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Free Transmission Problems

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We study transmission problems with free interfaces from one random medium to another. Solutions are required to solve distinct partial differential equations, L_+ and L_-, within their positive and negative sets respectively. A corresponding flux balance from one phase to another is also imposed. We establish existence and L^∞ bounds of solutions. We also prove that variational solutions are non-degenerate and develop the regularity theory for solutions of such free boundary problems.

Keywords: Transmission problems, elliptic equations, smoothness properties of solutions, free boundary theory.

1 Contributed Talk

I would very glad if I participate presentinfg contributed talk. I intend to dissert about "Free Transmission Problems". This was the content of my PhD the thesis wich was awarded honor 2015 Carlos Gutierrez Prize by the Brazilian Mathematical Society. This work was publishd in the reputed jornal "Communications in Mathematical Physics". The work concerns about a two phase free boundary problem with measurable medias and measurable nonhomogeneous terms in each equation of each phase. The behavior of the solution across the free transmission boundary was proved to be smooth, more precisyly assimptoptically Lipschitz continous.

Let me give a brief description of my Ph.D. thesis. In this thesis I work in the framework of a Lipschitz domain Ω and given functions of symmetric matrices $A_+(x)$, $A_-(x)$ satisfying the ellipticity condition

$$\lambda|\xi|^2 \leq \langle A_\pm(x)\,\xi, \xi\rangle \leq \Lambda|\xi|^2,$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. We study minimizers of the energy functional

$$\begin{aligned} \mathcal{F}_{A_\pm,\lambda_\pm,f_\pm}(v) \quad := & \int\limits_{\Omega\cap\{v>0\}} \left\{\frac{1}{2}\langle A_+(x)\nabla v(x), \nabla v(x)\rangle + \lambda_+(x) + f_+ v\right\} dx \\ + & \int\limits_{\Omega\cap\{v\leq 0\}} \left\{\frac{1}{2}\langle A_-(x)\nabla v(x), \nabla v(x)\rangle + \lambda_-(x) + f_- v\right\} dx, \end{aligned}$$

where $0 < \lambda \leq \Lambda$ are fixed constants, $\gamma\colon \Omega \to \mathbb{R}$ is a continuous, strictly positive function and $0 < \lambda_- < \lambda_+ < \infty$ are different constants. Finally we assume the sources functions $f_+, f_- \in L^q(\Omega)$, with $q > n/2$.

It must be observed that the energy functional $\mathcal{F}_{A_\pm,\lambda_\pm,f_\pm}$ described above can also be viewed as a generalization of the Alt-Caffarelli-Friedman's work (1984) following the famous 1981 paper by Alt and Caffarelli on the one-phase version of the problem. The Alt-Caffarelli-Friedman work corresponds to $A_+ = A_- = \mathrm{Id}$, $f_+ = f_- = 0$. A physical model related to the above is a cube of ice that melts in a heated inhomogeneous medium.

We prove that local minima of $\mathcal{F}$ satisfy in a weak sense

$$\nabla \cdot (A_+(x)\nabla u(x)) = f_+(x) \text{ in } \{u > 0\}$$

and

$$\nabla \cdot (A_-(x)\nabla u(x)) = f_-(x) \text{ in } int\{u \leq 0\}.$$

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Further, we obtain existence, uniform boundedness, geometric non degeneracy properties, and universal Hölder regularity of solutions of this problem. The $C^{0,\alpha}_{\text{loc}}$, $\forall\alpha < 1$ regularity result is optimal, since even when $A_+ = A_- = A(x)$ is continuous, the solutions to the homogeneous equation

$$\text{div}(A(x)\nabla u) = 0 \tag{1.1}$$

may fail to be Lipschitz continuous. Notwithstanding, we deliver an asymptotic optimal $C^{0,1-}$ regularity estimate for minima of $\mathcal{F}_{A,f_+,f_-}$ for $f_-, f_+ \in L^q$, $q \geq n$. This is a Cordes-Nirenberg type estimate, that is, given $\alpha \in (0,1)$ we can obtain interior $C^{0,\alpha}$ regularity, provided that the matrix A does not oscillate much in relation to a constant matrix.

An important observation is that in the case $A_+ \neq A_-$ (even if A_+ and A_- are constant), one cannot establish an Alt-Caffarelli type monotonicity formula for the homogeneous $\mathcal{F}_{A_+,A_-,0,0}$ functional. As a consequence, even in the simplest heterogeneous scenario, Lipschitz estimates may not be valid for local minima. In that perspective, the asymptotic $C^{0,1-}$ estimate proven, under the assumption that $\|A_+ - A_-\|_{L^2}$ is small, is of optimal nature. The proximity assumption on A_+ e A_- is in agreement with the physical intuition that the more the fluid and the ice mix, the more the medium tends to be close to homogeneous.

2 Main Results

The DeGiorge-Nash-Moser theorem for this free boudary theory.

Theorem 2.1 (Universal Hölder regularity)**.** *Let u be a minimum of the problem* (1.1)*, with $f \in L^q(\Omega), q > n/2$. Then, $u \in C^\tau_{(\Omega)}$, for some $0 < \tau = \tau(\lambda, \Lambda, q) < 1$. Furthermore, for any $\Omega' \Subset \Omega$, there exists a constant K depending only Ω' and the data of the problem such that*

$$\|u\|_{C^\tau(\Omega')} \leq K.$$

A Cordes-Niremberg type estimates. Assymptotically sharp regularity.

Theorem 2.2 (Improved Hölder regularity)**.** *Under the conditions above conditions, given any $\alpha \in (0,1)$ and $\Omega' \Subset \Omega$, there exists an $\epsilon > 0$, depending only upon n, λ, Λ, ω, Ω, Ω' and α, such that if*

$$\|A_+ - A_-\|_{L^2(\Omega')} \leq \epsilon \tag{2.2}$$

then $u \in C^{0,\alpha}(\Omega')$.

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