}$, let $\ell_0(\bar{X}) := \sqrt{nN} - \sum_{1\leq j\leq n} X_j$ and, for $i = 1, \dots, n$, $\ell_i(\bar{X}) := X_i + \sqrt{N}$. Then, for $i = 0, \dots, n$, we have that $\ell_i \in M(N - \|\bar{X}\|^2)$, where $\|\bar{X}\|^2 = \sum_{1\leq j\leq n} X_j^2$.*

Proof: By an explicit computation, we see that

$$\sqrt{nN} - \sum_{1\le j\le n} X_j = \frac{1}{2\sqrt{nN}}\Big((\sqrt{nN} - \sum_{1\le j\le n} X_j)^2 + \sum_{1\le j<j'\le n}(X_j - X_{j'})^2\Big) + \frac{\sqrt{n}}{2\sqrt{N}}(N - \sum_{1\le j\le n} X_j^2).$$

Also, for $i = 1, \dots, n$:

$$X_i + \sqrt{N} = \frac{1}{2\sqrt{N}}\Big((X_i + \sqrt{N})^2 + \sum_{j\ne i} X_j^2\Big) + \frac{1}{2\sqrt{N}}\Big(N - \sum_{1\le j\le n} X_j^2\Big).$$

This shows that $\ell_0, \ell_1, \dots, \ell_n \in M(N - \|\bar{X}\|^2)$. □

We are now able to prove the main result of the paper.

Proof of Theorem 1: We continue to use the notation introduced before.

Let $h \in \mathbb{R}[\bar{X}, Y, Z]$ be as in Proposition 3. We will apply Pólya's theorem, as stated in Lemma 5, to the polynomials $h_{(y,z)}(\bar{X}) := h(\bar{X}, y, z)$ for $(y, z) \in \mathbf{C}$ and the simplex Δ defined in (1).

The vertices of Δ are

$$\begin{aligned} v_0 &:= (-\sqrt{N}, \dots, -\sqrt{N}),\\ v_i &:= v_0 + (0, \dots, \underbrace{(n+\sqrt{n})\sqrt{N}}_{i\text{-th coord.}}, \dots, 0) \quad \text{for } i = 1, \dots, n, \end{aligned}$$

and its barycentric coordinates are given by

$$\begin{aligned} \ell_0(\bar{X}) &:= \frac{1}{(n+\sqrt{n})\sqrt{N}}(\sqrt{nN} - \sum_{1\le j\le n} X_j),\\ \ell_i(\bar{X}) &:= \frac{1}{(n+\sqrt{n})\sqrt{N}}(X_i + \sqrt{N}) \quad \text{for } i = 1, \dots, n. \end{aligned}$$

For each fixed $(y, z) \in \mathbf{C}$, the polynomial $h_{(y,z)}(\bar{X})$ satisfies

$$h_{(y,z)}(\bar{x}) > \frac{f^\bullet}{2} > 0 \quad \text{for every } \bar{x} \in \Delta.$$

Since the size of the coefficients of $h_{(y,z)}$ as polynomials in $\bar{X}$ and their minimum values on Δ are uniformly bounded for $(y, z) \in \mathbf{C}$, by Lemma 5, there exists $\kappa \gg 0$ such that

$$h(\bar{X}, Y, Z) = \sum_{|\beta|\le\kappa} b_\beta(Y, Z)\bar{\ell}(\bar{X})^\beta$$

with $b_\beta \in \mathbb{R}[Y, Z]$ and $b_\beta(y, z) > 0$ for every $(y, z) \in \mathbf{C}$. In addition, for each β, since h is homogeneous in (Y, Z) and b_β is a linear combination of the coefficients of h (seen as a polynomial in $\bar{X}$), then b_β is a homogeneous polynomial. This implies that $b_\beta(y, z) > 0$ for every $(y, z) \in \mathbb{R} \times \mathbb{R}_{\ge 0} \setminus \{0\}$; in particular, $b_\beta(y, 1) > 0$ for every $y \in \mathbb{R}$ and therefore $b_\beta(Y, 1)$ is a sum of squares in $\mathbb{R}[Y]$.

Finally, from the equality

$$h(\bar{X}, Y, 1) = q(Y)^M f(\bar{X}, Y) -$$
$$-\lambda \sum_{1\le i\le s} \alpha_i (Y^2+1)^{\frac{r+e_i}{2}} g_i(\bar{X}, Y)(\alpha_i(Y^2+1)^{\frac{e_i}{2}} g_i(\bar{X}, Y) - q(Y)^{\frac{m_i+e_i}{m_0}})^{2k} q(Y)^{M_i}$$

we have:

$$q(Y)^M f(\bar{X},Y) = \lambda \sum_{1\le i\le s} \alpha_i(Y^2+1)^{\frac{r+e_i}{2}}(\alpha_i(Y^2+1)^{\frac{e_i}{2}}g_i(\bar{X},Y) - q(Y)^{\frac{m_i+e_i}{m_0}})^{2k}q(Y)^{M_i}g_i(\bar{X},Y)$$

$$+\sum_{|\beta|\le\kappa} b_\beta(Y,1)\bar{\ell}(\bar{X})^\beta.$$

It is clear that for $i=1,\dots,s$,

$$\alpha_i(Y^2+1)^{\frac{r+e_i}{2}}(\alpha_i(Y^2+1)^{\frac{e_i}{2}}g_i(\bar{X},Y) - q(Y)^{\frac{m_i+e_i}{m_0}})^{2k}q(Y)^{M_i}g_i(\bar{X},Y) \in M(\mathbf{g}).$$

On the other hand, by Lemma 6,

$$\ell_0(\bar{X}),\dots,\ell_n(\bar{X}) \in M(N-\|\bar{X}\|^2)$$

and, taking into account that $M(N-\|\bar{X}\|^2)$ is closed under multiplication (since it is generated by a single polynomial), the same holds for all the products $\bar{\ell}(\bar{X})^\beta = \ell_0(\bar{X})^{\beta_0}\cdots\ell_n(\bar{X})^{\beta_n}$. By the assumption $N-\|\bar{X}\|^2 \in M(\mathbf{g})$, we deduce that $b_\beta(Y,1)\bar{\ell}(\bar{X})^\beta \in M(\mathbf{g})$ for every β with $|\beta|\le\kappa$. We conclude that $q(Y)^M f(\bar{X},Y) \in M(\mathbf{g})$. □

Remark 7 *The value of M in Theorem 1 is the same as in Proposition 3, therefore it can be bounded as mentioned in Remark 4.*

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