On a generalization of Lazer-Leach conditions for a system of second order ODE's

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Abstract

We study the existence of periodic solutions for a nonlinear second order system of ordinary differential equations. Assuming suitable Lazer-Leach type conditions, we prove the existence of at least one solution applying topological degree methods.

1 Introduction

We study the nonlinear system of second order differential equations for a vector function $u : [0, 2\pi] \to \mathbb{R}^N$ satisfying

$$u'' + m^2u + g(u) = p(t) \qquad 0 < t < 2\pi \tag{1}$$

under periodic boundary conditions:

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \tag{2}$$

We shall assume that $m \neq 0$ is an integer, $p \in L^1(0, 2\pi)$, and that the nonlinearity g is continuous and bounded. Thus, (1-2) is a resonant problem, since the kernel of the operator $L_m u := u'' + m^2u$ over the space of 2π-periodic functions is non-trivial. This situation is referred in the literature as *resonance at a higher order eigenvalue*: indeed, if one considers the eigenvalue problem $-u'' = \lambda u$ under periodic conditions, a simple computation shows that $\lambda_m = m^2 \in \mathbb{N}_0$. Let us recall that the case $m = 0$ for a scalar equation has a solution if one assumes the well known Landesman-Lazer conditions, which have been firstly obtained in [2] for a resonant elliptic second order scalar equation under Dirichlet conditions (for a survey on Landesman-Lazer conditions see e.g. [5]). Roughly speaking, these conditions state that if $\overline{p}$ (the average of p) lies between the limits at $\pm\infty$ of the nonlinearity g, then the problem admits at least one solution. This condition may be regarded as a degree condition over the sphere $S^0 = \{-1, 1\}$, in the following sense: if for $v = \pm 1$ we define $g_{\pm 1} = g(\pm\infty)$, then the function $\theta : S^0 \to S^0$ given by $\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$ is well defined and changes sign, and in consequence it has non-zero degree.

Thus, the following result, adapted from a theorem given by Nirenberg in [6] for elliptic systems, may be regarded as a natural extension of Landesman-Lazer theorem:

Theorem 1.1 *Assume that the radial limits $g_v := \lim_{r\to+\infty} g(rv)$ exist uniformly respect to $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then (1-2) with $m = 0$ has at least one T-periodic solution if the following conditions hold:*

- $g_v \neq \overline{p} := \frac{1}{T}\int_0^T p(t)dt$ *for any* $v \in S^{N-1}$.
- *The degree of the mapping $\theta : S^{N-1} \to S^{N-1}$ given by*

$$\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$$

is different from 0.

We remark that the average $\overline{p}$ can be regarded as the projection of the forcing term p into the kernel of the linear operator L_0, which consists in the set of constant functions, naturally identified with $\mathbb{R}^N$.

In contrast with the above-described case, the situation when $m \neq 0$ makes it necessary to deal with a $2N$-dimensional kernel, namely:

$$\mathrm{Ker}(L_m) = \left\{\cos(mt)\alpha + \sin(mt)\beta : (\alpha, \beta) \in \mathbb{R}^{2N}\right\} := V_m.$$

Thus, one might expect that a Landesman-Lazer type condition corresponding to this case can be expressed in terms of the projection of p into V_m or, equivalently, in terms of the m-th Fourier coefficients of p. For $N = 1$, it has been shown by D.E. Leach and A. Lazer that this is, indeed, the case (see [3]):

Theorem 1.2 *Let $N = 1$ and assume that $g \in C(\mathbb{R})$ has limits at infinity. Moreover, let α_p and β_p denote the m-th Fourier coefficients of p. Then, if*

$$\sqrt{\alpha_p^2 + \beta_p^2} < \frac{2}{\pi}\left|g(+\infty) - g(-\infty)\right|, \tag{3}$$

problem (1-2) admits at least one 2π-periodic solution.

The aim of this paper is to obtain a generalization of Lazer-Leach theorem for $N > 1$. It is worth to observe that some extra difficulty should be expected when one attempts to extend the result to a system of equations. For example, for the scalar case it is not necessary to assume that the limits $g(\pm\infty)$ exist; however, the same argument cannot be implemented for a system. When $m = 0$, an interesting example has been given in [7], showing that the existence of radial limits of g is in some sense necessary. More precisely, the authors have shown a system for which no periodic solution exists, although the following conditions are fulfilled for some $R > 0$:

- $g(u) \neq \overline{p}$ for $|u| \geq R$.
- The degree of the mapping $\theta_R : S^{N-1} \to S^{N-1}$ given by

$$\theta_R(v) = \frac{g(Rv) - \overline{p}}{|g(Rv) - \overline{p}|}$$

is different from 0.

Despite this example, we shall show that the assumption on the existence of radial limits can be replaced by a weaker condition (see condition 3.1 below).

Applying topological degree methods [4], we shall obtain solutions of (1-2) under appropriate conditions of Lazer-Leach type. In particular, if the nonlinearity g has uniform radial limits at infinity, these conditions involve the m-th Fourier coefficients of some suitable extension of g to the infinite sphere. However, unlike in Nirenberg's result, our condition 3.1 below does not assume that all radial limits exist, but only (upper) limits, in some specific directions. This kind of condition has been introduced in [1] in the case of resonance at the first eigenvalue for a ϕ-Laplacian system.

2 Preliminaries

Let H be the space of absolutely continuous 2π-periodic vector functions $u : [0, 2\pi] \to \mathbb{R}^N$, namely

$$H = H^1_{per}(0, 2\pi) := \left\{u \in H^1([0, 2\pi], \mathbb{R}^N) : u(0) = u(2\pi)\right\}$$

provided with the usual norm $\|u\| := \|u\|_{H^1}$, and let

$$D = H^2_{per}(0, 2\pi) := \left\{u \in H \cap H^2([0, 2\pi], \mathbb{R}^N) : u'(0) = u'(2\pi)\right\}.$$

The operator $L_m : D \to L^2([0, 2\pi], \mathbb{R}^N)$ is defined as in the introduction, and its kernel V_m may be described as

$$V_m := \mathrm{Ker}(L_m) = \{u_w : w = (\alpha, \beta) \in \mathbb{R}^{2N}\},$$

where

$$u_w(t) := \cos(mt)\alpha + \sin(mt)\beta.$$

For convenience, let $J : \mathbb{R}^{2N} \to V_m$ denote the isomorphism given by $J(w) = u_w$. The m-th Fourier coefficients of a function $u \in L^1([0, 2\pi], \mathbb{R}^N)$ shall be denoted respectively by α_u and β_u, i.e.

$$\alpha_u = \frac{1}{\pi}\int_0^{2\pi} \cos(mt)u(t)\,dt,$$

$$\beta_u = \frac{1}{\pi}\int_0^{2\pi} \sin(mt)u(t)\,dt.$$

Furthermore, if $w(u) = (\alpha_u, \beta_u)$, then the orthogonal projection $\mathcal{P}$ of the space $L^2([0, 2\pi], \mathbb{R}^N)$ onto V_m can be defined as $\mathcal{P}u = J(w(u)) = u_{w(u)}$.

In particular, the projection of p is given by

$$u_{w(p)} = \cos(mt)\alpha_p + \sin(mt)\beta_p.$$

A straightforward computation (or, equivalently, the fact that L_m is symmetric with respect to the inner product of L^2) shows that the range of L_m is the orthogonal complement of V_m, namely:

$$R(L_m) = \left\{\varphi \in L^2([0, 2\pi], \mathbb{R}^N) : \int_0^{2\pi} \cos(mt)\varphi(t)\,dt = \int_0^{2\pi} \sin(mt)\varphi(t)\,dt = 0\right\}.$$

Thus, we may define a right inverse $\mathcal{K} : R(L) \to H$ of the operator L_m, given by $\mathcal{K}\varphi = u$, where $u \in D$ is the unique solution of the problem

$$\begin{cases} u'' + m^2 u = \varphi \\ \mathcal{P}u = 0. \end{cases}$$

Moreover, we have the following standard estimate:

Lemma 2.1 *There exists a constant c such that*

$$\|u - \mathcal{P}u\|_{H^2} \leq c\|L_m(u)\|_{L^2}$$

for each $u \in D$.

Remark 2.2 *From the previous lemma and the embedding $H^2 \hookrightarrow H^1$ it becomes immediate that $\mathcal{K}$ is compact.*

3 Main result

In the sequel, we shall assume that the following condition is satisfied:

Condition 3.1 *There exists an open covering $\{U_j\}_{j=1...,K}$ of the unit sphere $S^{2N-1} \subset \mathbb{R}^{2N}$, and vectors $w_j = (\alpha^j, \beta^j) \in S^{2N-1}$ such that for each $w \in U_j$ the limit*

$$\overline{g}_{w,j}(t) := \limsup_{s\to+\infty} \langle g(su_w(t)), u_{w_j}(t)\rangle$$

is upper semi-continuous in w for almost every t, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^N$.

Theorem 3.2 *Assume that condition 3.1 holds, and that:*

1. *For each $w \in S^{2N-1}$ there exists $j \in \{1, \ldots, K\}$ such that*

$$\frac{1}{\pi}\int_0^{2\pi} \overline{g}_{w,j}(t)\, dt < \langle \alpha_p, \alpha^j\rangle + \langle \beta_p, \beta^j\rangle.$$

2. *For every $R \gg 0$ the Brouwer degree $deg_B(G, B_R(0), 0)$ is different from zero, where $G : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is the mapping defined by*

$$G(w) = w(p) - J^{-1}\mathcal{P}(g \circ u_w).$$

Then problem (1-2) admits at least one solution.

Remark 3.3 *It follows from the above definitions that G may be defined in terms of the m-th Fourier coefficients of the function $\varphi(t) = g(u_w(t))$, namely:*

$$G(w) = (\alpha_p - \alpha_{g\circ u_w}, \beta_p - \beta_{g\circ u_w})$$

Theorem 3.2 has an immediate consequence if we assume that radial limits $g_v = \lim_{s\to+\infty} g(sv)$ exist uniformly for $v \in S^{N-1}$. Indeed, in this case, we may define for each $t \in [0, 2\pi]$ and each $w \in S^{2N-1}$ the limit

$$g_w(t) := \lim_{s\to+\infty} g(su_w(t)). \tag{4}$$

It is worth to notice that $u_w(t)$ might eventually be 0 for finite values of t, in which case $g_w(t) = g(0)$. However, this "singular set" of values of t does not play any role when using the standard Lebesgue convergence theorems for the integral. On the other hand, if $u_w(t) \neq 0$, then $g_w(t)$ is continuous as a function of w: to prove this, it suffices to fix a constant $c > 0$ such that $|u_{\tilde{w}}(t)| \geq c > 0$ for $\tilde{w}$ in a neighborhood W of w. Then, $g(su_{\tilde{w}}(t)) = g(s|u_{\tilde{w}}(t)|\tilde{v}) \to g_{\tilde{v}}$ as $s \to +\infty$ for $\tilde{v} = \frac{u_{\tilde{w}}(t)}{|u_{\tilde{w}}(t)|}$. Given $\varepsilon > 0$, fix s such that $|g(su_{\tilde{w}}(t)) - g_{\tilde{v}}| < \frac{\varepsilon}{3}$ for $\tilde{w} \in W$, then

$$|g_{\tilde{w}}(t) - g_w(t)| = |g_{\tilde{v}} - g_v| < \frac{2\varepsilon}{3} + |g(su_{\tilde{w}}(t)) - g(su_w(t))| < \varepsilon$$

for $\tilde{w}$ close enough to w. Thus, condition 3.1 is clearly satisfied for any family $\{(U_j, w_j)\}_{j=1,\dots,K}$ such that $\{U_j\}$ covers S^{2N-1}. Furthermore, the inequality in condition 1 of Theorem 3.2 is equivalent to:

$$\int_0^{2\pi} \langle g_w(t) - p(t), u_{w_j}(t)\rangle \, dt < 0.$$

Hence, if $g_w - p$ is not orthogonal to the kernel of L_m, that is to say

$$(\alpha_{g_w}, \beta_{g_w}) \neq (\alpha_p, \beta_p), \tag{5}$$

then there exists a vector $w_j \in S^{2N-1}$ such that the previous inequality holds in a neighborhood of w. By compactness, if (5) holds for any $w \in S^{2N-1}$, then condition 1 is satisfied.

In this setting, the previous theorem can be expressed, as Nirenberg's result, in terms of a condition on the extension of g to the infinite sphere or, more precisely, in terms of the m-th Fourier coefficient of this extension. Indeed, for $w \in S^{2N-1}$ we have:

$$\lim_{s\to+\infty} G(sw) = (\alpha_p - \alpha_{g_w}, \beta_p - \beta_{g_w}) \neq 0,$$

and thus the mapping $\theta : S^{2N-1} \to S^{2N-1}$ given by

$$\theta(w) = \frac{(\alpha_p - \alpha_{g_w}, \beta_p - \beta_{g_w})}{|(\alpha_p - \alpha_{g_w}, \beta_p - \beta_{g_w})|}$$

is well defined. Hence we obtain:

Corollary 3.1 *Assume that the radial limits g_v exist uniformly for $v \in S^{N-1}$, and for each $w \in S^{2N-1}$ define the function $g_w(t)$ by (4). Further, assume that:*

1. $(\alpha_{g_w}, \beta_{g_w}) \neq (\alpha_p, \beta_p)$ *for any* w.

2. $deg(\theta) \neq 0$.

Then (1-2) admits at least one solution.

Remark 3.4 *In particular, when* $N = 1$ *for* $w = (\alpha, \beta) \in S^1$ *one has that* $u_w(t) = \cos(mt - \omega)$, *where* $\alpha = \cos(\omega)$ *and* $\beta = \sin(\omega)$. *It follows that*

$$g(su_w(t)) \to \begin{cases} g(+\infty) & \text{if } t \in I_\omega^+ \\ g(-\infty) & \text{if } t \in I_\omega^-, \end{cases}$$

where $I_\omega^+ = \{t \in [0, 2\pi] : \cos(mt - \omega) > 0\}$, $I_\omega^- = \{t \in [0, 2\pi] : \cos(mt - \omega) < 0\}$. *Hence*

$$g_w(t) = g(+\infty)\chi_{I_\omega^+}(t) + g(-\infty)\chi_{I_\omega^-}(t),$$

except for a finite number of values of t. *After computation, it follows that*

$$\int_{I_\omega^+} e^{imt}\, dt = e^{i\omega} \int_{I_\omega^+} e^{i(mt-\omega)}\, dt = e^{i\omega} \int_{I_0^+} e^{imt}\, dt = e^{i\omega} \int_{-\pi/2}^{\pi/2} e^{it}\, dt = 2e^{i\omega},$$

and thus

$$\int_{I_\omega^\pm} \cos(mt)\, dt = \pm 2\cos(\omega) = \pm 2\alpha,$$

$$\int_{I_\omega^\pm} \sin(mt)\, dt = \pm 2\sin(\omega) = \pm 2\beta.$$

Hence

$$\lim_{s \to +\infty} G(sw) = (\alpha_p, \beta_p) - \frac{2}{\pi}(g(+\infty) - g(-\infty))(\alpha, \beta),$$

from which the original result by Lazer and Leach is retrieved.

It is worth noting that our setting allows a natural interpretation of this result in terms of the complex integral. Indeed, from the previous computations it is clear that the degree of the function $\theta : S^1 \to S^1$ *given by* $\lim_{s\to+\infty} \frac{G(sw)}{|G(sw)|}$ *is equivalent to the index of the curve* $\gamma : [0, 2\pi] \to \mathbb{C}$ *defined by* $\gamma(t) = \frac{2}{\pi}(g(+\infty) - g(-\infty))e^{it}$ *at the point* $z_0 = \alpha_p + i\beta_p$. *From (3),* $|z_0| < |\gamma(t)|$, *and it follows that*

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt = \pm 1.$$

4 Proof of the main result

For $\lambda \in [0, 1]$, define the Fredholm operator $F_\lambda : H \to H$ given by $F_\lambda u = u - T_\lambda u$, where the (compact) operator T_λ is defined by:

$$T_\lambda u = \mathcal{P}u + \mathcal{P}(p - g(u)) + \lambda\mathcal{K}\big(p - g(u) - \mathcal{P}(p - g(u))\big).$$

Let us note that, for $\lambda > 0$, u is a zero of F_λ if and only if $u \in D$, and $L_m(u) = \lambda(p - g(u))$.

Indeed, if $F_\lambda u = 0$, it follows that $u = T_\lambda u \in D$, and applying $\mathcal{P}$ on both sides it follows that $\mathcal{P}(p - g(u)) = 0$. Moreover, as $L_m\mathcal{P} = 0$ we deduce that $L_m(u) = \lambda L_m\mathcal{K}\big(p - g(u)\big) = \lambda(p - g(u))$. Conversely, if $L_m(u) = \lambda(p - g(u))$ and $u \in D$, then using the fact that L_m is symmetric for the inner product of L^2, it follows that $\mathcal{P}(p - g(u)) = 0$, and then $u = \mathcal{P}u + \lambda\mathcal{K}(p - g(u)) = T_\lambda u$.

Thus, we search for a zero of F_1. We claim that, for some appropriate $R \gg 0$, the function F_λ does not vanish on $\partial\Omega$, with $\Omega = B_R(0)$. Then the Leray-Schauder degree of F_λ at 0 is defined, and $deg_{LS}(F_1, \Omega, 0) = deg_{LS}(F_0, \Omega, 0)$. Moreover, from the definition of the degree, and the fact that

$$F_0 u = u - \mathcal{P}(u + p - g(u)),$$

it follows that $deg_{LS}(F_0, \Omega, 0) = deg_B(F_0|_{V_m}, \Omega \cap V_m, 0)$. But, for $u \in V_m$ it is clear that $F_0 u = -\mathcal{P}(p - g(u))$. Hence, if we write $u = u_w$ with $w \in \mathbb{R}^{2N}$, it follows that

$$J^{-1}F_0 J(w) = -\big((\alpha_p, \beta_p) - J^{-1}\mathcal{P}(g(u_w))\big) = -G(w).$$

Hence, up to a $(-1)^N$ factor, the degree of F_1 at 0 over Ω can be identified with the Brouwer degree of G at 0 over a large ball of $\mathbb{R}^{2N}$. Thus, after the claim is proved, the result follows from assumption 2.

In order to prove the claim, let us suppose first that $F_{\lambda_n} u^n = 0$ for some $\lambda_n \in (0, 1]$ and $\|u^n\|_H \to \infty$. Then $L_m(u^n) = \lambda_n(p - g(u^n))$, and from Lemma 2.1 we deduce that

$$\|u^n - \mathcal{P}u^n\| \le c\|p - g(u^n)\|_{L^2} \le C$$

for some constant C, and hence $\|\mathcal{P}u^n\|_H \to \infty$.

Writing $\mathcal{P}u^n = \cos(mt)\alpha^n + \sin(mt)\beta^n = u_{w^n}(t)$, with $w^n = (\alpha^n, \beta^n) \to \infty$ in $\mathbb{R}^{2N}$, and taking a subsequence if necessary, we may assume that $\frac{w^n}{|w^n|} \to w \in S^{2N-1}$.

Let $j \in \{1, \ldots, K\}$ be chosen as in condition 1, and let $z^n(t) := \frac{u^n(t)}{|w^n|}$. Then we may write

$$z^n(t) = \frac{u^n(t) - \mathcal{P}u^n(t)}{|w^n|} + \frac{\mathcal{P}u^n(t)}{|w^n|},$$

and using the embedding of $H^1([0, 2\pi], \mathbb{R}^N)$ into $C([0, 2\pi], \mathbb{R}^N)$ and the continuity of $\mathcal{P}$, we conclude that if $n \to \infty$ then $z^n(t) \to u_w(t)$. From the upper semi-continuity of $\overline{g}_{w,j}$ with respect to w, for almost every t we have:

$$\limsup_{n\to\infty}\langle g(u^n(t)), u_{w_j}(t)\rangle = \limsup_{n\to\infty}\langle g(|w^n|z^n(t)), u_{w_j}(t)\rangle \le \overline{g}_{w,j}(t).$$

Moreover, as $L_m(u^n) = \lambda_n(p - g(u^n))$

$$0 = \int_0^{2\pi} \langle L_m(u^n), u_{w_j}\rangle = \lambda_n \int_0^{2\pi} \langle p - g(u^n), u_{w_j}\rangle.$$

This implies that

$$\pi\left(\langle\alpha_p,\alpha^j\rangle+\langle\beta_p,\beta^j\rangle\right)=\limsup_{n\to\infty}\int_0^{2\pi}\langle g(u^n),u_{w_j}\rangle\leq\int_0^{2\pi}\overline{g}_{w,j},$$

a contradiction. On the other hand, if $F_0u^n=0$ for $\|u^n\|_H\to\infty$, then $u^n=u_{w^n}\in V_m$, with $w^n\to\infty$ in $\mathbb{R}^{2N}$. It follows that $0=F_0u^n=-\mathcal{P}(p-g(u^n))$, and proceeding as before a contradiction yields. Thus, the proof is complete.

5 An example: a weakly coupled system

As an application of Theorem 3.2, consider the system

$$u_i''+m^2u_i+\tilde{g}_i(u_i)+h_i(u)=p_i(t)\qquad i=1,\ldots,N,$$

where $\tilde{g}_i$ has limits at infinity, and $h_i(u)\to 0$ uniformly as $|u_i|\to+\infty$. It is worth to notice that, in this case, radial limits of $g=\tilde{g}+h$ do not necessarily exist for those $v\in S^{N-1}$ such that $v_i=0$ for some i, since $h_i(sv)$ does not necessarily converge as $s\to+\infty$.

However, condition 3.1 is satisfied: for $w=(\alpha,\beta)\in S^{2N-1}$, fix i such that the i-th coordinate of α or β is different from 0. Then taking $z\in S^{2N-1}\cap span\{e_i,e_{N+i}\}$, where e_k is the k-th canonical vector of $\mathbb{R}^N$, it follows that

$$\langle g(su_w(t)),u_z(t)\rangle=\cos(mt-\omega)[\tilde{g}_i(su_i)+h_i(su_w(t))],$$

with $u_i=\alpha_i\cos(mt)+\beta_i\sin(mt)=\rho_i\cos(mt-\omega_i)$ for some $\rho_i>0$, and some $\omega_i\in[0,2\pi)$. As in Remark 3.4,

$$\tilde{g}_i(su_i)\to\tilde{g}_i(+\infty)\chi_{I^+_{\omega_i}}(t)+\tilde{g}_i(-\infty)\chi_{I^-_{\omega_i}}(t)$$

as $s\to+\infty$, and we deduce that

$$G(sw)\to(\alpha_p,\beta_p)-\frac{2}{\pi}\sum_{i=1}^N[\tilde{g}_i(+\infty)-\tilde{g}_i(-\infty)](\alpha_ie_i+\beta_ie_{N+i})$$

as $s\to+\infty$. From the product rule of the degree, it is easy to check that if

$$\sqrt{(\alpha_p)_i^2+(\beta_p)_i^2}<\frac{2}{\pi}\left|\tilde{g}_i(+\infty)-\tilde{g}_i(-\infty)\right|$$

for $i=1,\ldots,N$, then condition 2 in Theorem 3.2 is satisfied.

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