Detailed Asymptotic of Eigenvalues on Time Scales

Pablo Amster\textsuperscript{1,2}, Pablo De Nápoli\textsuperscript{1,2}, and Juan Pablo Pinasco\textsuperscript{1,2,3}

\textsuperscript{1}Departamento de Matemática, Facultad de Ciencias Exactas y Naturales Universidad de Buenos Aires Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina
\textsuperscript{2}Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina
\textsuperscript{3}Universidad Nacional de General Sarmiento J. M. Gutierrez 1150, Los Polvorines (1613) Prov. Buenos Aires, Argentina.
e-mail addresses: pamster@dm.uba.ar – pdenapo@dm.uba.ar – jpinasco@dm.uba.ar

Abstract

Let $T = \{a_n\}_{n} \cup \{0\}$ be a time scale with zero Minkowski (or box) dimension, where $\{a_n\}_{n}$ is a monotonically decreasing sequence converging to zero, and $a_1 = 1$. In this paper we find an upper bound for the eigenvalue counting function of the linear problem $-u^{\Delta\Delta} = \lambda u^\sigma$, with Dirichlet boundary conditions. We obtain that the $n^{th}$-eigenvalue is bounded below by $4a_{n+1}^{-2}$. We show that the bound is optimal for the $q-$ difference equations arising in quantum calculus.

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1 Introduction

In our previous work [3] we have studied the eigenvalue counting function $N(\lambda, T)$ defined as

$$N(\lambda, T) = \# \{ k : \lambda_k \leq \lambda \}$$

where $\lambda_k$ is the $k^{th}$ eigenvalue of problem

$$\begin{cases} -u^{\Delta \Delta} = \lambda u^\sigma \\ u(0) = u^\rho(1) = 0 \end{cases}$$

on a time scale $T \subset [a, b]$. Here, $\Delta$ stands for the derivative on the time scale $T$, and $\sigma$ and $\rho$ are the jump operators $\sigma(t) = \inf \{ s \in T : s > t \}$, $\rho(t) = \sup \{ s \in T : s < t \}$ as usual (see [1] for the properties of calculus and differential equations on time scales). The existence of a sequence of eigenvalues for problem (1) was proved in [2], [6], and even more general boundary conditions were considered.

The main result in [3] consists in an upper bound for $N(\lambda, T)$ as $\lambda$ goes to infinity,

$$N(\lambda, T) \leq C \lambda^{d(T)/2},$$

where $d(T)$ is the Minkowski (or box) dimension of $T$ in $\mathbb{R}$, which measures the growth of the number of intervals of length $\varepsilon$ needed to cover $T$ when $\varepsilon$ goes to zero (that is, we need $O(\varepsilon^{-d(T)})$ intervals, see [8] for details, or Section 2 in [3]).

When $d(T) = 0$, our result is less precise. We proved that

$$\frac{N(\lambda, T)}{\lambda^\delta} \to 0 \quad \text{when} \quad \lambda \to \infty$$

for any $\delta > 0$. Of course, in this case we cannot expect a bound of order $\lambda^{d(T)/2} = O(1)$ unless the number of eigenvalues is finite. On the other hand, the constant $C$ in (2) is related to the Minkowski content of $T$, $M_d(T)$ (roughly, it is a kind of measure of $T$ in dimension $d(T)$, although the Minkowski content is not $\sigma-$additive; for self similar sets like the Cantor set, it coincides with the Hausdorff measure on the respective dimension). When $d = 0$ the Minkowski content coincides with the cardinal measure, and $M_0(T) = \infty$ whenever $T$ has infinitely many points. So, equation (2) only confirms that problem (1) has infinitely many eigenvalues unless $T = \{ x_1, \ldots, x_n \}$, and equation (3) says that for any given exponent $a > 0$, if $d = 0$ and $M_0(T) = \infty$, then $\lambda_k \gg k^a$ as $k \to \infty$.

Therefore, we are left with the problem of refining the bound of $N(\lambda, T)$ in this case, which is equivalent to find lower bounds for the eigenvalues. We believe that this problem deserves special attention, since the calculus in time scales unifies the discrete theory of difference equations with the continuous calculus for differential equations, and the sharp transition from finite to infinitely many eigenvalues occurs in the class of zero dimensional sets.

In this work we focus in the case in which $T \subset [0, 1]$ is a sequence $\{ a_n \}$ of points converging -after a translation, if necessary- to 0, and we obtain an upper bound for $N(\lambda, T)$ depending on the rate of convergence to zero of $\{ a_n \}$. In much
the same way, we can consider countable sets with finitely many accumulation points having zero Minkowski dimension.

At first sight, this can be regarded as a particular case of relative interest, but in fact it is a critical case: we may recall the work of Besicovich and Taylor [5], where the Minkowski dimension and content of any closed set $A \subset [0, 1]$ was characterized in terms of the lengths of the intervals of the complementary set $A^c = [0, 1] \setminus A = \cup_i I_i$, by using the critical exponent $s_d$ defined as

$$s_d = \inf \{ s : \sum_i |I_i|^s < \infty \}.$$

Moreover, it was proved in [5] that the Minkowski dimension does not change after rearrangements of intervals on $[0, 1]$. For example, the Cantor set and any set in $[0, 1]$ which is the complement of $2^{k-1}$ intervals of length $3^{-k}$ (like the sequence of points $\{0, 1/3, 1/3 + 1/9, 1/3 + 2/9, \ldots \}$), have the same Minkowski dimension $\ln(2)/\ln(3)$. Let us also observe that for any $d \in (0, 1)$, there exists a sequence $\{a_n\}_n$ converging to zero with Minkowski dimension equal to $d$, since the set $\mathbb{T}_a = \{ n^{-a} : n \in \mathbb{N}, 0 < a < \infty \}$ has Minkowski dimension $d(\mathbb{T}_a) = (1 + a)^{-1}$.

Finally, let us recall that the case of zero dimensional sets includes the so-called $q$–difference equations, where

$$\mathbb{T} = q^{-\mathbb{N}_0} = \{ q^{-k} : q > 1, k \in \mathbb{N}_0 \} \cup \{0\}.$$ 

This is an important class of time scales which has independent interest, see the book of Kac and Cheung [10] on quantum calculus, and the comprehensive review of the literature (with more than 900 references) in the thesis of Thomas Ernst [7].

Our main result is the following theorem:

**Theorem 1.1** Let $a : [1, \infty) \to [0, 1]$ be a strictly decreasing function with $a(1) = 1$ and $a_n \to 0$ when $n \to \infty$. Let $\mathbb{T}$ be the set $\{a_n\}_n \cup \{0\}$, where $a(n) = a_n$, and let $\{\lambda_n\}_n$ be the sequence of eigenvalues of problem (1). Then,

$$N(\lambda, \mathbb{T}) = \# \{ n : \lambda_n \leq \lambda \} \leq a^{-1}(2\lambda^{-1/2}) + 1.$$ 

In Section §2 we give a proof of Theorem 1.1 which is based on the Lyapunov inequality proved by [9], although we use it in a different way than in [3]. Indeed, we show in Remark (2.4) below that our proof here gives very bad estimates for the sets with positive dimension considered in our previous work. However, we will show in Section §3 that the estimate is optimal for $q$–difference equations.

## 2 Proof of Theorem 1.1

Let us recall the Lyapunov inequality obtained in [9]:
Theorem 2.1 (Theorem 1.1 of [9]) Suppose that $q > 0$ and
\[
\left[\int_a^{\sigma^2(b)} \Delta t\right]^{\sigma(b)} \int_a^{q(b)} q(t) \Delta t \leq 4.
\]
Then, $u^{\Delta\Delta} + q(t)u^\sigma = 0$ is disconjugate on $[a, \sigma^2(b)]$.

Let us note that the integral involves $\sigma^2(b)$. In [3] we replaced it by $b$, considering the eigenvalue problem for $\hat{b} = \rho^2(b)$. Note that the boundary condition $u(\rho(b)) = 0$ implies that the number of (generalized) zeros between $\hat{b}$ and $\sigma^2(b)$ is lower than 3, so replacing $q(t)$ by $\lambda_5$, it is seen that
\[
\lambda_5 \left(\int_a^{b} \Delta t\right)^2 \leq 4
\]
holds for $b - a$ sufficiently small. Now, since the eigenfunction corresponding to $\lambda_5$ has 5 generalized zeros in $(a, b)$, and not more than one of them lies on $(a, \rho^2(b))$, the number of eigenvalues on small intervals is bounded by 4.

However, this approach is not necessary here, and we shall make use of Theorem 2.1 in a more direct way. An important auxiliary tool is the following property of the zeros of eigenfunctions proved in [2]: the eigenfunction $u_n$ corresponding to the $n$th eigenvalue has exactly $n + 1$ generalized zeros.

Proof [Proof of Theorem 1.1] Let $T = \{a_n\}_n \cup \{0\}$, and fix $\lambda > 1$.
We choose $n_\lambda$ such that
\[
\lambda \cdot a_{n_\lambda - 1}^2 \leq 4 < \lambda \cdot a_{n_\lambda - 2}^2;
\]
and let us consider the solution $u_\lambda$ of problem
\[
-u^{\Delta\Delta} = \lambda u^\sigma
\]
\[
u(0) = 0, \quad u^\Delta = 1.
\]

The number of zeros of $u_\lambda$ in $[0, 1]$ is bounded above by $n_\lambda - 1$: since
\[
\lambda \left(\int_0^{a_{n_\lambda - 1}} \Delta t\right) \left(\int_0^{\rho n_\lambda} \Delta t\right) \leq \lambda \cdot a_{n_\lambda - 1}^2 \leq 4,
\]
u_\lambda is disconjugate on $[0, a_{n_\lambda - 1}]$ and cannot have two or more generalized zeros; moreover, there are $n_\lambda - 2$ points of $T$ outside this interval, and hence it cannot have more than $n_\lambda - 2$ generalized zeros on $[a_{n_\lambda}, 1]$.

Clearly, $\lambda < \lambda_{n_{\lambda - 1}}$, the eigenvalue of order $n_{\lambda - 1}$, since the corresponding eigenfunction has $n_{\lambda}$ generalized zeros, and we obtain the upper bound
\[
N(\lambda, T) \leq n_\lambda - 1.
\]
Moreover, from the inequality
\[
4 < \lambda \cdot a_{n_\lambda - 2}^2,
\]
and let us consider the solution $u_\lambda$ of problem
\[
-u^{\Delta\Delta} = \lambda u^\sigma
\]
\[
\lambda \left(\int_0^{a_{n_\lambda - 1}} \Delta t\right) \left(\int_0^{\rho n_\lambda} \Delta t\right) \leq \lambda \cdot a_{n_\lambda - 1}^2 \leq 4,
\]
u_\lambda is disconjugate on $[0, a_{n_\lambda - 1}]$ and cannot have two or more generalized zeros; moreover, there are $n_\lambda - 2$ points of $T$ outside this interval, and hence it cannot have more than $n_\lambda - 2$ generalized zeros on $[a_{n_\lambda}, 1]$.

Clearly, $\lambda < \lambda_{n_{\lambda - 1}}$, the eigenvalue of order $n_{\lambda - 1}$, since the corresponding eigenfunction has $n_{\lambda}$ generalized zeros, and we obtain the upper bound
\[
N(\lambda, T) \leq n_\lambda - 1.
\]
Moreover, from the inequality
\[
4 < \lambda \cdot a_{n_\lambda - 2}^2,
\]
we obtain
\[ 2\lambda^{-1/2} > a_{n \lambda - 2}, \]
and
\[ a^{-1}(2\lambda^{-1/2}) > n \lambda - 2, \]
since \( a^{-1} \) is decreasing.
Hence,
\[ N(\lambda, T) \leq n \lambda - 1 \leq a^{-1}(2\lambda^{-1/2}) + 1, \]
and the proof is complete. \( \square \)

We have the following

**Corollary 2.2** Let \( \lambda_k \) be the \( k \)th eigenvalue of problem (1) on \( T = \{a_n\}_n \cup \{0\} \).
Then,
\[ \frac{4}{a_{k-1}^2} \leq \lambda_k \]
as \( k \to \infty \).

**Proof:** Let us note that
\[ k = N(\lambda_k, T) \leq a^{-1}(2\lambda_k^{-1/2}) + 1, \]
and we obtain, since \( a \) is decreasing,
\[ a(k - 1) \geq 2\lambda_k^{-1/2} \]
\[ \frac{4}{a_{k-1}^2} \leq \lambda_k. \]

Let us give two examples.

**Example 2.3** if \( T = \{n^{-a}\}_n \cup \{0\}, \ 0 < a < \infty \), we have
\[ N(\lambda, T) \leq (2\lambda^{-1/2})^{-1/a} + 1 = C\lambda^{1/2a} + 1. \]

**Remark 2.4** Example 2.3 shows that our estimate can be very bad for time scales with \( d(T) > 0 \), and our argument in this work cannot replace the one in [3]. Let us recall that the set \( T = \{n^{-a}\}_n \cup \{0\} \) has Minkowski dimension equal to \((a + 1)^{-1}\), and the upper bound obtained previously was
\[ N(\lambda, T) \leq C\lambda^{1/2(a+1)}. \]

**Example 2.5** if \( T = 2^{-N_0} = \{2^{-n}\}_n \cup \{0\}, \) we have
\[ N(\lambda, T) \leq -\log_2(2\lambda^{-1/2}) + 1 = \frac{1}{2} \log_2(\lambda). \]

This example can be generalized to \( q^{-N_0} \), and the corresponding upper bound is \( \log_q(\lambda)/2 \). In the next section we find a lower bound growing at the same rate.
3 Optimaly of the order of growth for $q-$calculus

In this section, we obtain a logarithmic lower bound of $N(\lambda, \mathbb{T})$ when $\mathbb{T} = 2^{-n_0}$. In particular, this shows that the result given in Theorem 1.1 is optimal in this case.

We shall make use of the computations performed in the recent work of Bohner and Hudson [4] of the eigenvalues on $\mathbb{T} = \{1, q, \ldots, q^n\}$, which give the values:

$$\lambda_m = \left(\frac{q + 1}{2\cos(m\pi/n)}\right)^2 - 2,$$

where $1 \leq m \leq (n - 1)/2$ if $n$ is odd, and $1 \leq m \leq (n - 2)/2$ if $n$ is even.

We shall combine this explicit formula for $\lambda_m$ with the following scaling lemma:

**Proposition 3.1** Let $\lambda_m, u_m(t)$ be the $m$th eigenpair of problem (1) on $\mathbb{T} = \{q^{-n}\}_{n \in \mathbb{N}_0} \cup \{0\}$. Then, $q^{2k}\lambda_m, u_m(q^k t)$ is the $m$th eigenpair of problem (1) on $\mathbb{T}_k = \{q^{-n}\}_{n \geq k} \cup \{0\}$.

**Proof:** The result follows from a simple change of variable. □

We will apply now the Dirichlet-Neumann bracketing (see [3]) by splitting the time scale $\mathbb{T}$ in two parts $\mathbb{T}_n = \{2^{-j}\}_{j \geq n} \cup \{0\}$ and $\tilde{\mathbb{T}} := \{2^{-n+1}, \ldots, 2^{-1}, 1\}$.

Let us recall that the eigenvalues are obtained by the following minimization of the Rayleigh quotient:

$$\lambda_{k+1} = \min_{u \in H^1_{0,\Delta}(\mathbb{T}), u \perp \varphi_1, \ldots, \varphi_k} \frac{\int_a^b u^{\Delta}u^2 \Delta t}{\int_a^b |u^2| \Delta t},$$

where $\varphi_1, \ldots, \varphi_k$ are the first $k$ eigenfunctions (see Theorem 2 in [2], Theorem 3.10 in [6]). Equivalently, we have

$$\lambda_k = \inf_{L_k \subset H^1_{0,\Delta}} \sup_{u \in L_k} \frac{\int_a^b u^{\Delta}u^2 \Delta t}{\int_a^b |u^2| \Delta t},$$

where $L_k$ runs over all the $k$-dimensional subspaces of a given space $H^1_{0,\Delta}$. Clearly, the following inclusion of Sobolev spaces holds,

$$H^1_{0,\Delta}(\mathbb{T}_n) \oplus H^1_{0,\Delta}(\tilde{\mathbb{T}}) \subset H^1_{0,\Delta}(\mathbb{T}),$$

since the functions can be glued together continuously due to the Dirichlet boundary condition.

So, the Rayleigh quotient on the left hand side is always greater or equal than the Rayleigh quotient on $H^1_{0,\Delta}(\mathbb{T})$. That is, the eigenvalues of problem (1) on $\mathbb{T}_n$ and $\tilde{\mathbb{T}}$ ordered in an increasing sequence, are greater than the corresponding ones on $\mathbb{T}$.

Hence, when $\lambda \gg 0$, we have

$$N(\lambda, \mathbb{T}_n) + N(\lambda, \tilde{\mathbb{T}}) \leq N(\lambda, \mathbb{T}).$$
The first term is bounded below by zero for every $n$. Now, we choose $n$ depending on $\lambda$, and we estimate $N(\lambda, \tilde{T})$ in the following way. Let us choose $n$ such that

$$2^{4n} < \lambda \leq 2^{4n+4}. \quad (4)$$

From the previous formula and Proposition 3.1, we have the following expression for the eigenvalues:

$$\lambda_m(\tilde{T}) = 2^{2n} \left[ \left( \frac{3}{2\cos(m\pi/n)} \right)^2 - 2 \right]$$

We are interested only in those values of $m$ such that $\lambda_m \leq \lambda$, namely:

$$2^{2n} \left[ \left( \frac{3}{2\cos(m\pi/n)} \right)^2 - 2 \right] \leq \lambda.$$

Since $\lambda_m$ is increasing with $m$, we have, for $m = (n-1)/2$,

$$\frac{\pi}{2n} - \frac{\pi^2}{8n^2} \leq \cos(m\pi/n) \leq \frac{\pi}{2n} + \frac{\pi^2}{8n^2},$$

that is, $\cos(m\pi/n)^{-1} = O(n)$, and asymptotically we have

$$\lambda_m(\tilde{T}) \simeq 2^{2n} \left[ \left( \frac{3n}{\pi} \right)^2 - 2 \right]$$

when $\lambda$ (and hence $n$) is large. Therefore, for $n \gg 0$

$$2^{2n} \left[ \left( \frac{3n}{\pi} \right)^2 - 2 \right] \leq 2^{4n} < \lambda.$$

So, we have $n/2 - 1$ (resp. $(n-1)/2$) eigenvalues on $\tilde{T}$ for $n$ even (resp., odd), and the lower bound

$$n/2 - 1 \leq N(\lambda, \tilde{T}).$$

Hence, we obtained a logarithmic lower bound for $N(\lambda, \tilde{T})$, since equation (4) implies

$$4n < \log_2(\lambda) \leq 4n + 4,$$

that is,

$$\frac{1}{4} \log_2(\lambda) - 1 \leq n,$$

which shows that the order of growth obtained in Theorem 1.1 is optimal for $\tilde{T} = 2^{-N_0}$. 

7
References


