On Nirenberg type conditions for higher order systems on time scales

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Abstract

We study the existence of periodic solutions for a nonlinear system of n-th order differential equations on time-scales. Assuming a suitable Nirenberg type condition, we prove the existence of at least one solution of the problem using Mawhin's coincidence degree.

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1 Introduction

In this work, we investigate the existence of solutions $y : [0, \sigma^n(T)]_{\mathbb{T}} \to \mathbb{R}^N$ to the following nonlinear system of n-th order differential equations on time scales

$$y^{\Delta^n} = f(t, y, \dots, y^{\Delta^{n-1}}), \qquad t \in [0, T]_{\mathbb{T}}; \tag{1}$$

under the time-scales periodic conditions:

$$y(0) = y(\sigma^n(T)),\ y^{\Delta}(0) = y^{\Delta}(\sigma^{n-1}(T)),\ \dots,\ y^{\Delta^{n-1}}(0) = y^{\Delta^{n-1}}(\sigma(T)). \tag{2}$$

We shall assume that the nonlinearity $f : [0, T]_{\mathbb{T}} \times \mathbb{R}^{n \cdot N} \to \mathbb{R}^N$ is continuous and bounded with respect to y. However, in contrast with the systems of equations of pendulum type, in this work f will be typically a non-periodic function of y. More precisely, we shall study problem (**??**) under a generalization of the so-called Nirenberg condition for $n = 2$ which, in turn, generalizes the well known Landesman-Lazer conditions for the case $N = 1$.

There exists a vast literature on Landesman-Lazer type conditions for resonant problems, starting at the pioneering work [**?**] for a (scalar) second order elliptic differential equation under Dirichlet conditions (for a survey on conditions of this kind see e.g. [**?**]). In [**?**], Nirenberg extended the Landesman-Lazer conditions to a system of second order elliptic equations. Nirenberg's result can be adapted for a system of periodic ODE's in the following way:

Theorem 1.1 *Let $p \in C([0, T], \mathbb{R}^N)$ and let $g : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and bounded. Further, assume that the radial limits $g_v := \lim_{r \to +\infty} g(rv)$ exist uniformly with respect to $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then the problem*

$$y'' + g(y) = p(t)$$

has at least one T-periodic solution if the following conditions hold:

1. *$g_v \neq \overline{p} := \frac{1}{T} \int_0^T p(t)dt$ for any $v \in S^{N-1}$.*
2. *The degree of the mapping $\theta : S^{N-1} \to S^{N-1}$ given by*

$$\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$$

is non-zero.

In this work, we generalize several aspects of this result. On the one hand, we do not deal with a system of classical ordinary differential equations but, more generally, with a system of dynamical equations on time scales. Let us recall that the concept of time scale, also known as measure chain, was introduced by Hilger in [**?**], with the aim of unifying continuous and discrete calculus. Thus, for a function $y : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T} \subset \mathbb{R}$ is an arbitrary closed set, a general derivative y^{Δ} is defined, in such a way that if $\mathbb{T} = \mathbb{R}$ (continuous case) then y^{Δ} is the usual derivative (i.e. $y^{\Delta} = y'$), and if $\mathbb{T} = \mathbb{Z}$, then the discrete derivative is retrieved, namely $y^\Delta = \Delta y$. For a detailed introduction to the theory of time scales see e.g. [?, ?, ?]. It is worth to remark that, although the field of boundary value problems for dynamic equations in time scales had a rapid growth in the last years, not much literature concerning *resonant* problems is known. A previous work dealing with this kind of situation on a time scale is [?], where a second order multi-point boundary value problem is studied. We may mention also [?], where Landesman-Lazer conditions for a second order periodic problem on time scales are obtained by variational methods.

On the other hand, our system consists of higher order equations, for which some of the standard tools of the theory of second order operators (e.g. maximum and comparison principles) are not applicable.

Finally, the nonlinearity f is more general, since it may also depend on the derivatives of y. In particular, even for the classical case $\mathbb{T} = \mathbb{R}$, this fact implies that the problem has non-variational structure, and motivates the use of topological methods instead: more precisely, we shall apply Mawhin's coincidence degree theory (see e.g. [?]). This powerful tool has been applied to many resonant boundary value problems. An application for a resonant problem on time scales are given in [?]; for periodic conditions, the continuation method has been firstly used in [?], and also in [?].

We shall assume that $f : [0,T]_{\mathbb{T}} \times \mathbb{R}^{n.N} \to \mathbb{R}^N$ is continuous and satisfies the linear growth condition

$$|f(t, y_0, \cdots, y_{n-1})| \leq \varepsilon \sup_{1\leq j\leq n-1} |y_j| + M \tag{3}$$

for some ε to be specified, and some arbitrary constant M. In this situation, our condition concerning the existence of radial limits of the nonlinearity takes the following form:

Condition (F)

For each t the limit

$$\lim_{s\to+\infty} f(t, sv, y_1, \cdots, y_{n-1}) := f_v(t) \tag{4}$$

exists uniformly with respect to $v \in S^{N-1}$ and $|y_j| \leq \frac{CM}{1-C\varepsilon}$ for $j = 1, \ldots, n-1$ where the constant $C > 0$ is defined in Lemma **??** below.

Thus, our main result reads:

Theorem 1.2 *Assume that condition* (**F**) *holds. Then the boundary value problem (**??**)-(**??**) admits at least one solution, provided that*

1. $\overline{f}_v := \frac{1}{\sigma(T)} \int_0^T f_v(t)\, \Delta t \neq 0$ *for any* $v \in S^{N-1}$.

2. *The degree of the mapping* $\theta : S^{N-1} \to S^{N-1}$ *given by*

$$\theta(v) = \frac{\overline{f}_v}{|\overline{f}_v|}$$

is non-zero.

For completeness, let us summarize the main aspects of coincidence degree theory. Let $\mathbb{V}$ and $\mathbb{W}$ be real normed spaces, $L : \mathrm{Dom}(L) \subset \mathbb{V} \to \mathbb{W}$ a linear Fredholm mapping of index 0, and $N : \mathbb{V} \to \mathbb{W}$ continuous. Moreover, set two continuous projectors $\pi_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ and $\pi_{\mathbb{W}} : \mathbb{W} \to \mathbb{W}$ such that $\mathrm{R}(\pi_{\mathbb{V}}) = \mathrm{Ker}(L)$ and $\mathrm{Ker}(\pi_{\mathbb{W}}) = \mathrm{R}(L)$, and an isomorphism $J : \mathrm{R}(\pi_{\mathbb{W}}) \to \mathrm{Ker}(L)$. It is readily seen that

$$L_{\pi_{\mathbb{V}}} := L|_{\mathrm{Dom}(L)\cap\mathrm{Ker}(\pi_{\mathbb{V}})} : \mathrm{Dom}(L) \cap \mathrm{Ker}(\pi_{\mathbb{V}}) \to \mathrm{R}(L)$$

is one-to-one; denote its inverse by $K_{\pi_{\mathbb{V}}}$. If Ω is a bounded open subset of $\mathbb{V}$, N is called $L-$compact on Ω if $\pi_{\mathbb{W}} N(\Omega)$ is bounded and $K_{\pi_{\mathbb{V}}}(I - \pi_{\mathbb{W}})N : \Omega \to \mathbb{V}$ is compact.

The following continuation theorem is due to Mawhin [**?**]:

Theorem 1.3 *Let L be a Fredholm mapping of index zero and N be $L-$compact on a bounded domain $\Omega \subset \mathbb{V}$. Suppose*

1. *$Lx \neq \lambda Nx$ for each $\lambda \in (0, 1]$ and each $x \in \partial\Omega$.*
2. *$\pi_{\mathbb{W}} Nx \neq 0$ for each $x \in \mathrm{Ker}(L) \cap \partial\Omega$.*
3. *$d(J\pi_{\mathbb{W}} N, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, where d denotes the Brouwer degree.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{Dom}(L) \cap \Omega$.

2 Proof of Theorem ??

Set $\mathbb{V} \subset C_{rd}([0, \sigma^n(T)]_{\mathbb{T}}, \mathbb{R}^N)$ be given by

$$\mathbb{V} = \{y : \exists\, y^{\Delta^{n-j}} \in C_{rd}([0, \sigma^j(T)]_{\mathbb{T}}, \mathbb{R}^N) \text{ for } j = 1, \ldots, n-1, \text{ and } y \text{ satisfies } (\textbf{??})\}$$

equipped with the norm

$$\|y\|_{\mathbb{V}} := \sup_{0\le j\le n-1} \|y^{\Delta^j}\|_{C_{rd}([0,\sigma^j(T)]_{\mathbb{T}},\mathbb{R}^N)}.$$

Moreover, let

$$D = \{y \in \mathbb{V} : \exists\, y^{\Delta^n} \in C_{rd}([0, T]_{\mathbb{T}}, \mathbb{R}^N)\},$$
$$\mathbb{W} = C_{rd}([0, T]_{\mathbb{T}}, \mathbb{R}^N),$$

and define the operators $L : D \to \mathbb{W}$, $N : \mathbb{V} \to \mathbb{W}$ given by

$$Ly = y^{\Delta^n}, \qquad Ny = f(\cdot, y, \cdots, y^{\Delta^{n-1}}).$$

A simple computation shows that $\mathrm{Ker}(L) = \mathbb{R}^N$, and

$$\mathrm{R}(L) = \{\varphi \in \mathbb{W} : \int_0^{\sigma(T)} \varphi(t)\Delta t = 0\}.$$

Thus, we may define the projectors

$$\pi_{\mathbb{V}}(y) = y(0), \qquad \pi_{\mathbb{W}}(\varphi) = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} \varphi(t)\Delta t,$$

and consider $J : \mathrm{R}(\pi_{\mathbb{W}}) \to \mathrm{Ker}(L)$ as the identity of $\mathbb{R}^N$.

It is immediate to prove that N is continuous; furthermore, if $\varphi \in \mathrm{R}(L)$, then $K_{\pi_{\mathbb{V}}}(\varphi)$ is the unique solution $y \in D$ of the problem $y^{\Delta^n} = \varphi$ satisfying $y(0) = 0$.

Remark 2.1 *The inverse operator $K_{\pi_{\mathbb{V}}}$ may be established in a more precise way. Indeed, if $y \in \mathbb{V}$ satisfies $y^{\Delta^n} = \varphi$, then*

$$y^{\Delta^{n-1}}(t) = c_1 + \int_0^t \varphi(s)\Delta s := c_1 + I(\varphi)(t)$$

where the constant c_1 is uniquely determined by the boundary condition $y^{\Delta^{n-2}}(0) = y^{\Delta^{n-2}}(\sigma^2(T))$; namely

$$c_1 = -\frac{1}{\sigma^2(T)} \int_0^{\sigma^2(T)} I(\varphi)(s)\Delta s.$$

Inductively, it follows that

$$y(t) = P(t) + I^n(\varphi)(t),$$

where P is a generalized polynomial of order $n-1$ (i.e. an n-th order antiderivative of 0), and the coefficients of P are uniquely determined by the successive integrals of φ.

The proof of the following lemma is immediate from the previous remark:

Lemma 2.2 *There exists a constant C such that*

$$\|K_{\pi_{\mathbb{V}}}(\varphi)\|_{C^{n-1}_{rd}} \leq C\|\varphi\|_{\mathbb{W}}$$

for any $\varphi \in \mathrm{R}(L)$.

If y belongs to a bounded set $\Omega \subset \mathbb{V}$, then $\varphi = (I - \pi_{\mathbb{W}})Ny$ is bounded, and from Arzelá theorem and the previous lemma we deduce that $K_{\pi_{\mathbb{V}}}(I - \pi_{\mathbb{W}})N$ is compact. Thus, the L-compactness of N follows.

We claim that the solutions $y \in D$ of the equation $Ly = \lambda Ny$ with $0 < \lambda \leq 1$ are a priori bounded for the $\mathbb{V}$-norm. Indeed, otherwise there exists a sequence $\{y_k\} \subset D$ such that

$$y_k^{\Delta^n} = \lambda_k f(t, y_k, \cdots, y_k^{\Delta^{n-1}})$$

with $0 < \lambda_k \leq 1$ and $\|y_k\|_{\mathbb{V}} \to \infty$. Writing

$$y_k(t) = y_k(0) + K(\lambda_k N y_k)$$

it follows that

$$\|y_k - y_k(0)\|_{C_{rd}^{n-1}} \leq C\|\lambda_k f(t, y_k, \cdots, y_k^{\Delta^{n-1}})\|_{\mathbb{W}} \leq C\varepsilon \sup_{1\leq j\leq n-1} \|y_k^{\Delta^j}\|_{C_{rd}} + CM.$$

Thus, if $\varepsilon < \min\{\frac{1}{C}, 1\}$, it follows that

$$\|y_k - y_k(0)\|_{C_{rd}^{n-1}} \leq \frac{CM}{1 - C\varepsilon}.$$

Then

$$|y_k^{\Delta^j}(t)| \leq \frac{CM}{1 - C\varepsilon} \qquad \text{for } j = 1, \ldots n-1,\ 0 \leq t \leq \sigma^j(T),$$

and it follows that $|y_k(0)| \to \infty$. Taking a subsequence, we may assume that $\frac{y_k(0)}{|y_k(0)|} \to u$ for some $u \in S^{N-1}$, whence $z_k(t) := \frac{y_k(t)}{\|y_k\|_{\mathbb{V}}}$ also converges to u. Integrating the equation, we obtain that

$$0 = \int_0^{\sigma(T)} y^{\Delta^n}(t)\ \Delta t = \lambda_k \int_0^{\sigma(T)} f(t, y_k, \cdots, y_k^{\Delta^{n-1}})\ \Delta t.$$

Thus, writing $y_k = \|y_k\|_{\mathbb{V}}.z_k$, and using the dominated convergence theorem for the time scales integral (see [**?**]), we deduce from condition **(F)** that

$$\int_0^{\sigma(T)} f_u(t)\ \Delta t = 0,$$

a contradiction.

Thus, the first condition in Theorem **??** is fulfilled taking $\Omega = B_R(0)$ with R large enough.

Furthermore, if $y \in \mathrm{Ker}(L) \cap \partial\Omega$, then

$$\pi_{\mathbb{W}} N y = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} f(t, y, 0, \cdots, 0)\ \Delta t,$$

and by the degree condition **??** it is easy to verify that the second and the third conditions of Theorem **??** are fulfilled.

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